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Extended Itô calculus for symmetric Markov processes

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Chen, Fitzsimmons, Kuwae and Zhang (*Ann. Probab.* **36** (2008) 931–970) have established an Itô formula consisting in the development of $F(u(X))$ for a symmetric Markov process X, a function u in the Dirichlet space of X and any $\mathcal{C}^2$-function F. We give here an extension of this formula for u locally in the Dirichlet space of X and F admitting a locally bounded Radon–Nikodym derivative. This formula has some analogies with various extended Itô formulas for semi-martingales using the local time stochastic calculus. But here the part of the local time is played by a process $(\Gamma_t^a, a \in \mathbb{R}, t \geq 0)$ defined thanks to Nakao's operator (*Z. Wahrsch. Verw. Gebiete* **68** (1985) 557–578).

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1. Introduction and main results

For any real-valued semimartingale $Y = (Y_0 + M_t + N_t)_{t\geq 0}$ (M martingale and N bounded variation process) and any function F in $\mathcal{C}^2(\mathbb{R})$, the classical Itô formula

$$\begin{aligned} F(Y_t) = F(Y_0) &+ \int_0^t F'(Y_s)\,\mathrm{d}M_s + \int_0^t F'(Y_s)\,\mathrm{d}N_s + \frac{1}{2}\int_0^t F''(Y_s)\,\mathrm{d}\langle M^c\rangle_s \\ &+ \sum_{s\leq t}\{F(Y_s) - F(Y_{s-}) - F'(Y_{s-})\Delta Y_s\} \end{aligned} \tag{1.1}$$

provides both an explicit expansion of $(F(Y_t))_{t\geq 0}$ and its stochastic structure of semimartingale.

Let now E be a locally compact separable metric space , m a positive Radon measure on E, and X a m-symmetric Hunt process. Under the assumption that the associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ of X is regular, Fukushima has showed that for any function u in $\mathcal{F}$, the additive functional (abbreviated as AF) $(u(X_t) - u(X_0))_{t\geq 0}$ admits the following unique decomposition:

$$u(X_t) = u(X_0) + M_t^u + N_t^u \qquad \mathbb{P}_x\text{-a.e. for quasi-every } x \text{ in } E, \tag{1.2}$$

where M^u is a martingale AF of finite energy and N^u is a continuous AF of zero energy.

Although $u(X)$ is not in general a semimartingale, Nakao [14] and Chen *et al.* [3] have proved that (1.1) is still valid with $u(X)$, M^u and N^u replacing, respectively, Y, M and N. This is done thanks to the construction of a new stochastic integral with respect to N^u, which takes the place of the well-defined Lebesgue–Stieltjes integral for the bounded variation processes. As the classical Itô formula (1.1), this Itô formula for symmetric Markov processes requires the use of $\mathcal{C}^2$-functions.

For the semimartingale case, there exist extended versions of (1.1) relaxing this regularity condition. This extensions are based on the replacement of the fourth and fifth terms of the right-hand side of (1.1) by an alternative expression requiring only the existence of F' and some integrability condition on F' (see, e.g., [7–9]). The integrability condition insures also the existence of the other terms of (1.1).

The question of relaxing the regularity condition on F in the formula of Nakao and Chen *et al.* is a more complex question. Indeed the integral $\int_0^t F'(u(X_s))\,\mathrm{d}N_s^u$ is well-defined only when $F'(u)$ belongs to $\mathcal{F}_{\text{loc}}$, the set of functions locally in $\mathcal{F}$. As in [3], $u \in \mathcal{F}_{\text{loc}}$ means that there exists a nest of finely open Borel sets $\{G_k\}_{k\in\mathbb{N}}$ and a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e. on G_k. As an example, in the case X is a Brownian motion, this condition implies that the second derivative F'' exists at least as a weak derivative. Nevertheless, in the general case, we know that for any function F element of $\mathcal{C}^1(\mathbb{R})$ with bounded derivative, $F(u)$ belongs to $\mathcal{F}$ and the process $F(u(X))$ hence admits a Fukushima decomposition. We can thus hope to obtain an Itô formula for $\mathcal{C}^1$-functions F that would express each element of the decomposition of $F(u(X))$ in terms of F, u, N^u and M^u. Our purpose here is to establish such a formula. The obtained formula is actually established for the functions F with locally bounded Radon Nikodym derivative and u element of $\mathcal{F}_{\text{loc}}$.

Before introducing this extended Itô formula for symmetric Markov processes, remark that one can easily obtain an extended Itô formula in case $u(X)$ is a semimartingale. Indeed, under the assumption that X has an infinite life time, we note (see (3.4) in [3]) that $u(X)$ is then a reversible semimartingale and that one can hence make use of [7] or [10] to develop $F(u(X))$. But in general, $u(X)$ is not a semimartingale.

The extended Itô formula for symmetric Markov processes presented here is based on the construction for a fixed $t > 0$, of a stochastic integral of deterministic functions with respect to the process $(\Gamma_t^a(u))_{a\in\mathbb{R}}$, defined as follows.

For u in $\mathcal{F}$, let $M^{u,c}$ be the continuous part of M^u. For any real a and $t \geq 0$, we set

$$Z_t^a(u) = \int_0^t 1_{\{u(X_s)\leq a\}}\,\mathrm{d}M_s^{u,c}$$

and define Γ^a by

$$\Gamma^a(u) = (\Gamma_t^a(u))_{t\geq 0} = (\Gamma(Z^a(u))_t)_{t\geq 0} = \Gamma(Z^a(u)),$$

where Γ is the operator on the space of martingale AF with finite energy constructed by Nakao [14] (its definition is recalled in Section 2). The process $(\Gamma_t^a(u))_{t\geq 0}$ is hence an additive functional with zero energy.

In Section 2, we will see that the definition of $\Gamma^a(u)$ can be extended to functions u in $\mathcal{F}_{\text{loc}}$. In that case, the process $M^{u,c}$ is a continuous martingale AF on $[\![0,\zeta[\![$ locally of finite energy and the process $(\Gamma_t^a(u))_{t\geq 0}$ is an AF on $[\![0,\zeta[\![$ locally with zero energy.

As shown by the Tanaka formula (1.4) below, the doubly-indexed process $(\Gamma_t^a(u), a\in\mathbb{R}, t \geq 0)$ plays almost the part of a local time process for $u(X)$. In Section 5, this analogy with local time will be fully clarified under some stronger assumption on u.

To introduce the obtained Itô formula, we need the objects presented by the following lemma. We denote by $(N(x, \mathrm{d}y), H)$ a Lévy system for X (See Definition A.3.7 of [12]), by ν_H the Revuz's measure of H and by ζ the life time of X.

Lemma 1.1. *Let $u \in \mathcal{F}$ (resp., $u \in \mathcal{F}_{\mathrm{loc}}$).There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and such that for any locally absolutely continuous function F from $\mathbb{R}$ into $\mathbb{R}$ with a locally bounded Radon–Nikodym derivative, the following two processes are well-defined.*

$$M_t^d(F,u) = \lim_{n\to\infty}\Bigg\{\sum_{s\le t}\{F(u(X_s)) - F(u(X_{s-}))\}1_{\{\varepsilon_n<|u(X_s)-u(X_{s-})|<1\}}1_{\{s<\zeta\}} - \int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\{F(u(y)) - F(u(X_s))\}N(X_s,\mathrm{d}y)\,\mathrm{d}H_s\Bigg\}$$

$$A_t(F,u) = \lim_{n\to\infty}\int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\{F(u(y)) - F(u(X_s))\}N(X_s,\mathrm{d}y)\,\mathrm{d}H_s.$$

The above limits are uniform on any compact of $[0,\infty)$ (resp., $[0,\zeta)$) $\mathbb{P}_x$-a.e. for q.e. $x \in E$. Moreover, $(M_t^d(F,u))_{t\ge0}$ is a local martingale AF (resp., AF on $[\![0,\zeta[\![$) with locally finite energy and $(A_t(F,u))_{t\ge0}$ is a continuous AF (resp., AF on $[\![0,\zeta[\![$) locally with 0 energy.

With the notation of Lemma 1.1, we have the following Itô formula.

Theorem 1.2. *Let $u \in \mathcal{F}$ (resp., $u \in \mathcal{F}_{\mathrm{loc}}$). For any locally absolutely continuous function F from $\mathbb{R}$ into $\mathbb{R}$ with a locally bounded Radon–Nikodym derivative F' such that $F(0) = 0$, the process $(F(u(X_t), t \in [0,\infty))$ (resp., $t \in [0,\zeta)$) admits the following decomposition $\mathbb{P}_x$-a.e. for q.e. $x \in E$*

$$F(u(X_t)) = F(u(X_0)) + M_t(F,u) + Q_t(F,u) + V_t(F,u), \tag{1.3}$$

where $M(F,u)$ is a local martingale AF (resp., AF on $[\![0,\zeta[\![$) locally of finite energy, $Q(F,u)$ is an AF (resp., AF on $[\![0,\zeta[\![$) locally of zero energy, and $V(F,u)$ is a bounded variation process, respectively, given by:

$$M_t(F,u) = M_t^d(F,u) + \int_0^t F'(u(X_s))\,\mathrm{d}M_s^{u,c},$$

$$Q_t(F,u) = \int_{\mathbb{R}} F'(z)\,\mathrm{d}_z\Gamma_t^z(u) + A_t(F,u),$$

$$V_t(F,u) = \sum_{s\le t}\{F(u(X_s)) - F(u(X_{s-}))\}1_{\{|u(X_s)-u(X_{s-})|\ge1\}}1_{\{s<\xi\}} - F(u(X_{\xi-}))1_{\{t\ge\xi\}}.$$

Note that for u element of $\mathcal{F}$ and F in $\mathcal{C}^2(\mathbb{R})$, (1.3) provides the Itô formula of Chen *et al.* [3] together with the identity connecting integration with respect to $(N_t^u)_{t\ge0}$ and integration with respect to $(\Gamma_t^a(u))_{a\in\mathbb{R}}$ for smooth enough functions.

As a consequence of Theorem 1.2, we obtain the following Tanaka formula for Γ^a_t:

$$\begin{aligned}\Gamma^a_t(u) = \big(u(X_0)-a\big)^- - \big(u(X_t)-a\big)^- + \int_0^t 1_{\{u(X_{s-})\le a\}}\,\mathrm{d}M^{u,c}_s \\ + \lim_{n\to\infty}\sum_{s\le t}\big\{\big(u(X_s)-a\big)^- - \big(u(X_{s-})-a\big)^-\big\}1_{\{|u(X_s)-u(X_{s-})|>\varepsilon_n\}},\end{aligned} \tag{1.4}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is the sequence of Lemma 1.1 and the limit is uniform on any compact $\mathbb{P}_x$-a.e. for q.e. $x\in E$. Using Tanaka's formula for semi-martingales (see [15]), we obtain that when $u(X)$ is a martingale, $-2\Gamma^a(u)$ is the local time process of $u(X)$ at level a. This is the case when $u(x)=x$ and X is a symmetric Lévy process.

Formula (1.3) is hence reminiscent of various extensions of Itô formula involving stochastic integrals with respect to local time, as for example the extensions given in [2] for some martingales, [5] for the Brownian Motion, [6] and [9] for Lévy processes with Brownian component and [16] for Lévy processes without Brownian component. Note that in case the martingale part of $u(X)$ has no continuous component, the process $\Gamma^a(u)$ is identically equal to 0. But (1.3) still represents an improvement of Fukushima's decomposition since (1.3) requires only u in $\mathcal{F}_{\mathrm{loc}}$ and F with a locally bounded Radon–Nikodym derivative.

Integration with respect to $(\Gamma^a_t(u))_{a\in\mathbb{R}}$ is constructed in Section 3 and the Itô formula (1.3) is established in Section 4.

In Section 5, we will show that, when $\Gamma(M^{u,c})$ is of bounded variation, $u(X)$ admits a local time process $(L^a_t, a\in\mathbb{R}, t<\zeta)$ satisfying an occupation time formula of the same type as the occupation time formula for the semimartingales and in this case, the process of locally zero energy $Q(F,u)$ can be rewritten as:

$$Q_t(F,u) = -\frac{1}{2}\int_{\mathbb{R}} F'(z)\,\mathrm{d}_z L^z_t + \int_0^t F'(u(X_s))\,\mathrm{d}\Gamma(M^{u,c})_s + A_t(F,u), \qquad t<\zeta.$$

Finally in Section 6 we give a multidimensional version of Theorem 1.2.

2. Preliminaries on *m*-symmetric Hunt processes

Let E be a locally compact separable metric space, m a positive Radon measure on E such that $\mathrm{Supp}[m]=E$, Δ be a point outside E and $E_\Delta = E\cup\Delta$. Let $X=\{\Omega,\mathcal{F}_\infty,\mathcal{F}_t,X_t,\theta_t,\zeta,\mathbb{P}_x, x\in E_\Delta, t\ge 0\}$ be a m-symmetric Hunt Processes such that its associated Dirichlet space $(\mathcal{E},\mathcal{F})$ is regular on $L^2(E;m)$. We may take as Ω the space $D([0,\infty[\to E_\Delta)$ of càdlàg functions from $[0,\infty[$ to E_Δ, for which Δ is a cemetery (i.e., if $\omega(t)=\Delta$, then $\omega(s)=\Delta$ for any $s>t$) and denote by θ_t the operator $\omega(s)\to\theta_t\omega(s):=\omega(t+s)$. Every element u of $\mathcal{F}$ admits a quasi-continuous m-version. In the sequel, the functions in $\mathcal{F}$ are always represented by their quasi-continuous m-versions. We use the term "quasi everywhere" or "q.e." to mean "except on an exceptional set."

We say that a subset Ξ of Ω is a defining set of a process $A=(A_t)_{t\ge 0}$ with values in $[-\infty,\infty]$, if for any $\omega\in\Xi$, $t,s\ge 0$: $\theta_t\Xi\subset\Xi$, $A_0(\omega)=0$, $A_\cdot(\omega)$ is càdlàg and finite on $[0,\zeta[$,

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$$

and $A_t(\omega_\Delta) = 0$, where ω_Δ is the constant path equal to Δ. A $(\mathcal{F}_t)$-adapted process is an additive functional if it has a defining set $\Xi \in \mathcal{F}_\infty$ admitting an exceptional set, that is, $\mathbb{P}_x(\Xi) = 1$ for q.e. $x \in E$.

An $(\mathcal{F}_t)$-adapted process is an additive functional on $[\![0, \zeta[\![$ or a local additive functional if it satisfies all the conditions to be an additive functional except that the additive property $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$ is required only for $t + s < \zeta(\omega)$.

Let $\mathcal{F}^m_\infty$ (resp., $\mathcal{F}^m_t$) be the $\mathbb{P}_m$-completion of $\sigma\{X_s, 0 \le s < \infty\}$ (resp., $\sigma\{X_s, 0 \le s \le t\}$). An $(\mathcal{F}_t)$-adapted process is an additive functional admitting m-null set if it has a defining set $\Xi \in \mathcal{F}^m_\infty$ such that $\mathbb{P}_x(\Xi) = 1$ for m-a.e. $x \in E$.

The abbreviations AF, PAF, CAF, PCAF and MAF stand respectively for "additive functional," "positive additive functional," "continuous additive functional," "positive continuous additive functional" and "martingale additive functional," respectively. Let $\mathring{\mathcal{M}}$ and $\mathcal{N}_c$ denote, respectively, the space of MAF's of finite energy and the space of continuous additive functionals of zero energy N such that $\mathbb{E}_x(|N_t|) < \infty$ q.e. for each $t > 0$. Moreover, $\mathring{\mathcal{M}}^c$ denotes the subset of continuous elements of $\mathring{\mathcal{M}}$ and $\mathring{\mathcal{M}}^d$ denotes the subset of purely discontinuous elements of $\mathring{\mathcal{M}}$.

For $u \in \mathcal{F}$, the elements M^u and N^u of the Fukushima's decomposition (1.2) are elements of, respectively, $\mathring{\mathcal{M}}$ and $\mathcal{N}_c$. We denote by $M^{u,c}$, $M^{u,j}$ and $M^{u,\kappa}$, respectively, the continuous part, the jump part and the killing part of M^u (see Section 5.3 of [12]). This three martingales are elements of $\mathring{\mathcal{M}}$.

Let Γ the linear operator from $\mathring{\mathcal{M}}$ to $\mathcal{N}_c$ constructed by Nakao [14] in the following way. It is shown in [14] that for every $Z \in \mathring{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}(w, v) + (w, v)_m = \tfrac{1}{2}\mu_{\langle M^v + M^{v,\kappa}, Z\rangle}(E) \qquad \text{for every } v \in \mathcal{F},$$

where $(w, v)_m = \int_E w(x)v(x)m(\mathrm{d}x)$ and $\mu_{\langle M^v + M^{v,\kappa}, Z\rangle}$ is the smooth signed measure corresponding to $\langle M^v + M^{v,\kappa}, Z\rangle$ by the Revuz correspondence. The process $\Gamma(Z)$ is then defined by:

$$\Gamma_t(Z) = N^w_t - \int_0^t w(X_s)\,\mathrm{d}s.$$

This operator satisfies: $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Thus N^u admits the decomposition:

$$N^u = {}^cN^u + {}^jN^u + {}^\kappa N^u, \tag{2.1}$$

where for $p \in \{c, j, \kappa\}$: ${}^pN^u = \Gamma(M^{u,p})$.

For a Borel subset B of $E \cup \{\Delta\}$, it is known that $\tau_B = \inf\{t > 0\colon X_t \notin B\}$ and $\sigma_B = \inf\{t > 0\colon X_t \in B\}$ are $(\mathcal{F}_t)$-stopping times.

An increasing sequence of Borel sets $\{G_k\}$ in E is called a nest if

$$\mathbb{P}_x\Big(\lim_{k\to\infty} \tau_{G_k} = \zeta\Big) = 1 \qquad \text{for q.e. } x \in E.$$

Let $\mathcal{D}$ be a class of AF's. We say that an AF (resp., AF on $[\![0, \zeta[\![$) is locally in $\mathcal{D}$ and write $A \in \mathcal{D}_{\mathrm{loc}}$ (resp., $A \in \mathcal{D}_{f\text{-loc}}$) if there exists a sequence $\{A^n\}$ in $\mathcal{D}$ and an increasing sequence of stopping times T_n with $T_n \to \infty$ (resp., a nest $\{G_n\}$ of finely open Borel sets) such that $\mathbb{P}_x$-a.e. for q.e. $x \in E$, $A_t = A_t^n$ for $t < T_n$ (resp., $t < \tau_{G_n}$).

Let $\{A^n\}$ be a sequence in $\mathcal{D}$ such that for $k > n$, $\mathbb{P}_x$-a.e. for q.e. $x \in E$, $A_t^k = A_t^n$ for $t < \tau_{G_n}$, then it is clear that the process

$$A_t := \begin{cases} A_t^n & \text{for } t < \tau_{G_n}, \\ 0 & \text{for } t \geq \zeta \end{cases}$$

is a well-defined element of $\mathcal{D}_{f\text{-loc}}$. A Borel function f from E into $\mathbb{R}$ is said to be locally in $\mathcal{F}$ (and denoted as $f \in \mathcal{F}_{\mathrm{loc}}$), if there is a nest of finely open Borel sets $\{G_k\}$ and a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e. on G_k. This is equivalent to (see Lemma 3.1(ii) in [3]) there is a nest of closed sets $\{D_k\}$ and a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{F}_b$ such that $f = f_k$ q.e. on D_k. For a such f,

$$M_t^{f,c} := \begin{cases} M_t^{f_k,c} & \text{for } t < \sigma_{E\setminus G_k}, \\ 0 & \text{for } t \geq \lim_{k\to\infty} \sigma_{E\setminus G_k} \end{cases}$$

is well defined and belongs to $\overset{\circ}{\mathcal{M}}_{f\text{-loc}}$ because, for $n > k$, $M_t^{f_n,c} = M_t^{f_k,c}$ $\forall t \leq \sigma_{E\setminus G_k}$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$. Indeed, the last property is shown in Lemma 5.3.1 in [12] for τ_{G_k} instead of $\sigma_{E\setminus G_k}$, we conclude with the following observation:

For a CAF A, and a Borel set $G \subset E$, $\mathbb{P}_x$-a.e. for q.e. $x \in E$:

$$A_t = 0 \qquad \text{for } t < \tau_G \quad \Leftrightarrow \quad A_t = 0 \qquad \text{for } t < \sigma_{E\setminus G}. \tag{2.2}$$

Every $f \in \mathcal{F}_{\mathrm{loc}}$ admits a quasi-continuous m-version, so we may assume that all $f \in \mathcal{F}_{\mathrm{loc}}$ are quasi-continuous and we set $f(\Delta) = 0$.

We use the following notation for a locally bounded measurable function f and a $(\mathcal{F}_t)_{t\geq 0}$-semimartingale M:

$$(f * M)_t = \int_0^t f(X_{s-})\,\mathrm{d}M_s.$$

We will use repeatedly the following fact (see Theorem 5.6.2 in [12]):

For any F in $\mathcal{C}^1(\mathbb{R}^d)$ (d is a positive integer) and $u_1, \ldots, u_d$ in $\mathcal{F}_b$, the composite function $Fu = F(u_1, \ldots, u_d)$ belongs to $\mathcal{F}_{\mathrm{loc}}$ and

$$M^{Fu,c} = \sum_{i=1}^d F_{x_i}(u) * M^{u_i,c}. \tag{2.3}$$

Chen *et al.* [3] have extended Nakao's definition of the operator Γ to the set of locally square-integrable MAF. We keep using the letter Γ for this extension without possible confusion since thanks to Theorem 3.6 of [3] on the set $\overset{\circ}{\mathcal{M}}$, both definitions given in [3] and [14] agree $\mathbb{P}_m$-a.e. on $[\![0, \zeta[\![$. For a continuous locally square-integrable MAF M, $\Gamma(M)$ is defined to be the following CAF admitting m-null set on $[\![0, \zeta[\![$:

$$\Gamma_t(M) = -\tfrac{1}{2}(M_t + M_t \circ r_t) \qquad \text{for } t \in [0, \zeta[, \tag{2.4}$$

where the operator r_t is defined by

$$r_t(\omega)(s) = \omega((t-s)-)1_{\{0\le s\le t\}} + \omega(0)1_{\{s>t\}} \qquad \text{for a path } \omega \in \{t < \zeta\}$$

and $r_t(\omega) := \omega_\Delta$ for a path $\omega \in \{t \ge \zeta\}$.

The continuity of $\Gamma(M)$ $\mathbb{P}_m$-a.e. on $[0, \zeta[$ is a consequence of Theorem 2.18 in [3].

For f a bounded element of $\mathcal{F}$ and M in $\overset{\circ}{\mathcal{M}}$, Nakao has defined the stochastic integral of $f(X)$ with respect to $\Gamma(M)$. We use here the extension of this definition set by Chen *et al.* [3] for f in $\mathcal{F}_{\mathrm{loc}}$ and M continuous locally square-integrable MAF as follows:

$$f * \Gamma(M)_t = \int_0^t f(X_{s-})\,\mathrm{d}\Gamma_s(M) := \Gamma_t(f * M) - \frac{1}{2}\langle M^{f,c}, M\rangle_t. \tag{2.5}$$

It is a CAF admitting m-null set on $[\![0, \zeta[\![$.

When $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{F}_{\mathrm{loc}}$ the integral $f * \Gamma(M)_t$ can be well defined $\mathbb{P}_x$-a.e. for q.e. $x \in E$. In particular, the process $(f * \Gamma(M)_t)_{t\ge 0}$ is a local CAF of X (Lemma 4.6 of [3]).

The argument developed by Chen *et al.* to write "q.e. $x \in E$" instead of "m-a.e. $x \in E$" in the proof of their Lemma 4.6 in [3], is sufficient to establish Lemma 2.1 below.

Lemma 2.1. *Let A be an AF of X (resp., AF on $[\![0, \zeta[\![$). Let G be a measurable subset of E_Δ (resp., $G \subset E$) and $\Xi := \{\omega \in \Omega\colon\ A_t \ge 0, \forall t < \tau_G\}$, then $\mathbb{P}_x(\Xi) = 1$ for m-a.e. $x \in E$ if and only if $\mathbb{P}_x(\Xi) = 1$ for q.e. $x \in E$.*

Lemma 2.2. *Let $\{D_n\}$ be a nest of closed sets and $\sigma := \lim_{n\to\infty} \sigma_{E\setminus D_n}$. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of $\overset{\circ}{\mathcal{M}}{}^c$ such that for $n < k$, $\mathbb{P}_x$-a.e. for q.e. $x \in E$, $M_t^n = M_t^k$ if $t < \sigma_{E\setminus D_n}$. Define a continuous locally square-integrable MAF M by*:

$$M_t = \begin{cases} M_t^n & \text{on } t < \sigma_{E\setminus D_n}, \\ 0 & \text{on } t \ge \sigma. \end{cases}$$

Then $\Gamma_t(M)$ can be well defined for all t in $[0, \infty)$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$, by setting

$$\Gamma_t(M) = \begin{cases} \Gamma_t(M^n) & \text{on } t < \sigma_{E\setminus D_n}, \\ 0 & \text{on } t \ge \sigma. \end{cases} \tag{2.6}$$

Moreover, $\Gamma(M)$ belongs to $\mathcal{N}_{c,f\text{-loc}}$.

For f element of $\mathcal{F}_{\mathrm{loc}}$, (2.5) shows then that $f * \Gamma(M)$ is a well defined CAF on $[\![0, \zeta[\![$.

Proof of Lemma 2.2. A consequence of the m-symmetry assumption on X is that the measure $\mathbb{P}_m$, when restricted to $\{t < \zeta\}$ is invariant under r_t, so we have $\mathbb{P}_m$-a.e. on $t < \zeta$: $M_t \circ r_t = M_t^n \circ r_t$ if $t \le \tau_{D_n} \circ r_t$, but since D_n is closed, for any $\omega \in \Omega$ and $t < \zeta(\omega)$: $t \le \tau_{D_n}(\omega) \Leftrightarrow t \le \tau_{D_n}(r_t\omega)$. Hence, it follows from (2.4) that (2.6) hold $\mathbb{P}_m$-a.e. on $[\![0, \tau_{D_n}[\![$. This shows also, with Lemma 2.1 that if $l > n$, $\mathbb{P}_x$-a.e. for q.e. $x \in E$: $\Gamma_t(M^n) = \Gamma_t(M^l)$ for $t \le \tau_{D_n}$ (and consequently for $t \le \sigma_{E\setminus D_n}$ by (2.2)). Hence, the right-hand side of (2.6) is well defined as a CAF belongs to $\mathcal{N}_{c,f\text{-loc}}$. □

Remark 2.3. Lemma 2.2 shows that for any $u \in \mathcal{F}_{\mathrm{loc}}$, ${}^cN^u := \Gamma(M^{u,c})$ is an element of $\mathcal{N}_{c,f\text{-loc}}$.

The above Lemma 2.1 and Theorem 4.1 of [3] lead to the following lemma.

Lemma 2.4. *Let M be an element of $\overset{\circ}{\mathcal{M}}$ such that $\Gamma(M)$ is of bounded variation on each compact interval of $[0,\zeta[$. Then for every element f of $\mathcal{F}_{\mathrm{loc}}$, $\mathbb{P}_x$-a.e. q.e. for $x \in E$, on $t < \zeta$, $\int_0^t f(X_s)\,\mathrm{d}\Gamma_s(M)$ coincides with the Lebesgue–Stieljes integral of $f(X)$ with respect to $\Gamma(M)$.*

For the reader's convenience, we recall the following result which is Theorem 5.2.1 of [12] and Theorem 3.2 of [14], the last statement can be seen directly from their proofs. By $e(M)$, we denote the energy of M.

Theorem 2.5. *Let $\{M^n\colon n \in \mathbb{N}\}$ be a e-Cauchy sequence of $\overset{\circ}{\mathcal{M}}$. There exists a unique element M of $\overset{\circ}{\mathcal{M}}$ such that $e(M^n - M)$ converges to zero. The subsequence n_k such that there exists $C \in \mathbb{R}_+$ such that for every k in $\mathbb{N}$: $e(M - M^{n_k}) < C2^{-4k}$, satisfies: $\mathbb{P}_x$-a.e. for q.e. $x \in E$, $M_t^{n_k}$ and $\Gamma_t(M^{n_k})$ converge uniformly on any finite interval of t to M_t and $\Gamma_t(M)$, respectively.*

3. Integration with respect to Γ^z

We fix a function u of $\mathcal{F}_{\mathrm{loc}}$. Let $\{D_k\}_{k\in\mathbb{N}}$ be a nest of closed sets and $(u_k)_{k\in\mathbb{N}}$ be a sequence of bounded elements of $\mathcal{F}$ associated to u such that $u = u_k$ q.e. on D_k. Let $\sigma := \lim_{n\to\infty} \sigma_{E\setminus D_n}$. For any real number a, define $Z^a = Z^a(u)$ by

$$Z_t^a = \begin{cases} \displaystyle\int_0^t 1_{\{u_k(X_{s-})\le a\}}\,\mathrm{d}M_s^{u_k,c} & \text{for } t \le \sigma_{E\setminus D_k},\\ 0 & \text{for } t \ge \sigma. \end{cases}$$

Z^a is a MAF on $[\![0,\zeta[\![$ locally of finite energy. In particular, when u belongs to $\mathcal{F}$, Z^a is in $\overset{\circ}{\mathcal{M}}{}^c$ for any real a. By Lemma 2.2, $\Gamma(Z^a)$ is well-defined and belongs to $\mathcal{N}_{c,f\text{-loc}}$.

Remark 3.1. For u element of $\mathcal{F}$, we can choose D_k such that

$$\sigma = \lim_{k\to\infty} \sigma_{E\setminus D_k} = \infty, \qquad \mathbb{P}_x\text{-a.e. for q.e. } x \in E. \tag{3.1}$$

Indeed, in this case, take $u_k := (-k) \vee u \wedge k$ and $G_k := \{x\colon |u(x)| < k\}$, then it follows from the strict continuity of u that $\lim_{k\to\infty} \sigma_{E\setminus G_k} = \infty$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$. Therefore, the nest of closed sets $\{F_k\}_{k\in\mathbb{N}}$ built in the proof of Lemma 3.1(ii) in [3] satisfies the property (3.1) and $u = u_k$ q.e. on F_k. Choose then, $\{D_k\} = \{F_k\}$.

Definition 3.2. *The process $(\Gamma_t^a, a \in \mathbb{R}, t \ge 0)$ is defined by $\Gamma_t^a = \Gamma_t^a(u) = \Gamma_t(Z^a)$.*

Consider an elementary function f, that is, there exists two finite sequences $(z_i)_{0\le i\le n}$ and $(f_i)_{0\le i\le n-1}$ of real numbers such that:

$$f(z)=\sum_{i=0}^{n-1} f_i 1_{(z_i,z_{i+1}]}(z).$$

For such a function integration with respect to $\Gamma_t=\{\Gamma_t^z; z\in\mathbb{R}\}$ is defined to be the following CAF on $[\![0,\zeta[\![$:

$$\int_{\mathbb{R}} f(z)\,\mathrm{d}_z\Gamma_t^z=\sum_{i=0}^{n-1} f_i(\Gamma_t^{z_{i+1}}-\Gamma_t^{z_i}). \tag{3.2}$$

Thanks to the linearity property of the operator Γ we have for any elementary function f:

$$\int_{\mathbb{R}} f(z)\,\mathrm{d}_z\Gamma_t^z=\Gamma_t\left(\int_0^\cdot f(u(X_s))\,\mathrm{d}M_s^{u,c}\right).$$

For any $k\in\mathbb{N}$, we define the norm $\|\cdot\|_k$ on the set of measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ by

$$\|f\|_k=\left(\int_E f^2(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x)\right)^{1/2}. \tag{3.3}$$

Let $\mathcal{I}_k$ be the set of measurable functions from $\mathbb{R}$ into $\mathbb{R}$ such that $\|f\|_k<\infty$.

On $\mathcal{I}=\bigcap_{k\in\mathbb{N}}\mathcal{I}_k$, we define a distance d by setting:

$$d(f,g)=[f-g],$$

where

$$[f]=\sum_{k=1}^{\infty}2^{-k}(1\wedge\|f\|_k). \tag{3.4}$$

Note that $\mathcal{I}$ contains the measurable locally bounded functions and that the set of elementary functions is dense in $(\mathcal{I},d)$. Indeed, by a monotone class argument, we can show that if f is bounded, for any $n\in\mathbb{N}$, there exists f_n elementary such that $\sup_{k\le n}\|f-f_n\|_k\le 2^{-n}$. Hence,

$$\sum_{n=1}^{\infty}[f-f_n]\le\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}2^{-k}(1\wedge\|f-f_n\|_k)+2^{-n}\right)<2.$$

Consequently it is sufficient to show that the set of bounded functions is dense in $\mathcal{I}$. By dominated convergence, $\lim_{n\to\infty}[f-(-n)\vee f\wedge n]=0$ for any $f\in\mathcal{I}$.

Let f be an element of $\mathcal{I}$. The MAF W^k defined by: $W_t^k=\int_0^t f(u_k(X_s))\,\mathrm{d}M_s^{u^k,c}$, has finite energy since: $e(W^k)=\frac{1}{2}\|f\|_k^2$. Hence,

$$fu*M_s^{u,c}:=\begin{cases} fu_k*M_s^{u_k,c} & \text{for } t<\sigma_{E\setminus D_k},\\ 0 & \text{for } t\ge\sigma,\end{cases}$$

belongs to $\overset{\circ}{\mathcal{M}}{}^{c}_{f\text{-loc}}$ ($\overset{\circ}{\mathcal{M}}{}^{c}_{\text{loc}}$ if $u \in \mathcal{F}$) and by Lemma 2.2, $\Gamma(fu * M^{u,c})$ is well defined and is an element of $\mathcal{N}_{c,f\text{-loc}}$ ($\mathcal{N}_{c,\text{loc}}$ if $u \in \mathcal{F}$).

Theorem 3.3. *The application defined by* (3.2) *on the set of elementary functions can be extended to the set $\mathcal{I}$. This extension, denoted by $\int f(z)\,\mathrm{d}_z\Gamma^z$, for f in $\mathcal{I}$, satisfies*:

(i) $\int f(z)\,\mathrm{d}_z\Gamma^z_t = \Gamma_t(fu * M^{u,c})\ \forall t \geq 0$, $\mathbb{P}_x$*-a.e. for q.e.* $x \in E$.

(ii) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence $\mathcal{I}$. Assume that*: $[f_n - f] \to 0$. *Then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(\int f_{n_k}(z)\,\mathrm{d}_z\Gamma^z_t)_{k\in\mathbb{N}}$ converges uniformly on any compact of* $[0,\zeta)$ ($[0,\infty)$ *if $u \in \mathcal{F}$*) *to $\int f(z)\,\mathrm{d}_z\Gamma^z_t$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$.*

Proof. Elementary functions are dense in $\mathcal{I}$ and (i) holds for elementary functions. It is sufficient to prove that that if $[f_n - f]$ converge to zero, there exists a subsequence n_k such that for any $p \in \mathbb{N}$, $\Gamma(f_{n_k}u * M^{u,c})$ converges to $\Gamma(fu * M^{u,c})$ uniformly on any compact of $[0, \sigma_{E\setminus D_p}[$. Let n_k be such that $[f_{n_k} - f] < 2^{-4k}$ and $p \in \mathbb{N}$, hence $\|f - f_{n_k}\|_p \leq 2^p 2^{-4k}$ for any $k > p/4$ and it follows from Theorem 2.5 that $\Gamma(f_{n_k}u_p * M^{u_p,c})$ converges uniformly on any compact to $\Gamma(fu_p * M^{u_p,c})$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$. But thanks to (2.6), $\Gamma(f_{n_k}u_p * M^{u_p,c})$ and $\Gamma(fu_p * M^{u_p,c})$ agrees on $t < \sigma_{E\setminus D_p}$ with $\Gamma(f_{n_k}u * M^{u,c})$ and $\Gamma(fu * M^{u,c})$, respectively, $\mathbb{P}_x$-a.e. for q.e. $x \in E$. □

We finish this section with a characterization of the set $\mathcal{I}$ when u belongs to $\mathcal{F}$. Let $\mathcal{E}^{(c)}$ be the local part in the Beurling–Deny decomposition for $\mathcal{E}$ (see Theorem 3.2.1 of [12]). $\mathcal{E}^{(c)}$ has the local property, hence with the same argument used to proof Theorems 5.2.1 and 5.2.3 of [1], there exists a function U in $L^1(\mathbb{R}, \mathrm{d}x)$ such that for any function F in $\mathcal{C}^1$ with bounded derivatives f:

$$\mathcal{E}^{(c)}(F(u), F(u)) = \frac{1}{2}\int_{\mathbb{R}} f^2(x)U(x)\,\mathrm{d}x.$$

Then thanks to (2.3) and Lemma 3.2.3 of [12],

$$\int_E f^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x) = \int_{\mathbb{R}} f^2(x)U(x)\,\mathrm{d}x,$$

hence it follows by a monotone class argument that for any measurable positive function f we have:

$$\int_E f(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x) = \int_{\mathbb{R}} f(x)U(x)\,\mathrm{d}x. \tag{3.5}$$

Lemma 3.4. *For u element of $\mathcal{F}$, the set $\mathcal{I}$ coincides with the set $L^1_{\text{loc}}(\mathbb{R}, U(x)\,\mathrm{d}x)$, where the function U is defined by* (3.5).

Proof. For k integer, the function u_k is defined be $(-k) \vee u \wedge k$. Associate U_k to u_k as U is associated to u. We have then: $\|f\|_k^2 = \int_{\mathbb{R}} f^2(x)U_k(x)\,\mathrm{d}x$ for any measurable function f.

In order to proof Lemma 3.4, it is sufficient to prove that: $U_k(x) = 1_{[-k,k]}U(x)$ for a.e. x in $\mathbb{R}$.

Let f be a continuous function with support in $[-k,k]$ and set $F(x) := \int_0^x f(z)\,\mathrm{d}z$. We have hence: $F(u(x)) = F(u_k(x))$ for any x in E and therefore $f(u_k) * M^{u_k,c} = f(u) * M^{u,c}$, indeed thanks to (2.3) both martingales coincides with $M^{Fu_k,c}$ $(= M^{Fu,c})$.

We have therefore: $\int_E f^2(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x) = \int_E f^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)$. This shows that

$$\int_{\mathbb{R}} f^2(x)U_k(x)\,\mathrm{d}x = \int_{\mathbb{R}} f^2(x)U(x)\,\mathrm{d}x$$

for any function f continuous with compact support in $[-k,k]$, hence $U_k(x) = U(x)$ for a.e. x in $[-k,k]$.

Now if g is a continuous positive function with support in $\mathbb{R}\setminus[-k,k]$ then:

$$\int_{\mathbb{R}} g(x)U_k(x)\,\mathrm{d}x = \int_E g(u_k(x))\mu_{\langle M^{u_k,c}\rangle}(\mathrm{d}x) = 0$$

therefore $U_k(x) = 0$ for a.e. x in $\mathbb{R}\setminus[-k,k]$. This finishes the proof. □

4. Itô formula

In this section, we first prove Lemma 1.1 and then Theorem 1.2.

Proof of Lemma 1.1. Let u be an element of $\mathcal{F}_{\mathrm{loc}}$, thanks to the proof of Lemma 3.1 of [3], there exists a nest of finely open Borel sets $\{\mathcal{G}_k\}_{k\in\mathbb{N}}$ and a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{F}$ such that $u(x) = u_k(x)$ for q.e. $x \in \mathcal{G}_k$ and $\|u_k\|_\infty < k$. Let $\phi \in L^1(E;m)$ such that $0 < \phi \le 1$ and for any k let

$$h_k(x) := \mathbb{E}_x\left(\int_0^{\sigma_{E\setminus\mathcal{G}_k}} \mathrm{e}^{-t}\phi(X_t)\,\mathrm{d}t\right),$$

$G_k := \{x \in E\colon\ h_k(x) > k^{-1}\}$ and $g_k(x) := 1 \wedge (kh_k(x))$. For any k, $G_k \subset \mathcal{G}_k$, thus $u(x) = u_k(x)$ for q.e. $x \in G_k$. Moreover, by the proof of Lemma 3.8 of [13], $\{G_k\}_{k\in\mathbb{N}}$ is a nest and we have: $0 \le g_k \le 1$, $g_k(x) = 1$ q.e. on G_k, $g_k(x) = 0$ on $E\setminus\mathcal{G}_k$. Since h_k is quasi-continuous, we can suppose that each G_k is finely open (Theorem 4.6.1 of [12]). For any $k \in \mathbb{N}$, we have:

$$\begin{aligned}
&\int_{G_k}\int_{\{|u(x)-u(y)|<1\}} |u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x) \\
&\quad = \int_{G_k} |g_k(x)|^2 \int_{\{|u(x)-u(y)|<1\}} |u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x) \\
&\quad \le 2\int_{G_k}\int_{\{|u(x)-u(y)|<1\}} |g_k(x)-g_k(y)|^2|u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x) \\
&\qquad + 2\int_{\mathcal{G}_k\times\mathcal{G}_k\cap\{|u(x)-u(y)|<1\}} |g_k(y)|^2|u(x)-u(y)|^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)
\end{aligned}$$

$$\leq 2\int_{E\times E}|g_k(x)-g_k(y)|^2N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)$$
$$+2\int_{E\times E}|u_k(x)-u_k(y)|^2N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)$$
$$\leq 4\mathcal{E}(g_k,g_k)+4\mathcal{E}(u_k,u_k)<\infty.$$

Therefore, if for any $\varepsilon>0$, we set:

$$S_\varepsilon=\sum_{k=1}^{\infty}2^{-k}\biggl(1\wedge\int_{G_k}\int_{\{|u(x)-u(y)|<\varepsilon\}}|u(x)-u(y)|^2N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)\biggr).$$

We have then $\lim_{\varepsilon\to 0}S_\varepsilon=0$. We choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $S_{\varepsilon_n}<2^{-4n}$.

Let F be a locally absolutely continuous function with a locally bounded Radon–Nikodym derivative f. For k in $\mathbb{N}$, define (F_k) by

$$F_k(x)=F(x)1_{[-k-1,k+1]}(x)+F(k+1)1_{[k+1,\infty)}(x)+F(-k-1)1_{(-\infty,-k-1]}(x).$$

Note that F_k has a bounded Radon–Nikodym derivative: $f_k=f1_{[-k-1,k+1]}$.

For a function $\beta:E^2\to\mathbb{R}$, define:

$$A_t(\beta,n):=\int_0^t\int_{\{\varepsilon_n<|u(y)-u(X_s)|<1\}}\beta(y,X_s)N(X_s,\mathrm{d}y)\,\mathrm{d}H_s\quad\text{and}$$
$$M^d(\beta,n)\ =\sum_{s\leq t}\beta(X_s,X_{s-})1_{\{\varepsilon_n<|u(X_{s-})-u(X_s)|<1\}}1_{\{s<\xi\}}-A_t(\beta,n).$$

Denote by $M^d(F,u,n)$ (resp., $M^d(F,u,n,k)$) the process $M^d(\beta,n)$ for $\beta(y,x)=F(u(y))-F(u(x))$ (resp., $\beta(y,x)=(F(u(y))-F(u(x))1_{G_k}(x)$). Similarly, define $A^d(F,u,n)$ and $A(F,n,u,k)$.

We just have to prove that $\mathbb{P}_x$-a.e. for q.e. $x\in E$, the limits $\lim_{n\to\infty}M^d(F,u,n)$ and $\lim_{n\to\infty}A(F,u,n)$ exist uniformly on any compact of $[0,\sigma_{E\setminus G_k}[$. We have: $M_t^d(F,u,n)=M_t^d(F_k,u,n,k)$ and $A_t(F,u,n)=A_t(F_k,u,n,k)$ on $[0,\sigma_{E\setminus G_k}[$. For every k, the process $M^d(F_k,u,n,k)$ belongs to $\overset{\circ}{\mathcal{M}}$ and for $4n>k$, we have

$$e\bigl(M^d(F_k,u,n+1,k)-M^d(F_k,u,n,k)\bigr)\leq c_k2^k2^{-4n},$$

where $c_k=\|f_k\|_\infty$. Indeed, from the definition of ε_n:

$$e\bigl(M^d(F_k,u,n+1,k)-M^d(F_k,u,n,k)\bigr)$$
$$=\frac{1}{2}\int_{G_k\times E}\bigl(F_k(u(x))-F_k(u(y))\bigr)^21_{\{\varepsilon_{n+1}\leq|u(x)-u(y)|<\varepsilon_n\}}N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)$$
$$\leq c_k\int_{G_k\times E}|u(x)-u(y)|^21_{\{|u(x)-u(y)|<\varepsilon_n\}}N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)$$
$$\leq c_k2^k2^{-4n}$$

thus, the convergence of $M^d(F,u,n)$ follows from Theorem 2.5. Still thanks to Theorem 2.5, the convergence of $A(F,u,n)$ can be seen as a consequence of:

$$\Gamma(M_t^d(F_k,u,n,k)) = A_t(F_k,u,n,k), \qquad \mathbb{P}_x\text{-a.e. for q.e. } x \in E. \tag{4.1}$$

To prove (4.1), we note that $(A_t(F_k,u,n,k))_{t\geq 0}$ is of bounded variation, so $A_t(F_k,u,n,k)\circ r_t = A_t(F_k,u,n,k)$ $\mathbb{P}_m$-a.e. on $t<\zeta$ (Theorem 2.1 of [11]). Hence, making use of the operator Λ defined in [3], instead of Γ, we first obtain:

$$\Lambda(M_t^d(F_k,u,n,k)) = A_t(F_k,u,n,k), \qquad \mathbb{P}_m\text{-a.e. for q.e. } x \in E \text{ on } [\![0,\zeta[\![.$$

Finally by Theorem 3.6 in [3] and Lemma 2.1, (4.1) holds, $\mathbb{P}_x$-a.e. for q.e. $x\in E$ on $[\![0,\zeta[\![$, and therefore on $[\![0,\infty[\![$ thanks to the continuity of $\Gamma(M_t^d(F_k,u,n,k))$ and $A_t(F_k,u,n,k)$.

It is clear that $M^d(F,u)\in\mathring{\mathcal{M}}_{f\text{-loc}}$ and $A(F,u)\in\mathcal{N}_{c,f\text{-loc}}$. Moreover, for u element of $\mathcal{F}$, we can take $G_n=\{x\colon |u(x)|<n\}$ for any n. In this case, from the strict continuity of u we have, $\mathbb{P}_x(\lim_{n\to\infty}\sigma_{E\setminus G_n}=\infty)=1$ for q.e. $x\in E$, thus the convergence of $M^d(F,u,n)$ and $A(F,u,n)$ are uniformly on any compact of $[0,\infty)$. Thus, we obtain: $M^d(F,u)\in\mathring{\mathcal{M}}_{\text{loc}}$ and $A(F,u)\in\mathcal{N}_{c,\text{loc}}$. □

Remark 4.1. (i) If $u\in\mathcal{F}$ and f is bounded, then $M^d(F,u)\in\mathring{\mathcal{M}}$ and $\Gamma(M^d(F,u))=A(F,u)$.
(ii) With the notation of the proof of Lemma 1.1, it holds that if $u_k=u$ q.e. on G_k:

$$M_t^d(F,u)+A_t(F,u) = M_t^d(F_k,u_k)+A_t(F_k,u_k) \qquad \text{for } t\in[0,\sigma_{E\setminus G_k}[,\ \mathbb{P}_x\text{-a.e. for q.e. } x\in E.$$

Proof of Theorem 1.2. We use the notation of the proof of Lemma 1.1. Thus, if $u\in\mathcal{F}$, we take $G_n:=\{x\colon |u(x)|<n\}$, $n\in\mathbb{N}$. Let F be a locally absolutely continuous function F with a locally bounded Radon–Nikodym derivative f.

Let I_t be the difference of the left-hand side and the right-hand side of (1.3). For any k, we define I_t^k as I_t with u_k and f_k replacing u and f, respectively. Hence, $I_t=I_t^k$ for $t<\sigma_{E\setminus G_k}$, $\mathbb{P}_x$-a.e. for q.e. $x\in E$. Since $\sigma_{E\setminus G_n}\wedge\zeta$ converges to ζ if $u\in\mathcal{F}_{\text{loc}}$ and $\sigma_{E\setminus G_n}$ converges to ∞ if $u\in\mathcal{F}$, it is sufficient to prove (1.3) on $[0,\sigma_{E\setminus G_k}[$ for any $k\in\mathbb{N}$. Consequently, we can assume (and we do) that u is an element of $\mathcal{F}_b$ and f is bounded.

If f is continuous, thanks to (2.3), $F(u)\in\mathcal{F}$ and $M^{Fu,c}=fu*M^{u,c}$ and we have the Fukushima decomposition:

$$F(u(X_t)) = F(u(X_0)) + fu*M_t^{u,c} + \Gamma(fu*M^{u,c})_t + M_t^{u,d} + \Gamma(M^{u,d})_t.$$

We obtain (1.3) from Lemma 3.3(i) and Remark 4.1(i).

If f is not necessarily continuous, let g be in $L^1(\mathbb{R})$ be a strictly positive function on $\mathbb{R}$ such that g and $1/g$ are locally bounded. Define the norms $\|\cdot\|$ and $\|\cdot\|_*$ on the Borel measurable functions as follows:

$$\|h\|_* = \left(\int_E h^2(u(x))\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)\right)^{1/2},$$

$$\|h\| = \|h\|_* + \int |h(x)|g(x)\,\mathrm{d}x$$
$$+ \left(\int_{E\times E-\delta} |u(x)-u(y)| \int_{u(x)\wedge u(y)}^{u(x)\vee u(y)} h(z)^2\,\mathrm{d}z N(x,\mathrm{d}y)\nu_H(\mathrm{d}x) \right)^{1/2}.$$

Since u is in $\mathcal{F}$, we have $\|f\| < \infty$. By a monotone class argument, one shows that there exists a sequence of bounded continuous functions $(f_n)_{n\in\mathbb{N}}$ with compact support such that $\|f_n - f\|$ converges to 0 as n tends to infinity. We set $F_n(x) = \int_0^x f_n(z)\,\mathrm{d}z$.

In order to show (1.3), we will show that there exists a subsequence n_k such that the terms in the expansion (1.3) for F_{n_k} converge as $k\to\infty$ to the corresponding expression with f replacing f_{n_k}. The convergence of $F_n(u(X_t)) - F_n(u(X_0)) - V_t(F_n,u)$ to $F(u(X_t)) - F(u(X_0)) - V_t(F,u)$ is a consequence of the pointwise convergence of F_n to F, indeed, for any $x\in\mathbb{R}$,

$$|F_n(x) - F(x)| \le \int_{-x^-}^{x^+} |f_n(z) - f(z)|\,\mathrm{d}z \le \sup_{|\lambda|\le|x|} \frac{1}{g(\lambda)} \int_{-\infty}^{\infty} |f_n(z) - f(z)|g(z)\,\mathrm{d}z \to 0.$$

The existence of a subsequence $\{n_k\}$ such that $\int_0^t f_{n_k}(u(X_s))\,\mathrm{d}M_s^{u,c}$ and $\int_{\mathbb{R}} f_{n_k}(z)\,\mathrm{d}_z\Gamma_t^z(u)$ converge to $\int_0^t f(u(X_s))\,\mathrm{d}M_s^{u,c}$ and $\int_{\mathbb{R}} f(z)\,\mathrm{d}_z\Gamma_t^z(u)$, respectively, is a consequence of the fact that $e(fu * M^{u,c} - f_n u * M^{u,c}) = \frac{1}{2}\|f - f_n\|_* \to 0$ as $n\to\infty$, and Theorem 2.5. Thanks to Theorem 2.5 and Remark 4.1(i), it is then sufficient to show that $e(M(F_n,u) - M(F,u))$ converges to zero as $n\to\infty$. But

$$\begin{aligned} e(M - M^n) &\le \frac{1}{2}\int_{E\times E-\delta} \bigl(F(u(x)) - F_n(u(x)) - F(u(y)) \\ &\qquad + F_n(u(y))\bigr)^2 N(x,\mathrm{d}y)\nu_H(\mathrm{d}x) \\ &\le \frac{1}{2}\|f - f_n\|_*^2 \to 0 \qquad \text{as } n\to\infty. \end{aligned}$$ □

As an example, for $F(z) = z$ and u in $\mathcal{F}_{\mathrm{loc}}$, one obtains a Fukushima decomposition for the process $u(X)$. This case can be seen as a refinement of Lemma 2.2 in [4].

5. Local time

We fix an element u of $\mathcal{F}_{\mathrm{loc}}$. The associated process ${}^cN^u$ has been defined in (2.1) by ${}^cN^u = \Gamma(M^{u,c})$. By Remark 2.3, ${}^cN^u$ is a CAF locally of zero energy or merely a CAF of zero energy when u belongs to F. We suppose that u satisfies the additional assumption that ${}^cN^u$ is of bounded variation on $[0,\zeta)$, that is, there exists two PCAF's $A^{(1)}$ and $A^{(2)}$ such that $\mathbb{P}_x$-a.e. for q.e. $x\in E$:

$$^cN_t^u = A_t^{(1)} - A_t^{(2)} \qquad \forall t\in[0,\zeta). \tag{5.1}$$

We remind that a measure ν on E is a smooth signed measure on E if there exists a nest $\{F_k\}$ such that for each k, $1_{F_k}.\nu$ is a finite signed Borel measure charging no set of zero capacity and further ν charges no Borel subset of $E \setminus \bigcup_{k=1}^{\infty} F_k$. Such nest is said to be associated to ν. For a closed set $F \subset E$, we set:

$$\mathcal{F}_{b,F} = \{u \in \mathcal{F}_b \colon u = 0 \text{ q.e. on } E \setminus F\}.$$

We also need the following definition:

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_m.$$

Lemma 5.1. *The process ${}^cN^u$ is of bounded variation if and only if there exists a smooth signed measure ν on E with associated nest $\{F_k\}$ such that*

$$\mathcal{E}^{(c)}(u, v) = \langle v, \nu \rangle, \qquad \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b,F_k}.$$

Proof. From Theorem 5.2.4 of [12], ${}^cN^u$ is the only AF of zero energy such that for any $h \in \mathcal{F}$,

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{h.m}[{}^cN_t^u] = -e(M^{u,c}, M^{h,c}) = -\mathcal{E}^{(c)}(u, h).$$

On the other hand, since: $|\mathcal{E}^{(c)}(u, h)| \le (\mathcal{E}^{(c)}(u, u))^{1/2} (\mathcal{E}_1(h, h))^{1/2}$, there exists a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}^{(c)}(u, h) = \mathcal{E}_1(w, h) \qquad \text{for any } h \in \mathcal{F}.$$

Hence, $\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{h.m}[N_t^w - \int_0^t w(X_s)\,\mathrm{d}s] = -\mathcal{E}^{(c)}(u, h)$ for all $h \in \mathcal{F}$. This implies that the AF $N^w - \int_0^{\cdot} w(X_s)\,\mathrm{d}s$ is equivalent to ${}^cN^u$. Consequently, ${}^cN^u$ is of bounded variation if and only if N^w is of bounded variation. But thanks to Theorem 5.4.2 of [12], this last condition is equivalent to the existence of a smooth signed measure ν with an associated nest $\{F_k\}$ such that

$$\mathcal{E}_1(w, v) = \langle v, \nu \rangle \qquad \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b,F_k}.$$

□

5.1. Definition of local time

Definition 5.2. *The local time at a of $u(X)$, denoted by $L_t^a = L_t^a(u)$ is the following CAF on* $[\![0, \zeta[\![$:

$$\frac{1}{2} L_t^a = -\Gamma(Z^a)_t + \int_0^t 1_{\{u(X_{s-}) \le a\}}\,\mathrm{d}^c N_s^u \qquad \textit{for } t \in [0, \zeta).$$

The name "local time" is justified by Proposition 5.3 and Corollary 5.4 below.

Proposition 5.3. *There exists a $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$-measurable version of the local time process $\{\tilde{L}^a_t; a\in\mathbb{R}, t\geq 0\}$ such that $\mathbb{P}_m$-a.e. we have the occupation time density formula*:

$$\int_{\mathbb{R}} f(x)\tilde{L}^x_t\,\mathrm{d}x=\int_0^t f(u(X_s))\,\mathrm{d}\langle M^{u,c}\rangle_s \qquad \textit{for any } f \textit{ Borel bounded and } t<\zeta.$$

Proof. We start with the case when u is a bounded element of $\mathcal{F}$. From (2.4) we have: $\mathbb{P}_m$-a.e. on $[\![0,\zeta[\![$: $L^a_t=Z^a_t+Z^a_t\circ r_t+2\int_0^t 1_{\{u(X_{s-})\leq a\}}\,\mathrm{d}N^{u,c}_s$. Moreover, thanks to Theorem 63, Chapter IV of [15], there exists a function $\tilde{Z}(a,t,\omega)$ in $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$, such that for each $a\in\mathbb{R}$, $\tilde{Z}(a,t,w)$ is a continuous $(\mathcal{F}^m_t)$-adapted version of the stochastic integral Z^a, and thanks to Lemma 2.10 and Theorem 2.18 of [3], $\tilde{Z}(a,t,w)\circ r_t(\omega)\in\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$ is a continuous $(\mathcal{F}^m_t)$-adapted version of $Z^a_t\circ r_t$ for each $a\in\mathbb{R}$. Besides, we can take $\int_0^t 1_{\{u(X_{s-})\leq a\}}\,\mathrm{d}N^{u,c}_s$ jointly continuous in t and right continuous in a, $\mathbb{P}_m$-a.e. on $[\![0,\zeta[\![\times\mathbb{R}$. Thus, we have constructed a version $\{\tilde{L}^a_t, a\in\mathbb{R}, t\in[0,\zeta[\}$ of $\{L^a_t, a\in\mathbb{R}, t\in[0,\zeta)\}$ which is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}^m_\infty$-measurable.

Let f be a continuous positive element of $L^1(\mathbb{R})$. Using the proof presented in [15] of Fubini's theorem for stochastic integrals (Theorem 64, Chapter IV of [15]), we know that $\int_{\mathbb{R}}\tilde{Z}(z,t,\omega)f(z)\,\mathrm{d}z$ is a well-defined Lebesgue integral since $\mathbb{P}_m$-a.e.:

$$\int_{\mathbb{R}}|\tilde{Z}(z,t,\omega)|f(z)\,\mathrm{d}z<\infty \qquad \text{for all } t.$$

Moreover, still thanks to this theorem, $\int_{\mathbb{R}}\tilde{Z}(z,t,\omega)f(z)\,\mathrm{d}z$ is a continuous $\mathbb{P}_m$-version of $\int_0^t F(u(X_s))\,\mathrm{d}M^{u,c}_s$, where $F(z)=\int_z^\infty f(\lambda)\,\mathrm{d}\lambda$. Consequently, for $t>0$, $\mathbb{P}_m$-a.e. on $\{t<\zeta\}$, $\int_{\mathbb{R}}|\tilde{Z}(z,t,r_t(\omega))|f(z)\,\mathrm{d}z<\infty$ and $\int_{\mathbb{R}}\tilde{Z}(z,t,r_t(\omega))f(z)\,\mathrm{d}z$ is a continuous $\mathbb{P}_m$-version of $\int_0^t F(u(X_s))\,\mathrm{d}M^{u,c}_s\circ r_t$.

Since $(\int_0^t 1_{\{u(X_s-)\leq a\}}\,\mathrm{d}N^{u,c}_s)_{a\in\mathbb{R}}$ is of bounded variation on $\{t<\zeta\}$, we obtain $\mathbb{P}_m$-a.e. on $\{t<\zeta\}$: $\int_{\mathbb{R}}f(z)|\tilde{L}^z_t|\,\mathrm{d}z<\infty$ and

$$\int_{\mathbb{R}}f(z)\tilde{L}^z_t\,\mathrm{d}z=\int_0^t F(u(X_s))\,\mathrm{d}M^{u,c}_s+\int_0^t F(u(X_s))\,\mathrm{d}M^{u,c}_s\circ r_t+2\int_0^t F(u(X_s))\,\mathrm{d}N^{u,c}_s$$

which leads to

$$\int_{\mathbb{R}}f(z)\tilde{L}^z_t\,\mathrm{d}z=-2\Gamma(Fu*M^{u,c})_t+2\int_0^t F(u(X_s))\,\mathrm{d}N^{u,c}_s. \tag{5.2}$$

Now thanks to (2.3), Fu belongs to $\mathcal{F}_{\mathrm{loc}}$ and $M^{Fu,c}_t=-\int_0^t f(u(X_s))\,\mathrm{d}M^{u,c}_s$. Thus,

$$\langle M^{Fu,c},M^{u,c}\rangle_t=-\int_0^t f(u(X_s))\,\mathrm{d}\langle M^{u,c}\rangle_s.$$

Thanks to Lemma 2.4 we have $\mathbb{P}_m$-a.e. on $\{t<\zeta\}$:

$$\int_0^t F(u(X_s))\,\mathrm{d}^c N^u_s=\int_0^t F(u(X_s))\,\mathrm{d}\Gamma(M^{u,c})_s.$$

On the other hand, the definition of the integral with respect to $\Gamma(M^{u,c})$ (Chen *et al.* [3]) gives:

$$\int_0^t F(u(X_s))\,\mathrm{d}\Gamma(M^{u,c})_s = \Gamma(Fu * M^{u,c})_t + \frac{1}{2}\int_0^t f(u(X_s))\,\mathrm{d}\langle M^{u,c}\rangle_s$$

which together with (5.2) lead to

$$\int_{\mathbb{R}} f(z)\tilde{L}_t^z\,\mathrm{d}z = \int_0^t f(u(X_s))\,\mathrm{d}\langle M^{u,c}\rangle_s, \qquad \mathbb{P}_m\text{-a.e. on}\{t<\zeta\}. \tag{5.3}$$

Actually, the set of null $\mathbb{P}_m$-measure on which (5.3) could fail can be chosen independently of f. Indeed, the set of continuous functions with compact support, is a separable topological space for the metric of uniform convergence.

We show now that the set of null $\mathbb{P}_m$-measure on which (5.3) could fail does not depend on t either. We have thanks to (5.3)

$$\mathbb{P}_m\text{-a.e. on } \{t<\zeta\}, \qquad \tilde{L}_t^z \geq 0 \text{ for d}z\text{-a.e. } z \tag{5.4}$$

hence by a monotone class argument, (5.3) holds $\mathbb{P}_m$-a.e. on $\{t<\zeta\}$ for any f Borel bounded. It remains to show that (5.3) holds $\mathbb{P}_m$-a.e. on $[\![0,\zeta[\![$. To do so, it is sufficient to show that the left-hand side of (5.3) is continuous in t.

It follows from Theorem 2.18 in [3] that for any z, $\tilde{Z}(z,t,r_t(\omega))$ is continuous and has the additivity property $\mathbb{P}_m$-a.e. for on $[\![0,\zeta[\![$. Hence, thanks to (5.4) for dz-a.e. z, $\tilde{L}_t^z$ is increasing. One shows then by monotone convergence that for any positive Borel function f, $t\to\int_{\mathbb{R}} f(z)\tilde{L}_t^z\,\mathrm{d}z$ is continuous $\mathbb{P}_m$-a.e. on $[\![0,\zeta[\![$.

For a function u in $\mathcal{F}_{\mathrm{loc}}$, take an nest of closed sets $\{D_k\}$ and a sequence $(u_k)_{n\in\mathbb{N}}$ of bounded elements of $\mathcal{F}$ such that $u=u_k$ for q.e. $x\in E$. For any $k\in\mathbb{N}$, let $\tilde{L}_t^z(u_k)$ be the version $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_\infty^m$-measurable of local time obtained above. Then $\tilde{L}_t^z := \tilde{L}_t^z(u_k)$ on $t<\tau_{D_k}$ is a $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}_\infty^m$-measurable version of L_t^z and satisfies the occupation time density formula on $[0,\tau_{D_k}[$, for any $k\in\mathbb{N}$, so it satisfies it on $[0,\zeta[$. □

Corollary 5.4. *For any real a, L^a is a PCAF and $\mathbb{P}_x$-a.e. for q.e. $x\in E$, the measure in t, $\mathrm{d}_t L_t^a$ is carried by the set $\{s\colon u(X_{s-})=u(X_s)=a\}$.*

Proof. We use u_k and $\{D_k\}$ defined as in the end of the proof of Proposition 5.3. Since we need to show the assertion of Cororally 5.3 only on $[0,\tau_{D_k}[$, we can assume that u is a bounded element of $\mathcal{F}$. It follows from the occupation time density formula and the $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}_+)\otimes F_\infty^m$-measurability of $\tilde{L}$, that there exists a subset R of $\mathbb{R}$ of Lebesgue's measure zero, such that for any a outside of R: $\mathbb{P}_m$-a.e. $\tilde{L}_t^a\geq 0$ on $[\![0,\zeta[\![$. Consequently, L^a has the same property. This property holds for any $a\in\mathbb{R}$. Indeed for any real a, take a sequence $(a_n)_{n\in\mathbb{N}}\subset\mathbb{R}\setminus R$ such that $a_n\downarrow a$. We have: $e(Z^{a_n}-Z^a)=\int 1_{\{a<u(x)\leq a_n\}}\mu_{\langle M^{u,c}\rangle}(\mathrm{d}x)$, which converges to 0 as n tends to ∞ by dominated convergence. Thus, thanks to Theorem 2.5 (taking a subsequence if necessary) $\Gamma(Z^{a_n})$ converges to $\Gamma(Z^a)$ uniformly on any finite interval of t, $\mathbb{P}_m$-a.e. On the other hand, for $\mathbb{P}_m$-a.e. $w\in\Omega$, $\int_0^t 1_{\{u(X_s)\leq a_n\}}\,\mathrm{d}N_s^{u,c}(\omega)$ converges to $\int_0^t 1_{\{u(X_s)\leq a\}}\,\mathrm{d}N_s^{u,c}(\omega)$ for any $t<\zeta(\omega)$. Consequently, we obtain for $\mathbb{P}_m$-a.e. $\omega\in\Omega$, $L_t^a(\omega)\geq 0$ for any $t<\zeta(\omega)$.

It follows from Lemma 2.1 that for any real a, L^a is a PCAF on $[\![0,\zeta[\![$. By Remark 2.2 in [3], it can be extended to a PCAF.

Now defining $f(x)=(x-a)^4$ and $h(x)=(x-a)^4 1_{\{x\le a\}}$, it follows from (2.3) that fu and hu belong to $\mathcal{F}_{\mathrm{loc}}$. Moreover, we have:

$$M_t^{fu,c}=4\int_0^t \big(u(X_s)-a\big)^3\,\mathrm{d}M_s^{u,c} \quad \text{and} \quad M_t^{hu,c}=4\int_0^t \big(u(X_s)-a\big)^3 1_{\{u(X_s)\le a\}}\,\mathrm{d}M_s^{u,c}$$

thus, $\langle M^{fu,c},Z^a\rangle=\langle M^{hu,c},M^{u,c}\rangle$, and from the definition of the stochastic integral (2.5) we have that $\mathbb{P}_m$-a.e. on $\{t<\zeta\}$

$$\int_0^t \big(u(X_s)-a\big)^4\,\mathrm{d}\Gamma(Z^a)_s=\int_0^t \big(u(X_s)-a\big)^4 1_{\{u(X_s)\le a\}}\,\mathrm{d}\Gamma(M^{u,c})_s.$$

By Lemmas 2.1 and 2.4, we finally obtain: $\int_0^t (u(X_s)-a)^4\,\mathrm{d}L_s^a=0$ $\mathbb{P}_x$-a.e. for q.e. $x\in E$. □

5.2. Integration with respect to local time

We fix u an element of $\mathcal{F}$ satisfying (5.1) and set: $l_t^a=\int_0^t 1_{\{u(X_{s-})\le a\}}\,\mathrm{d}N_s^{u,c}$. Hence, the local time at a of $u(X)$ satisfies:

$$L^a=-2\Gamma^a+2l^a.$$

For any $\omega\in\Omega$ and $t<\zeta(\omega)$, the function $z\to l_t^z(\omega)$ is of bounded variation. The application defined for the elementary functions by

$$f\to\sum_{i=0}^{n-1} f_i\big(l_t^{z_{i+1}}-l_t^{z_i}\big),\qquad t<\zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ as a Lebesgue–Stieljes integral and we have:

$$\int_{\mathbb{R}} f(z)\,\mathrm{d}_z l_t^z=\int_0^t f(u(X_s))\,\mathrm{d}N_s^{u,c},\qquad t<\zeta.$$

Using the stochastic integral with respect to Γ, the application defined for the elementary functions by

$$f\to\sum_{i=0}^{n-1} f_i\big(L_t^{z_{i+1}}-L_t^{z_i}\big),\qquad t<\zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from $\mathbb{R}$ into $\mathbb{R}$ and we have:

$$-\frac{1}{2}\int_{\mathbb{R}} f(z)\,\mathrm{d}_z L_t^z=\int_{\mathbb{R}} f(z)\,\mathrm{d}_z\Gamma_t^z-\int_0^t f(u(X_s))\,\mathrm{d}N_s^{u,c},\qquad t<\zeta.$$

6. Multidimensional case

In this section, we need the following notation. For $d \in \mathbb{N}$, $x = (x^1, \dots, x^d)$, $y = (y^1, \dots, y^d) \in \mathbb{R}^d$, we set $x \le y$ (resp., $x < y$) if and only of $x^i \le y^i$ (resp., $x^i < y^i$) for each $i = 1, \dots, d$ and $]x, y] = \{z \in \mathbb{R}^d\colon\ x < z \le y\}$. The vector $\hat{x}$ is obtained from x by elimination of its coordinate x^d, that is, $\hat{x} = (x^1, \dots, x^{d-1})$, $\widehat{]x, y]} = \{z \in \mathbb{R}^{d-1}\colon\ \hat{x} < z \le \hat{y}\}$.

Let φ be a measurable function from $\mathbb{R}^d$ into $\mathbb{R}$. We define integration of simple functions with respect to φ as follows. For f a simple function, that is, there exists $x, y \in \mathbb{R}^d$ such that $f(z) = 1_{]x,y]}(z)$ for all $z \in \mathbb{R}^d$:

$$\text{if } d = 1: \qquad \int_{\mathbb{R}} f(z)\,\mathrm{d}\varphi(z) = \varphi(y) - \varphi(x),$$

$$\text{if } d > 1: \qquad \int_{\mathbb{R}^d} f(z)\,\mathrm{d}\varphi(z) = \int_{\mathbb{R}^{d-1}} 1_{\widehat{]x,y]}}(z)\,\mathrm{d}\varphi(z, y^d) - \int_{\mathbb{R}^{d-1}} 1_{\widehat{]x,y]}}(z)\,\mathrm{d}\varphi(z, x^d).$$

As an example, if there exist functions h_i, $1 \le i \le d$ such that $\varphi(z) = \prod_{i=1}^d h(z_i)$, then $\int_{\mathbb{R}^d} f(z)\,\mathrm{d}\varphi(z) = \prod_{i=1}^d (h_i(y^i) - h_i(x^i))$.

We extend this integration to the elementary functions $f : \mathbb{R}^d \to \mathbb{R}$ (i.e., $f(z) = \sum_{i=1}^n a_i f_i(z)$ where f_i, $1 \le i \le n$, are simple functions and a_i, $1 \le i \le n$, are real numbers) by setting

$$\int_{\mathbb{R}^d} f(z)\,\mathrm{d}\varphi(z) = \sum_{i=1}^d a_i \int_{\mathbb{R}^d} f_i(z)\,\mathrm{d}\varphi(z).$$

An elementary function has many representations as linear combination of simple functions, but as in the Riemann integration theory, the integral does not depend on the choice of its representation.

Let u be in $\mathcal{F}^d_{\mathrm{loc}}$ where $\mathcal{F}^d_{\mathrm{loc}} = \{(u^1, u^2, \dots, u^d)\colon\ u^i \in \mathcal{F}_{\mathrm{loc}}, 1 \le i \le d\}$. Let $\{D_k\}_{k\in\mathbb{N}}$ be a nest of closed set, $\sigma := \lim_{k\to\infty} \sigma_{E\setminus D_k}$ and $(u_k)_{k\in\mathbb{N}}$ a sequence of bounded elements of $\mathcal{F}^d$ such that $u = u_k$ q.e. on D_k.

For any a in $\mathbb{R}^d$ and i in $\{1, 2, \dots, d\}$, we define $Z^a(u^i)$ and $\Gamma^a(u^i)$, respectively, in $\mathring{\mathcal{M}}^c_{f\text{-loc}}$ and $\mathcal{N}_{c,f\text{-loc}}$ by

$$Z^a_t(u^i) = \begin{cases} \displaystyle\int_0^t 1_{\{u_k(X_{s-}) \le a\}}\,\mathrm{d}M^{u^i_k,c}_s & \text{for } t \le \sigma_{E\setminus D_k}, \\ 0 & \text{for } t \ge \sigma, \end{cases}$$

$$\Gamma^a(u^i) = \Gamma(Z^a(u^i)).$$

Thanks to the linearity property of Γ, we have for any elementary function f:

$$\int_{\mathbb{R}^d} f(z)\,\mathrm{d}_z\Gamma^z_t(u^i) = \Gamma_t\left(\int_0^t f(u(X_s))\,\mathrm{d}M^{u^i,c}_s\right).$$

We extend (3.3) of Section 3 from $d = 1$ to $d \geq 1$, by defining for $k \in \mathbb{N}$, the norm $\|\cdot\|_k$ on the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$

$$\|f\|_k := \sum_{i=1}^{d} \left(\int_E f^2(u_k(x)) \mu_{\langle M^{u_k^i,c} \rangle}(\mathrm{d}x) \right)^{1/2}$$

and we define the set $\mathcal{I}$ with the metric $[\cdot,\cdot]$ as in (3.4) of Section 3. The set of elementary functions is dense in $\mathcal{I}$. We have the following version of Lemma 3.3.

Lemma 6.1. *The applications $f \to \int_{\mathbb{R}^d} f(z)\,\mathrm{d}_z\Gamma_t^z(u^i)$ $(1 \leq i \leq d)$ defined on the set of elementary functions, can be extended to the set $\mathcal{I}$. This extensions, denoted by $\int_{\mathbb{R}^d} \mathrm{d}_z\Gamma^z(u^i)$, satisfy*:

(i) $\int_{\mathbb{R}^d} f(z)\,\mathrm{d}_z\Gamma_t^z(u^i) = \Gamma(fu * M^{u^i,c})_t$ $\forall t \geq 0$, *$\mathbb{P}_x$-a.e. for q.e. $x \in E$.*

(ii) *For $(f_n)_{n\in\mathbb{N}}$ sequence of $\mathcal{I}$ such that $[f_n - f] \to 0$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\int f_{n_k}(z)\,\mathrm{d}_z\Gamma_t^z(u^i)$ converges uniformly on any compact of $[0,\zeta)$ $([0,\infty)$ if $u \in \mathcal{F}^d)$ to $\int f(z)\,\mathrm{d}_z\Gamma_t^z(u^i)$ for every $1 \leq i \leq d$ $\mathbb{P}_x$-a.e. for q.e. $x \in E$.*

With can prove a multidimensional version of Lemma 1.1 with the same arguments used in its proof. We have the following multidimensional Itô formula.

Proposition 6.2. *Let u be an element of $\mathcal{F}^d$ (resp., $\mathcal{F}^d_{\mathrm{loc}}$) and $F : \mathbb{R}^d \to \mathbb{R}$ a continuous function admitting locally bounded Radon–Nikodym derivatives $f_i = \partial F/\partial x_i$, $1 \leq i \leq d$, satisfying the following condition for any $1 \leq i \leq d$ and $k \in \mathbb{N}$*

$$\lim_{h\to 0} \int_E \bigl\{ f_i\bigl(u_k(x) + h\bigr) - f_i(u_k(x)) \bigr\}^2 \mu_{\langle M^{u_k^i,c} \rangle}(\mathrm{d}x) = 0. \tag{6.1}$$

Then, $\mathbb{P}_x$-a.e. for q.e. $x \in E$, the process $F(u(X_t))$, $t \in [0,\infty)$ (resp., $[0,\zeta)$) admits the decomposition

$$F(u(X_t)) = F(u(X_0)) + M_t(F,u) + Q_t(F,u) + V_t(F,u), \tag{6.2}$$

where $M(F,u) \in \mathring{\mathcal{M}}_{\mathrm{loc}}$, (resp., $\mathring{\mathcal{M}}_{f\text{-loc}}$) $Q(F,u) \in \mathcal{N}_{c,\mathrm{loc}}$ (resp., $\mathcal{N}_{c,f\text{-loc}}$) and $V(F,u)$ is a bounded variation process given by:

$$M_t(F,u) = M_t^d(F,u) + \sum_{i=1}^{d} \int_0^t f_i(u(X_s))\,\mathrm{d}M_s^{u^i,c},$$

$$Q_t(F,u) = \sum_{i=1}^{d} \int_{\mathbb{R}} f_i(z)\,\mathrm{d}_z\Gamma_t^z(u_i) + A_t(F,u),$$

$$V_t(F,u) = \sum_{s\leq t} \{F(u(X_s)) - F(u(X_{s-}))\} 1_{\{|u(X_s)-u(X_{s-})|\geq 1\}} 1_{\{s<\xi\}} - F(u(X_{\xi-})) 1_{\{t\geq\xi\}}.$$

Proof. As in the proof of Theorem 1.2, we can assume that u is a bounded element of $\mathcal{F}$ and each f_i is bounded. For $\phi:\mathbb{R}^d \to \mathbb{R}$ an infinitely differentiable function with compact support, the function F_n defined by $F_n(z) := \int_{\mathbb{R}^d} F(z + y/n)\phi(y)\,\mathrm{d}y$ converges pointwise to $F(z)$. Setting: $f_{n,i} = \partial F_n/\partial x_i$ we obtain thanks to (6.1):

$$\lim_{n\to\infty} \int_E [f_{n,i}(u(x)) - f_i(u(x))]^2 \mu_{\langle M^{u^i,c}\rangle}(\mathrm{d}x) = 0.$$

The rest of the proof follows step by step the proof of Theorem 1.2. □

In the case where $E = \mathbb{R}^d$ and $\mathcal{E}^{(c)}$ is given by

$$\mathcal{E}^{(c)} = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a_{ij}(x)\,\mathrm{d}x,$$

where for every (i, j), a_{ij} is a bounded measurable function. The coordinates functions $\pi_i(x) = x_i$, $1 \le i \le d$, belong to $\mathcal{F}_{\mathrm{loc}}$ and $M = (M^{\pi_1,c}, \dots, M^{\pi_d,c})$ is a martingale additive functional with quadratic covariation $\langle M^i, M^j\rangle_s = \int_0^t a_{ij}(X_s)\,\mathrm{d}s$, hence, $\mu_{\langle M^{i,c}\rangle}(\mathrm{d}x) = a_{ii}(x)\,\mathrm{d}x$, and the condition (6.1) holds for any locally bounded measurable function.

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References

[1] Bouleau, N. and Hirsch, F. (1991). *Dirichlet Forms and Analysis on Wiener Space*. *de Gruyter Studies in Mathematics* **14**. Berlin: de Gruyter. MR1133391

[2] Bouleau, N. and Yor, M. (1981). Sur la variation quadratique des temps locaux de certaines semi-martingales. *C. R. Acad. Sci. Paris Sér. I Math.* **292** 491–494. MR0612544

[3] Chen, Z.Q., Fitzsimmons, P.J., Kuwae, K. and Zhang, T.S. (2008). Stochastic calculus for symmetric Markov processes. *Ann. Probab.* **36** 931–970. MR2408579

[4] Chen, Z.Q., Fitzsimmons, P.J., Takeda, M., Ying, J. and Zhang, T.S. (2004). Absolute continuity of symmetric Markov processes. *Ann. Probab.* **32** 2067–2098. MR2073186

[5] Eisenbaum, N. (2000). Integration with respect to local time. *Potential Anal.* **13** 303–328. MR1804175

[6] Eisenbaum, N. (2006). Local time–space stochastic calculus for Lévy processes. *Stochastic Process. Appl.* **116** 757–778. MR2218334

[7] Eisenbaum, N. (2007). Local time–space calculus for reversible semimartingales. In *Séminaire de Probabilités XL*. *Lecture Notes in Math.* **1899** 137–146. Berlin: Springer. MR2409003

[8] Eisenbaum, N. and Kyprianou, A.E. (2008). On the parabolic generator of a general one-dimensional Lévy process. *Electron. Commun. Probab.* **13** 198–209. MR2399282

[9] Eisenbaum, N. and Walsh, A. (2009). An optimal Itô formula for Lévy processes. *Electron. Commun. Probab.* **14** 202–209. MR2507749

[10] Errami, M., Russo, F. and Vallois, P. (2002). Itô's formula for $C^{1,\lambda}$-functions of a càdlàg process and related calculus. *Probab. Theory Related Fields* **122** 191–221. MR1894067

[11] Fitzsimmons, P.J. (1995). Even and odd continuous additive functionals. In *Dirichlet Forms and Stochastic Processes* (*Beijing*, 1993) (Z.M. Ma, M. Rockner and J.A. Yan, eds.) 139–154. Berlin: de Gruyter. MR1366430

[12] Fukushima, M., Ōshima, Y. and Takeda, M. (1994). *Dirichlet Forms and Symmetric Markov Processes. de Gruyter Studies in Mathematics* **19**. Berlin: de Gruyter. MR1303354

[13] Kuwae, K. (2008). Maximum principles for subharmonic functions via local semi-Dirichlet forms. *Canad. J. Math.* **60** 822–874. MR2432825

[14] Nakao, S. (1985). Stochastic calculus for continuous additive functionals of zero energy. *Z. Wahrsch. Verw. Gebiete* **68** 557–578. MR0772199

[15] Protter, P.E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed., version 2.1. *Stochastic Modelling and Applied Probability* **21**. Berlin: Springer. MR2273672

[16] Walsh, A. (2011). Local time–space calculus for symmetric Lévy processes. *Stochastic Process. Appl.* **121** 1982–2013.

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