Bernoulli **19**(5A), 2013, 1965–1999
DOI: 10.3150/12-BEJ437

Testing monotonicity of a hazard: Asymptotic distribution theory

PIET GROENEBOOM[*,**] and GEURT JONGBLOED[†,‡]

Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands. E-mail: [*]P.Groeneboom@tudelft.nl; [†]G.Jongbloed@tudelft.nl, url: [**]http://dutiosc.twi.tudelft.nl/~pietg/; [‡]http://dutiosc.twi.tudelft.nl/~geurt/.

Two test statistics are introduced to test the null hypotheses that the sampling distribution has an increasing hazard rate on a specified interval $[0, a]$. These statistics are empirical L_1-type distances between the isotonic estimates, which use the monotonicity constraint, and either the empirical distribution function or the empirical cumulative hazard. They measure the excursions of the empirical estimates with respect to the isotonic estimates, owing to local non-monotonicity. Asymptotic normality of the test statistics, if the hazard is strictly increasing on $[0, a]$, is established under mild conditions. This is done by first approximating the global empirical distance by a distance with respect to the underlying distribution function. The resulting integral is treated as sum of increasingly many local integrals to which a central limit theorem can be applied. The behavior of the local integrals is determined by a canonical process, the difference between the stochastic process $x \mapsto W(x) + x^2$, where W is standard two-sided Brownian motion, and its greatest convex minorant.

Keywords: convex minorant; failure rate; global asymptotics; Hungarian embedding

1. Introduction

One way of characterizing a distribution of an absolutely continuous random variable X that is particularly useful in reliability theory and survival analysis is by its hazard rate h_0. Suppose that X models the failure time of a certain device. The interpretation of the hazard rate is that for small $\varepsilon > 0$, $\varepsilon h_0(x)$ reflects the probability of failure of the device in the time interval $(x, x + \varepsilon]$ given that the device was still unimpaired at time x (assuming that h_0 is continuous at x). Put differently, $h_0(x)$ represents the level of instantaneous risk of failure of the device at time x, given that it still works at time x. A high value reflects high risk, and a low value reflects low risk. Lifetimes of devices that are subject to aging can be described by distributions with increasing hazard rate. Locally decreasing hazard rates can be used to model lifetimes of devices that become more reliable with age during a certain time period.

It is especially this clear interpretation of these qualitative properties of a hazard rate that makes this function a natural characteristic of a survival distribution. The problem of estimating a hazard rate nonparametrically under qualitative (or shape) restrictions gained attention in the 1960s (see the overview in [7,19] and [21]). Also the problem of testing the null hypothesis of constant hazard (exponentiality) against monotonicity of the hazard was studied intensively (see, e.g., [17]). Only quite recently was another "shape-constrained" rather than parametric null hypothesis studied, namely the null hypothesis that the underlying hazard rate is increasing against the alternative that it is not. [3] studied local versions of the test statistic of [17] to test this hypothesis, and [11] introduced and studied a test based on the "biased bootstrap" concept. [2] used the supremum distance between two estimators of the cumulative hazard rate as the test statistic.

In this paper, we consider two integral-type test statistics for the hypothesis that a hazard rate, h_0, is monotone on an interval, $[0,a]$, for some known $a > 0$. We restrict ourselves to the increasing case; the case of locally decreasing hazard can be considered analogously. Experiments conducted by [8] indicated that testing the hypothesis based on our test statistic, using a bootstrap procedure to determine critical values outperforms the test proposed by [11] and [2] for different reasons. The test proposed by [2] is quite conservative, and the test proposed by [11] is anticonservative (also explaining in part its high power). (For more details, see [8].) In this paper, we focus on the asymptotic distribution theory for the test statistics, especially under the assumption that h_0 is strictly increasing on $[0,a]$.

We now introduce our test statistics. Based on an i.i.d. sample $X_1,\dots,X_n$ from the distribution associated with H_0, the most natural nonparametric estimator for H_0, without assuming anything on H_0, is the empirical cumulative hazard function given by

$$\mathbb{H}_n(x) = \begin{cases} -\log\bigl\{1-\mathbb{F}_n(x)\bigr\}, & x \in [0, X_{(n)}), \\ \infty, & x \ge X_{(n)}, \end{cases}$$

where $\mathbb{F}_n$ denotes the empirical distribution function based on $X_1, X_2, \dots, X_n$. Under the assumption that H_0 is convex on $[0,a]$, the cumulative hazard can be estimated by the greatest convex minorant, $\hat{H}_n$, of the empirical cumulative hazard function $\mathbb{H}_n$ on the interval $[0,a]$. Using these two estimators, the following test statistic emerges:

$$T_n = \int_{[0,a]} \bigl\{\mathbb{H}_n(x-) - \hat{H}_n(x)\bigr\}\,\mathrm{d}\mathbb{F}_n(x). \tag{1.1}$$

Note that this is the empirical L_1 distance between the two aforementioned estimators for the cumulative hazard function with resp[ect to the empirical measure $\mathrm{d}\mathbb{F}_n$, and that $T_n \ge 0$, because $\hat{H}_n$ is a minorant of $\mathbb{H}_n$. If H_0 is concave on $[0,a]$, then both estimators for H_0 will be close to H_0 and T_n will tend to be small (converge to zero a.s. for $n \to \infty$). In contrast, if h_0 has a region in $[0,a]$ in which it is not increasing, then $\mathbb{H}_n$ will capture this "non-convexity" of H_0 and converge to H_0 on this region, whereas $\hat{H}_n$ will converge to the convex minorant of H_0 on $[0,a]$. Note that $T_n = 0$ if and only if $\hat{H}_n$ coincides with the linear interpolation of the points $(x_{(i)}, \mathbb{H}_n(x_{(i)}-))$ on the range of the data falling in $[0,a]$. One could say that $T_n = 0$ if $\mathbb{H}_n$ is "as convex as it can be on $[0,a]$," being an increasing right- continuous step function. This is the reason for taking $\mathbb{H}_n(x-)$ instead of $\mathbb{H}_n(x)$ in (1.1). Similar reasoning can be followed for another test statistic,

$$U_n = \int_{[0,a)} \bigl\{\mathbb{F}_n(x-) - \hat{F}_n(x)\bigr\}\,\mathrm{d}\mathbb{F}_n(x) \qquad \text{where } \hat{F}_n(x) = 1 - \exp\bigl(-\hat{H}_n(x)\bigr). \tag{1.2}$$

An advantage of this statistic compared with T_n is that U_n is less sensitive to possible problems that can occur with large values of $\mathbb{H}_n$.

The main result of this paper concerns the asymptotic distribution of T_n and U_n. Suppose that h_0 satisfies:

Condition 1. *h_0 is strictly positive on $[0,a]$, with a strictly positive continuous derivative h_0' on $(0,a)$, which also has a strictly positive right limit at 0 and a strictly positive left limit at a.*

Then

$$n^{5/6}\{T_n - ED_n\} \xrightarrow{\mathcal{D}} N\big(0,\sigma^2_{H_0}\big) \quad \text{and} \quad n^{5/6}\{U_n - EU_n\} \xrightarrow{\mathcal{D}} N\big(0,\sigma^2_{F_0}\big), \tag{1.3}$$

where D_n is a modified version of T_n; see Theorems 5.1 and 5.2. Here $\sigma^2_{H_0}$ and $\sigma^2_{F_0}$ are constants depending on f_0. Results of a similar flavor were established by, for example, [15] for the difference between the empirical distribution function and its concave majorant. Note that the asymptotic variances of the test statistics depend on the unknown underlying hazard rate. Thus, the limit results cannot be applied immediately to compute approximate critical values. As mentioned earlier, [8] proposed a bootstrap procedure to approximate critical values for the test statistics. First, the underlying hazard rate h_0 is estimated under the null hypothesis. Then samples from the corresponding distribution are drawn and bootstrap realizations of the test statistic obtained. These latter realizations can be used to approximate appropriate critical values. In fact, the proof of [8] that shows that the bootstrap works is based on the limit results obtained in this paper.

The basic idea of the proof of the limit theorems is to approximate the integral in the test statistic by a sum of increasingly many local integrals, using the crucial localization Lemma 3.4, and then apply a central limit theorem to the components that arise in this way. The behavior of the local integrals is determined by a canonical process, the difference between a Brownian motion with parabolic drift and its convex minorant. Relevant properties of this process are derived in Section 2. In Section 3, a statistic related to T_n (where the integral is taken with respect to F_0 rather than $\mathbb{F}_n$) is closely approximated by an integral involving the independent increments of Brownian motion. Moreover, the resulting integral is represented as a sum of local integrals using a "big blocks separated by small blocks" construction as introduced by [20]. The local integrals over the big blocks reduce to the processes considered in Section 2. Finally, because the local integrals are based on the independent increments of a Brownian motion process, a central limit theorem can be applied to obtain the first result in the spirit of (1.3), but still with integrating measure $\mathrm{d}F_0$ rather than $\mathrm{d}\mathbb{F}_n$. The asymptotic distribution of the statistic related to U_n (with $\mathrm{d}F_0$ instead of $\mathrm{d}\mathbb{F}_n$) is derived in Section 4. In Section 5, the main results of the paper are established by showing that the differences between the integrals with respect to $\mathrm{d}\mathbb{F}_n$ and $\mathrm{d}F_0$ are sufficiently small to pass on the asymptotic normality results obtained in Sections 3 and 4 to the original test statistics. Moreover, a result is proved for the case where the underlying hazard rate is constant on $[0,a]$, showing that this leads to completely different asymptotics. This also explains the aforementioned conservative behavior of the tests considered by [2], in which critical values are obtained using the exponential distribution (with flat hazard rate).

2. Local asymptotic process and its integral

A key factor in proving (1.3) is that the 'global integral' can be approximated by a sum of increasingly many 'local integrals.' By normalization, these local integrals are all related to the

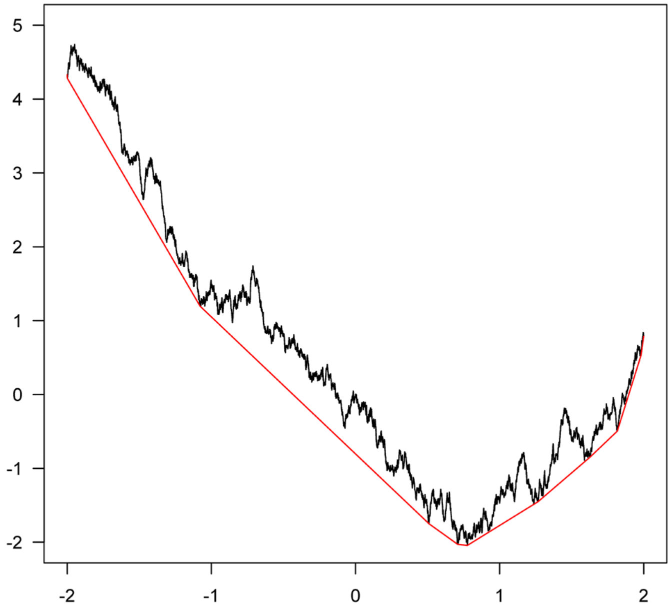


Figure 1. The greatest convex minorant of $W(x) + x^2$, restricted to $[-2, 2]$.

integral of a canonical local asymptotic process. Consider the process

$$x \mapsto V(x) = W(x) + x^2, \qquad x \in \mathbb{R}, \tag{2.1}$$

with W standard two-sided Brownian motion on $\mathbb{R}$, and let C be the greatest convex minorant of V on $\mathbb{R}$. Then, for $c > 0$, define the functional Q_c as the integral of the 'canonical process' $x \mapsto V(x) - C(x)$ over the interval $[0, c]$:

$$Q_c = \int_0^c \{V(x) - C(x)\}\,\mathrm{d}x. \tag{2.2}$$

A picture of the process V and its greatest convex minorant, restricted to the interval $[-2, 2]$ is provided in Figure 1.

In this section, we first derive asymptotic properties of Q_c for $c \to \infty$ in Theorem 2.1. In Theorem 2.2, we show that changing the integration bounds in the definition of Q_c in a specific way and changing the definition of V slightly does not essentially affect the asymptotic properties of Q_c. Because we use the asymptotic result later in the paper in conjunction with a local rescaling argument, we also prove a slightly more general result allowing for this in Theorem 2.3.

The basic result of this section is as follows:

Theorem 2.1.

$$c^{-1/2}\{Q_c - cE|C(0)|\} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \qquad c \to \infty,$$

where $C(0)$ is the value of the greatest convex minorant C of the process V at 0 and

$$\sigma^2 = 2\int_0^\infty \operatorname{covar}\bigl(-C(0), V(x) - C(x)\bigr)\,\mathrm{d}x. \tag{2.3}$$

All moments of $c^{-1/2}\{Q_c - cE|C(0)|\}$ exist, and in particular, the fourth moment is uniformly bounded in c and converges to the fourth moment of the normal $N(0,\sigma^2)$ distribution as $c \to \infty$.

In the proof we use the following lemma, which is proved in the Appendix.

Lemma 2.1.

(i) *For the process V defined in (2.1), there exist positive constants c and c' such that for all $u \ge 0$,*

$$P\Bigl(\min_{x\notin[-u,u]} V(x) \le 0\Bigr) \le c\mathrm{e}^{-c'u^3}.$$

(ii) *Let $\tau(a)$ be defined by:*

$$\tau(a) = \arg\min_{x\in\mathbb{R}}\bigl\{W(x) + (x-a)^2\bigr\}.$$

For each fixed a, $\tau(a)$ is almost surely unique, and the process $a \mapsto \tau(a), a \in \mathbb{R}$ is stationary. Moreover, there exist constants $c_1, c_2 > 0$, such that for events A and B satisfying

$$A \in \sigma\bigl\{\tau(a) : a \le 0\bigr\} \quad \text{and} \quad B \in \sigma\bigl\{\tau(a) : a \ge m\bigr\},$$

we have that

$$\bigl|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\bigr| \le c_1 e^{-c_2 m^3}. \tag{2.4}$$

(iii) *The process*

$$V(x) - C(x), \qquad x \in \mathbb{R},$$

is stationary and that there exist constants $c_1, c_2 > 0$, such that for events A and B satisfying

$$A \in \sigma\bigl\{V(x) - C(x) : x \le 0\bigr\} \quad \text{and} \quad B \in \sigma\bigl\{V(x) - C(x) : x \ge m\bigr\},$$

we have

$$\bigl|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\bigr| \le c_1 e^{-c_2 m^3}. \tag{2.5}$$

Proof of Theorem 2.1. By part (iii) of Lemma 2.1, the process

$$V(x) - C(x), \qquad x \in \mathbb{R}, \tag{2.6}$$

is stationary. In fact, the process touches 0 at changes in the slope of C and behaves between these touches of 0 as an excursion of a Brownian motion path above a parabola of the form

$$\phi(x) = s - (x-a)^2, \qquad x \in \mathbb{R},$$

where ϕ is a parabola touching two local minima of Brownian motion, and where the (random) values a and s depend on the Brownian motion path. Defining

$$D_k = \int_k^{k+1} \{V(x) - C(x)\}\,\mathrm{d}x, \qquad k \in \mathbb{Z},$$

we get a stationary sequence of random variables. Fubini and the stationarity of (2.6) give

$$ED_k = \int_k^{k+1} E\{V(x) - C(x)\}\,\mathrm{d}x = \int_k^{k+1} E\{V(0) - C(0)\}\,\mathrm{d}x = E|C(0)|,$$

where we also use that $V(0) = 0$. Moreover, all moments of D_k exist. This follows from the fact that $\max_{x\in[0,1]}\{V(x) - C(x)\}$ has a distribution with tails that die out faster than exponentially. To see this, note that $\forall u \geq 0$,

$$\begin{aligned}
\mathbb{P}\Big\{\max_{x\in[0,1]}\{V(x) - C(x)\} \geq M\Big\} &\leq \mathbb{P}\Big\{\max_{x\in[0,1]} V(x) \geq \frac{1}{2}M\Big\} + \mathbb{P}\Big\{\min_{x\in\mathbb{R}} C(x) \leq -\frac{1}{2}M\Big\} \\
&\leq \mathbb{P}\Big\{\max_{x\in[0,1]} W(x) \geq \frac{1}{2}M - 1\Big\} + \mathbb{P}\Big\{\min_{x\in\mathbb{R}} V(x) \leq -\frac{1}{2}M\Big\} \\
&\leq \sqrt{\frac{2}{\pi}}\int_{M/2-1}^{\infty} e^{-x^2/2}\,\mathrm{d}x + \mathbb{P}\Big\{\min_{x\in[-u,u]} W(x) \leq -\frac{1}{2}M\Big\} \\
&\quad + \mathbb{P}\Big\{\min_{x\notin[-u,u]} V(x) \leq 0\Big\}.
\end{aligned} \tag{2.7}$$

The first term on the right-hand side is bounded by $c\exp\{-c'M^2/4\}$ for some $c, c' > 0$. By part (i) of Lemma 2.1, for the third term we have

$$P\Big(\min_{x\notin[-u,u]} V(x) \leq 0\Big) \leq ce^{-c'u^3}$$

for constants $c, c' > 0$. For the second term in (2.7), by Brownian scaling, we get

$$\begin{aligned}
\mathbb{P}\Big\{\min_{x\in[-u,u]} W(x) \leq -M/2\Big\} &= \mathbb{P}\Big\{\min_{x\in[-1,1]} W(ux) \leq -M/2\Big\} \\
&= \mathbb{P}\Big\{\min_{x\in[-1,1]} u^{-1/2}W(ux) \leq -u^{-1/2}M/2\Big\} \\
&= \mathbb{P}\Big\{\max_{x\in[-1,1]} W(x) \geq u^{-1/2}M/2\Big\} \\
&\leq 2\sqrt{\frac{2}{\pi}}\int_{u^{-1/2}M/2}^{\infty} e^{-x^2/2}\,\mathrm{d}x \leq 2\sqrt{\frac{2}{\pi}}\frac{2\sqrt{u}}{M}\exp\Big\{-\frac{1}{8}M^2/u\Big\}.
\end{aligned}$$

Thus, taking $u = \sqrt{M}$ in the second and third terms in (2.7) and observing that the first term is of lower order, we obtain

$$\mathbb{P}\Big\{\max_{x\in[0,1]}\big\{V(x) - C(x)\big\} \geq M\Big\} \leq c_1 e^{-c_2 M^{3/2}} \tag{2.8}$$

for constants $c_1, c_2 > 0$.

By part (iii) of Lemma 2.1, the process $x \mapsto V(x) - C(x)$ is strongly mixing. Thus, we can apply Theorem 18.5.3 of [13], page 347, yielding that

$$c^{-1/2}\big\{Q_c - cE\big|C(0)\big|\big\} \xrightarrow{\mathcal{D}} N\big(0, \sigma^2\big) \qquad \text{where } \sigma^2 = \mathrm{var}(D_0) + 2\sum_{k=1}^{\infty} \mathrm{covar}(D_0, D_k).$$

Using the stationarity of the process (2.6) again, we obtain (2.3). The last statement of the theorem follows from (2.8) and (2.5). □

To successfully apply the "big blocks separated by small blocks" method in the sequel, the following extension of Theorem 2.1 is needed. It shows that the theorem essentially goes through if the convex minorant is taken over a finite interval and the region of integration is altered slightly.

Theorem 2.2. *Let C_c be the greatest convex minorant on $[0, c]$ of the process V defined by (2.1). Note that C_c is not the restriction of C to $[0, c]$, because C is defined globally on $\mathbb{R}$ and C_c is the greatest convex minorant of the process V on $[0, c]$ and is defined only on $[0, c]$.*

(i) *For $c > 4$, let the interval I_c be defined by $I_c = [\sqrt{c}, c - \sqrt{c}]$. Then, for σ^2 as defined in Theorem 2.1,*

$$\begin{aligned} &c^{-1/2}\bigg\{\int_{I_c}\big\{V(x) - C_c(x)\big\}\,\mathrm{d}x - E\int_{I_c}\big\{V(x) - C_c(x)\big\}\,\mathrm{d}x\bigg\} \\ &\quad\xrightarrow{\mathcal{D}} N\big(0, \sigma^2\big), \qquad c \to \infty. \end{aligned} \tag{2.9}$$

(ii) *Relation (2.9) also holds if the interval I_c is given by $I_c = [0, c - \sqrt{c}]$ or $I_c = [\sqrt{c}, c]$.*

(iii) *For any choice of I_c in* (i) *or* (ii), *the fourth moment of*

$$c^{-1/2}\bigg\{\int_{I_c}\big\{V(x) - C_c(x)\big\}\,\mathrm{d}x - E\int_{I_c}\big\{V(x) - C_c(x)\big\}\,\mathrm{d}x\bigg\}$$

is uniformly bounded in c, and converges to the fourth moment of a normal $N(0, \sigma^2)$ distribution, as $c \to \infty$.

Proof. (i) The probability that C_c differs from C on the interval I_c is less than or equal to $k_1 \exp\{-k_2 c^{3/2}\}$ for constants $k_1, k_2 > 0$. The proof of this is analogous to the proof of Lemma 3.4 in the next section. Thus, if K_c denotes the event that $C_c \not\equiv C$ on I_c, then we get, using stationarity of $x \mapsto (V(x) - C(x))^2$ and the fact that $EC(0)^2 < \infty$,

$$E\int_{I_c} \left|C_c(x) - C(x)\right| \mathrm{d}x \le \left\{\int_{I_c} E\left\{V(x) - C(x)\right\}^2 \mathrm{d}x\right\}^{1/2} \mathbb{P}(K_c)^{1/2}$$
$$\le \left\{|I_c| EC(0)^2\right\}^{1/2} k_1 \exp\left\{-k_2 c^{3/2}\right\}$$
$$= O\left(c^{1/2} e^{-kc^{3/2}}\right), \qquad c \to \infty,$$

for some $k > 0$. Thus,

$$c^{-1/2}\left\{\int_{I_c} \left\{V(x) - C_c(x)\right\} \mathrm{d}x - E\int_{I_c} \left\{V(x) - C_c(x)\right\} \mathrm{d}x\right\}$$
$$= c^{-1/2}\left\{\int_{I_c} \left\{V(x) - C(x)\right\} \mathrm{d}x - E\int_{I_c} \left\{V(x) - C(x)\right\} \mathrm{d}x\right\}$$
$$+ O_p\left(ce^{-kc^{3/2}}\right),$$

and the statement now follows.

(ii) We can repeat the argument on the interval $[0, \sqrt{c}]$, and apply the argument used in (i) on the subinterval $I_c' = [c^{1/4}, \sqrt{c} - c^{1/4}]$ (but leaving C_c as defined in (i)). This yields

$$c^{-1/4}\left\{\int_{I_c'} \left\{V(x) - C_c(x)\right\} \mathrm{d}x - E\int_{I_c'} \left\{V(x) - C_c(x)\right\} \mathrm{d}x\right\} \xrightarrow{\mathcal{D}} N\left(0, \sigma^2\right), \qquad c \to \infty,$$

implying that

$$c^{-1/2}\left\{\int_{I_c'} \left\{V(x) - C_c(x)\right\} \mathrm{d}x - E\int_{I_c'} \left\{V(x) - C_c(x)\right\} \mathrm{d}x\right\} \xrightarrow{p} 0, \qquad c \to \infty.$$

Moreover,

$$c^{-1/2}\int_{[0,c^{1/4}]} E\left|V(x) - C_c(x)\right| \mathrm{d}x = O\left(c^{-1/4}\right), \qquad c \to \infty.$$

The statement now follows for the first choice of the interval I_c in (ii). For the second choice of I_c, the argument is similar.

(iii) Let I_c be as in (i). Then

$$c^{-2}E\left\{\int_{I_c} \left\{V(x) - C_c(x)\right\} \mathrm{d}x - E\int_{I_c} \left\{V(x) - C_c(x)\right\} \mathrm{d}x\right\}^4$$
$$= c^{-2}E\left\{\int_{I_c} \left\{V(x) - C(x)\right\} \mathrm{d}x - E\int_{I_c} \left\{V(x) - C(x)\right\} \mathrm{d}x\right\}^4$$
$$+ O\left(e^{-kc^{3/2}}\right)$$

for some $k > 0$, and the statement now follows from Theorem 2.1, (2.8), and (2.5), along with the fact that $c^2(c - 2\sqrt{c})^{-2} \to 1$ as $c \to \infty$. If, for example, $I_c = [0, c - \sqrt{c}]$, then we write

$$\int_{I_c} \{V(x) - C_c(x)\} \, dx - E \int_{I_c} \{V(x) - C_c(x)\} \, dx = A_c + B_c,$$

where

$$A_c = \int_{[0,\sqrt{c}]} \{V(x) - C_c(x)\} \, dx - E \int_{[0,\sqrt{c}]} \{V(x) - C_c(x)\} \, dx$$

and

$$B_c = \int_{[\sqrt{c}, c-\sqrt{c}]} \{V(x) - C_c(x)\} \, dx - E \int_{[\sqrt{c}, c-\sqrt{c}]} \{V(x) - C_c(x)\} \, dx.$$

Thus, we get

$$\begin{aligned} & c^{-2} E \left\{ \int_{I_c} \{V(x) - C_c(x)\} \, dx - E \int_{I_c} \{V(x) - C_c(x)\} \, dx \right\}^4 \\ & \quad = c^{-2} E B_c^4 + c^{-2} \{4 E B_c^3 A_c + 6 E B_c^2 A_c^2 + 4 E B_c A_c^3 + E A_c^4\}. \end{aligned}$$

We have

$$c^{-2} E A_c^4 = \left(\frac{\sqrt{c}}{c} \right)^2 c^{-1} E A_c^4 = O(c^{-1})$$

and similarly, using the Cauchy–Schwarz inequality,

$$c^{-2} E B_c A_c^3 = E c^{-3/2} A_c^3 c^{-1/2} B_c \le \sqrt{E c^{-3} A_c^6} \sqrt{E c^{-1} B_c^2} = O(c^{-3/4}).$$

Continuing in this way, we find that the only non-vanishing term is $c^{-2} E B_c^4$. The statement now follows from what we proved for $I_c = [\sqrt{c}, c - \sqrt{c}]$. □

Finally, we also need the following extension of Theorem 2.2, allowing for some scaling in the arguments of the various processes and coefficients.

Theorem 2.3. *Let F_c, G_c, and H_c be twice-differentiable increasing functions on $[0, c]$, with continuous derivatives f_c, g_c, and h_c, respectively, satisfying*

$$F_c(x) = f_c(0) x \big(1 + o(1)\big), \qquad G_c(x) = g_c(0) x \big(1 + o(1)\big), \qquad H_c(x) = \tfrac{1}{2} h_c'(0) x^2 \big(1 + o(1)\big)$$

as $c \to \infty$, where the $o(1)$ term is uniform in x. We assume that $f_c(0)$, $g_c(0)$, $h_c(0)$, and $h_c'(0)$ are positive and stay away from 0 and ∞ as $c \to \infty$, where $h_c'(0)$ denotes the right derivative of h_c at 0. Let C_c be the greatest convex minorant on $[0, c]$ of the process

$$V_c(x) = H_c(x) + W\big(G_c(x)\big), \qquad x \in [0, c].$$

In addition, let S_c be defined by

$$S_c(x) = V_c(x) - C_c(x), \qquad x \in [0, c].$$

(i) *For $c > 4$, define the interval $I_c = [\sqrt{c}, c - \sqrt{c}]$. Then, as $c \to \infty$,*

$$\begin{aligned} c^{-1} E \int_{I_c} S_c(x)\,\mathrm{d}F_c(x) &\sim \frac{g_c(0)^{2/3} f_c(0)}{(h_c'(0)/2)^{1/3}} E\bigl|C(0)\bigr|, \\ \mathrm{var}\biggl(c^{-1/2} \int_{I_c} S_c(x)\,\mathrm{d}F_c(x)\biggr) &\sim \sigma_c^2, \end{aligned} \tag{2.10}$$

where

$$\sigma_c^2 = \frac{g_c(0)^{5/3} f_c(0)^2}{(h_c'(0)/2)^{4/3}} \sigma^2, \tag{2.11}$$

and C and σ^2 are defined as in Theorem 2.1. Moreover, the fourth moment of

$$c^{-1/2} \int_{I_c} \bigl\{S_c(x) - E S_c(x)\bigr\}\,\mathrm{d}F_c(x)$$

is uniformly bounded and satisfies

$$E\biggl(c^{-1/2} \int_{I_c} \bigl\{S_c(x) - E S_c(x)\bigr\}\,\mathrm{d}F_c(x)\biggr)^4 \sim M_c^{(4)}, \qquad c \to \infty, \tag{2.12}$$

where $M_c^{(4)}$ denotes the fourth moment of a normal $N(0, \sigma_c^2)$ distribution.

(ii) *Relations (2.10) and (2.12) also hold if the interval I_c is given by $I_c = [0, c - \sqrt{c}]$, $I_c = [\sqrt{c}, c]$ or $I_c = [0, c]$.*

Proof. Because the proof proceeds along the lines of the proofs of Theorems 2.1 and 2.2, we focus only on the new type of scaling present in the process

$$x \mapsto \tfrac{1}{2} h_c'(0) x^2 + W\bigl(g_c(0)x\bigr), \qquad x \in [0, c],$$

which replaces the process V defined in (2.1) on $[0, c]$.

Let $a, b > 0$. By Brownian scaling, the process

$$x \mapsto ax^2 + W(bx), \qquad x \in [0, c], \tag{2.13}$$

has the same distribution as the process

$$x \mapsto a^{-1/3} b^{2/3} \bigl\{\bigl(a^{2/3} b^{-1/3} x\bigr)^2 + W\bigl(a^{2/3} b^{-1/3} x\bigr)\bigr\}, \qquad x \in [0, c]. \tag{2.14}$$

Thus, if $C_{a,b}$ is the greatest convex minorant of the process given in (2.13) and $\tilde{C}_{a,b}$ of the process given in (2.14), then we get

$$\begin{aligned}
&\int_0^c \{ax^2 + W(bx) - C_{a,b}(x)\} f_c(0)\,\mathrm{d}x \\
&\stackrel{\mathcal{D}}{=} a^{-1/3}b^{2/3} f_c(0) \int_0^c \bigl\{\bigl(a^{2/3}b^{-1/3}x\bigr)^2 + W\bigl(a^{2/3}b^{-1/3}x\bigr) - a^{1/3}b^{-2/3}\tilde{C}_{a,b}(x)\bigr\}\,\mathrm{d}x \\
&= \frac{bf_c(0)}{a} \int_0^{a^{2/3}b^{-1/3}c} \bigl\{u^2 + W(u) - a^{1/3}b^{-2/3}\tilde{C}_{a,b}\bigl(a^{-2/3}b^{1/3}u\bigr)\bigr\}\,\mathrm{d}u \\
&= \frac{bf_c(0)}{a} \int_0^{a^{2/3}b^{-1/3}c} \bigl\{u^2 + W(u) - C_c(u)\bigr\}\,\mathrm{d}u,
\end{aligned}$$

where C_c is the greatest convex minorant of the process

$$u \mapsto u^2 + W(u), \qquad u \in \bigl[0, a^{2/3}b^{-1/3}c\bigr].$$

Thus, for $c \to \infty$,

$$c^{-1} E \int_0^c \{ax^2 + W(bx) - C_{a,b}(x)\} f_c(0)\,\mathrm{d}x \sim \frac{b^{2/3} f_c(0)}{a^{1/3}} E\bigl|C(0)\bigr|.$$

Given that $a = \frac{1}{2}h_c'(0)$, $b = g_c(0)$, (2.10) follows. Moreover,

$$\begin{aligned}
&\operatorname{var}\biggl(c^{-1/2} \int_0^c \{ax^2 + W(bx) - C_{a,b}(x)\} f_c(0)\,\mathrm{d}u\biggr) \\
&\quad= \frac{b^2 f_c(0)^2}{a^2 c} \operatorname{var}\biggl(\int_0^{a^{2/3}b^{-1/3}c} \bigl\{u^2 + W(u) - C_c(u)\bigr\}\,\mathrm{d}u\biggr) \\
&\quad= \frac{b^{5/3} f_c(0)^2}{a^{4/3}} \operatorname{var}\biggl(\frac{1}{\sqrt{b^{-1/3}a^{2/3}c}} \int_0^{a^{2/3}b^{-1/3}c} \bigl\{u^2 + W(u) - C_c(u)\bigr\}\,\mathrm{d}u\biggr) \\
&\quad\sim \frac{b^{5/3} f_c(0)^2}{a^{4/3}} \sigma^2
\end{aligned}$$

for $c \to \infty$. Taking $a = \frac{1}{2}h_c'(0)$, $b = g_c(0)$ now yields (2.11). □

3. Embedding and central limit result for T_n-type statistic

In this section, we establish a central limit result for the quantity

$$n^{5/6} \int_0^a \bigl\{\mathbb{H}_n(x) - \hat{H}_n(x) - \mu_n\bigr\}\,\mathrm{d}F_0(x) = n^{5/6} \int_0^a \bigl\{\mathbb{H}_n(x-) - \hat{H}_n(x) - \mu_n\bigr\}\,\mathrm{d}F_0(x), \tag{3.1}$$

where μ_n denotes a centering sequence to be specified below. This result is the first step in obtaining the limit result for T_n defined in (1.1), where the integral is taken with respect to $\mathrm{d}\mathbb{F}_n$ rather than $\mathrm{d}F_0$. To derive the limiting distribution of (3.1), we first replace the process $\mathbb{H}_n(x) - \hat{H}_n(x)$ by

$$x \mapsto H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}} - \tilde{H}_n(x), \qquad x \in [0, a],$$

where E_n is the empirical process $\sqrt{n}\{\mathbb{F}_n - F_0\}$ and $\tilde{H}_n$ is the greatest convex minorant of the process

$$x \mapsto H_0(x) + \frac{E_n(x)}{\sqrt{n}\{1 - F_0(x)\}}, \qquad x \in [0, a]. \tag{3.2}$$

We next use the strong approximation of the empirical process by a Brownian bridge B_n, yielding the approximation

$$x \mapsto H_0(x) + \frac{n^{-1/2} B_n(F_0(x))}{1 - F_0(x)}, \qquad x \in [0, a],$$

to the process (3.2). This process is distributed as

$$x \mapsto H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)}\right), \qquad x \in [0, a], \tag{3.3}$$

where W is standard Brownian motion on $[0, \infty)$. Next, the interval $[0, a]$ is split into so-called "big blocks" separated by small blocks. The local contributions to the integral over the big blocks can be treated using the results of Section 2.

The first lemma to be proven states a contraction property for convex minorants that will be used repeatedly in the sequel. It is related to Marshall's lemma in the theory of isotonic regression.

Lemma 3.1. *Let f and g be bounded functions on an interval $I \subset \mathbb{R}$, and let C_f and C_g be their greatest convex minorants, respectively. Then*

$$\sup_{x \in I} \bigl|C_f(x) - C_g(x)\bigr| \le \sup_{x \in I} \bigl|f(x) - g(x)\bigr|.$$

Proof. Given that $f \ge g - \sup_{u \in I} |f(u) - g(u)|$ and that $g \ge C_g$ by definition, it follows that $f \ge C_g - \sup_{u \in I} |f(u) - g(u)|$. Because the right-hand side is convex, this means that it is a convex minorant of f on I. Thus, it lies below the greatest convex minorant C_f of f on I,

$$C_f(x) \ge C_g(x) - \sup_{u \in I} \bigl|f(u) - g(u)\bigr|, \qquad x \in I.$$

Because this inequality also holds with f and g interchanged, the result follows. $\square$

We now consider the functional

$$\begin{aligned}\int_{[0,a]}&\{\mathbb{H}_n(x)-\hat{H}_n(x)\}\,\mathrm{d}F_0(x)\\&=\int_{[0,a]}\left\{H_0(x)-\log\left(1-\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}\right)-\hat{H}_n(x)\right\}\mathrm{d}F_0(x),\end{aligned}\tag{3.4}$$

where $E_n=\sqrt{n}\{\mathbb{F}_n-F_0\}$ is the empirical process. The following lemma allows us to dispense with the logarithms.

Lemma 3.2. *Let* $\tilde{H}_n$ *be the greatest convex minorant of the process given in* (3.2), *where* $F_0(a)<1$. *Then*:

(i)

$$\int_{[0,a]}\left|\mathbb{H}_n(x)-H_0(x)-\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}\right|\mathrm{d}F_0(x)=O_p\bigl(n^{-1}\bigr).$$

(ii)

$$\int_{[0,a]}\bigl|\hat{H}_n(x)-\tilde{H}_n(x)\bigr|\,\mathrm{d}F_0(x)=O_p\bigl(n^{-1}\bigr).$$

Proof. (i) Let A_n denote the event

$$\left|\sup_{x\in[0,a]}\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}\right|\le\frac{1}{2}.$$

Then, by a well-known result in large deviation theory (Chernoff's theorem), we have $\mathbb{P}(A_n^c)=\mathrm{O}(e^{-nc})$ for a constant $c>0$. If A_n occurs, then we can expand the logarithm, which yields

$$-\log\left\{1-\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}\right\}=\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}+n^{-1}O\Bigl(\sup_{x\in[0,a]}\bigl|E_n(x)\bigr|\Bigr),$$

and (i) now follows.

(ii) This follows from Lemma 3.1 and the argument of the proof of (i). □

Below, we prove that

$$\begin{aligned}n^{5/6}\int_{[0,a]}&\left\{H_0(x)+\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}-\tilde{H}_n(x)\right.\\&\left.-E\left\{H_0(x)+\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}-\tilde{H}_n(x)\right\}\right\}\mathrm{d}F_0(x)\end{aligned}$$

converges in distribution to a normal distribution, which, together with Lemma 3.2, implies that

$$n^{5/6}\int_{[0,a]}\left\{\mathbb{H}_n(x)-\hat{H}_n(x)-E\left\{H_0(x)+\frac{E_n(x)}{\sqrt{n}\{1-F_0(x)\}}-\tilde{H}_n(x)\right\}\right\}\mathrm{d}F_0(x)$$

converges to the same normal distribution.

Remark 3.1. We avoid taking the expectation of $\mathbb{H}_n(x) - \hat{H}_n(x)$, because $\mathbb{H}_n$ is infinite with a positive (but vanishing) probability on $[0, a]$, as is $\hat{H}_n$. This occurs when the empirical distribution function $\mathbb{F}_n$ reaches the value 1 on $[0, a]$.

By Theorem 3 of [14], we can construct Brownian bridges B_n on the same sample space as $\mathbb{F}_n$ such that

$$Y_n = \sup_{x\in[0,a]} \frac{n^{1/2}|E_n(x) - B_n(F_0(x))|}{2 \vee \log n}$$

is a random variable with with $EY_n \le C < \infty$ for all n. Thus, for $n \ge 2$,

$$0 \le E \sup_{x\in[0,a]} n^{-1/2} \left| \frac{E_n(x)}{1 - F_0(x)} - \frac{B_n(F_0(x))}{1 - F_0(x)} \right| \le \frac{EY_n \log n}{n(1 - F_0(a))} = O\left(\frac{\log n}{n}\right). \tag{3.5}$$

We now have the following result.

Lemma 3.3. *Let $\tilde{E}_n$ be defined by*

$$\tilde{E}_n(x) = \frac{B_n(F_0(x))}{1 - F_0(x)}, \qquad x \in [0, a], \tag{3.6}$$

where B_n is as defined above. Let C_n^B be the greatest convex minorant of

$$x \mapsto H_0(x) + n^{-1/2}\tilde{E}_n(x), \qquad x \in [0, a]. \tag{3.7}$$

Then

$$\begin{aligned}
&\int_{[0,a]} \{\mathbb{H}_n(x) - \hat{H}_n(x)\}\,\mathrm{d}F_0(x) \\
&\quad = \int_{[0,a]} \{H_0(x) + n^{-1/2}\tilde{E}_n(x) - C_n^B(x)\}\,\mathrm{d}F_0(x) + O_p\left(\frac{\log n}{n}\right).
\end{aligned} \tag{3.8}$$

Proof. The result follows immediately from (3.5) and Lemmas 3.1 and 3.2(i). □

Now note that the process defined in (3.7) has the same distribution as the process

$$x \mapsto V_n(x) \stackrel{\text{def}}{=} H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)}\right), \qquad x \in [0, a], \tag{3.9}$$

where W is standard Brownian motion on $\mathbb{R}_+$. Therefore, if C_n is the greatest convex minorant of the process V_n on $[0, a]$, then we have

$$\int_{[0,a]} \{H_0(x) + n^{-1/2}\tilde{E}_n(x) - C_n^B(x)\}\,\mathrm{d}F_0(x) \stackrel{\mathcal{D}}{=} \int_{[0,a]} \{V_n(x) - C_n(x)\}\,\mathrm{d}F_0(x). \tag{3.10}$$

For (3.10), the asymptotic distribution is given in Theorem 3.1 below.

Theorem 3.1. *Let h_0 satisfy Condition 1, and let S_n be defined by*

$$S_n(x) = H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x) = V_n(x) - C_n(x), \qquad x \in [0, a], \tag{3.11}$$

where C_n is the greatest convex minorant of V_n defined in (3.9). Let D_n be defined by

$$D_n = \int_0^a S_n(x)\,\mathrm{d}F_0(x). \tag{3.12}$$

Finally, let $C(0)$ and σ^2 be defined as in Theorem 2.1. Then

$$n^{5/6}\{D_n - ED_n\} \xrightarrow{\mathcal{D}} N\big(0, \sigma_{H_0}^2\big), \qquad n \to \infty,$$

where

$$n^{2/3} ED_n \to E\big|C(0)\big| \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)}\right)^{1/3} \mathrm{d}H_0(t) \tag{3.13}$$

and

$$\sigma_{H_0}^2 = 2^{4/3}\sigma^2 \int_0^a \frac{h_0(t)^2 \{h_0(t) f_0(t)\}^{1/3}}{h_0'(t)^{4/3}}\,\mathrm{d}H_0(t). \tag{3.14}$$

For the proof of Theorem 3.1, we divide the interval $[0, a]$ into m_n intervals $I_{n,k}$ with (equal) length of order $n^{-1/3}\log n$ (big blocks), separated by intervals $J_{n,k}$ ($k = 2, 3, \dots, m_n$) with length of order $2n^{-1/3}\sqrt{\log n}$ (small blocks). The small interval $J_{n,1}$ to the left of $I_{n,1}$ has half the length of the other separating blocks, as has the small interval J_{n,m_n+1} to the right of I_{n,m_n}. Thus,

$$[0, a] = J_{n,1} \cup I_{n,1} \cup J_{n,2} \cup I_{n,2} \cup \cdots \cup J_{n,m_n} \cup I_{n,m_n} \cup J_{n,m_n+1}.$$

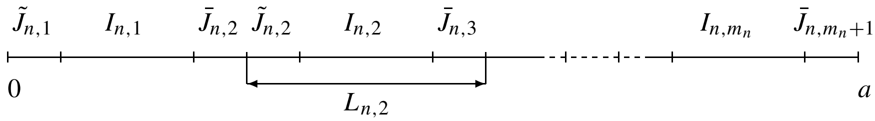
For $k = 2, 3, \dots, m_n$, let $\tilde{J}_{n,k}$ be the interval with the same right endpoint as $J_{n,k}$ with half the length of $J_{n,k}$, and take $\tilde{J}_{n,1} = J_{n,1}$. For $k = 1, 2, \dots, m_n - 1$ let $\bar{J}_{n,k+1}$ be the interval with the same left endpoint as $J_{n,k+1}$ with half the length of $J_{n,k+1}$ and $\tilde{J}_{n,m_n+1} = J_{n,m_n+1}$. Then

$$[0, a] = \tilde{J}_{n,1} \cup I_{n,1} \cup \bar{J}_{n,2} \cup \tilde{J}_{n,2} \cup I_{n,2} \cup \cdots \cup \tilde{J}_{n,m_n} \cup I_{n,m_n} \cup \bar{J}_{n,m_n+1},$$

where all I intervals have the same length, of order $n^{-1/3}\log n$, and the J intervals have the same length of (smaller) order $n^{-1/3}\sqrt{\log n}$. Finally, let the interval $L_{n,k}$ be defined by

$$\begin{aligned} &L_{n,k} = \tilde{J}_{n,k} \cup I_{n,k} \cup \bar{J}_{n,k+1} = [a_{nk}, a_{n,k+1}), \qquad k = 1, 2, \dots, m_n \\ &\text{yielding } [0, a) = \bigcup_{k=1}^{m_n} L_{n,k}. \end{aligned} \tag{3.15}$$

Note that $m_n \sim an^{1/3}/\log n$, and see the figure for the structure of the partition.



The (key) localization lemma below, which is proven in the Appendix, shows that on intervals $I_{n,k}$, the global convex minorant of V_n (defined in (3.9)) over $[0, a]$ coincides with high probability with the restriction to $I_{n,k}$ of the local convex minorant of the process V_n on the interval $L_{n,k}$.

Lemma 3.4. *Let h_0 satisfy Condition 1. Then*:

(i) *The probability that there exists a k, $1 \le k \le m_n$ such that the greatest convex minorant C_n of V_n is different on the interval $I_{n,k}$ from the restriction to $I_{n,k}$ of the* (*local*) *greatest convex minorant of V_n on $L_{n,k}$, is bounded above by*

$$c_1 \exp\{-c_2(\log n)^{3/2}\}$$

for constants $c_1, c_2 > 0$, uniformly in n.

(ii) *The probability that there exists a k, $1 \le k \le m_n$, such that C_n has no change of slope in an interval $\bar{J}_{n,k}$ or $\tilde{J}_{n,k}$ is also bounded by*

$$c_1 \exp\{-c_2(\log n)^{3/2}\}$$

for constants c_1, c_2, uniformly in n.

For each $n \ge 1$ and $1 \le k \le m_n$, define independent standard Brownian motions $W_{n1}, \dots, W_{n,m_n}$ and consider the processes

$$x \mapsto H_0(x) - H_0(a_{nk}) + n^{-1/2} W_{nk}\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})}\right), \qquad x \in L_{n,k}.$$

Denote the greatest convex minorants of these processes (on $L_{n,k}$) by C_{nk}. Furthermore, define the processes S_{nk} by

$$\begin{aligned} S_{nk}(x) &= H_0(x) - H_0(a_{nk}) \\ &\quad + n^{-1/2} W_{nk}\left(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})}\right) - C_{nk}(x), \qquad x \in L_{n,k}. \end{aligned} \tag{3.16}$$

Lemma 3.5. *Assume that the conditions of Theorem 3.1 are satisfied. Moreover, let $C(0)$ be defined as in Theorem 2.1 and let $\sigma_{H_0}^2$ be defined as in Theorem 3.1. Then*

$$n^{5/6} \sum_{k=1}^{m_n} \int_{I_{n,k}} \{S_{nk}(x) - ES_{nk}(x)\}\, \mathrm{d}F_0(x) \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \qquad n \to \infty,$$

where (*see* (3.13))

$$n^{2/3}\sum_{k=1}^{m_n}\int_{I_{n,k}} ES_{nk}(x)\,\mathrm{d}F_0(x)\to E\big|C(0)\big|\int_0^a\left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{1/3}\mathrm{d}H_0(t),\qquad n\to\infty. \tag{3.17}$$

Proof. Let $c_n=n^{1/3}|L_{n,k}|\sim\log n$ and $I_{n,k}=[a_{nk}+n^{-1/3}\sqrt{c_n},a_{nk}+n^{-1/3}(c_n-\sqrt{c_n})]$. We then have

$$\begin{aligned}
&n\int_{I_{n,k}}\big\{S_{nk}(x)-ES_{nk}(x)\big\}\,\mathrm{d}F_0(x)\\
&\quad=\int_{\sqrt{c_n}}^{c_n-\sqrt{c_n}}\bigg\{n^{1/6}W_{nk}\bigg(\frac{F_0(a_{nk}+n^{-1/3}x)}{1-F_0(a_{nk}+n^{-1/3}x)}-\frac{F_0(a_{nk})}{1-F_0(a_{nk})}\bigg)\\
&\qquad\qquad-n^{2/3}C_{nk}\big(a_{nk}+n^{-1/3}x\big)\\
&\qquad\qquad-E\bigg\{n^{1/6}W_{nk}\bigg(\frac{F_0(a_{nk}+n^{-1/3}x)}{1-F_0(a_{nk}+n^{-1/3}x)}-\frac{F_0(a_{nk})}{1-F_0(a_{nk})}\bigg)\\
&\qquad\qquad\qquad-n^{2/3}C_{nk}\big(a_{nk}+n^{-1/3}x\big)\bigg\}\bigg\}f_0\big(a_{nk}+n^{-1/3}x\big)\,\mathrm{d}x.
\end{aligned}$$

Here we use the fact that the (first two) deterministic terms in (3.16) drop out because of subtraction of the expectation. This implies that

$$\begin{aligned}
&n\int_{I_{n,k}}\big\{S_{nk}(x)-ES_{nk}(x)\big\}\,\mathrm{d}F_0(x)\\
&\stackrel{\mathcal{D}}{=}\int_{\sqrt{c_n}}^{c_n-\sqrt{c_n}}\bigg\{n^{1/6}W\bigg(\frac{F_0(a_{nk}+n^{-1/3}x)}{1-F_0(a_{nk}+n^{-1/3}x)}-\frac{F_0(a_{nk})}{1-F_0(a_{nk})}\bigg)-C_{nk}'(x)\\
&\qquad\qquad-E\bigg\{n^{1/6}W\bigg(\frac{F_0(a_{nk}+n^{-1/3}x)}{1-F_0(a_{nk}+n^{-1/3}x)}-\frac{F_0(a_{nk})}{1-F_0(a_{nk})}\bigg)-C_{nk}'(x)\bigg\}\bigg\}\\
&\qquad\qquad\times f_0\big(a_{nk}+n^{-1/3}x\big)\,\mathrm{d}x,
\end{aligned}$$

where C_{nk}' is the greatest convex minorant of the process

$$\begin{aligned}
x\mapsto{}&n^{2/3}\big\{H_0\big(a_{nk}+n^{-1/3}x\big)-H_0(a_{nk})-n^{-1/3}xh_0(a_{nk})\big\}\\
&+n^{1/6}W\bigg(\frac{F_0(a_{nk}+n^{-1/3}x)}{1-F_0(a_{nk}+n^{-1/3}x)}-\frac{F_0(a_{nk})}{1-F_0(a_{nk})}\bigg),\qquad x\in[0,c_n].
\end{aligned}$$

Here we use the fact that adding a linear function to a function does not change the difference between this function and its greatest convex minorant. Note that the integrals on $I_{n,k}$ depend only on the increments of the Brownian motion process on the corresponding disjoint intervals

$L_{n,k}$ and thus are independent. For the individual integrals, we are close to the situation of Theorem 2.3, with, for $c_n \to \infty$ on $[0, c_n]$ (note that n is determined by c_n, $n = e^{c_n}$),

$$\begin{aligned} F_{c_n}(x) &= n^{1/3}\big\{F_0\big(a_{nk} + n^{-1/3}x\big) - F_0(a_{nk})\big\} = f_0(a_{nk})x\big(1 + \mathrm{o}(1)\big), \\ H_{c_n}(x) &= n^{2/3}\big\{H_0\big(a_{nk} + n^{-1/3}x\big) - H_0(a_{nk}) - n^{-1/3}xh_0(a_{nk})\big\} \\ &= \tfrac{1}{2}h_0'(a_{nk})x^2\big(1 + \mathrm{o}(1)\big) \end{aligned}$$

and

$$G_{c_n}(x) = n^{1/3}\left\{\frac{F_0(a_{nk} + n^{-1/3}x)}{1 - F_0(a_{nk} + n^{-1/3}x)} - \frac{F_0(a_{nk})}{1 - F_0(a_{nk})}\right\} = \frac{f_0(a_{nk})x}{(1 - F_0(a_{nk}))^2}\big(1 + \mathrm{o}(1)\big).$$

This yields

$$\mathrm{var}\left(\frac{n}{\sqrt{c_n}}\int_{I_{n,k}} S_{nk}(x)\,\mathrm{d}F_0(x)\right) \sim \sigma_{nk}^2, \qquad n \to \infty,$$

uniformly in $k = 1, \dots, m_n$, where

$$\begin{aligned} \sigma_{nk}^2 &= \frac{(f_0(a_{nk})/\{1 - F_0(a_{nk})\}^2)^{5/3} f_0(a_{nk})^2}{(h_0'(a_{nk})/2)^{4/3}}\sigma^2 \\ &= \frac{(h_0(a_{nk}))^{10/3} f_0(a_{nk})^{1/3}}{(h_0'(a_{nk})/2)^{4/3}}\sigma^2 \\ &= \frac{2^{4/3}h_0(a_{nk})^3\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{4/3}}\sigma^2, \end{aligned}$$

and σ^2 is defined as in Theorem 2.1. Likewise, with $C(0)$ as defined in Theorem 2.1,

$$\frac{n}{c_n}\int_{I_{n,k}} ES_{nk}(x)\,\mathrm{d}F_0(x) \sim \frac{2^{1/3}h_0(a_{nk})\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}E|C(0)|}{h_0'(a_{nk})^{1/3}}.$$

Because the fourth moments of

$$\frac{n}{\sqrt{c_n}}\int_{I_{n,k}} \big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x)$$

are uniformly bounded by Theorem 2.3, for each $\varepsilon > 0$, using Chebyshev's inequality, we get

$$\sum_{k=1}^{m_n} \mathbb{P}\left\{m_n^{-1/2}\left|\frac{n}{\sqrt{c_n}}\int_{I_{n,k}} \big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x)\right| \ge \varepsilon\right\} \to 0, \qquad n \to \infty.$$

Using the fact that $m_n^{-1} \sim a^{-1}n^{-1/3}\log n$ and that the intervals $I_{n,k}$ have lengths of order $n^{-1/3}\log n$, we get

$$m_n^{-1}\sum_{k=1}^{m_n}\sigma_{nk}^2 \;\sim\; m_n^{-1}\sum_{k=1}^{m_n}\frac{2^{4/3}h_0(a_{nk})^3\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{4/3}}\sigma^2$$
$$\longrightarrow \frac{2^{4/3}\sigma^2}{a}\int_0^a \frac{h_0(t)^3\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{4/3}}\,\mathrm{d}t.$$

Because $m_n = an^{1/3}/c_n$, the normal convergence criterion on page 316 of [16] now gives

$$n^{5/6}\sum_{k=1}^{m_n}\int_{I_{n,k}}\big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x)$$
$$= m_n^{-1/2}\sum_{k=1}^{m_n}\frac{n\sqrt{a}}{\sqrt{c_n}}\int_{I_{n,k}}\big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x) \xrightarrow{\mathcal{D}} N\big(0,\sigma_{H_0}^2\big).$$

Also note that

$$m_n^{-1/2}\sum_{k=1}^{m_n}\frac{n}{c_n^{1/2}}\int_{I_{n,k}} ES_{nk}(x)\,\mathrm{d}F_0(x) \sim m_n^{-1/2}c_n^{1/2}\sum_{k=1}^{m_n}\frac{2^{1/3}h_0(a_{nk})\{h_0(a_{nk})f_0(a_{nk})\}^{1/3}}{h_0'(a_{nk})^{1/3}}$$
$$\sim \sqrt{m_nc_n}\,E\big|C(0)\big|\int_0^a \frac{2^{1/3}h_0(t)\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{1/3}}\,\mathrm{d}t$$
$$= n^{1/6}\int_0^a\left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{1/3}\mathrm{d}H_0(t). \qquad \square$$

We can now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemmas 3.4 and 3.5, we have

$$n^{5/6}\sum_{k=1}^{m_n}\int_{I_{n,k}}\big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x) \xrightarrow{\mathcal{D}} N\big(0,\sigma_{H_0}^2\big), \qquad n\to\infty.$$

For similar reasons, we have

$$n^{5/6}\sum_{k=1}^{m_n}\int_{L_{n,k}\setminus I_{n,k}}\big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x) \xrightarrow{p} 0, \qquad n\to\infty,$$

where we use Theorem 2.3. (This is the essence of the "big blocks, small blocks" method.) The result now follows, because

$$D_n = n^{5/6}\sum_{k=1}^{m_n}\int_{L_{n,k}}\big\{S_{nk}(x) - ES_{nk}(x)\big\}\,\mathrm{d}F_0(x). \qquad \square$$

The corollary below follows from Lemma 3.3, (3.10), and Theorem 3.1.

Corollary 3.1. *Let h_0 satisfy Condition 1. Then*

$$n^{5/6}\left\{\int_0^a \{\mathbb{H}_n(x) - \hat{H}_n(x)\}\,\mathrm{d}F_0(x) - ED_n\right\} \xrightarrow{\mathcal{D}} N\big(0, \sigma_{H_0}^2\big), \qquad n \to \infty,$$

where ED_n and $\sigma_{H_0}^2$ are defined as in Theorem 3.1.

4. Central limit result for U_n-type statistics

To derive the asymptotic distribution of the statistic U_n defined in (1.2) and used in the simulations in [8], we first consider the statistic

$$\int_{[0,a]} \{\mathbb{F}_n(x) - \hat{F}_n(x)\}\,\mathrm{d}F_0(x), \tag{4.1}$$

which is analogous to the statistic discussed in the preceding section but has $\mathbb{F}_n(x) - \hat{F}_n(x)$ as an integrand instead of $\mathbb{H}_n(x) - \hat{H}_n(x)$. Note that this is not U_n; the difference is in the integrating measure ($\mathrm{d}F_0$ instead of $\mathrm{d}\mathbb{F}_n$). If E_n again denotes the empirical process, and arguing as in the proof of Lemma 3.2(i), then we have

$$\begin{aligned}\mathbb{F}_n(x) = 1 - \exp\{-\mathbb{H}_n(x)\} &= 1 - \exp\left\{-H_0(x) + \log\left\{1 - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\}\right\}\\ &= 1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\} + O_p\big(n^{-1}\big),\end{aligned}$$

uniformly for $x \in [0, a]$. Thus, defining, as in Lemma 3.2, $\tilde{H}_n$ as the greatest convex minorant of the process given in (3.2), by Lemma 3.2, we get

$$\begin{aligned}\mathbb{F}_n(x) - \hat{F}_n(x) &= \exp\{-\tilde{H}_n(x)\} - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)}\right\} + O_p\big(n^{-1}\big)\\ &= \exp\{-\tilde{H}_n(x)\}\left\{1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}E_n(x)}{1 - F_0(x)} + \tilde{H}_n(x)\right\}\right\} + O_p\big(n^{-1}\big).\end{aligned}$$

Next, replacing $E_n(x)$ by $B_n(F_0(x))$ as in Lemma 3.3, where (B_n) are the approximating Brownian bridges, and C_n^B is the greatest convex minorant of the process (3.7), we get

$$\begin{aligned}\mathbb{F}_n(x) - \hat{F}_n(x) = \exp\{-C_n^B(x)\}&\left\{1 - \exp\left\{-H_0(x) - \frac{n^{-1/2}B_n(F_0(x))}{1 - F_0(x)} + C_n^B(x)\right\}\right\}\\ &+ O_p\left(\frac{\log n}{n}\right).\end{aligned}$$

Again using the results of the preceding section, and denoting by C_n the greatest convex minorant of the process defined in (3.9), this implies that

$$\begin{aligned} &\mathbb{F}_n(x) - \hat{F}_n(x) \\ &\quad \stackrel{\mathcal{D}}{=} \exp\{-C_n(x)\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2} W\left(\frac{F_0(x)}{1-F_0(x)}\right) + C_n(x)\right\}\right\} \\ &\qquad + O_p\left(\frac{\log n}{n}\right). \end{aligned} \tag{4.2}$$

Based on this representation, we now first consider the following approximation to (4.2):

$$\begin{aligned} &\exp\{-H_0(x)\}\left\{H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x)\right\} \\ &\quad = \{1 - F_0(x)\}\left\{H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x)\right\} \\ &\quad = \{1 - F_0(x)\} S_n(x) \end{aligned}$$

for $x \in [0, a]$, where S_n as defined in Theorem 3.1. For the integral of this process with respect to $\mathrm{d}F_0$, we have the following result.

Lemma 4.1. *Let h_0 satisfy Condition 1. Moreover, let C_n, S_n, and V_n be defined as in Theorem 3.1, and let $D_n^{F_0}$ be defined by*

$$D_n^{F_0} = \int_0^a S_n(x)\{1 - F_0(x)\}\,\mathrm{d}F_0(x). \tag{4.3}$$

Then, for $n \to \infty$,

$$n^{5/6}\{D_n^{F_0} - ED_n^{F_0}\} \xrightarrow{\mathcal{D}} N(0, \sigma_{F_0}^2) \qquad \text{with } \sigma_{F_0}^2 = \sigma^2 \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)}\right)^{4/3} \mathrm{d}F_0(t),$$

where σ^2 is defined as in Theorem 2.1 and

$$ED_N^{F_0} \sim n^{-2/3} E|C(0)| \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)}\right)^{1/3} \mathrm{d}F_0(t), \qquad n \to \infty.$$

Proof. The difference with Theorem 3.1 is that $\mathrm{d}F_0(t)$ is replaced by $\{1 - F_0(t)\}\,\mathrm{d}F_0(t)$ in the integral. This means that instead of ED_n, we get

$$ED_N^{F_0} \sim n^{-2/3} E|C(0)| \int_0^a \frac{2^{1/3} h_0(t)\{h_0(t) f_0(t)\}^{1/3}\{1 - F_0(t)\}}{h_0'(t)^{1/3}}\,\mathrm{d}t, \qquad n \to \infty,$$

and instead of $\sigma^2_{H_0}$, we get

$$\sigma^2_{F_0} = 2^{4/3}\sigma^2 \int_0^a \frac{h_0(t)^3\{h_0(t)f_0(t)\}^{1/3}}{h_0'(t)^{4/3}}\{1 - F_0(t)\}^2\,\mathrm{d}t$$
$$= \sigma^2 \int_0^a \left(\frac{2h_0(t)f_0(t)}{h_0'(t)}\right)^{4/3} \mathrm{d}F_0(t). \qquad \square$$

Based on Lemma 4.1, we now derive the asymptotic distribution of (4.1).

Theorem 4.1. *Let h_0 satisfy Condition 1. Moreover, let S_n' be defined by $S_n'(x) = \mathbb{F}_n(x) - \hat{F}_n(x)$, $x \in [0, a]$, where $\hat{F}_n$ is as defined in (1.2), and let D_n' be defined by*

$$D_n' = \int_0^a S_n'(x)\,\mathrm{d}F_0(x). \tag{4.4}$$

Then, with $\sigma^2_{F_0}$ defined as in Lemma 4.1,

$$n^{5/6}\{D_n' - ED_n'\} \xrightarrow{\mathcal{D}} N\bigl(0, \sigma^2_{F_0}\bigr), \qquad n \to \infty.$$

Proof. This is, in a sense, an application of the delta method. By (4.2), we can replace $\mathbb{F}_n - \hat{F}_n$ by

$$\exp\{-C_n(x)\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) + C_n(x)\right\}\right\}.$$

Using notation of the same type as in the proof of Lemma 3.5, we also have that

$$\int_0^a E\{H_0(x) - C_n(x)\}^2\,\mathrm{d}F_0(x)$$
$$\sim \sum_{k=1}^{m_n} \int_0^{c_n} E\{H_0(a_{nk} + n^{-1/3}u) - H_0(a_{nk}) - C_n(a_{nk} + n^{-1/3}u) + C_n(a_{nk})\}^2$$
$$\times f_0(a_{nk})\,\mathrm{d}u$$
$$\sim n^{-5/3} \sum_{k=1}^{m_n} \int_0^{c_n} E\left\{\frac{1}{2}h_0'(a_{nk})u^2 - C_{nk}(u)\right\}^2 f_0(a_{nk})\,\mathrm{d}u,$$

where C_{nk} is the greatest convex minorant of the process

$$x \mapsto \frac{1}{2}h_0'(a_{nk})u^2 + W\left(\frac{h_0(a_{nk})u}{1 - F_0(a_{nk})}\right), \qquad u \in [0, c_n].$$

By Brownian scaling, we get

$$\int_0^{c_n} E\left\{\frac{1}{2}h_0'(a_{nk})u^2 - C_{nk}(u)\right\}^2 f_0(a_{nk})\,\mathrm{d}u$$
$$\sim c_n f_0(a_{nk})\left(\frac{1}{2}h_0'(a_{nk})\right)^{2/3}\left(\frac{h_0(a_{nk})}{1-F_0(a_{nk})}\right)^{4/3} EC(0)^2,$$

where C is the greatest convex minorant of $x \mapsto W(x) + x^2$, $x \in \mathbb{R}$. Thus, we find that

$$\begin{aligned}&\int_0^a E\{H_0(x) - C_n(x)\}^2\,\mathrm{d}F_0(x)\\ &\quad\sim n^{-4/3} EC(0)^2 \int_0^a \left(\frac{1}{2}h_0'(t)\right)^{2/3}\left(\frac{h_0(t)}{1-F_0(t)}\right)^{4/3}\mathrm{d}F_0(t).\end{aligned} \tag{4.5}$$

We also have that

$$\begin{aligned}&\int_0^a E\left\{H_0(x) + n^{-1/2} W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x)\right\}^2 \mathrm{d}F_0(x)\\ &\quad\sim n^{-4/3} EC(0)^2 \int_0^a \left(\frac{1}{2}h_0'(t)\right)^{2/3}\left(\frac{h_0(t)}{1-F_0(t)}\right)^{4/3}\mathrm{d}F_0(t).\end{aligned} \tag{4.6}$$

Thus, by (4.5) and (4.6),

$$\begin{aligned}&\int_0^a \exp\{-C_n(x)\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x)}{1-F_0(x)}\right) + C_n(x)\right\}\right\}\mathrm{d}F_0(t)\\ &\quad= \int_0^a \{1 - F_0(t)\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x)}{1-F_0(x)}\right) + C_n(x)\right\}\right\}\mathrm{d}F_0(t)\\ &\qquad + O_p(n^{-4/3})\\ &\quad= \int_0^a \{1 - F_0(t)\}\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x)\right\}\mathrm{d}F_0(t) + O_p(n^{-4/3}),\end{aligned}$$

where we also use the Cauchy–Schwarz inequality in the first equality.

Similarly, for the expectation, we get

$$\int_0^a E\{\mathbb{F}_n(x) - \hat{F}_n(x)\}\,\mathrm{d}F_0(x) = ED_n^{F_0} + O\left(\frac{\log n}{n}\right),$$

where $D_n^{F_0}$ is defined by (4.3). This is seen in the following way. Because we assume that $F_0(a) < 1$, by Chernoff's theorem (as in the proof of Lemma 3.2), we have

$$\mathbb{P}\{1 - \mathbb{F}_n(a) < \tfrac{1}{2}\{1 - F_0(a)\}\} \le e^{-nc}$$

for $c > 0$, and thus, defining the event $A_n = \{1 - \mathbb{F}_n(a) \geq \frac{1}{2}\{1 - F_0(a)\}\}$, we get

$$\begin{aligned}
&\int_0^a E\{\mathbb{F}_n(x) - \hat{F}_n(x)\}\,\mathrm{d}F_0(x)\\
&= \int_0^a E\{\mathbb{F}_n(x) - \hat{F}_n(x)\}1_{A_n}\,\mathrm{d}F_0(x) + O(e^{-nc})\\
&= \int_0^a E\{e^{-\hat{H}_n(x)} - e^{-\mathbb{H}_n(x)}\}1_{A_n}\,\mathrm{d}F_0(x) + O(e^{-nc})\\
&= \int_0^a E\{1 - \mathbb{F}_n(x)\}\{e^{-\{\hat{H}_n(x) - \mathbb{H}_n(x)\}} - 1\}1_{A_n}\,\mathrm{d}F_0(x) + O(e^{-nc})\\
&= \int_0^a \{1 - F_0(x)\}E\{e^{-\{\hat{H}_n(x) - \mathbb{H}_n(x)\}} - 1\}1_{A_n}\,\mathrm{d}F_0(x) + O(n^{-1})\\
&= \int_0^a \{1 - F_0(x)\}E\left\{H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1 - F_0(x)}\right) - C_n(x)\right\}\mathrm{d}F_0(x) + O\left(\frac{\log n}{n}\right)\\
&= ED_n^{F_0} + O\left(\frac{\log n}{n}\right).
\end{aligned}$$

The result now follows from Lemma 4.1. □

5. Asymptotic distribution of the test statistics

We are now in the position to prove the main results for the original test statistics T_n and U_n. In view of Theorems 3.1 and 4.1, this essentially amounts to proving that the integrating measure in the statistics may be changed from $\mathrm{d}F_0$ to $\mathrm{d}\mathbb{F}_n$.

Theorem 5.1. *Let D_n be defined as in Theorem 3.1, and let the conditions of Theorem 3.1 be satisfied. Then*

$$n^{5/6}\left\{\int_0^a \{\mathbb{H}_n(x-) - \hat{H}_n(x)\}\,\mathrm{d}\mathbb{F}_n(x) - ED_n\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{H_0}^2), \qquad n \to \infty,$$

where ED_n and $\sigma_{H_0}^2$ are defined as in Theorem 3.1.

Theorem 5.2. *Let the conditions of Lemma 4.1 be satisfied, and let $\sigma_{F_0}^2$ and $C(0)$ be defined as in Lemma 4.1. Then, as $n \to \infty$,*

$$\begin{aligned}
&n^{5/6}\left\{\int_0^a \{\mathbb{F}_n(x-) - \hat{F}_n(x)\}\,\mathrm{d}\mathbb{F}_n(x) - \int_0^a E\{\mathbb{F}_n(x-) - \hat{F}_n(x)\}\,\mathrm{d}F_0(x)\right\}\\
&\quad \xrightarrow{\mathcal{D}} N(0, \sigma_{F_0}^2)
\end{aligned}$$

and

$$\int_0^a E\{\mathbb{F}_n(x-) - \hat{F}_n(x)\} \, dF_0(x) \sim n^{-2/3} E|C(0)| \int_0^a \left(\frac{2h_0(t) f_0(t)}{h_0'(t)}\right)^{1/3} dF_0(t).$$

We prove only Theorem 5.1, given that the proof of Theorem 5.2 proceeds along similar lines.

Proof of Theorem 5.1. Again using Lemma 3.1, we get

$$\int_0^a \{\mathbb{H}_n(x-) - \hat{H}_n(x)\} \, d\mathbb{F}_n(x) \stackrel{\mathcal{D}}{=} \int_0^a \left\{ H_0(x) + n^{-1/2} W_n\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x) \right\} d\mathbb{F}_n(x) + O_p\left(\frac{\log n}{n}\right),$$

where C_n is the greatest convex minorant of V_n, defined as in (3.9), with W_n replacing W. The process W_n is distributed as a standard Brownian motion on $[0, a]$, and $W_n \circ (F_0/(1-F_0))$ is given by

$$W_n\left(\frac{F_0(x)}{1-F_0(x)}\right) = \frac{B_n(F_0(x))}{1-F_0(x)}, \qquad x \in [0, a],$$

where B_n is coupled to the empirical process as in Lemma 3.3.

We need only show that

$$\int_0^a \{V_n(x) - C_n(x)\} \, d(\mathbb{F}_n - F_0)(x) = o_p(n^{-5/6}), \tag{5.1}$$

because we then have

$$\begin{aligned}
&\int_0^a \{\mathbb{H}_n(x-) - \hat{H}_n(x)\} \, d\mathbb{F}_n(x) \\
&\quad = \int_0^a \{V_n(x) - C_n(x)\} \, d\mathbb{F}_n(x) + O_p\left(\frac{\log n}{n}\right) \\
&\quad = \int_0^a \{V_n(x) - C_n(x)\} \, dF_0(x) + \int_0^a \{V_n(x) - C_n(x)\} \, d(\mathbb{F}_n - F_0)(x) + O_p\left(\frac{\log n}{n}\right) \\
&\quad = \int_0^a \{V_n(x) - C_n(x)\} \, dF_0(x) + o_p(n^{-5/6}).
\end{aligned}$$

To show that this relation holds, we follow a method somewhat similar to that used by [15] but that uses the Brownian motion representation instead of the empirical process and does not bring the derivative of the greatest convex minorant into play.

The p-variation of a function f on the interval $I = [0, a]$ is defined by

$$\nu_p(f; I) = \sup\left\{ \sum_{i=1}^m |f(x_i) - f(x_{i-1})|^p : x_0 = 0 < x_1 < \cdots < x_m = a \right\}.$$

The p-variation norm of f on I is defined by

$$\|f\|_{[p]} = v_p(f; I)^{1/p} + \sup_{x \in I} \bigl|f(x)\bigr|.$$

By Theorem II.3.27 of [1], for $p, q > 0$ and $1/p + 1/q > 1$, we have

$$\left|\int_{[0,a]} \bigl\{V_n(x) - C_n(x)\bigr\}\, \mathrm{d}(\mathbb{F}_n - F_0)(x)\right| \le c\|V_n - C_n\|_{[p]}\|\mathbb{F}_n - F_0\|_{[q]} \tag{5.2}$$

for a constant $c > 0$. Moreover, by Theorems I.6.1 and I.6.2 [1] and Theorem 3.2 of [18], we have

$$\|\mathbb{F}_n - F_0\|_{[q]} = \begin{cases} O_p\bigl(n^{(1-q)/q}\bigr), & q \in [1, 2), \\ O_p\bigl(n^{-1/2}\sqrt{L\bigl(L(n)\bigr)}\bigr), & q = 2, \\ O_p\bigl(n^{-1/2}\bigr), & q > 2, \end{cases} \tag{5.3}$$

where $L(n) = 1 \vee \log n$.

Let $\tau_1, \ldots, \tau_m$ be the points of jump of the derivative c_n of C_n on $[0, a]$, and let $\tau_0 = 0$, $\tau_{m+1} = a$. The function C_n is linear on the intervals $[\tau_i, \tau_{i+1}]$, and V_n behaves on such an interval as an excursion above its greatest convex minorant C_n, with the same values as V_n at the endpoints of the interval. Thus, for $p > 2$, by Lemma 4 of [12], we have

$$v_p\bigl(V_n - C_n; [0, a]\bigr) \le 2^{p-1} \sum_{k=1}^{m+1} v_p\bigl(V_n - C_n; [\tau_{i-1}, \tau_i]\bigr) = 2^{p-1} \sum_{i=1}^{m+1} v_p\bigl(\tilde{V}_n; [\tau_{i-1}, \tau_i]\bigr),$$

where, using the fact that the linear part drops out in taking the comparison with the greatest convex minorant,

$$\begin{aligned} \tilde{V}_n(x) = n^{-1/2}\biggl\{ & W\biggl(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\biggr) \\ & - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} W\biggl(\frac{F_0(\tau_i)}{1 - F_0(\tau_i)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\biggr)\biggr\} \\ & + H_0(x) - H_0(\tau_{i-1}) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}}\bigl\{H_0(\tau_i) - H_0(\tau_{i-1})\bigr\}, \qquad x \in [\tau_{i-1}, \tau_i]. \end{aligned}$$

By part (ii) of Lemma 3.4, we have $E \max_i(\tau_i - \tau_{i-1}) = \mathrm{O}(n^{-1/3} \log n)$. Let u_i be the midpoint of the interval $[\tau_{i-1}, \tau_i]$ and let f_{H_0} by defined by

$$f_{H_0}(x) = H_0(x) - H_0(\tau_{i-1}) - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}}\bigl\{H_0(\tau_i) - H_0(\tau_{i-1})\bigr\}, \qquad x \in [\tau_{i-1}, \tau_i].$$

Then

$$f_{H_0}(x) = -\tfrac{1}{2}h_0'(u_i)\{x - \tau_{i-1}\}\{\tau_i - x\}\bigl\{1 + \mathrm{o}_p(1)\bigr\},$$

where $x \mapsto \{x - \tau_{i-1}\}\{\tau_i - x\}$ is increasing on $[\tau_{i-1}, u_i]$ and decreasing on $[u_i, \tau_i]$, and

$$\nu_p\big(f_{H_0}; [\tau_{i-1}, \tau_i]\big) \sim 2^{1-p} h_0'(u_i)^p \{u_i - \tau_{i-1}\}^p \{\tau_i - u_i\}^p,$$

(see, e.g., (3.4) of [12]). Thus, for any $p > 2$,

$$\begin{aligned}
\sum_{i=1}^{m+1} \nu_p\big(f_{H_0}; [\tau_{i-1}, \tau_i]\big) &\sim 2^{1-p} \sum_{i=1}^{m+1} h_0'(u_i)^p \{u_i - \tau_{i-1}\}^p \{\tau_i - u_i\}^p \\
&= 2^{1-3p} \sum_{i=1}^{m+1} h_0'(u_i)^p \{u_i - \tau_{i-1}\}^{2p} \\
&\le 2^{-3p} \max_i \{u_i - \tau_{i-1}\}^{2p-1} \sum_{i=1}^{m+1} h_0'(u_i)^p \{\tau_i - \tau_{i-1}\} \\
&\sim 2^{-5p+1} \max_i \{\tau_i - \tau_{i-1}\}^{2p-1} \int_0^a h_0'(u)^p \,\mathrm{d}u \\
&= O_p\big(n^{-(2p-1)/3} (\log n)^{(2p-1)/2}\big).
\end{aligned}$$

Note that the O_p-term becomes $\mathrm{O}_p(n^{-1}(\log n)^{3/2})$ for $p = 2$.

For the Brownian part,

$$\begin{aligned}
B_{nk}(x) \stackrel{\text{def}}{=} n^{-1/2} \bigg\{ & W\bigg(\frac{F_0(x)}{1 - F_0(x)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\bigg) \\
& - \frac{x - \tau_{i-1}}{\tau_i - \tau_{i-1}} W\bigg(\frac{F_0(\tau_i)}{1 - F_0(\tau_i)} - \frac{F_0(\tau_{i-1})}{1 - F_0(\tau_{i-1})}\bigg)\bigg\}
\end{aligned}$$

for $p > 2$, we find that

$$\sum_{i=1}^{m+1} \nu_p\big(B_{nk}; [\tau_{i-1}, \tau_i]\big) = O_p\big(n^{-p/2}\big)$$

by the fact that almost all Brownian motion paths are Hölder continuous of any order $< 1/2$.

Thus, we find that

$$\|V_n - C_n\|_{[p]} = O_p\big(n^{-1/2} (\log n)^{(2p-1)/(2p)}\big) \tag{5.4}$$

for any $p > 2$. Thus, (5.2), (5.3), and (5.4) imply that

$$\int_{[0,a]} \big\{V_n(x) - C_n(x)\big\} \,\mathrm{d}(\mathbb{F}_n - F_0)(x) = O_p\big(n^{-1+\varepsilon}\big)$$

for arbitrarily small $\varepsilon > 0$. □

We end this section with a result for the situation that the hazard is nondecreasing, but not strictly nondecreasing. Clearly, the asymptotics are quite different from the case where h_0 increases strictly on $[0,a]$; the rate of convergence drops from $n^{5/6}$ to $n^{1/2}$, and the asymptotic distribution is not normal. Cases where h_0 has intervals in $[0,a]$ where it is constant lead to similar results, because these intervals will dominate the asymptotic behavior of the test statistic.

Theorem 5.3. *Let $\hat{F}_n$ and $\mathbb{F}_n$ be defined as in Theorem 5.2, and let U_n as in (1.2). Let U be given by*

$$U = \int_0^a \{1 - F_0(x)\}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C(x)\right\} \mathrm{d}F_0(x),$$

where W is standard Brownian motion on $[0,\infty)$ and C is the greatest convex minorant of

$$x \mapsto W\left(\frac{F_0(x)}{1-F_0(x)}\right), \qquad x \in [0,a]. \tag{5.5}$$

Suppose that the underlying hazard h_0 is constant on $[0,a]$. Then

$$n^{1/2}U_n \xrightarrow{\mathcal{D}} U, \qquad n \to \infty.$$

Proof. The proof follows lines that by now are familiar. We first consider

$$U_n' = \int_{[0,a]} \{\mathbb{F}_n(x-) - \hat{F}_n(x)\} \mathrm{d}F_0(x).$$

By (4.2), we can replace $\mathbb{F}_n - \hat{F}_n$ by

$$\exp\{-C_n(x)\}\left\{1 - \exp\left\{-H_0(x) - n^{-1/2}W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C_n(x)\right\}\right\},$$

where C_n is the greatest convex minorant of the process

$$x \mapsto H_0(x) + n^{-1/2}W\left(\frac{F_0(x)}{1-F_0(x)}\right), \qquad x \in [0,a],$$

with a remainder term of order $O_p((\log n)/n)$. Using the delta method, as in the proof of Theorem 4.1, we can replace this (apart from a remainder term of order $O_p(n^{-1})$) by

$$n^{-1/2}\{1 - F_0(x)\}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - C(x)\right\}, \qquad x \in [0,a],$$

where C is the greatest convex minorant of the process (5.5) and H_0 is linear on $[0,a]$. The statement for U_n now follows by an application of [1], as in the proof of Theorem 5.1. $\square$

Remark 5.1. The limit behavior in Theorem 5.3 can be analyzed using the methods of [4], where the concave majorant of Brownian motion without drift is characterized via a Poisson process of jump locations and Brownian excursions.

Appendix

Proof of Lemma 2.1. Part (i). Let $u > 0$. Then, for $x \geq u$,

$$\begin{aligned} V(x) &= W(x) + (x-u)^2 + 2u(x-u) + u^2 \\ &\geq W(x) + (x-u)^2 + u^2 \\ &= W(u) + u^2 + W(x) - W(u) + (x-u)^2. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{P}\Big(\min_{x\geq u} V(x) \leq 0\Big) &\leq \mathbb{P}\Big(\min_{x\geq u} W(u) + u^2 + W(x) - W(u) + (x-u)^2 \leq 0\Big) \\ &= \mathbb{P}\Big(W(u) + u^2 + \min_{x\geq u} W(x) - W(u) + (x-u)^2 \leq 0\Big) \\ &\leq \mathbb{P}\bigg(W(u) \leq -\frac{1}{2}u^2\bigg) + \mathbb{P}\bigg(\min_{x\geq u} W(x) - W(u) + (x-u)^2 \leq -\frac{1}{2}u^2\bigg). \end{aligned}$$

The process

$$x \mapsto W(x) - W(u) + (x-u)^2, \qquad x \geq u,$$

behaves in the same way as the process $t \mapsto V(t), t \geq 0$, but starts at x instead of 0. By Corollary 2.1 of [9], we have that for all $z > 0$,

$$\mathbb{P}\Big\{\min_{t\in\mathbb{R}} V(t) \leq -z\Big\} \sim 2 \cdot 3^{-1/2} \exp\big\{-8z^{3/2}/\sqrt{27}\big\}, \qquad z \to \infty, \tag{A.1}$$

implying that there exist positive constants c_1 and c_2 such that for all $u \geq 0$

$$\mathbb{P}\bigg(\min_{x\geq u} W(x) - W(u) + (x-u)^2 \leq -\frac{1}{2}u^2\bigg) \leq c_1 \exp\big\{-c_2 u^3\big\}.$$

We also have, for all $u > 0$,

$$\mathbb{P}\bigg\{W(u) < -\frac{1}{2}u^2\bigg\} = \mathbb{P}\bigg\{W(u)/\sqrt{u} < -\frac{1}{2}u^{3/2}\bigg\} \leq \frac{\exp\{-u^3/8\}}{u^{3/2}\sqrt{\pi/2}},$$

implying that there exist positive constants c_3 and c_4 such that for all $u \geq 0$,

$$\mathbb{P}\big\{W(u) < -\tfrac{1}{2}u^2\big\} \leq c_3 \exp\big\{-c_4 u^3\big\}.$$

Combining these upper bounds with the fact that the process V running to the left from 0 behaves in the same way as the process V running to the right from 0, part (i) now follows.

Part (ii). The (stationary) process $a \mapsto \tau(a) - a$ is studied in [5] and [6]. Theorem 2.5 in [6] shows that $\{\tau(a) : a \in \mathbb{R}\}$ is a Markovian pure jump process. Moreover, it states that given

$\tau(a-) = x$, the jump density at time a is given by

$$u \mapsto \frac{2g(x-a+u)up(u)}{g(x-a)\phi(x-a)}, \qquad u > 0, \tag{A.2}$$

and the conditional distribution function of the waiting time to the next jump is given by

$$F_{x-a}(b-a) = 1 - \exp\left\{-\int_{u=x-b}^{x-a} \phi(u)\,\mathrm{d}u\right\}, \qquad b-a > 0. \tag{A.3}$$

The functions g, p, and ϕ are specified in terms of Airy functions and power series in [6], and have the properties

$$\phi(t) \sim 2t^2, \qquad t \to -\infty, \qquad \phi(t) \sim \frac{1}{t}, \qquad t \to \infty,$$
$$p(t) \sim \left(2\pi t^3\right)^{-1/2}, \qquad t \downarrow 0, \qquad p(t) \sim 2e^{2^{1/3}\tilde{a}_1 t}$$

as $t \to \infty$, where $\tilde{a}_1$ denotes the largest 0 of Ai on the negative half-line and

$$g(x) = \frac{1}{2^{2/3}\pi} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}ux}}{\mathrm{Ai}(\mathrm{i}2^{-1/3}u)}\,\mathrm{d}u. \tag{A.4}$$

A picture of g using this representation is shown in Figure 2.

The meaning of this result is that we can generate the process V by first generating the stationary process $\{\tau(a) : a \in \mathbb{R}\}$, which is done by first generating $\tau(0)$ according to its (known) distribution, and then generating the points $\tau(a), a > 0$ and $\tau(a), a < 0$, using the waiting time distribution between jumps (A.3). Note that the distribution of the jump sizes is both space-

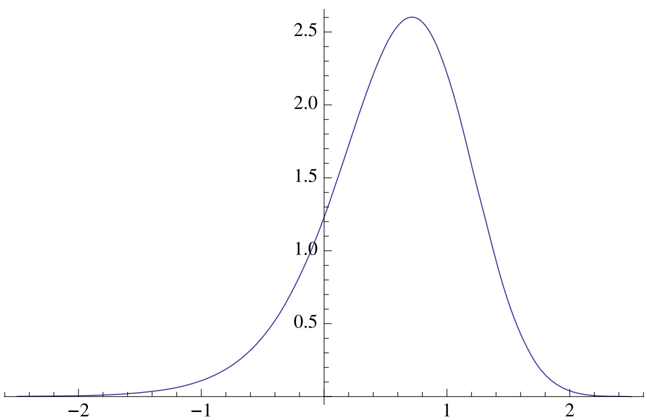


Figure 2. The function g, defined by (A.4).

dependent and time-dependent, not exponential. By part (iii) of Lemma 2.5 of [6], we have

$$\phi(-u) \sim 2u^2, \qquad u \to \infty.$$

This yields, for fixed $x, a \in \mathbb{R}$,

$$\int_{u=x-b}^{x-a} \phi(u)\,\mathrm{d}u \sim \frac{2}{3}(b-x)^3 \sim \frac{2}{3}b^3, \qquad b \to \infty,$$

implying that

$$\log\bigl\{1 - F_{x-a}(b-a)\bigr\} \sim -\tfrac{2}{3}b^3, \qquad b \to \infty,$$

in accordance with

$$\log\bigl(1 - \mathbb{P}\bigl\{\bigl|\tau(a) - a\bigr| > t\bigr\}\bigr) \sim -\tfrac{2}{3}t^3, \qquad t \to \infty;$$

see Corollary 3.4, part (iii), in [5]. Part (ii), particularly (2.4), now follow.

Part (iii). After generating the points $\tau(a)$, we can generate excursions of the Brownian path above the pieces of the greatest convex minorant, given by the jump times a_i, where we take a_1 as the first jump time to the right of 0 and number the jump times to the left and right from here. The slopes of the greatest convex minorant are in fact given by $\ldots, 2a_{-1}, 2a_0, 2a_1, \ldots$. The distribution of the excursions depends only on the duration of the intervals between successive jumps and the slope of the line segment of the convex minorant between these points. The only thing left to do is to pin down the paths at some point, and we do that by letting each path be 0 at time 0.

This construction reveals that the process of excursions $\{V(x) - C(x) : x \in \mathbb{R}\}$ inherits its stationarity from the stationarity of the point process $\{\tau(a) : a \in \mathbb{R}\}$, and that

$$A \in \sigma\bigl\{V(x) - C(x) : x \le 0\bigr\} \quad \text{and} \quad B \in \sigma\bigl\{V(x) - C(x) : x \ge m\bigr\} \tag{A.5}$$

are independent, given a jump time of the process of slopes of the greatest convex minorant between 0 and m and the slopes of the segments to the left and right of the vertex of the greatest convex minorant at this jump time. This implies that there also exist positive constants c_1 and c_2 such that

$$\bigl|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)\bigr| \le c_1 e^{-c_2 m^3}$$

for events A and B as defined in (A.5), noting that, by part (ii), the probability that there is no change of slope on the interval $[0, m]$ (meaning that the process τ has no jump in the interval $[0, m]$) is $\mathrm{O}(\exp\{-cm^3\})$ for a $c > 0$. □

Proof of Lemma 3.4. (i). The interval $I_{n,k}$ is bounded on the left by the interval $\tilde{J}_{n,k}$ and on the right by the interval $\bar{J}_{n,k+1}$. The intervals $\tilde{J}_{n,k}$ and $\bar{J}_{n,k+1}$ both have length of order $n^{-1/3}\sqrt{\log n}$. If the greatest convex minorant C_n of V_n on $[0, a]$ has changes of slope in the intervals $\tilde{J}_{n,k}$ and $\bar{J}_{n,k+1}$, the greatest convex minorant of V_n on $[0, a]$, restricted to the interval $I_{n,k}$, coincides with the greatest convex minorant C_{nk} of V_n on $L_{n,k}$, restricted to the interval $I_{n,k}$. Thus, we have to find bounds for the probability that the greatest convex minorant of V_n on $[0,a]$ has no changes of slope in $\tilde{J}_{nk}$ or $\bar{J}_{n,k+1}$. To do this, we follow the method used by [10], page 96.

Let a_{nk} and b_{nk} be the left and right endpoints of $\bar{J}_{n,k+1}$, respectively, and let u_{nk} be its midpoint. If

$$c_n(a_{nk}) < h_0(u_{nk}) < c_n(b_{nk}), \tag{A.6}$$

where c_n is the left-continuous slope of C_n, then C_n has a change of slope in the interval $\bar{J}_{n,k+1}$. Note that for $x \ge b_{nk}$, using the assumed smoothness of H_0 and $\inf_{[0,a]} h_0'(x) = 2\kappa > 0$,

$$\begin{aligned} V_n(x) - V_n(u_{nk}) \ge n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - W\left(\frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \\ + h_0(u_{nk})(x-u_{nk}) + \kappa(x-u_{nk})^2. \end{aligned} \tag{A.7}$$

Now consider the event that

$$c_n(b_{nk}) \le h_0(u_{nk}), \tag{A.8}$$

and let τ_{nk} be the first point of jump of c_n to the right of b_{nk}. Then

$$c_n(x) \le h_0(u_{nk}), \qquad x < \tau_{nk},$$

and thus,

$$V_n(\tau_{nk}) - V_n(x) \le C_n(\tau_{nk}) - C_n(x) = \int_x^{\tau_{nk}} c_n(y)\,\mathrm{d}y \le h_0(u_{nk})(\tau_{nk} - x), \qquad x < \tau_{nk}.$$

Using (A.7) and the stationarity of Brownian motion, this means that the probability of (A.8) is bounded above by

$$\begin{aligned} &\mathbb{P}\{V_n(\tau_{nk}) - V_n(u_{nk}) \le h_0(u_{nk})(\tau_{nk} - u_{nk})\} \\ &\quad \le \mathbb{P}\{\exists x \ge b_{nk} : V_n(x) - V_n(u_{nk}) \le h_0(u_{nk})(x - u_{nk})\} \\ &\quad \le \mathbb{P}\left\{\exists x \ge b_{nk} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)}\right) - W\left(\frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \le -\kappa(x-u_{nk})^2\right\} \\ &\quad = \mathbb{P}\left\{\exists x \ge b_{nk} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \le -\kappa(x-u_{nk})^2\right\}. \end{aligned} \tag{A.9}$$

We can see that this probability will become exponentially small. Toward this end, define the following covering of $[b_{nk}, a]$:

$$K_{nkj} \stackrel{\text{def}}{=} [t_{nkj}, t_{nk,j+1}] \stackrel{\text{def}}{=} \left[b_{nk} + jn^{-1/3}, b_{nk} + (j+1)n^{-1/3}\right] = [b_{nk}, a]$$

for $0 \le j \le \lfloor n^{1/3}(a - b_{nk}) \rfloor$, where the right endpoint of the last interval is taken to be a. Then the probability in (A.9) can be bounded above by

$$\sum_{j=0}^{\lfloor n^{1/3}(a-b_{nk})\rfloor} \mathbb{P}\left\{\exists x \in K_{nkj} : n^{-1/2}\left\{W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right)\right\} \le -\kappa(x-u_{nk})^2\right\}. \tag{A.10}$$

Denoting the probabilities in this sum by p_{nkj}, we get

$$\begin{aligned} p_{nkj} &\le \mathbb{P}\left\{\sup_{x\in K_{nkj}} W\left(\frac{F_0(x)}{1-F_0(x)} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})}\right) \ge \kappa\sqrt{n}(t_{nkj}-u_{nk})^2\right\} \\ &\le \mathbb{P}\left\{\sup_{0\le z\le F_0(t_{nk,j+1})/(1-F_0(t_{nk,j+1}))-F_0(u_{nk})/(1-F_0(u_{nk}))} W(z) \ge \kappa\sqrt{n}(t_{nkj}-u_{nk})^2\right\}. \end{aligned}$$

Because $t_{nk,j+1} \in [b_{nk}, a]$ for all j's under consideration,

$$0 \le \frac{F_0(t_{nk,j+1})}{1-F_0(t_{nk,j+1})} - \frac{F_0(u_{nk})}{1-F_0(u_{nk})} \le \frac{(F_0(t_{nk,j+1}) - F_0(u_{nk}))}{(1-F_0(a))^2} \le \lambda(t_{nk,j+1}-u_{nk})$$

for some $0 < \lambda < \infty$, we obtain, for a standard normal random variable Z,

$$\begin{aligned} p_{nkj} &\le \mathbb{P}\left\{\sup_{0\le z\le \lambda(t_{nk,j+1}-u_{nk})} W(z) \ge \kappa\sqrt{n}(t_{nkj}-u_{nk})^2\right\} \\ &= \mathbb{P}\left\{|Z| \ge \frac{\kappa\sqrt{n}(t_{nkj}-u_{nk})^2}{\sqrt{\lambda(t_{nk,j+1}-u_{nk})}}\right\} \\ &\le \mathbb{P}\left\{|Z| \ge \tilde{\kappa}\sqrt{n}(t_{nkj}-u_{nk})^{3/2}\right\} \\ &\le \frac{1}{2}\exp\left\{-\frac{1}{2}n\tilde{\kappa}^2(t_{nkj}-u_{nk})^3\right\}. \end{aligned}$$

Using that $t_{nkj} - u_{nk} = b_{nk} - u_{nk} + jn^{-1/3}$ and $b_{nk} - u_{nk} \sim \frac{1}{2}n^{-1/3}\sqrt{\log n}$, we get

$$p_{nkj} \le \exp\left\{-\frac{1}{2}\tilde{\kappa}\left((\log n)^{3/2} + j^3\right)\right\} \quad \Longrightarrow \quad \sum_{j=0}^{\lfloor n^{1/3}(a-b_{nk})\rfloor} p_{nkj} \le \rho\exp\left\{-\rho'(\log n)^{3/2}\right\}$$

for some $\rho, \rho' > 0$. In combination with (A.9) and (A.10), this bounds the probability of (A.8) from above. Because a similar bound holds for the probability of the event $c_n(a_{nk}) \ge h_0(u_{nk})$, the probability that (A.6) does not hold for a specific k, is bounded by a bound of the same structure. Moreover, because this upper bound does not depend on k and $m_n \sim an^{1/3}/\log n$, the probability that there exists a $1 \le k \le m_n$ for which (A.6) does not hold satisfies the same bound (with slight change in ρ and ρ'), this proves (i). Part (ii) is an immediate consequence of (i). □

Acknowledgements

We thank a referee for suggestions that have improved the readability of our paper.

References

[1] Dudley, R.M. and Norvaiša, R. (1999). *Differentiability of Six Operators on Nonsmooth Functions and p-Variation*. *Lecture Notes in Math*. **1703**. Berlin: Springer. With the collaboration of Jinghua Qian. MR1705318

[2] Durot, C. (2008). Testing convexity or concavity of a cumulated hazard rate. *IEEE Transactions on Reliability* **57** 465–473.

[3] Gijbels, I. and Heckman, N. (2004). Nonparametric testing for a monotone hazard function via normalized spacings. *J. Nonparametr. Stat*. **16** 463–477. MR2073036

[4] Groeneboom, P. (1983). The concave majorant of Brownian motion. *Ann. Probab.* **11** 1016–1027. MR0714964

[5] Groeneboom, P. (1989). Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields* **81** 79–109. MR0981568

[6] Groeneboom, P. (2011). Vertices of the least concave majorant of Brownian motion with parabolic drift. *Electron. J. Probab.* **16** 2234–2258. MR2861676

[7] Groeneboom, P. and Jongbloed, G. (2012). Smooth and non-smooth estimates of a monotone hazard. In *From Probability to Statistics and Back: High-Domensional Models and Processes. IMS Collections* **9** 174–196. Beachwood, OH: IMS.

[8] Groeneboom, P. and Jongbloed, G. (2012). Isotonic L_2-projection test for local monotonicity of a hazard. *J. Statist. Plann. Inference*. **142** 1644–1658. MR2903377

[9] Groeneboom, P. and Temme, N.M. (2011). The tail of the maximum of Brownian motion minus a parabola. *Electron. Commun. Probab.* **16** 458–466. MR2831084

[10] Groeneboom, P. and Wellner, J.A. (1992). *Information Bounds and Nonparametric Maximum Likelihood Estimation*. *DMV Seminar* **19**. Basel: Birkhäuser. MR1180321

[11] Hall, P. and Van Keilegom, I. (2005). Testing for monotone increasing hazard rate. *Ann. Statist*. **33** 1109–1137. MR2195630

[12] Huang, Y.C. and Dudley, R.M. (2001). Speed of convergence of classical empirical processes in p-variation norm. *Ann. Probab.* **29** 1625–1636. MR1880235

[13] Ibragimov, I.A. and Linnik, Y.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff. With a supplementary chapter by I.A. Ibragimov and V.V. Petrov, Translation from the Russian edited by J.F.C. Kingman. MR0322926

[14] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrsch. Verw. Gebiete* **32** 111–131. MR0375412

[15] Kulikov, V.N. and Lopuhaä, H.P. (2008). Distribution of global measures of deviation between the empirical distribution function and its concave majorant. *J. Theoret. Probab.* **21** 356–377. MR2391249

[16] Loève, M. (1963). *Probability Theory*, 3rd ed. Princeton: Van Nostrand. MR0203748

[17] Proschan, F. and Pyke, R. (1967). Tests for monotone failure rate. In *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability* (*Berkeley*, *Calif.*, 1965/66), *Vol. III*: *Physical Sciences* 293–312. Berkeley, CA: Univ. California Press. MR0224232

[18] Qian, J. (1998). The p-variation of partial sum processes and the empirical process. *Ann. Probab.* **26** 1370–1383. MR1640349

[19] Robertson, T., Wright, F.T. and Dykstra, R.L. (1988). *Order Restricted Statistical Inference*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Chichester: Wiley. MR0961262

[20] Rosenblatt, M. (1956). A central limit theorem and a strong mixing condition. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **42** 43–47. MR0074711

[21] Singpurwalla, N.D. and Wong, M.Y. (1983). Estimation of the failure rate—a survey of nonparametric methods. I. Non-Bayesian methods. *Comm*. *Statist*. *Theory Methods* **12** 559–588. MR0696809

Received May 2011 and revised February 2012