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Parisian ruin probability for spectrally negative Lévy processes

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In this note we give, for a spectrally negative Lévy process, a compact formula for the Parisian ruin probability, which is defined by the probability that the process exhibits an excursion below zero, with a length that exceeds a certain fixed period r. The formula involves only the scale function of the spectrally negative Lévy process and the distribution of the process at time r.

Keywords: Lévy process; Parisian ruin; risk process; ruin probability

1. Introduction

Let $X = \{X_t, t \geq 0\}$ be a spectrally negative Lévy process on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\colon t \geq 0\}, \mathbb{P})$; that is, X is a stochastic process issued from the origin which has stationary and independent increments and cadlag paths that have no positive jump discontinuities. To avoid degenerate cases, we exclude the case where X has monotone paths. As a strong Markov process, we shall endow X with probabilities $\{\mathbb{P}_x\colon x \in \mathbb{R}\}$, such that under $\mathbb{P}_x$, we have $X_0 = x$ with probability one. Further, $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$. Recall that $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$. For background on spectrally negative Lévy processes, we refer to Section 8 of [7].

In this paper we deal with the quantity κ_r with $r > 0$, which is defined by

$$\kappa_r = \inf\{t > r\colon\ t - g_t > r\}, \qquad \text{where } g_t = \sup\{0 \leq s \leq t\colon\ X_s \geq 0\}.$$

Hereby we make the convention that $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. The stopping time κ_r is the first time the process has stayed below zero for a consecutive period of length greater than r, and here we are interested in the probability that such an excursion occurs. The number r is referred to as the delay. Such *Parisian* stopping times have been studied by Chesney *et al.* [3] in the context of barrier options in mathematical finance. Dassios and Wu [5] introduced κ_r in actuarial risk theory: the process X under $\mathbb{P}_x$ with $x \geq 0$ is then used as a model for the surplus process of an insurance company with initial capital x, and the company is said to be Parisian ruined if an excursion as described above occurs. We therefore call κ_r the Parisian ruin time and the probability of the event $\{\kappa_r < \infty\}$ under $\mathbb{P}_x$ the Parisian ruin probability. Note that in risk theory, the classical term ruin is referred to as the event that the surplus process reaches a strictly negative level.

In another paper Dassios and Wu [6] gave a formula for the Parisian ruin probability and, more generally, the Laplace transform of the distribution of the Parisian ruin time in the case

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where X is a Brownian motion plus drift and in the case where X is the classical compound Poisson risk process with exponentially distributed claims. Recently, Czarna and Palmowski [4] gave a description of the Parisian ruin probability for a general spectrally negative Lévy process; hereby they split the analysis into two cases: one in which the underlying Lévy process has paths of bounded variation and one in which the process has paths of unbounded variation. In particular, their result allowed them to reproduce the formulas of Dassios and Wu [6] in the two aforementioned cases, and to tackle also the case where X is the classical compound Poisson risk process perturbed by Brownian motion and with exponentially distributed claims.

Another relevant paper is the one of Landriault *et al.* [8]. Here the authors study, for a spectrally negative Lévy process of bounded variation, a somewhat different type of Parisian stopping time, in which, loosely speaking, the deterministic, fixed delay r is replaced by an independent exponential random variable with a fixed parameter $p > 0$. To be a little bit more precise, each time the process starts a new excursion below zero, a new independent exponential random variable with parameter p is considered, and the stopping time of interest, let us denote it by $\kappa_{\exp(p)}$, is defined as the first time when the length of the excursion is bigger than the value of the accompanying exponential random variable. Although in insurance the stopping time $\kappa_{\exp(p)}$ is arguably less interesting than κ_r; working with exponentially distributed delays allowed the authors to obtain relatively simple expressions, for example, the Laplace transform of $\kappa_{\exp(p)}$ in terms of the so-called (q-)scale functions of X. In order to avoid a misunderstanding, we emphasize that, in the definition of $\kappa_{\exp(p)}$, by [8], there is not a single underlying exponential random variable, but a whole sequence (each attached to a separate excursion below zero); therefore $\mathbb{P}_x(\kappa_{\exp(p)} \in \mathrm{d}z)$ does not equal $\int_0^\infty p\mathrm{e}^{-pr}\mathbb{P}_x(\kappa_r \in \mathrm{d}z)$.

As the main result of our paper, we provide an expression for the Parisian ruin probability, which is considerably simpler than the one of Czarna and Palmowski [4], and which simultaneously holds for spectrally negative Lévy processes of bounded and unbounded variation. Before stating this result, we introduce a little extra notation.

The Laplace exponent of X is denoted by $\psi(\theta)$, that is,

$$\psi(\theta) = \log \mathbb{E}[\mathrm{e}^{\theta X_1}],$$

which is well defined for $\theta \geq 0$. We further introduce the scale function W of X (cf. [7], Section 8), which is the strictly increasing, continuous function uniquely defined on $[0, \infty)$ through its Laplace transform which is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W(x)\,\mathrm{d}x = \frac{1}{\psi(\theta)}, \qquad \theta > 0. \tag{1}$$

We extend W to the whole real line by setting $W(x) = 0$ for $x < 0$.

We now give the promised expression for the Parisian ruin probability. Hereby we assume that X drifts to infinity (equivalently, $\mathbb{E}[X_1] > 0$) as otherwise Parisian ruin happens with probability one. Recall that r stands for a strictly positive real number.

Theorem 1. *Assume* $\mathbb{E}[X_1] > 0$. *Then for any* $x \in \mathbb{R}$,

$$\mathbb{P}_x(\kappa_r < \infty) = 1 - \mathbb{E}[X_1]\frac{\int_0^\infty W(x+z)z\mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z)}. \tag{2}$$

We remark that the methodology of the proof of Theorem 1 can also be used to derive the Laplace transform of κ_r. However, since the proof would be considerably longer and the resulting expression much harder to (e.g., numerically) evaluate, we stick to presenting only the Parisian ruin probability.

In the next section we introduce some additional notation and recall some facts about spectrally negative Lévy processes, which are needed for the proof of Theorem 1 given in Section 3. We conclude by giving some specific examples for which the Parisian ruin probability can be expressed in a somewhat more explicit form.

2. Preliminaries

We define for $a \in \mathbb{R}$,

$$\tau_a^+ = \inf\{t > 0\colon\ X_t > a\}, \qquad \tau_a^- = \inf\{t > 0\colon\ X_t < a\}.$$

Note that since 0 is regular for the upper half-line for X (cf. [7], Section 8.1), we have, by the strong Markov property, $\tau_a^+ = \inf\{t > 0\colon\ X_t \geq a\}$.

If $\mathbb{E}[X_1] > 0$ (equivalently, $\psi'(0) > 0$), then

$$\mathbb{P}_x(\tau_0^- < \infty) = 1 - \mathbb{E}[X_1]W(x); \tag{3}$$

cf. [7], equation (8.7). We see in particular that, in this case, $W(x)$ is bounded by $1/\mathbb{E}[X_1]$.

The scale function W is absolutely continuous on $(0, \infty)$ (cf. [7], Lemma 8.2), and we denote a version of its density by W'. Further, $W(0) > 0$ if X has paths of bounded variation, and $W(0) = 0$ if X has paths of unbounded variation; cf. [7], Lemma 8.6. We further recall the following well-known expression for the Laplace transform of the first passage time above a:

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau_a^+}] = \mathrm{e}^{\Phi(\theta)(x-a)}, \qquad x \leq a, \theta \geq 0, \tag{4}$$

where $\Phi(\theta) = \sup\{\lambda \geq 0\colon\ \psi(\lambda) = \theta\}$; cf. [7], Section 8.1. We will also use Kendall's identity (cf. [1], Corollary VII.3), which relates the distribution of a spectrally negative Lévy process to the distribution of its upward passage time τ_z^+,

$$r\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\,\mathrm{d}z = z\mathbb{P}(X_r \in \mathrm{d}z)\,\mathrm{d}r. \tag{5}$$

Last, we need the following three identities.

Lemma 2. *For* $\theta > 0$,

$$\mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}}\mathrm{e}^{\Phi(\theta)X_{\tau_0^-}}\big] = \frac{\theta}{\Phi(\theta)}\int_0^\infty \mathrm{e}^{-\Phi(\theta)y}W'(x+y)\,\mathrm{d}y, \tag{6}$$

$$\int_0^\infty \mathrm{e}^{-\theta r}\int_y^\infty \frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)\,\mathrm{d}r = \frac{1}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)y}, \qquad y \geq 0, \tag{7}$$

$$\int_0^\infty W(z)\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) = 1. \tag{8}$$

Proof. Using the known identity (cf. [10], equation (35))

$$\mathbb{E}_x\big[\mathrm{e}^{pX_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\big]=\mathrm{e}^{px}-\psi(p)\mathrm{e}^{px}\int_0^x \mathrm{e}^{-pz}W(z)\,\mathrm{d}z-\frac{\psi(p)}{p}W(x),\qquad p>0,$$

by (1) and a change of variables and an integration by parts, we get

$$\begin{aligned}\mathbb{E}_x\big[\mathrm{e}^{\Phi(\theta)X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\big]&=\mathrm{e}^{\Phi(\theta)x}\left(1-\theta\int_0^x \mathrm{e}^{-\Phi(\theta)z}W(z)\,\mathrm{d}z\right)-\frac{\theta}{\Phi(\theta)}W(x)\\&=\mathrm{e}^{\Phi(\theta)x}\theta\int_x^\infty \mathrm{e}^{-\Phi(\theta)z}W(z)\,\mathrm{d}z-\frac{\theta}{\Phi(\theta)}W(x)\\&=\theta\int_0^\infty \mathrm{e}^{-\Phi(\theta)y}W(x+y)\,\mathrm{d}y-\frac{\theta}{\Phi(\theta)}W(x)\\&=\frac{\theta}{\Phi(\theta)}\int_0^\infty \mathrm{e}^{-\Phi(\theta)y}W'(x+y)\,\mathrm{d}y.\end{aligned}$$

This proves the first identity. For the second, we have, by Kendall's identity (5), Tonelli and (4),

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r}\int_y^\infty \frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\,\mathrm{d}r&=\int_0^\infty \mathrm{e}^{-\theta r}\int_y^\infty \mathbb{P}(\tau_z^+\in\mathrm{d}r)\,\mathrm{d}z\\&=\int_y^\infty \mathrm{e}^{-\Phi(\theta)z}\,\mathrm{d}z=\frac{1}{\Phi(\theta)}\mathrm{e}^{-\Phi(\theta)y}.\end{aligned}$$

In order to prove the last identity, we use again Kendall's identity (5), Tonelli and (4) combined with (1) to get

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r}\int_0^\infty W(z)\frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)\,\mathrm{d}r&=\int_0^\infty \mathrm{e}^{-\theta r}\int_0^\infty W(z)\mathbb{P}(\tau_z^+\in\mathrm{d}r)\,\mathrm{d}z\\&=\int_0^\infty \mathrm{e}^{-\Phi(\theta)z}W(z)\,\mathrm{d}z=\frac{1}{\theta}.\end{aligned}$$

Hence (8) follows by Laplace inversion (note that the left-hand side of (8) is continuous in r because a Lévy process is continuous in probability, and W is bounded and continuous). Note that (8) can also directly be deduced from [1], Corollary VII.16. □

3. Proof of Theorem 1

In the case where X has paths of unbounded variation, we will use a limiting argument in the proof. For this reason we introduce for $\varepsilon\geq 0$ the stopping time κ_r^ε, which is defined by

$$\kappa_r^\varepsilon=\inf\{t>r:\ t-g_t^\varepsilon>r,X_{t-r}<0\},\qquad \text{where } g_t^\varepsilon=\sup\{0\leq s\leq t:\ X_s\geq\varepsilon\}.$$

The stopping time κ_r^ε is the first time that an excursion starting when X gets below zero, ending before X gets back up to ε and having length greater than r, has occurred. Note that $\kappa_r^0 = \kappa_r$. We have for $x < 0$, by the strong Markov property and the absence of upward jumps,

$$\begin{aligned}\mathbb{P}_x(\kappa_r^\varepsilon < \infty) &= \mathbb{P}_x(\tau_\varepsilon^+ > r) + \mathbb{E}_x[\mathbb{P}_x(\tau_\varepsilon^+ \le r, \kappa_r^\varepsilon < \infty | \mathcal{F}_{\tau_\varepsilon^+})] \\ &= \mathbb{P}_x(\tau_\varepsilon^+ > r) + \mathbb{P}_x(\tau_\varepsilon^+ \le r)\mathbb{P}_\varepsilon(\kappa_r^\varepsilon < \infty) \\ &= 1 - \mathbb{P}_x(\tau_\varepsilon^+ \le r)\big(1 - \mathbb{P}_\varepsilon(\kappa_r^\varepsilon < \infty)\big).\end{aligned} \tag{9}$$

Using the above we have, for $x \ge 0$, again by the strong Markov property,

$$\begin{aligned}\mathbb{P}_x(\kappa_r^\varepsilon < \infty) &= \mathbb{E}_x[\mathbb{P}_x(\kappa_r^\varepsilon < \infty | \mathcal{F}_{\tau_0^-})] \\ &= \mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathbb{P}_{X_{\tau_0^-}}(\kappa_r^\varepsilon < \infty)\big] \\ &= \mathbb{P}_x(\tau_0^- < \infty) - \big(1 - \mathbb{P}_\varepsilon(\kappa_r^\varepsilon < \infty)\big)\mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathbb{P}_{X_{\tau_0^-}}(\tau_\varepsilon^+ \le r)\big].\end{aligned} \tag{10}$$

We proceed by finding an expression in terms of the scale function W and the law of X_r for the expectation in the right-hand side of (10). By spatial homogeneity, we can write, for $x \ge 0$,

$$\begin{aligned}\mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathbb{P}_{X_{\tau_0^-}}(\tau_\varepsilon^+ \le r)\big] &= \int_{[0,\infty)} \mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\}} \mathbb{P}_{-z}(\tau_\varepsilon^+ \le r)\big] \\ &= \int_{[0,\infty)} \mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty, -X_{\tau_0^-} \in \mathrm{d}z\}} \mathbb{P}(\tau_{\varepsilon+z}^+ \le r)\big].\end{aligned}$$

Since we have for $\theta, z > 0$ by an integration by parts and (4),

$$\int_0^\infty \mathrm{e}^{-\theta r} \mathbb{P}(\tau_{\varepsilon+z}^+ \le r)\,\mathrm{d}r = \frac{1}{\theta}\mathbb{E}[\mathrm{e}^{-\theta \tau_{\varepsilon+z}^+}] = \frac{1}{\theta}\mathrm{e}^{-\Phi(\theta)(z+\varepsilon)},$$

we deduce in combination with Tonelli, (6) and (7), that for $\theta > 0$ and $x \ge 0$,

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-\theta r} \mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathbb{P}_{X_{\tau_0^-}}(\tau_\varepsilon^+ \le r)\big]\,\mathrm{d}r \\ &\quad = \frac{1}{\theta}\mathrm{e}^{-\Phi(\theta)\varepsilon} \int_0^\infty \mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- < \infty\}} \mathrm{e}^{\Phi(\theta) X_{\tau_0^-}}\big] \\ &\quad = \int_0^\infty \frac{\mathrm{e}^{-\Phi(\theta)(y+\varepsilon)}}{\Phi(\theta)} W'(x+y)\,\mathrm{d}y \\ &\quad = \int_0^\infty W'(x+y) \int_0^\infty \mathrm{e}^{-\theta r} \int_{y+\varepsilon}^\infty \frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)\,\mathrm{d}r\,\mathrm{d}y.\end{aligned} \tag{11}$$

Hence by Tonelli and Laplace inversion (noting that both sides of the equation below are right-continuous in r), for $x \geq 0$,

$$\begin{aligned}\mathbb{E}_x\big[\mathbf{1}_{\{\tau_0^- <\infty\}}\mathbb{P}_{X_{\tau_0^-}}(\tau_\varepsilon^+ \leq r)\big] &= \int_\varepsilon^\infty \int_0^{z-\varepsilon} W'(x+y)\,\mathrm{d}y \frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z) \\ &= \int_\varepsilon^\infty [W(x+z-\varepsilon) - W(x)]\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z).\end{aligned} \tag{12}$$

The next step is to prove (2) for $x = 0$. For this we split the analysis into two cases, $W(0) > 0$ and $W(0) = 0$. First we consider the case $W(0) > 0$ (or equivalently X has paths of bounded variation). Then by (3), $\mathbb{P}(\tau_0^- < \infty) < 1$ and thus $\mathbb{E}[\mathbf{1}_{\{\tau_0^- <\infty\}}\mathbb{P}_{X_{\tau_0^-}}(\tau_0^+ \leq r)] < 1$. Using (10) and (12) with $x = \varepsilon = 0$ and (3) and (8), we get

$$\begin{aligned}\mathbb{P}(\kappa_r < \infty) &= \frac{\mathbb{P}(\tau_0^- < \infty) - \int_0^\infty [W(z) - W(0)](z/r)\mathbb{P}(X_r \in \mathrm{d}z)}{1 - \int_0^\infty [W(z) - W(0)](z/r)\mathbb{P}(X_r \in \mathrm{d}z)} \\ &= \frac{-W(0)\mathbb{E}[X_1] + W(0)\int_0^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z)}{W(0)\int_0^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z)} \\ &= \frac{\int_0^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z) - \mathbb{E}[X_1]}{\int_0^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z)}.\end{aligned}$$

Now we deal with the second case, that is, we assume $W(0) = 0$. Let $\varepsilon > 0$. Then $\mathbb{P}_\varepsilon(\tau_0^- < \infty) < 1$ (cf. (3)) and consequently using (10) and (12) with $x = \varepsilon$ and (3), we deduce

$$\begin{aligned}&\mathbb{P}_\varepsilon(\kappa_r^\varepsilon < \infty) \\ &\quad= \frac{\mathbb{P}_\varepsilon(\tau_0^- < \infty) - \int_\varepsilon^\infty [W(z) - W(\varepsilon)](z/r)\mathbb{P}(X_r \in \mathrm{d}z)}{1 - \int_\varepsilon^\infty [W(z) - W(\varepsilon)](z/r)\mathbb{P}(X_r \in \mathrm{d}z)} \\ &\quad= \frac{1/W(\varepsilon) - \mathbb{E}[X_1] + \int_\varepsilon^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z) - \int_\varepsilon^\infty (W(z)/W(\varepsilon))(z/r)\mathbb{P}(X_r \in \mathrm{d}z)}{1/W(\varepsilon) + \int_\varepsilon^\infty (z/r)\mathbb{P}(X_r \in \mathrm{d}z) - \int_\varepsilon^\infty (W(z)/W(\varepsilon))(z/r)\mathbb{P}(X_r \in \mathrm{d}z)}.\end{aligned} \tag{13}$$

We now want to compute the limit as $\varepsilon \downarrow 0$ of both sides of (13). To this end, recalling that $0 < \mathbb{E}[X_1] < \infty$, we have by (8), an integration by parts and l'Hôpital,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \frac{1 - \int_\varepsilon^\infty W(z)(z/r)\mathbb{P}(X_r \in \mathrm{d}z)}{W(\varepsilon)} \\ &\quad= \lim_{\varepsilon\downarrow 0} \int_0^\varepsilon \frac{W(z)}{W(\varepsilon)}(z/r)\mathbb{P}(X_r \in \mathrm{d}z) \\ &\quad= \lim_{\varepsilon\downarrow 0}\left(\int_0^\varepsilon \frac{y}{r}\mathbb{P}(X_r \in \mathrm{d}y) - \int_0^\varepsilon \frac{W'(z)}{W(\varepsilon)}\int_0^z \frac{y}{r}\mathbb{P}(X_r \in \mathrm{d}y)\,\mathrm{d}z\right) \\ &\quad= 0 - \lim_{\varepsilon\downarrow 0} \frac{W'(\varepsilon)\int_0^\varepsilon (y/r)\mathbb{P}(X_r \in \mathrm{d}y)}{W'(\varepsilon)} = 0.\end{aligned} \tag{14}$$

For the limit as $\varepsilon \downarrow 0$ of the left-hand side of (13), we introduce, for $\varepsilon > 0$, the stopping time

$$\widetilde{\kappa}_r^{\varepsilon} = \inf\{t > r\colon\ t - g_t > r, X_{t-r} < -\varepsilon\}, \qquad \text{where } g_t = \sup\{0 \le s \le t\colon\ X_s \ge 0\}.$$

We easily see that for $0 < \varepsilon' < \varepsilon$, $\{\widetilde{\kappa}_r^{\varepsilon} < \infty\} \subset \{\widetilde{\kappa}_r^{\varepsilon'} < \infty\}$ and $\bigcup_{\varepsilon>0}\{\widetilde{\kappa}_r^{\varepsilon} < \infty\} = \{\kappa_r < \infty\}$. Hence, by spatial homogeneity,

$$\lim_{\varepsilon\downarrow 0} \mathbb{P}_{\varepsilon}(\kappa_r^{\varepsilon} < \infty) = \lim_{\varepsilon\downarrow 0} \mathbb{P}(\widetilde{\kappa}_r^{\varepsilon} < \infty) = \mathbb{P}(\kappa_r < \infty),$$

and, combined with (13) and (14), this leads to

$$\mathbb{P}(\kappa_r < \infty) = \frac{\int_0^{\infty} (z/r)\mathbb{P}(X_r \in \mathrm{d}z) - \mathbb{E}[X_1]}{\int_0^{\infty} (z/r)\mathbb{P}(X_r \in \mathrm{d}z)}. \tag{15}$$

Hence, recalling (8), we have shown in both cases that (2) holds for $x = 0$. Now plugging (3) and the above expression for $\mathbb{P}(\kappa_r < \infty)$ into (10) and using (12) with $\varepsilon = 0$, we arrive at (2) for any $x \ge 0$.

We now prove that (2) also holds for $x < 0$. For $x < 0$, setting $\varepsilon = 0$ in (9) and applying (2) (with $x = 0$) in combination with (8), we get

$$\mathbb{P}_x(\kappa_r < \infty) = 1 - \mathbb{E}[X_1] \frac{r\mathbb{P}_x(\tau_0^+ \le r)}{\int_0^{\infty} z\mathbb{P}(X_r \in \mathrm{d}z)}.$$

The proof will be completed once we show that for $x < 0$

$$\mathbb{P}_x(\tau_0^+ \le r) = \frac{1}{r}\int_0^{\infty} W(x+z) z\mathbb{P}(X_r \in \mathrm{d}z).$$

The above identity follows by showing that the Laplace transforms in r of both sides are equal. To this end, note that by an integration by parts and (4), the Laplace transform of the left-hand side is equal to

$$\int_0^{\infty} \mathrm{e}^{-\theta r}\mathbb{P}_x(\tau_0^+ \le r)\,\mathrm{d}r = \frac{1}{\theta}\mathbb{E}[\mathrm{e}^{-\theta\tau_{-x}^+}] = \frac{1}{\theta}\mathrm{e}^{\Phi(\theta)x},$$

and by Tonelli, Kendall's identity (5), (4) and (1), the Laplace transform of the left-hand side equals

$$\begin{aligned}
\int_0^{\infty} \mathrm{e}^{-\theta r}\frac{1}{r}\int_0^{\infty} W(x+z) z\mathbb{P}(X_r \in \mathrm{d}z)\,\mathrm{d}r &= \int_0^{\infty} W(x+z)\int_0^{\infty} \mathrm{e}^{-\theta r}\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\,\mathrm{d}z \\
&= \int_0^{\infty} W(x+z)\mathrm{e}^{-\Phi(\theta)z}\,\mathrm{d}z \\
&= \frac{1}{\psi(\Phi(\theta))}\mathrm{e}^{\Phi(\theta)x} \\
&= \frac{1}{\theta}\mathrm{e}^{\Phi(\theta)x}.
\end{aligned}$$

Note that in the third equality we used that $x < 0$.

4. Examples

If one wants to evaluate formula (2), one needs to know the scale function W and the distribution of X_r. There are plenty of examples of spectrally negative Lévy processes for which the distribution at a fixed time has a closed-form expression and, thanks to recent developments (for the latest we refer to Section 3 of Chazal *et al.* [2]), there are also a lot of examples for which the scale function is known explicitly. Unfortunately, it seems that there are only a few examples for which both of them are known in closed-form. Below we give three examples for which this is the case and for which we give the corresponding expression for the Parisian ruin probability. In the other case, one can resort to numerically inverting the Laplace transform (which recall is explicitly given in terms of ψ) of one or both of the two quantities.

4.1. Brownian motion

Let $X_t = \mu t + \sigma B_t$, with $\mu, \sigma > 0$ and $\{B_t, t \geq 0\}$ a Brownian motion. Then $\psi(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2$, and we easily deduce from (1) that

$$W(x) = \frac{1}{\mu}\bigl(1 - \mathrm{e}^{-2(\mu/\sigma^2)x}\bigr), \qquad x \geq 0.$$

Hence by (2) for $x \geq 0$,

$$\mathbb{P}_x(\tau_r < \infty) = \mathrm{e}^{-2(\mu/\sigma^2)x} \frac{\int_0^\infty \mathrm{e}^{-2(\mu/\sigma^2)z} z\mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z)}.$$

Setting $x = 0$ in above formula, comparing with (15) and realizing that $\mathbb{E}[X_r] = r\mathbb{E}[X_1] = \mu r$, we see that

$$\int_0^\infty \mathrm{e}^{-2(\mu/\sigma^2)z} z\mathbb{P}(X_r \in \mathrm{d}z) = \int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z) - \mu r.$$

Noting that X_r has a normal distribution with mean μr and variance $\sigma^2 r$ and making the change of variables $y = \frac{z-\mu r}{\sigma\sqrt{r}}$, we get

$$\begin{aligned}
\int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z) &= \frac{1}{\sqrt{2\pi\sigma^2 r}} \int_0^\infty z\mathrm{e}^{-(z-\mu r)^2/(2\sigma^2 r)}\,\mathrm{d}z \\
&= \frac{\sigma\sqrt{r}}{\sqrt{2\pi}} \int_{-\mu\sqrt{r}/\sigma}^\infty y\mathrm{e}^{-y^2/2}\,\mathrm{d}y + \frac{\mu r}{\sqrt{2\pi}} \int_{-\mu\sqrt{r}/\sigma}^\infty \mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&= \frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\mu^2 r/(2\sigma^2)} + \mu r\mathcal{N}\biggl(\frac{\mu\sqrt{r}}{\sigma}\biggr),
\end{aligned}$$

where $\mathcal{N}$ is the cumulative distribution function of a standard normal random variable. Putting everything together results in

$$\mathbb{P}_x(\tau_r < \infty) = \mathrm{e}^{-2(\mu/\sigma^2)x}\biggl(\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\mu^2 r/(2\sigma^2)} - \mu r\mathcal{N}\biggl(-\frac{\mu\sqrt{r}}{\sigma}\biggr)\biggr) \Big/ \biggl(\frac{\sigma\sqrt{r}}{\sqrt{2\pi}}\mathrm{e}^{-\mu^2 r/(2\sigma^2)} + \mu r\mathcal{N}\biggl(\frac{\mu\sqrt{r}}{\sigma}\biggr)\biggr), \qquad x \geq 0,$$

which agrees with the formula found by Dassios and Wu [6].

4.2. Classical risk process with exponential claims

Let $X_t = ct - \sum_{i=1}^{N_t} C_i$, where $\{N_t, t \geq 0\}$ is a Poisson process with rate η and $C_1, C_2, \ldots$ are i.i.d. exponentially distributed random variables with parameter α independent of N_t. Its Laplace exponent is given by $\psi(\theta) = c\theta - \eta + \eta\frac{\alpha}{\theta+\alpha}$. Assume that $c > \eta/\alpha$, so that $\mathbb{E}[X_1] = \psi'(0) = c - \eta/\alpha > 0$. From (1) it easily follows that

$$W(x) = \frac{1}{c - \eta/\alpha}\biggl(1 - \frac{\eta}{c\alpha}\mathrm{e}^{(\eta/c-\alpha)x}\biggr), \qquad x \geq 0.$$

Recalling that a sum of i.i.d. exponential random variables equals a gamma random variable and utilizing the independence between C_i and N_t, we get

$$\begin{aligned}\mathbb{P}\biggl(\sum_{i=1}^{N_r} C_i \in \mathrm{d}y\biggr) &= \sum_{k=0}^{\infty}\mathbb{P}\biggl(\sum_{i=0}^{k} C_i \in \mathrm{d}y\biggr)\mathbb{P}(N_r = k)\\ &= \mathrm{e}^{-\eta r}\delta_0(\mathrm{d}y) + \sum_{k=1}^{\infty}\alpha^k\frac{y^{k-1}\mathrm{e}^{-\alpha y}}{(k-1)!}\frac{(\eta r)^k}{k!}\mathrm{e}^{-\eta r}\,\mathrm{d}y\\ &= \mathrm{e}^{-\eta r}\biggl(\delta_0(\mathrm{d}y) + \mathrm{e}^{-\alpha y}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!(m+1)!}y^m\,\mathrm{d}y\biggr),\end{aligned}$$

where $\delta_0(\mathrm{d}y)$ is the Dirac mass at 0. It follows that

$$\begin{aligned}\int_0^{\infty} z\mathbb{P}(X_r \in \mathrm{d}z) &= \int_0^{cr} z\mathrm{e}^{-\eta r}\biggl(\delta_0(cr - \mathrm{d}z) + \mathrm{e}^{-\alpha(cr-z)}\sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!(m+1)!}(cr - z)^m\,\mathrm{d}z\biggr)\\ &= \mathrm{e}^{-\eta r}\biggl(cr + \sum_{m=0}^{\infty}\frac{(\alpha\eta r)^{m+1}}{m!(m+1)!}\int_0^{cr}\mathrm{e}^{-\alpha y}(cr - y)y^m\,\mathrm{d}y\biggr)\\ &= \mathrm{e}^{-\eta r}\biggl(cr + \sum_{m=0}^{\infty}\frac{(\eta r)^{m+1}}{m!(m+1)!}\biggl[cr\Gamma(m+1, cr\alpha) - \frac{1}{\alpha}\Gamma(m+2, cr\alpha)\biggr]\biggr),\end{aligned}$$

where $\Gamma(a,x)=\int_0^x e^{-t}t^{a-1}\,dt$ is the incomplete gamma function. Putting everything together and using the same trick as in the previous example leads, for $x\ge 0$, to

$$\begin{aligned}
&\mathbb{P}_x(\kappa_r<\infty)\\
&\quad= e^{(\eta/c-\alpha)x}\frac{\int_0^\infty z\mathbb{P}(X_r\in dz)-(c-\eta/\alpha)r}{\int_0^\infty z\mathbb{P}(X_r\in dz)}\\
&\quad= e^{(\eta/c-\alpha)x}\Bigg(1-e^{\eta r}(c-\eta/\alpha)\\
&\qquad\qquad \Big/\Bigg(c+\sum_{m=0}^{\infty}\frac{(\eta r)^{m+1}}{m!(m+1)!}\Bigg[c\Gamma(m+1,cr\alpha)-\frac{1}{\alpha r}\Gamma(m+2,cr\alpha)\Bigg]\Bigg)\Bigg).
\end{aligned}$$

Although it does not seem easy to show directly that the above expression is equal to the one found by Dassios and Wu [6], one can check numerically that the two expressions match.

4.3. Stable process with index 3/2

Let $X_t=ct+Z_t$, where $c>0$ and $\{Z_t,t\ge 0\}$ is a spectrally negative α-stable process with $\alpha=3/2$. The Laplace exponent of X is given by $\psi(\theta)=c\theta+\theta^{3/2}$. One can, in a straightforward way, check via (1) that

$$W(x)=\frac{1}{c}\big[1-E_{1/2}\big(-c\sqrt{x}\big)\big],\qquad x\ge 0,$$

where $E_{1/2}(z)$ is the Mittag–Leffler function of order 1/2, that is, $E_{1/2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma((1/2)k+1)}$. It follows that, for $x\ge 0$,

$$\mathbb{P}_x(\kappa_r<\infty)=\frac{\int_0^\infty E_{1/2}(-c\sqrt{x+z})z\mathbb{P}(Z_r\in dz-cr)}{\int_0^\infty z\mathbb{P}(Z_r\in dz-cr)}.$$

Using the scaling property of stable processes (see, e.g., [1], Section VIII.1) and the expression for the density of the underlying stable distribution taken from Schneider [11], equations (3.4) and (3.5), we get the following expression for the distribution of Z_r:

$$\mathbb{P}(Z_r\in dy)=\mathbb{P}(r^{2/3}Z_1\in dy)=\begin{cases}\sqrt{\dfrac{3}{\pi}}r^{2/3}y^{-1}e^{-u/2}W_{1/2,1/6}(u)\,dy, & y>0,\\ -\dfrac{1}{2\sqrt{3\pi}}r^{2/3}y^{-1}e^{u/2}W_{-1/2,1/6}(u)\,dy, & y<0,\end{cases}$$

where $u=\frac{4}{27}r^{9/2}|y|^3$ and $W_{\kappa,\mu}(z)$ is Whittaker's W-function; see, for example, [9], Section 9.13.

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