

Consistent nonparametric Bayesian inference for discretely observed scalar diffusions

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We study Bayes procedures for the problem of nonparametric drift estimation for one-dimensional, ergodic diffusion models from discrete-time, low-frequency data. We give conditions for posterior consistency and verify these conditions for concrete priors, including priors based on wavelet expansions.

Keywords: Bayesian nonparametrics; drift function; posterior consistency; posterior distribution; stochastic differential equations; wavelets

1. Introduction

Consider the one-dimensional diffusion model

$$dX_t = b(X_t) dt + dW_t, \quad t \geq 0, \quad (1.1)$$

where W is a standard Brownian motion and b is an unknown drift function belonging to a class of functions \mathcal{B} . We will make assumptions on b , stated precisely in the next section, ensuring that (1.1) has a unique stationary solution X . The aim is to make inference about b on the basis of discrete-time observations $X_0, X_\Delta, \dots, X_{n\Delta}$, for some fixed sampling frequency $1/\Delta$.

Under appropriate conditions the solution to (1.1) is a positively recurrent, ergodic Markov process with a unique invariant probability distribution. Moreover, under mild regularity conditions the process has transition densities $p_b(t, x, y)$ relative to Lebesgue measure. In this case, we can employ a Bayes procedure for making inference about the drift function b . This involves putting a prior distribution Π on the set of drift functions \mathcal{B} and computing the posterior $\Pi(\cdot | X_0, X_\Delta, \dots, X_{n\Delta})$. If the initial distribution is the invariant probability measure with density π_b , the posterior measure of a measurable set $B \subset \mathcal{B}$ is given by

$$\Pi(B | X_0, \dots, X_{n\Delta}) = \frac{\int_B \pi_b(X_0) \prod_{i=1}^n p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta}) \Pi(db)}{\int_{\mathcal{B}} \pi_b(X_0) \prod_{i=1}^n p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta}) \Pi(db)}.$$

(We assume of course the necessary measurability to ensure that this is well defined.)

This immediately reveals a practical complication, since the transition densities of a diffusion process can typically not be computed explicitly. Several approaches have been proposed

in the literature to circumvent this problem. These include for instance simulation-based methods for approximating the transition densities, or Y. Aït-Sahalia's closed-form expansions, cf. for example, Jensen and Poulsen [17] for an overview. A method that has been proven to be particularly useful for dealing with Bayes procedures is to view the continuous segments of the diffusion process between the observations as missing data and to employ a Gibbs sampling scheme. Practically this involves repeatedly simulating diffusion bridges to generate the missing data and drawing from the posterior distribution of b given the augmented, continuous data $(X_t: t \in [0, n\Delta])$. Several schemes have been devised to simulate the diffusion bridges, see, for example, Elerian *et al.* [8], Eraker [9], Roberts and Stramer [20], Beskos *et al.* [2], Gollightly and Wilkinson [13] and Chib *et al.* [5]. Drawing from the continuous-data posterior can be done by more conventional methods, because contrary to the discrete-observations likelihood, the continuous-data likelihood has a known closed form expression given by Girsanov's theorem.

For parametric models, where the drift function is known up to a Euclidean parameter θ that has to be estimated, the outlined approach has been shown to provide an effective method for dealing with discretely observed diffusions. The approach is however not essentially limited to a parametric setup. The methodology has great potential to be developed into a practically feasible methodology in nonparametric settings as well. It is however very well known that in Bayesian nonparametrics the choice of the prior distribution is crucial and posterior consistency is not automatically guaranteed (e.g., Diaconis and Freedman [7]). This motivates the study of posterior consistency for discretely observed diffusions carried out in this paper.

In the i.i.d.-setting, sufficient conditions for posterior consistency were first obtained by Schwartz [22]. See also Barron *et al.* [1], Ghosal and van der Vaart [10] and Shen and Wasserman [23]. Here we consider discrete observations from a diffusion model (1.1), which constitute a Markov chain. A number of recent papers have investigated the problem of posterior consistency or convergence rates for Markov data, cf. for example, Ghosal and van der Vaart [12], Ghosal and Tang [11], Tang and Ghosal [24]. The results in these papers do however not immediately lead to practically useful results for our setting. The problem lies again in the fact that in our case, the transition densities of the model are typically not analytically tractable. Since the conditions for consistency given for instance by Tang and Ghosal [24] involve the transition densities, they cannot be readily used to verify consistency for a given prior in our discretely observed diffusion model. The aim in this paper is to formulate conditions involving only the coefficients appearing in the stochastic differential equation (1.1). We achieve this by adapting the results of Tang and Ghosal [24] to the present setting. Basically, we need two assumptions. Firstly, if μ_0 denotes the true invariant probability measure of the process X , we require that the prior puts positive mass on balls $\{b \in \mathcal{B}: \|b - b_0\|_{2, \mu_0} < \varepsilon\}$ for each $\varepsilon > 0$ ($\|\cdot\|_{2, \mu_0}$ denotes a weighted L^2 -norm and b_0 denotes the true drift). This is a natural condition, since if the prior excludes the true drift, consistency can never be obtained. Secondly, we need an equicontinuity assumption (Definition 3.4), which limits the size, or rather the complexity, of the set of drift functions. Under these assumptions, we obtain posterior consistency (Theorem 3.5): the posterior measure of appropriately defined weak neighborhoods of the true drift function b_0 converges to 1 almost surely, as the number of observations n tends to infinity. This is the main result of the paper.

Ghosal and van der Vaart [12] give conditions from which the posterior rate of convergence for Markov chain data can be calculated. These conditions are a combination of a prior mass

condition and a testing condition. This testing condition requires that one can test the true drift function against balls of alternatives with exponentially decaying error probabilities. Such tests are not easily constructed in the present setup. Appropriate tests for Markov chains have been shown to exist under certain (lower) bounds on the transition probabilities (e.g., [3]). In our setup such bounds do however not seem to be valid in general. An interesting line of future research would be to extend or adapt the available testing results for Markov chains to the setting of discretely observed diffusions. This may not only give posterior consistency results in a stronger topology, but may pave the way for obtaining posterior rates of convergence as well. In the present paper, we completely avoid the construction of tests. Instead we employ martingale arguments in a similar fashion as Tang and Ghosal [24], who adapted the approach of Walker [25] to the Markov chain setting.

The remainder of the paper is organized as follows. In Section 2, preliminaries on the statistical model and Bayes procedure are outlined. The main consistency result of this paper is formulated in Section 3. Examples of priors that satisfy the requirements for consistency are given in Section 4. The paper ends with a proof of the main result and some concluding remarks. The Appendix contains a technical lemma.

1.1. Notation

$\|g\|_{p,\nu} = (\int |g|^p d\nu)^{1/p}$: L_p -norm relative to the measure ν .

$L^2(\mu)$: space of square integrable functions with respect to measure μ .

$C(A)$, $BC(A)$: space of continuous functions, space of bounded continuous functions defined on $A \subseteq \mathbb{R}$.

$C^s(\mathbb{R})$, for $s \in (0, 1)$: space of s -Hölder functions, that is,

$$C^s(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) : \|f\|_s = \sup_{x,h} \frac{|f(x+h) - f(x)|}{|h|^s} < \infty \right\}. \quad (1.2)$$

$\mathcal{L}(X)$: law of a random variable X .

\mathbb{P}_μ^b : law that the solution of the SDE (1.1), with $\mathcal{L}(X_0) = \mu$, generated on the canonical path space $C(\mathbb{R}_+)$.

μ_b : invariant measure.

\mathbb{P}_b : short-hand notation for $\mathbb{P}_{\mu_b}^b$.

\mathbb{P}_x^b : short-hand notation for $\mathbb{P}_{\delta_x}^b$, where δ_x denotes Dirac measure at $x \in \mathbb{R}$.

μ_0 : short-hand notation for μ_{b_0} .

π_b : density of invariant measure.

$(P_t^b)_{t \geq 0}$: transition semigroup associated with the diffusion.

$p_b(t, x, y)$: transition density.

2. Setup

2.1. Description of the diffusion model

In this section, we give a precise description of the diffusion model that we consider. Let $\mathcal{B} \subset C(\mathbb{R})$ be a collection of continuous functions on \mathbb{R} . For $b \in \mathcal{B}$ and a fixed number $c \in \mathbb{R}$, let the function $s_b : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$s_b(x) = \int_c^x \exp\left(-2 \int_c^y b(z) dz\right) dy.$$

We assume that

$$\lim_{x \downarrow -\infty} s_b(x) = -\infty, \quad \lim_{x \uparrow \infty} s_b(x) = \infty$$

for all $b \in \mathcal{B}$. The finiteness (or nonfiniteness) of these limits does not depend on the choice of c (see page 339 in Karatzas and Shreve [19]). It is classical that under these assumptions, we have that for every $x \in \mathbb{R}$ and $b \in \mathcal{B}$, the SDE

$$dX_t = b(X_t) dt + dW_t, \quad X_0 = x,$$

has a unique weak solution. Let \mathbb{P}_x^b denote the law that this solution generates on the canonical path space $C(\mathbb{R}_+)$. Then in the commonly used terminology of Itô and McKean [16] or Kallenberg [18], Chapter 23, the collection of laws $(\mathbb{P}_x^b : x \in \mathbb{R})$ constitutes a canonical, recurrent diffusion on the real line. In other words, for X the canonical process on $\Omega = C(\mathbb{R}_+)$ defined by $X_t(\omega) = \omega(t)$ we have the following:

- (i) Under \mathbb{P}_x^b the process X starts in x , that is, $\mathbb{P}_x^b(X_0 = x) = 1$ for all $x \in \mathbb{R}$.
- (ii) The process X is strong Markov.
- (iii) For all $x \in \mathbb{R}$, the process X is recurrent under \mathbb{P}_x^b .

For a probability measure μ on \mathbb{R} we define, as usual, $\mathbb{P}_\mu^b(B) = \int \mathbb{P}_x^b(B) \mu(dx)$ for a measurable set B . Then under \mathbb{P}_μ^b the law of X_0 equals μ and X is the weak solution of

$$dX_t = b(X_t) dt + dW_t, \quad \mathcal{L}(X_0) = \mu.$$

As the notation suggests, s_b is the scale function of the diffusion. The speed measure is denoted by m_b . In the present setting, it is the Borel measure on \mathbb{R} given by

$$m_b(dx) = \exp\left(2 \int_0^x b(z) dz\right) dx. \tag{2.1}$$

We assume that the speed measure is finite, that is, $m_b(\mathbb{R}) < \infty$. This ensures that the diffusion is positively recurrent and ergodic in the sense that for all $x \in \mathbb{R}$,

$$X_t \xrightarrow{\mathbb{P}_x^b} \mu_b \tag{2.2}$$

as $t \rightarrow \infty$, where $\mu_b = m_b/m_b(\mathbb{R})$ is the normalized speed measure (cf., e.g., Kallenberg [18], Theorem 23.15). We will write $\mu_0 = \mu_{b_0}$. The measure μ_b is the unique invariant probability measure of the diffusion. In particular, the process X is stationary under $\mathbb{P}_{\mu_b}^b$. It is easily verified that under our conditions, μ_b has a continuously differentiable Lebesgue density π_b . Moreover, it follows from (2.1) that we have the relation

$$b = \frac{\pi_b'}{2\pi_b}. \quad (2.3)$$

We denote the transition semigroup associated to the diffusion by $(P_t^b)_{t \geq 0}$. In other words, for a bounded measurable function f on \mathbb{R} and $x \in \mathbb{R}$ we have $P_t^b f(x) = \mathbb{E}_x^b f(X_t)$, where \mathbb{E}_x^b is the expectation associated to \mathbb{P}_x^b . The operator P_t^b maps the space $BC(\mathbb{R})$ of bounded, continuous functions on \mathbb{R} into itself (see, e.g., (the proof of) Theorem 23.13 of Kallenberg [18], or Rogers and Williams [21], Proposition V.50.1). A regular diffusion as we are considering is known to have positive transition densities with respect to its speed measure, cf., for example, Itô and McKean [16], Section 4.11. Since the speed measure has a positive Lebesgue density under our assumptions, we have in fact the existence of transition densities $p_b: (0, \infty) \times \mathbb{R} \times \mathbb{R} \rightarrow (0, \infty)$ such that for all bounded, measurable functions f , $x \in \mathbb{R}$ and $t > 0$,

$$P_t^b f(x) = \int_{\mathbb{R}} p_b(t, x, y) f(y) dy.$$

For more background on the theory of one-dimensional diffusions and relevant references to the literature, see, for instance, Borodin and Salminen [4].

2.2. Statistical model and Bayes procedure

Consider the setting described in the preceding section, that is, we have a collection $\mathcal{B} \subset C(\mathbb{R})$ such that every $b \in \mathcal{B}$ determines an SDE that generates an ergodic diffusion on \mathbb{R} . For $b \in \mathcal{B}$, let \mathbb{P}_b be defined by $\mathbb{P}_b = \mathbb{P}_{\mu_b}^b$. In other words, under \mathbb{P}_b the canonical process X on $C(\mathbb{R}_+)$ is the unique stationary solution of the SDE

$$dX_t = b(X_t) dt + dW_t.$$

We assume that for some fixed $\Delta > 0$ and a natural number n , we have $n + 1$ observations $X_0, X_\Delta, \dots, X_{n\Delta}$ from X under \mathbb{P}_{b_0} , for some “true” drift function $b_0 \in \mathcal{B}$. The aim is to infer the drift function b_0 from these data.

In our Bayesian approach, we assume that the model \mathcal{B} is a measurable subset of $C(\mathbb{R})$ and we put a prior distribution Π on it. Next, we consider the posterior distribution $\Pi(\cdot | X_0, \dots, X_{n\Delta})$ on \mathcal{B} , which is given by

$$\Pi(B | X_0, \dots, X_{n\Delta}) = \frac{\int_B \pi_b(X_0) \prod_{i=1}^n p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta}) \Pi(db)}{\int_{\mathcal{B}} \pi_b(X_0) \prod_{i=1}^n p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta}) \Pi(db)}.$$

In the next section, we provide sufficient conditions under which the posterior asymptotically concentrates its mass around the true drift function b_0 as $n \rightarrow \infty$.

3. Consistency

We are interested in conditions under which the posterior asymptotically concentrates its mass around the true drift function b_0 . More precisely, we want that under \mathbb{P}_{b_0} the posterior mass concentrates on arbitrarily small neighborhoods of b_0 . To ensure that neighborhoods of points $b \neq b_0$ do not receive posterior mass in the limit, the topology we use to define the neighborhoods should have some separation properties, it should for instance be Hausdorff.

We define a weak topology on \mathcal{B} through the transition operators P_Δ^b (see Section 2). This is justified by the following lemma, which states that identifying the drift parameter b is in our setting equivalent to identifying P_Δ^b .

Lemma 3.1. *If $P_t^b = P_t^{b'}$ for some $t > 0$, then $b = b'$.*

Proof. Fix an $x \in \mathbb{R}$ and $b \in \mathcal{B}$. By the semigroup property, the law of X_{nt} under \mathbb{P}_x^b is determined by P_t^b . Indeed, for f a bounded measurable function and n a natural number we have

$$\mathbb{E}_x^b f(X_{nt}) = (P_t^b)^n f(x).$$

On the other hand, ergodicity implies that the law of X_{nt} under \mathbb{P}_x^b converges weakly to the invariant distribution μ_b , cf. (2.2). It follows that P_t^b completely determines μ_b . By (2.3), μ_b completely determines b under our assumptions. \square

Now let ν be a finite Borel measure on the state space \mathbb{R} . For $b \in \mathcal{B}$, $f \in BC(\mathbb{R})$ and $\varepsilon > 0$, let

$$U_{f,\varepsilon}^b = \{b' \in \mathcal{B}: \|P_\Delta^{b'} f - P_\Delta^b f\|_{1,\nu} < \varepsilon\}.$$

Consider the topology on \mathcal{B} that is determined by the requirement that for $b \in \mathcal{B}$, the collection of sets

$$\{U_{f,\varepsilon}^b: f \in BC(\mathbb{R}), \varepsilon > 0\}$$

forms a sub-base for the neighborhood system at b . By definition, this means that any open neighborhood of $b \in \mathcal{B}$ is a union of finite intersections of the form $U_{f_1,\varepsilon_1}^b \cap \dots \cap U_{f_m,\varepsilon_m}^b$.

Although the topology is defined in a rather indirect fashion, it has the desired Hausdorff property, that is, different points in \mathcal{B} can be separated by disjoint open sets.

Lemma 3.2. *If ν assigns positive mass to all nonempty open intervals, then the topology on \mathcal{B} is Hausdorff.*

Proof. Consider two functions $b \neq b'$ in \mathcal{B} . By Lemma 3.1, we have $P_\Delta^b \neq P_\Delta^{b'}$ and hence there exists an $f \in BC(\mathbb{R})$ and an $x \in \mathbb{R}$ such that $P_\Delta^b f(x) \neq P_\Delta^{b'} f(x)$. By continuity there exists in fact a nonempty open interval $J \subset \mathbb{R}$ where the functions $P_\Delta^b f$ and $P_\Delta^{b'} f$ are different. By the assumption on ν , it follows that for some $\varepsilon > 0$,

$$\|P_\Delta^b f - P_\Delta^{b'} f\|_{1,\nu} > \varepsilon.$$

This implies that the neighborhoods $U_{f,\varepsilon/2}^b$ and $U_{f,\varepsilon/2}^{b'}$ are disjoint. \square

An alternative point of view on the topology that we use is obtained by considering the high-frequency limit $\Delta \rightarrow 0$. Let A_b be the generator of X under \mathbb{P}_b , that is, $A_b f = bf' + f''/2$ for a C^2 -function f . Then for small Δ ,

$$P_\Delta^{b_1} f - P_\Delta^{b_2} f \approx \Delta(A_{b_1} f - A_{b_2} f) = \Delta(b_1 - b_2) f'.$$

It follows that for small Δ , the constructed topology is close to the topology induced by the $L^1(\nu)$ -norm on the set of drift functions \mathcal{B} .

Having specified the topology, we can define *weak posterior consistency*, or just *consistency*.

Definition 3.3. *We have weak posterior consistency if for every open neighborhood U_{b_0} of b_0 , it holds that*

$$\Pi(b \notin U_{b_0} | X_0, X_\Delta, \dots, X_{n\Delta}) \rightarrow 0 \quad \mathbb{P}_{b_0}\text{-a.s.}$$

as $n \rightarrow \infty$. Note that the word “weak” refers to the topology, not to the mode of stochastic convergence.

Theorem 3.5 below is the main result of this section. It needs the following definition.

Definition 3.4. *We call a collection \mathcal{F} of real-valued functions on the real line locally uniformly equicontinuous if for every $\varepsilon > 0$ and every compact $K \subset \mathbb{R}$, there exists a $\delta > 0$ such that*

$$\sup_{f \in \mathcal{F}} \sup_{\substack{x, y \in K \\ |x-y| < \delta}} |f(x) - f(y)| < \varepsilon.$$

In Section 4 we give examples of locally uniformly equicontinuous collections of functions.

Theorem 3.5. *Suppose we have discrete-time data from the stationary solution to the stochastic differential equation*

$$dX_t = b(X_t) dt + dW_t, \quad t \geq 0.$$

Denote the invariant measure of the diffusion with drift b_0 by μ_0 . Let Π be a prior on the set of drift functions \mathcal{B} and suppose that \mathcal{B} is locally uniformly equicontinuous and $\sup_{b \in \mathcal{B}} \|b\|_\infty < \infty$. Then if

$$\Pi(b \in \mathcal{B}: \|b - b_0\|_{2, \mu_0} < \varepsilon) > 0 \quad \text{for all } \varepsilon > 0, \quad (3.1)$$

we have weak consistency (as in Definition 3.3).

In Bayesian practice, a model set \mathcal{B} is typically not specified explicitly. Usually some prior Π is simply chosen and the procedure is carried out. From this perspective, the theorem states that

if the chosen prior gives mass 1 to a set of functions that is uniformly bounded and locally uniformly equicontinuous, then we have weak consistency for every true b_0 in the $L^2(\mu_0)$ -support of the prior.

Prior mass conditions like (3.1) are standard in results on posterior consistency. Intuitively, it is reasonable that if we want the posterior to concentrate around b_0 asymptotically, the prior should put sufficient mass near b_0 too. The uniform boundedness and equicontinuity conditions limit the size of the support of the prior, which is reasonable as well. The conditions are somewhat restrictive, but due to technical reasons cannot be avoided in our approach. In settings where consistency can be derived using testing arguments, boundedness and equicontinuity conditions can typically be relaxed, and only need to be valid on certain subsets \mathcal{B}_n of the support \mathcal{B} of the prior with increasing prior probability. However, since we do not have the appropriate tests available in this case, we cannot follow such an approach unfortunately. On the other hand, computational approaches like the one of Beskos *et al.* [2] require in fact that both b and its derivative b' are uniformly bounded, which is more restrictive than the conditions of our consistency theorem.

The proof of the theorem is deferred to Section 5. In the next section, we first consider a number of concrete priors for which the assumptions of the theorem are verified.

4. Examples of concrete priors

The following example is perhaps of little practical relevance, but it shows already that there is an abundance of priors available that yield posterior consistency.

Example 4.1 (Discrete net priors). Let the collection of drift functions \mathcal{B} satisfy the requirements of Theorem 3.5. That is, \mathcal{B} is locally uniformly equicontinuous and $\sup_{b \in \mathcal{B}} \|b\|_\infty < \infty$. To construct the prior choose two probability distributions (p_n) and (q_n) on the positive integers such that $p_n, q_n > 0$ for n large enough, and a decreasing sequence of positive numbers $\varepsilon_n \downarrow 0$. For $m \geq 1$, let $\mathcal{B}_m = \{b|_{[-m,m]} : b \in \mathcal{B}\}$ be the set of restrictions of functions in \mathcal{B} to the interval $[-m, m]$. The functions in \mathcal{B}_m are uniformly equicontinuous and hence, by the Arzelà–Ascoli theorem, \mathcal{B}_m is totally bounded for the uniform norm. For every n , we fix a finite ε_n -net $\mathcal{B}_{m,\varepsilon_n}$ for \mathcal{B}_m , that is, $\mathcal{B}_{m,\varepsilon_n}$ is a finite set such that every element of \mathcal{B}_m is within uniform distance ε_n of some element of $\mathcal{B}_{m,\varepsilon_n}$. We extend every function in the net to the whole real line by setting it equal to 1 on $(-\infty, -m - 1]$ and to -1 on $[m + 1, \infty)$, and interpolating linearly in the intervals $[-m - 1, -m]$ and $[m, m + 1]$. A draw b from the prior Π is now generated as follows:

- (i) draw m from the probability distribution (p_m) ,
- (ii) draw n from the probability distribution (q_n) ,
- (iii) draw b uniformly from $\mathcal{B}_{m,\varepsilon_n}$.

In other words, if $\mathcal{B}_{m,\varepsilon_n} = \{b_1^{m,n}, \dots, b_{k_{m,n}}^{m,n}\}$, then

$$\Pi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{k_{m,n}} \frac{p_m q_n}{k_{m,n}} \delta_{b_k^{m,n}}.$$

By construction, Π assigns mass 1 to a countable set of functions that is uniformly bounded and locally uniformly continuous. Now consider $b_0 \in \mathcal{B}$ and $\varepsilon > 0$. We show that condition (3.1) is satisfied. For every $b \in \mathcal{B}$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} \|b - b_0\|_{2, \mu_0}^2 &= \int_{|x| \leq m} (b(x) - b_0(x))^2 d\mu_0(x) + \int_{|x| > m} (b(x) - b_0(x))^2 d\mu_0(x) \\ &\leq \|b - b_0\|_{m, \infty}^2 + 2 \int_{|x| > m} (b^2(x) + b_0^2(x)) d\mu_0(x) \\ &\leq \|b - b_0\|_{m, \infty}^2 + C\mu_0(|x| > m), \end{aligned}$$

where $\|\cdot\|_{m, \infty}$ denotes the uniform norm on $[-m, m]$ and $C = 2(1 + \sup_{b \in \mathcal{B}} \|b\|_{\infty}^2)$. Hence, for $m \in \mathbb{N}$ so large that $C\mu_0(|x| > m) < \varepsilon^2$, it holds that

$$\Pi(b: \|b - b_0\|_{2, \mu_0}^2 < 2\varepsilon^2) \geq \Pi(b: \|b - b_0\|_{m, \infty}^2 < \varepsilon^2).$$

For $n \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ and $q_n > 0$, we have, by construction,

$$\Pi(b: \|b - b_0\|_{m, \infty}^2 < \varepsilon^2) \geq \Pi(b: \|b - b_0\|_{m, \infty} < \varepsilon_n) \geq \frac{p_m q_n}{k_{m, n}} > 0.$$

This shows that condition (3.1) holds, and hence we have posterior consistency for this class of priors.

If $\mathcal{B} \subset C^s(\mathbb{R})$ for some $s \in (0, 1)$ and $\sup_{b \in \mathcal{B}} \|b\|_s < \infty$ (see (1.2)), then clearly \mathcal{B} satisfies the equicontinuity condition of Definition 3.4. In the following example, we use wavelet expansions to construct a consistent prior on drift functions which belong to such a class of Hölder functions.

Example 4.2 (Wavelets). Suppose $\{\varphi_k, \psi_{j, k}\}_{k \in \mathbb{Z}, j \geq 0}$ is an orthonormal wavelet basis, so that functions $f \in L^2(\mathbb{R})$ can be represented as

$$f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_k \rangle \varphi_k + \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} \langle f, \psi_{j, k} \rangle \psi_{j, k}$$

(the convergence being in $L^2(\mathbb{R})$). The functions $\psi_{j, k}$ are obtained from the mother wavelet function by translation and scaling: $\psi_{j, k}(\cdot) = 2^{j/2} \psi(2^j \cdot - k)$. Similarly, the φ_k are obtained from the father wavelet φ (also called scaling function) by translation: $\varphi_k(\cdot) = \varphi(\cdot - k)$.

It is well known that under appropriate smoothness conditions on ψ , the rate of decay of the wavelet coefficients characterizes the smoothness of the function f . Assume ψ is continuously differentiable and has compact support. Then $f \in C^s \cap L^2(\mathbb{R})$ if and only if $\|f\|_{\infty} < \infty$ and

- $|\langle f, \varphi_k \rangle| \leq C_f$ for all $k \in \mathbb{Z}$,
- $|\langle f, \psi_{j, k} \rangle| \leq C_f 2^{-j(s+1/2)}$ for all $j \geq 0$ and $k \in \mathbb{Z}$.

Moreover, C_f can be taken as the product of the Hölder norm of f and a constant (that does not depend of f). For a proof, we refer to Section 6.7 in Hernández and Weiss [15], see also Daubechies [6], Section 9.2. This characterization implies that for $s \in (0, 1)$ and $L > 0$, the collection

$$\mathcal{F}_{s,L} := \left\{ f \in L^2(\mathbb{R}): f = \sum_{k \in \mathbb{Z}} a_k \varphi_k + \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} b_{j,k} \psi_{j,k}, \right. \\ \left. \sup_k |a_k| + \sup_j \sup_k 2^{j(s+1/2)} |b_{j,k}| \leq L \right\}$$

consists of s -Hölder continuous functions with uniformly bounded Hölder norms.

In addition to the smoothness condition on ψ , we assume that the scaling function φ is bounded and compactly supported. This implies that the function $\theta_\varphi(x) = \sum_k |\varphi(x-k)|$ is such that $\text{ess sup}_x \theta_\varphi(x) < \infty$. This is a localization condition that is referred to as Condition (θ) in Härdle *et al.* [14] (page 77). By inequalities (9.34) and (9.35) on page 114 in Härdle *et al.* [14], Condition (θ) implies that the supremum norm of $\sum_k a_k \varphi_{0,k}$ is equivalent to the sup-norm on the sequence $\{a_k\}_k$. In addition, the supremum norm of $\sum_{j \geq 0} \sum_k b_{j,k} \psi_{j,k}$ is equivalent to the $\|\cdot\|_w$ -norm of the doubly indexed sequence $b = \{b_{j,k}\}_{j \geq 0, k}$, where

$$\|b\|_w = \sum_{j \geq 0} 2^{j/2} \sup_k |b_{j,k}|.$$

It follows in particular that the uniform norm of the functions in $\mathcal{F}_{s,L}$ is uniformly bounded.

To construct a prior on drift functions that is consistent for all true drift functions $b_0 \in \mathcal{F}_{s,L}$ we first construct an auxiliary prior Π' on the whole class $\mathcal{F}_{s,L}$ (which does not only charge drift functions of ergodic diffusions). Let J be a discrete random variable, supported on $\mathbb{N}_0 = \{0, 1, \dots\}$ and let $U_{j,k}, V_k$, for $j \in \mathbb{N}_0, k \in \mathbb{Z}$, be independent random variables, independent of J , from a distribution with a strictly positive, continuous density on its support $[-L, L]$. Define the prior Π' as the law of the random function

$$x \mapsto \sum_{k \in \mathbb{Z}} V_k \varphi_k + \sum_{j=0}^J \sum_{k \in \mathbb{Z}} \eta_j U_{j,k} \psi_{j,k}$$

on \mathbb{R} , where $\eta_j = 2^{-j(s+1/2)}$. To arrive at a prior on drift functions of ergodic diffusions we proceed as in the preceding example. We choose a probability distribution (p_m) on \mathbb{N} , with $p_m > 0$ for all m . A draw from the final prior Π is then constructed as follows:

- (i) Draw m from the probability distribution (p_m) .
- (ii) Independently of m , draw a random function from Π' and restrict it to $[-m, m]$.
- (iii) Extend the function to the whole real line by setting it equal to 1 on $(-\infty, -m-1]$ and to -1 on $[m+1, \infty)$, and interpolating linearly in the intervals $[-m-1, -m]$ and $[m, m+1]$.

By construction, Π assigns mass 1 to a set of drift functions satisfying the equicontinuity and uniform boundedness conditions of Theorem 3.5. To prove that this prior yields consistency for

$b_0 \in \mathcal{F}_{s,L}$ it remains to show that (3.1) holds. Let $\varepsilon > 0$ be fixed. Then exactly as in the preceding example, there exists an $m \in \mathbb{N}$ such that

$$\Pi(b: \|b - b_0\|_{2,\mu_0}^2 < 2\varepsilon^2) \geq \Pi(b: \|b - b_0\|_{m,\infty} < \varepsilon).$$

Since the right-hand side is further bounded from below by

$$\Pi'(b: \|b - b_0\|_{m,\infty} < \varepsilon) \sum_{n \geq m} p_n,$$

it now suffices to show that $\Pi'(b: \|b - b_0\|_{m,\infty} < \varepsilon) > 0$.

To see that this is true, let a_k^0 and $b_{j,k}^0$ be the wavelet coefficients of the true drift function b_0 and let

$$B = \sum_k V_k \varphi_k + \sum_{j=0}^J \sum_k \eta_j U_{j,k} \psi_{j,k}$$

be distributed according to Π' . Then

$$\begin{aligned} \|B - b_0\|_{m,\infty} &\leq \left\| \sum_k (V_k - a_k^0) \varphi_k \right\|_{m,\infty} \\ &\quad + \left\| \sum_{j=0}^J \sum_k (\eta_j U_{j,k} - b_{j,k}^0) \psi_{j,k} \right\|_{m,\infty} + \left\| \sum_{j>J} \sum_k b_{j,k}^0 \psi_{j,k} \right\|_{m,\infty}. \end{aligned} \quad (4.1)$$

The first term on the right is bounded by

$$\|\varphi\|_\infty \sum_{k \in K_m} |V_k - a_k^0|,$$

where K_m is a finite set of natural numbers, since φ is compactly supported.

Since $|a_k^0| \leq L$, the V_k have full support in $[-L, L]$ and K_m is finite, this quantity is bounded by $\varepsilon/3$ with positive probability. By the equivalence of norms mentioned above and the definition of $\mathcal{F}_{s,L}$, there exists a constant $c > 0$ such that the third term on the right of (4.1) is bounded by

$$c \sum_{j>J} 2^{j/2} \max_k |b_{j,k}^0| \leq c \sum_{j>J} 2^{j/2} L 2^{-j(s+1/2)} \leq c L 2^{-J_s}.$$

Hence, if we choose $J_0 \in \mathbb{N}$ such that $c L 2^{-J_0 s} \leq \varepsilon/3$, then the third term on the right of (4.1) is bounded by $\varepsilon/3$ with probability at least $\mathbb{P}(J = J_0) > 0$. On the event $\{J = J_0\}$, the second term on the right-hand side of (4.1) is bounded by a constant times

$$\sum_{j=0}^{J_0} 2^{j/2} \max_{k \in K'_m} |\eta_j U_{j,k} - b_{j,k}^0| \leq J_0 2^{J_0/2} \max_{j \leq J_0, k \in K'_m} |\eta_j U_{j,k} - b_{j,k}^0|.$$

The set K'_m is finite, since ψ is compactly supported. Since $|b_{j,k}^0| \leq \eta_j L$ and the $U_{j,k}$'s have full support in $[-L, L]$, the right-hand side of this display is less than $\varepsilon/3$ as well with positive probability. Combining the considerations above and using the fact that J , the V_k and the $U_{j,k}$ are all independent, we conclude that $\Pi'(b: \|b - b_0\|_{m,\infty} < \varepsilon) > 0$.

5. Proof of Theorem 3.5

Recall that under \mathbb{P}_b , the observations X_0, X_Δ, \dots form a discrete-time Markov chain with positive, continuous transition densities $p_b(\Delta, x, y)$ and a positive, continuous invariant density π_b . For $b \in \mathcal{B}$, we consider the associated Kullback–Leibler divergence

$$\text{KL}(b_0, b) = \int \int p_{b_0}(\Delta, x, y) \log \frac{p_{b_0}(\Delta, x, y)}{p_b(\Delta, x, y)} \pi_{b_0}(x) \, dx \, dy.$$

The following lemma shows that condition (3.1) of Theorem 3.5 implies that we have the Kullback–Leibler property relative to this distance measure.

Lemma 5.1. *Condition (3.1) of Theorem 3.5 implies that for every $\varepsilon > 0$, we have $\Pi(b: \text{KL}(b_0, b) < \varepsilon) > 0$.*

Proof. To prove the lemma we bound the quantity $\text{KL}(b_0, b)$ from above by a multiple of $\|b_0 - b\|_{2,\mu_0}^2$. For convenience we introduce the notation $K(P, Q) = \mathbb{E}_P \log dP/dQ$ for the Kullback–Leibler divergence between two probability measures P and Q on the same σ -field. The law of a random element Z under the underlying probability measure \mathbb{Q} is denoted by $\mathcal{L}(Z|\mathbb{Q})$.

Under \mathbb{P}_b , for every $b \in \mathcal{B}$, the pair (X_0, X_Δ) has joint density $(x, y) \mapsto \pi_b(x)p_b(\Delta, x, y)$ relative to Lebesgue measure. Hence, the Kullback–Leibler divergence between $\mathcal{L}((X_0, X_\Delta)|\mathbb{P}_{b_0})$ and $\mathcal{L}((X_0, X_\Delta)|\mathbb{P}_b)$ equals

$$\int \int \pi_{b_0}(x) p_{b_0}(\Delta, x, y) \log \frac{\pi_{b_0}(x) p_{b_0}(\Delta, x, y)}{\pi_b(x) p_b(\Delta, x, y)} \, dx \, dy = \text{KL}(b_0, b) + K(\mu_{b_0}, \mu_b).$$

Now (X_0, X_Δ) is a measurable functional of the continuous path $(X_t: t \in [0, \Delta])$. Hence, the Kullback–Leibler divergence between $\mathcal{L}((X_0, X_\Delta)|\mathbb{P}_{b_0})$ and $\mathcal{L}((X_0, X_\Delta)|\mathbb{P}_b)$ is bounded by the Kullback–Leibler divergence between the laws $\mathcal{L}((X_t: t \in [0, \Delta])|\mathbb{P}_{b_0})$ and $\mathcal{L}((X_t: t \in [0, \Delta])|\mathbb{P}_b)$ of the full path $(X_t: t \in [0, \Delta])$ under \mathbb{P}_{b_0} and \mathbb{P}_b . (To see this, observe that the likelihood for (X_0, X_Δ) is the conditional expectation of the likelihood for $(X_t: t \in [0, \Delta])$ and use the concavity of the logarithm and Jensen’s inequality.) By Girsanov’s theorem, the latter Kullback–Leibler divergence is given by

$$-\mathbb{E}_{b_0} \left(\log \frac{\pi_b(X_0)}{\pi_{b_0}(X_0)} + \int_0^\Delta (b - b_0)(X_s) \, dW_s - \frac{1}{2} \int_0^\Delta (b - b_0)^2(X_s) \, ds \right),$$

where W is a \mathbb{P}_{b_0} -Brownian motion. Using the stationarity of the process X under \mathbb{P}_{b_0} , we see that this equals

$$K(\mu_{b_0}, \mu_b) + \frac{\Delta}{2} \|b - b_0\|_{2,\mu_0}^2.$$

Hence, we find that $2\text{KL}(b_0, b) \leq \Delta \|b - b_0\|_{2, \mu_0}^2$. \square

For any sequence of measurable sets $C_n \subset \mathcal{B}$, we have that the posterior measure of C_n can be written as

$$\Pi(C_n | X_0, \dots, X_{\Delta n}) = \frac{\int_{C_n} L_n(b) \Pi(db)}{\int_{\mathcal{B}} L_n(b) \Pi(db)},$$

where

$$L_n(b) = \frac{\pi_b(X_0)}{\pi_{b_0}(X_0)} \prod_{i=1}^n \frac{p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta})}{p_{b_0}(\Delta, X_{(i-1)\Delta}, X_{i\Delta})}$$

is the likelihood ratio. Since we have the Kullback–Leibler property and our Markov chain satisfies a law of large numbers, the denominator in the expression for the posterior can be dealt with in the usual manner. This leads to the following result.

Lemma 5.2. *Suppose that for every $\varepsilon > 0$, we have $\Pi(b: \text{KL}(b_0, b) < \varepsilon) > 0$. If for a collection of measurable subsets $C_n \subset \mathcal{B}$ there exists some $c > 0$ such that*

$$e^{nc} \int_{C_n} L_n(b) \Pi(db) \rightarrow 0, \quad \mathbb{P}_{b_0}\text{-almost surely}, \quad (5.1)$$

then $\Pi(C_n | X_0, \dots, X_{\Delta n}) \rightarrow 0$, \mathbb{P}_{b_0} -almost surely.

Proof. By ergodicity, it \mathbb{P}_{b_0} -a.s. holds that

$$\frac{1}{n} \log L_n(b) = \frac{1}{n} \left(\log \frac{\pi_b(X_0)}{\pi_{b_0}(X_0)} + \sum_{i=1}^n \log \frac{p_b(\Delta, X_{(i-1)\Delta}, X_{i\Delta})}{p_{b_0}(\Delta, X_{(i-1)\Delta}, X_{i\Delta})} \right) \rightarrow -\text{KL}(b_0, b).$$

In particular, for $\eta > 0$ arbitrary and b such that $\text{KL}(b_0, b) < \eta$, it \mathbb{P}_{b_0} -a.s. holds that $\liminf_{n \rightarrow \infty} e^{n\alpha} L_n(b) \geq 1$ for all $\alpha > \eta$. It follows that \mathbb{P}_{b_0} -a.s.,

$$\liminf_{n \rightarrow \infty} e^{n\alpha} \int_{\mathcal{B}} L_n(b) \Pi(db) \geq \int_{b: \text{KL}(b_0, b) < \eta} \liminf_{n \rightarrow \infty} e^{n\alpha} L_n(b) \Pi(db) \geq \Pi(b: \text{KL}(b_0, b) < \eta),$$

and hence

$$\limsup_{n \rightarrow \infty} \Pi(C_n | X_0, \dots, X_{\Delta n}) \leq \frac{\limsup_{n \rightarrow \infty} e^{n\alpha} \int_{C_n} L_n(b) \Pi(db)}{\Pi(b: \text{KL}(b_0, b) < \eta)}.$$

In view of Lemma 5.1 and the fact that we can take $\alpha > 0$ arbitrarily small, this completes the proof. \square

We proceed with the proof of the theorem. By definition of the topology on \mathcal{B} it suffices to show that $\Pi(B | X_0, \dots, X_{\Delta n}) \rightarrow 0$, \mathbb{P}_{b_0} -almost surely, where

$$B = \{b \in \mathcal{B}: \|P_{\Delta}^b f - P_{\Delta}^{b_0} f\|_{1, \nu} > \varepsilon\},$$

with $\varepsilon > 0$ and f a continuous function on \mathbb{R} that is uniformly bounded by 1. We fix ε , f and the set B from this point on.

In view of Lemma A.1 the assumptions of Theorem 3.5, imply an equicontinuity property for the collections of functions

$$\{(P_{\Delta}^b f)1_K : b \in \mathcal{B}\},$$

for $K \subset \mathbb{R}$ a compact set. Arguing as in Tang and Ghosal [24], this allows us to derive the following useful intermediate result.

Lemma 5.3. *There exists a compact set $K \subset \mathbb{R}$, a positive integer N and bounded intervals I_1, \dots, I_N that cover K such that*

$$B \subset \bigcup_{j=1}^N B_j^+ \cup \bigcup_{j=1}^N B_j^-,$$

where

$$B_j^+ = \left\{ b \in B : P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x) > \frac{\varepsilon}{4\nu(K)} \forall x \in I_j \right\},$$

$$B_j^- = \left\{ b \in B : P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x) < -\frac{\varepsilon}{4\nu(K)} \forall x \in I_j \right\}$$

for $j = 1, \dots, N$.

Proof. Since ν is a finite Borel measure on the line there exists a compact subset $K \subset \mathbb{R}$ such that $\nu(K^c) \leq \varepsilon/4$. Let $\delta > 0$ and cover K with $N < \infty$ intervals with width $\delta/2$, denote the intervals by I_1, \dots, I_N . First, we show that $B \subset \bigcup_{j=1}^N B_j$, where

$$B_j = \left\{ b \in B : |P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x)| > \frac{\varepsilon}{4\nu(K)} \forall x \in I_j \right\}.$$

Suppose the inclusion is not true. Then there exists a $b \in B$ such that for each $j \in \{1, \dots, N\}$ there exists a point $z_j \in I_j$ such that

$$|P_{\Delta}^b f(z_j) - P_{\Delta}^{b_0} f(z_j)| \leq \frac{\varepsilon}{4\nu(K)}. \quad (5.2)$$

Now

$$\begin{aligned} \|P_{\Delta}^b f - P_{\Delta}^{b_0} f\|_{1,\nu} &= \int_K |P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x)| \nu(dx) + \int_{K^c} |P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x)| \nu(dx) \\ &\leq \nu(K) \max_j \max_{x \in I_j} |P_{\Delta}^b f(x) - P_{\Delta}^{b_0} f(x)| + 2\|f\|_{\infty} \nu(K^c) \end{aligned}$$

$$\begin{aligned} &\leq \nu(K) \max_j \max_{x \in I_j} (|P_\Delta^b f(x) - P_\Delta^b f(z_j)| + |P_\Delta^b f(z_j) - P_\Delta^{b_0} f(z_j)| \\ &\quad + |P_\Delta^{b_0} f(z_j) - P_\Delta^{b_0} f(x)|) + \varepsilon/2. \end{aligned}$$

By local uniform equicontinuity and Lemma A.1, we can find a δ such that the first term can be bounded by $\varepsilon/(8\nu(K))$. The second term can be bounded by (5.2). By continuity the third term can be bounded by $\varepsilon/(8\nu(K))$. Therefore, the preceding display can be bounded by ε , contradicting that $b \in B$. Thus $B \subset \bigcup_{j=1}^N B_j$.

Since the function $P_\Delta^b f - P_\Delta^{b_0} f$ is continuous and I_j is connected, we have that B_j is included in

$$\begin{aligned} &\left\{ b \in B: P_\Delta^b f(x) - P_\Delta^{b_0} f(x) > \frac{\varepsilon}{4\nu(K)} \forall x \in I_j \right\} \\ &\cup \left\{ b \in B: P_\Delta^b f(x) - P_\Delta^{b_0} f(x) < -\frac{\varepsilon}{4\nu(K)} \forall x \in I_j \right\} =: B_j^+ \cup B_j^-. \end{aligned}$$

This completes the proof of the lemma. \square

As a consequence of this lemma, the proof of the theorem is complete once we show that for $j = 1, \dots, N$,

$$\Pi(B_j^+ | X_0, \dots, X_{n\Delta}) \rightarrow 0, \quad \Pi(B_j^- | X_0, \dots, X_{n\Delta}) \rightarrow 0$$

\mathbb{P}_{b_0} -almost surely. We give the details for the sets B_j^+ , the argument for the sets B_j^- is completely analogous. Here, we follow the approach of [25]. We fix $j \in \{1, \dots, N\}$ and consider the stochastic process D defined by

$$D_n = \sqrt{\int_{B_j^+} L_n(b) \Pi(db)}.$$

We will show that \mathbb{P}_{b_0} -almost surely, D_n converges to 0 exponentially fast. According to Lemma 5.2 this is sufficient.

Note that since L_n is the likelihood, we have $\mathbb{E}_{b_0} D_n^2 = \Pi(B_j^+) < \infty$. Next, we are interested in the conditional expectation $\mathbb{E}_{b_0}(D_{n+1} | \mathcal{F}_n)$, where (\mathcal{F}_n) is the filtration generated by the Markov chain $(X_{n\Delta})_{n=0,1,\dots}$. Recall that the Hellinger distance $h(p, q)$ between two densities p, q relative to a dominating measure μ is defined by $h^2(p, q) = \int (\sqrt{p} - \sqrt{q})^2 d\mu$. It satisfies $h^2(p, q) = 2 - 2A(p, q)$, where $A(p, q) = \int \sqrt{pq} d\mu$ is the Hellinger affinity between p and q . Then with $p_{n,C}$ the random transition density

$$p_{n,C}(\Delta, x, y) = \frac{\int_C p_b(\Delta, x, y) L_n(b) \Pi(db)}{\int_C L_n(b) \Pi(db)},$$

we have

$$\begin{aligned}
 \mathbb{E}_{b_0}(D_{n+1}|\mathcal{F}_n) &= \mathbb{E}_{b_0}\left(\sqrt{\int_{B_j^+} \frac{p_b(\Delta, X_n, X_{n+1})}{p_{b_0}(\Delta, X_n, X_{n+1})} L_n(b) \Pi(db)} \middle| \mathcal{F}_n\right) \\
 &= \int \sqrt{\int_{B_j^+} \frac{p_b(\Delta, X_n, y)}{p_{b_0}(\Delta, X_n, y)} L_n(b) \Pi(db) p_{b_0}(\Delta, X_n, y)} dy \\
 &= \int \sqrt{\int_{B_j^+} p_b(\Delta, X_n, y) L_n(b) \Pi(db) p_{b_0}(\Delta, X_n, y)} dy \\
 &= D_n A_n,
 \end{aligned}$$

where $A_n = A(p_{n, B_j^+}(\Delta, X_n, \cdot), p_{b_0}(\Delta, X_n, \cdot))$. Next, we bound A_n . First, note that since $2\|f\|_\infty h(p, q) \geq |\int f(p - q) d\mu|$, we have $h^2(p, q) \geq \frac{1}{4}(\int f(p - q) d\mu)^2$ for functions f that are uniformly bounded by 1. Therefore,

$$A(p, q) = 1 - \frac{1}{2}h^2(p, q) \leq 1 - \frac{1}{8}\left(\int f(p - q) d\mu\right)^2.$$

Hence, to bound A_n it suffices to lower bound

$$\int f(y)[p_{n, B_j^+}(\Delta, X_n, y) - p_{b_0}(\Delta, X_n, y)] dy$$

which equals

$$\int_{B_j^+} \int f(y)[p_b(\Delta, X_n, y) - p_{b_0}(\Delta, X_n, y)] dy \frac{L_n(b)}{\int_{B_j^+} L_n(b) \Pi(db)} \Pi(db).$$

By the definition of B_j^+ in Lemma 5.3, if $X_n \in I_j$ the inner integral is lower bounded by $\varepsilon/(4\nu(K))$. This implies that

$$A_n \leq 1 - \frac{1}{8}\left(\frac{\varepsilon}{4\nu(K)}\right)^2 \mathbf{1}_{\{X_n \in I_j\}}.$$

Hence,

$$\mathbb{E}_{b_0}(D_{n+1}|\mathcal{F}_n) \leq D_n(1 - k\varepsilon^2 \mathbf{1}_{X_n \in I_j}),$$

where $k = 1/(128\nu(K)^2)$. We conclude that the process

$$M_n = D_n(1 - k\varepsilon^2)^{-\sum_{i=1}^{n-1} \mathbf{1}_{X_i \in I_j}}$$

is an (\mathcal{F}_n) -supermartingale under the measure \mathbb{P}_{b_0} (note that M_n is bounded by the integrable process $D_n(1 - k\varepsilon^2)^{-(n-1)}$, hence M_n is integrable). By Doob's martingale convergence theorem,

we have $M_n \rightarrow M_\infty$ almost surely, for some finite-valued random variable M_∞ . By ergodicity, we have $n^{-1} \sum_{i=1}^{n-1} 1_{X_i \in I_j} \rightarrow \mu_{b_0}(I_j) > 0$ almost surely. An application of Lemma 5.2 completes the proof.

6. Concluding remarks

In this paper, we obtain conditions for posterior consistency of nonparametric Bayesian drift estimation for low-frequency observations from a scalar ergodic diffusion. The main theorem and the subsequent examples provide several priors for which consistency is guaranteed. As discussed in the [Introduction](#), data augmentation techniques that have been proven to be effective in parametric settings, are in principle usable for numerical implementation of nonparametric models as well. Preliminary investigations indicate that practically feasible procedures can indeed be constructed, but more work on computational issues is necessary at the moment.

The results and proofs in this paper show that in this low-frequency observations setting, obtaining consistency relative to a rather weak topology is already quite involved. Very challenging but equally interesting would be the development of a testing approach to posterior consistency in this setting. It would allow to obtain consistency in stronger topologies, rates of contraction and relaxation of boundedness and equicontinuity conditions. For general diffusions this seems rather difficult, but some progress might be possible for diffusions on compact state spaces.

Appendix: An equicontinuity property of the transition operators

The concept of local uniform equicontinuity is given in Definition 3.4.

Lemma A.1. *If $\sup_{b \in \mathcal{B}} \|b\|_\infty < \infty$ and \mathcal{B} is locally uniformly equicontinuous, then for every $f \in BC(\mathbb{R})$ and $t > 0$, the collection $\{P_t^b f : b \in \mathcal{B}\}$ is locally uniformly equicontinuous as well.*

Proof. Let $K \subset \mathbb{R}$ be a compact set. For \mathbb{P}_x^0 the law of the Brownian motion starting in x we have, by Girsanov's theorem,

$$P_t^b f(x) = \mathbb{E}_x^0 f(X_t) \frac{d\mathbb{P}_x^b}{d\mathbb{P}_x^0} = \mathbb{E}_x^0 f(X_t) \exp\left(\int_0^t b(X_s) dX_s - \frac{1}{2} \int_0^t b^2(X_s) ds\right).$$

Under \mathbb{P}_x^0 the process X has the same law as $x + W$, for W a standard Brownian motion starting in 0. Hence, we get

$$P_t^b f(x) = \mathbb{E} f(x + W_t) L_x,$$

where

$$L_u = e^{l_u}, \quad l_u = \int_0^t b(u + W_s) dW_s - \frac{1}{2} \int_0^t b^2(u + W_s) ds.$$

It follows that

$$\begin{aligned}
 |P_t^b f(x) - P_t^b f(y)| &\leq \mathbb{E}|f(x + W_t)L_x - f(y + W_t)L_y| \\
 &\leq \mathbb{E}|f(x + W_t)||L_x - L_y| + \mathbb{E}|L_y||f(x + W_t) - f(y + W_t)| \\
 &=: I + II.
 \end{aligned}$$

We first bound the term I . By the fact that $|e^b - e^a| \leq |a - b|(e^a + e^b)$ and Cauchy–Schwarz,

$$\begin{aligned}
 |I|^2 &\leq \|f\|_\infty^2 (\mathbb{E}|L_x - L_y|)^2 \\
 &\leq \|f\|_\infty^2 (\mathbb{E}|l_x - l_y||L_x + L_y|)^2 \\
 &\leq \|f\|_\infty^2 \mathbb{E}|l_x - l_y|^2 \mathbb{E}(L_x + L_y)^2.
 \end{aligned}$$

We have

$$l_x - l_y = \int_0^t (b(x + W_s) - b(y + W_s)) dW_s - \frac{1}{2} \int_0^t (b^2(x + W_s) - b^2(y + W_s)) ds. \quad (\text{A.1})$$

For the first term on the right the Itô isometry gives, for $x, y \in K$,

$$\begin{aligned}
 &\mathbb{E} \left(\int_0^t (b(x + W_s) - b(y + W_s)) dW_s \right)^2 \\
 &= \mathbb{E} \int_0^t (b(x + W_s) - b(y + W_s))^2 ds \\
 &\leq \mathbb{E} \int_0^t (b(x + W_s) - b(y + W_s))^2 1_{\sup_{s \leq t} |W_s| \leq M} ds + 4t \|b\|_\infty^2 \mathbb{P} \left(\sup_{s \leq t} |W_s| > M \right) \\
 &\leq t \sup_{\substack{u, v \in K' \\ |u-v| \leq |x-y|}} |b(u) - b(v)|^2 + 4t \|b\|_\infty^2 \mathbb{P} \left(\sup_{s \leq t} |W_s| > M \right)
 \end{aligned}$$

for every $M > 0$, where $K' = \{x + y: x \in K, y \in [-M, M]\}$. The assumptions on \mathcal{B} imply that by choosing M large enough and $|x - y|$ small enough, the right-hand side can be made arbitrarily small, uniformly in \mathcal{B} . The second term on the right of (A.1) can be handled in the same manner, using also the fact that $|b^2(u) - b^2(v)| \leq 2\|b\|_\infty |b(u) - b(v)|$. To complete the bound for term I , we note that $\mathbb{E}(L_x + L_y)^2 \leq 2\mathbb{E}L_x^2 + 2\mathbb{E}L_y^2$ and we write

$$\begin{aligned}
 L_u^2 &= \exp \left(\int_0^t 2b(u + W_s) dW_s - \int_0^t b^2(u + W_s) ds \right) \\
 &= \exp \left(\int_0^t b^2(u + W_s) ds \right) \exp \left(\int_0^t 2b(u + W_s) dW_s - \frac{1}{2} \int_0^t (2b)^2(u + W_s) ds \right).
 \end{aligned}$$

The first factor on the right is bounded by $\exp(t\|b\|_\infty^2)$ and the second one is the time t value of a martingale that starts in 1. Hence,

$$\mathbb{E}L_u^2 \leq e^{t\|b\|_\infty^2}.$$

Finally, observe that by Cauchy–Schwarz and a bound derived above,

$$|II|^2 \leq e^{t\|b\|_\infty^2} \mathbb{E}|f(x + W_t) - f(y + W_t)|^2.$$

This completes the proof. □

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