Bernoulli **18**(4), 2012, 1099–1127
DOI: 10.3150/11-BEJ372

A Ferguson–Klass–LePage series representation of multistable multifractional motions and related processes

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The study of non-stationary processes whose local form has controlled properties is a fruitful and important area of research, both in theory and applications. In (*J. Theoret. Probab.* **22** (2009) 375–401), a particular way of constructing such processes was investigated, leading in particular to *multifractional multistable processes*, which were built using sums over Poisson processes. We present here a different construction of these processes, based on the Ferguson–Klass–LePage series representation of stable processes. We consider various particular cases of interest, including multistable Lévy motion, multistable reverse Ornstein–Uhlenbeck process, log-fractional multistable motion and linear multistable multifractional motion. We also show that the processes defined here have the same finite dimensional distributions as the corresponding processes built in (*J. Theoret. Probab.* **22** (2009) 375–401). Finally, we display numerical experiments showing graphs of synthesized paths of such processes.

Keywords: Ferguson–Klass–LePage series representation; localisable processes; multifractional processes; stable processes

1. Introduction

This work deals with a general method for building stochastic processes for which certain aspects of the local form are prescribed. We will mainly be interested here in local Hölder regularity and local intensity of jumps, but our construction allows in principle to control other properties that could be of interest. Our approach is in the same spirit as the one proposed in [8], but it uses different methods. In particular, in [8], multistable processes, that is localisable processes which are locally α-stable, but where the index of stability α varies with time, were constructed using sums over Poisson processes. We present here an alternative construction of such processes, based on the Ferguson–Klass–LePage series representation of stable stochastic processes [9,13,14]. This representation is a powerful tool for the study of various aspects of stable processes, see, for instance [3,19]. A comprehensive reference for the properties of this representation that will be needed here is [20].

Stochastic processes where the local Hölder regularity varies with a parameter t are interesting both from a theoretical and practical point of view. A well-known example is multifractional Brownian motion (m.B.m.), where the Hurst index h of fractional Brownian motion (f.B.m.) [11,

1350-7265

16] is replaced by a functional parameter $h(t)$, permitting the Hölder exponent to vary in a prescribed manner [1,2,10,17]. This allows in addition local regularity and long range dependence to be decoupled to give sample paths that are both highly irregular and highly correlated, a useful feature for instance in terrain or TCP traffic modeling.

However, local regularity, as measured by the Hölder exponent, is not the only local feature that is useful in theory and applications. Jump characteristics also need to be accounted for, for example, for studying processes with paths in $D(\mathbf{R})$ (the space of càdlàg functions, that is, functions which are continuous on the right and have left limits at all $t \in T$). This has applications for instance in the modeling of financial or medical data. Stable non-Gaussian processes yield relevant models in this case, with the stability index α controlling the distribution of jumps.

Just for the same reason why it is interesting to consider stochastic processes whose local Hölder exponent changes in a controlled manner, tractable models where the "jump intensity" α is allowed to vary in time are needed, for instance to obtain a more accurate description of some aspects of the local structure of functions in $D(\mathbf{R})$.

The approach described in this work allows in particular to construct processes where h and α evolve in time in a prescribed way. Having two functional parameters allows to finely tune the local properties of these processes. This may prove useful to model two distinct aspects of financial risk, textured images where both Hölder regularity and the distribution of discontinuities vary or to describe epileptic episodes in EEG where at time there may be only small jumps and at other very large ones. That the processes defined below have a varying jump intensity just means that they are tangent, in a sense to be described shortly, to a stable process with prescribed α. The varying local Hölder regularity is studied in [12], where an upper bound is given under general conditions, and an exact value is provided in the case of the multistable Lévy motion.

Let us now recall the definition of a localisable process [5,6]: $Y = \{Y(t)\colon\ t \in \mathbf{R}\}$ is h-localisable at u if there exists an $h \in \mathbf{R}$ and a nontrivial limiting process Y'_u such that

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^h} = Y'_u(t). \tag{1.1}$$

(Note Y'_u will in general vary with u.) When the limit exists, $Y'_u = \{Y'_u(t)\colon\ t \in \mathbf{R}\}$ is termed the *local form* or tangent process of Y at u (see [2,17] for similar notions). The limit (1.1) may be taken in several ways. In this work, we will only deal with the case where convergence occurs in finite dimensional distributions. When convergence takes place in distribution, the process is called *strongly localisable* (equality in distribution is denoted $\stackrel{d}{=}$).

As mentioned above, a now classical example is m.B.m. Y which "looks like" index-$h(u)$ f.B.m. close to time u but where $h(u)$ varies, that is

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^{h(u)}} = B_{h(u)}(t), \tag{1.2}$$

where B_h is index-h f.B.m. A generalization of m.B.m., where the Gaussian measure is replaced by an α-stable one, leads to multifractional stable processes [21].

The h-local form Y'_u at u, if it exists, must be h-self-similar, that is $Y'_u(rt) \stackrel{d}{=} r^h Y'_u(t)$ for $r > 0$. In addition, as shown in [5,6], under quite general conditions, Y'_u must also have stationary increments at almost all (a.a.) u at which there is strong localisability. Thus, typical local forms are self-similar with stationary increments (s.s.s.i.), that is, $r^{-h}(Y'_u(u+rt) - Y'_u(u)) \stackrel{d}{=} Y'_u(t)$ for all u and $r > 0$. Conversely, all s.s.s.i. processes are localisable. Classes of known s.s.s.i. processes include f.B.m., linear fractional stable motion and α-stable Lévy motion, see [20].

Similarly to [8], our method for constructing localisable processes is to make use of stochastic fields $\{X(t,v), (t,v) \in \mathbf{R}^2\}$, where the process $t \mapsto X(t,v)$ is localisable for all v. This field will allow to control the local form of a 'diagonal' process $Y = \{X(t,t)\colon\ t \in \mathbf{R}\}$. For instance, in the case of m.B.m., X will be a field of fractional Brownian motions, that is, $X(t,v) = B_{h(v)}(t)$, where h is a smooth function of v ranging in $[a,b] \subset (0,1)$. This is the approach that was used originally in [1] for studying m.B.m. From a heuristic point of view, taking the diagonal of such a stochastic field constructs a new process with local form depending on t by piecing together known localisable processes. In other words, we shall use random fields $\{X(t,v)\colon\ (t,v) \in \mathbf{R}^2\}$ such that for each v the local form $X'_v(\cdot,v)$ of $X(\cdot,v)$ at v is the desired local form Y'_v of Y at v. An easy situation is when, for each v, the process $\{X(t,v)\colon\ t \in \mathbf{R}\}$ is s.s.s.i. It is clear that, in this approach, the structure of $X(\cdot,v)$ for v in a neighbourhood of u is crucial to determine the local behaviour of Y near u. A simple way to control this structure is to define the random field as an integral or sum of functions that depend on t and v with respect to a single underlying random measure so as to provide the necessary correlations. General criteria that guarantee the transference of localisability from the $X(\cdot,v)$ to $Y = \{X(t,t)\colon\ t \in \mathbf{R}\}$ were obtained in [8]. We will use the following theorem.

Theorem 1.1. *Let U be an interval with u an interior point. Suppose that for some $0 < h < \eta$ the process $\{X(t,u), t \in U\}$ is h-localisable at $u \in U$ with local form $X'_u(\cdot,u)$ and*

$$\mathsf{P}\big(|X(v,v) - X(v,u)| \geq |v-u|^{\eta}\big) \to 0 \tag{1.3}$$

as $v \to u$. Then $Y = \{X(t,t)\colon\ t \in U\}$ is h-localisable at u with $Y'_u(\cdot) = X'_u(\cdot,u)$.

In the sequel, we consider certain random fields and use Theorem 1.1 to build localisable processes with interesting properties. As a particular case, we study multifractional multistable processes, where both the local regularity and intensity of jumps evolve in a controlled way.

The remaining of this article is organized as follows: we first collect some notations in Section 2. We then build localisable processes using a series representation that yields the necessary flexibility required for our purpose. We need to distinguish between the situations where the underlying space is finite (Section 3), or merely σ-finite (Section 4). In each case, we define a random field depending on a "kernel" f, and give conditions on f ensuring localisability of the diagonal process. We then consider in Section 5 some examples: multistable Lévy motion, multistable reverse Ornstein–Uhlenbeck process, log-fractional multistable motion and linear multistable multifractional motion. Section 6 is devoted to computing the finite dimensional distributions of our processes, and proving that they are the same as the ones of the corresponding processes constructed in [8]. Even though the processes constructed here and the ones from [8] coincide in law, they provide different representations of multistable processes, just as, in the stable case, both the Poisson series representation and the classical Ferguson–Klass–LePage representation have their own interest. It is also worthwhile to note that the conditions required on the kernel defining the process are different here and in [8]: we provide a specific example in

Section 5 where localisability can be proved using the Ferguson–Klass–LePage representation but not using the Poisson one. Finally, Section 7 displays graphs of certain localisable processes of interest. Before we proceed, we note that constructing localisable processes using a stochastic field composed of s.s.s.i. processes is obviously not the only approach that one can think of. See [7] for an example.

2. Notations

We refer the reader to the first chapters of [20] for basic notions on stable random variables and processes. In particular, recall that a process $\{X(t)\colon\ t \in T\}$, where T is generally a subinterval of $\mathbf{R}$, is called α-*stable* ($0 < \alpha \le 2$) if all its finite-dimensional distributions are α-stable. Many stable processes admit a stochastic integral representation as follows. Write $S_\alpha(\sigma, \beta, \mu)$ for the α-stable distribution with scale parameter σ, skewness β and shift-parameter μ; we will assume throughout that $\beta = 0$ and $\mu = 0$. Let $(E, \mathcal{E}, m)$ be a sigma-finite measure space, $m \not\equiv 0$. Taking m as the control measure, this defines an α-stable random measure M on E such that for $A \in \mathcal{E}$ we have $M(A) \sim S_\alpha(m(A)^{1/\alpha}, 0, 0)$. It is termed *symmetric* α-stable, or $S\alpha S$. Let

$$\mathcal{F}_\alpha \equiv \mathcal{F}_\alpha(E, \mathcal{E}, m) = \{f\colon\ f \text{ is measurable and } \|f\|_\alpha < \infty\},$$

where $\|\cdot\|_\alpha$ is the quasinorm (or norm if $1 < \alpha \le 2$) given by

$$\|f\|_\alpha = \left(\int_E |f(x)|^\alpha m(\mathrm{d}x)\right)^{1/\alpha}. \tag{2.1}$$

The stochastic integral of $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$ with respect to M then exists [20], Chapter 3:

$$I(f) = \int_E f(x) M(\mathrm{d}x) \sim S_\alpha(\|f\|_\alpha, 0, 0). \tag{2.2}$$

In particular,

$$\mathsf{E}|I(f)|^p = \begin{cases} c(\alpha, p)\|f\|_\alpha^p & (0 < p < \alpha), \\ \infty & (p \ge \alpha), \end{cases} \tag{2.3}$$

where $c(\alpha, p) < \infty$, see [20], Property 1.2.17. As said above, we will consider in this work only the case of symmetric α-stable measures. We believe most results should have a counterpart in the non-symmetric case, although the proofs would probably have to be significantly more involved. We use the following notations throughout the paper:

- $(\Gamma_i)_{i\ge 1}$ is a sequence of arrival times of a Poisson process with unit arrival time.
- $(V_i)_{i\ge 1}$ is a sequence of i.i.d. random variables with distribution $\hat{m}$ on a measure space (E, m). When $m(E) < \infty$, we always choose $\hat{m} = m/m(E)$. Otherwise, $\hat{m}$ is specified and is equivalent to the measure m in each case.
- $(\gamma_i)_{i\ge 1}$ is a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$.

The three sequences $(\Gamma_i)_{i\ge 1}$, $(V_i)_{i\ge 1}$, and $(\gamma_i)_{i\ge 1}$ are always assumed to be independent.

3. A Ferguson–Klass–LePage series representation of localisable processes in the finite measure space case

A well-known representation of stable random variables is the Ferguson–Klass–LePage series one [3,9,13,14,19]. It is well adapted for our purpose since it will allows for easy generalization to the case of varying α. In this work, we will use the following version of the theorem.

Theorem 3.1 ([20], Theorem 3.10.1). *Let $(E, \mathcal{E}, m)$ be a finite measure space where $m \not\equiv 0$, and M be a symmetric α-stable random measure with $\alpha \in (0,2)$ and finite control measure m. Then, for any $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$,*

$$\int_E f(x) M(\mathrm{d}x) \stackrel{d}{=} (C_\alpha m(E))^{1/\alpha} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} f(V_i), \tag{3.1}$$

where $C_\alpha = (\int_0^\infty x^{-\alpha} \sin(x)\,\mathrm{d}x)^{-1}$ (Theorem 3.10.1 in [20] is more general, as it extends to non-symmetric stable processes, that are not considered here). As mentioned above, a relevant feature of this representation for us is that the distributions of all random variables appearing in the sum are independent of α. We will use (3.1) to construct processes with varying α as described in the following theorem.

Theorem 3.2. *Let $(E, \mathcal{E}, m)$ be a finite measure space where $m \not\equiv 0$. Let α be a C^1 function defined on $\mathbf{R}$ and ranging in $(0,2)$. Let b be a C^1 function defined on $\mathbf{R}$. Let $f(t,u,\cdot)$ be a family of functions such that, for all $(t,u) \in \mathbf{R}^2$, $f(t,u,\cdot) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. Consider:*

$$X(t,u) = b(u)(m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} f(t,u,V_i). \tag{3.2}$$

Assume $X(\cdot,u)$ is localisable at u with exponent $h \in (0,1)$ and local form $X'_u(\cdot,u)$ and that there exists $\epsilon > 0$ such that:

(C1) *$v \to f(t,v,x)$ is differentiable on $B(u,\varepsilon)$ for any $t \in B(u,\varepsilon)$ and almost all $x \in E$ (the derivatives of f with respect to v are denoted by f'_v),*

(C2)

$$\sup_{t \in B(u,\varepsilon)} \int_E \sup_{w \in B(u,\varepsilon)} \left[|f(t,w,x) \log|f(t,w,x)||^{\alpha(w)} \right] \hat{m}(\mathrm{d}x) < \infty, \tag{3.3}$$

(C3)

$$\sup_{t \in B(u,\varepsilon)} \int_E \sup_{w \in B(u,\varepsilon)} \left(|f'_v(t,w,x)|^{\alpha(w)} \right) \hat{m}(\mathrm{d}x) < \infty. \tag{3.4}$$

Then $Y(t) \equiv X(t,t)$ is localisable at u with exponent h and local form $Y'_u(t) = X'_u(t,u)$.

Proof. First, note that the condition (C2) implies the following condition:

(C4) There exists $\varepsilon > 0$ such that:

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \big(|f(t,w,x)|^{\alpha(w)}\big)\hat{m}(\mathrm{d}x) < \infty. \tag{3.5}$$

Indeed, for all $t \in B(u,\varepsilon)$, $w \in B(u,\varepsilon)$ and $x \in E$,

$$\begin{aligned} |f(t,w,x)| &= |f(t,w,x)|\mathbf{1}_{|f(t,w,x)|<1/\mathrm{e}} + |f(t,w,x)|\mathbf{1}_{|f(t,w,x)|>\mathrm{e}} \\ &\quad + |f(t,w,x)|\mathbf{1}_{|f(t,w,x)|\in[1/\mathrm{e},\mathrm{e}]} \\ &\le 2|f(t,w,x)|\big|\log|f(t,w,x)|\big| + \mathrm{e}. \end{aligned}$$

Condition (C4) is then a consequence of the inequality $|a+b|^\delta \le \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$, valid for all real numbers a, b and all positive δ.

The function $w \mapsto C_{\alpha(w)}^{1/\alpha(w)}$ is C^1 since $\alpha(w)$ ranges in $(0,2)$. We shall denote $a(w) = b(w)(m(E))^{1/\alpha(w)} C_{\alpha(w)}^{1/\alpha(w)}$. The function a is thus also C^1. We want to apply Theorem 1.1. With that in view, we estimate, for $v \in B(u,\varepsilon)$ (the ball centered at u with radius ε),

$$X(v,v) - X(v,u) =: \sum_{i=1}^{\infty} \gamma_i \big(\Phi_i(v) - \Phi_i(u)\big) + \sum_{i=1}^{\infty} \gamma_i \big(\Psi_i(v) - \Psi_i(u)\big),$$

where

$$\begin{aligned} \Phi_i(w) &:= \Phi_i(v,w) := a(w) i^{-1/\alpha(w)} f(v,w,V_i), \\ \Psi_i(w) &:= \Psi_i(v,w) := a(w)\big(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\big) f(v,w,V_i). \end{aligned}$$

The reason for introducing the Φ_i and the Ψ_i is that the random variables Γ_i are not independent, which complicates their study. We shall decompose the sum involving the Φ_i into series of independent random variables which will be dealt with using the three series theorem. The sum involving the Ψ_i will be studied by taking advantage of the fact that, for large enough i, each Γ_i is "close" to i in some sense.

Let $c = \inf_{v\in B(u,\varepsilon)} \alpha(v)$, $d = \sup_{v\in B(u,\varepsilon)} \alpha(v)$. If $\inf_{v\in B(u,\varepsilon)} \alpha(v) = \sup_{v\in B(u,\varepsilon)} \alpha(v)$, we let instead $c = \inf_{v\in B(u,\varepsilon)} \alpha(v)$, $d = c + \delta$, for some $\delta > 0$. Note that, in both cases, by decreasing ε and δ, $d - c$ may be made arbitrarily small.

Thanks to the assumptions on a and f, Φ_i and Ψ_i are differentiable almost surely (a.s.):

$$\begin{aligned} \Phi_i'(w) &= a'(w) i^{-1/\alpha(w)} f(v,w,V_i) + a(w) i^{-1/\alpha(w)} f_w'(v,w,V_i) \\ &\quad + a(w)\frac{\alpha'(w)}{\alpha(w)^2} \log(i) i^{-1/\alpha(w)} f(v,w,V_i), \\ \Psi_i'(w) &= a'(w)\big(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\big) f(v,w,V_i) + a(w)\big(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\big) f_w'(v,w,V_i) \\ &\quad + a(w)\frac{\alpha'(w)}{\alpha(w)^2}\big(\log(\Gamma_i)\Gamma_i^{-1/\alpha(w)} - \log(i) i^{-1/\alpha(w)}\big) f(v,w,V_i). \end{aligned}$$

Notice that the functions Φ_i' and Ψ_i' depend on v. Consider now the function $h_i : x \to \Phi_i(x) - \Phi_i(u) - \frac{\Phi_i(v)-\Phi_i(u)}{v-u}(x-u)$ with $v \neq u$. Note that h_i is a random process a.s. $\mathcal{C}^1$ on $B(u,\varepsilon)$. Set $w_i = \min K_i$, where $K_i = \{x \in [u,v]\colon\ h_i'(x) = 0\}$. The mean value theorem yields that K_i is a nonempty closed set of $\mathbf{R}$.

Considering the function $k_i : x \to \Psi_i(x) - \Psi_i(u) - \frac{\Psi_i(v)-\Psi_i(u)}{v-u}(x-u)$, and the set $F_i = \{x \in [u,v]\colon\ k_i'(x) = 0\}$, we define also $x_i = \min F_i$.

Then there exists a sequence of independent measurable random numbers $w_i \in [u,v]$ (or $[v,u]$) and a sequence of measurable random numbers $x_i \in [u,v]$ (or $[v,u]$) such that:

$$X(v,u) - X(v,v) = (u-v)\sum_{i=1}^{\infty}(Z_i^1 + Z_i^2 + Z_i^3) + (u-v)\sum_{i=1}^{\infty}(Y_i^1 + Y_i^2 + Y_i^3),$$

where

$$\begin{aligned}
Z_i^1 &= \gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v, w_i, V_i),\\
Z_i^2 &= \gamma_i a(w_i) i^{-1/\alpha(w_i)} f_u'(v, w_i, V_i),\\
Z_i^3 &= \gamma_i a(w_i) \frac{\alpha'(w_i)}{\alpha(w_i)^2} \log(i) i^{-1/\alpha(w_i)} f(v, w_i, V_i),\\
Y_i^1 &= \gamma_i a'(x_i)\bigl(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\bigr) f(v, x_i, V_i),\\
Y_i^2 &= \gamma_i a(x_i)\bigl(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\bigr) f_u'(v, x_i, V_i),\\
Y_i^3 &= \gamma_i a(x_i) \frac{\alpha'(x_i)}{\alpha(x_i)^2}\bigl(\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i) i^{-1/\alpha(x_i)}\bigr) f(v, x_i, V_i).
\end{aligned}$$

Note that each w_i depends on a, f, α, u, v, V_i, but not on γ_i. This remark will be useful in the sequel. We establish now a lemma in order to control the series $\sum_{i=1}^{\infty} \mathsf{P}(|Z_i^1| > \lambda)$.

Lemma 3.1. *There exists a positive constant K such that for all $\lambda > 0$,*

$$\sum_{i=1}^{\infty} \mathsf{P}(|Z_i^1| > \lambda) \le \frac{K\,\mathsf{E}[\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}]}{\min(\lambda^c, \lambda^d)}.$$

Proof. Fix $\lambda > 0$. We may assume without loss of generality that $a' \not\equiv 0$, otherwise there is nothing to prove. One has:

$$\mathsf{P}(|Z_i^1| > \lambda) \le \mathsf{P}\biggl(|f(v, w_i, V_i)|^{\alpha(w_i)} > i\frac{\min(\lambda^c, \lambda^d)}{\sup_{w\in B(u,\varepsilon)}[|a'(w)|^{\alpha(w)}]}\biggr).$$

Note that, since a' is bounded on the compact interval $[u,v]$,

$$K := \sup_{w\in B(u,\varepsilon)} \bigl[|a'(w)|^{\alpha(w)}\bigr] < +\infty,$$

$$\mathsf{P}(|Z_i^1| > \lambda) \le \mathsf{P}\Big(K \sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} > i \min(\lambda^c, \lambda^d)\Big).$$

Thus,

$$\begin{aligned}\sum_{i=1}^{+\infty} \mathsf{P}(|Z_i^1| > \lambda) &\le \sum_{i=1}^{+\infty} \mathsf{P}\Big(K \sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} > i \min(\lambda^c, \lambda^d)\Big) \\ &\le \frac{K}{\min(\lambda^c,\lambda^d)} \mathsf{E}\Big[\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}\Big].\end{aligned}$$

□

Now we come back to the proof of Theorem 3.2. The remainder of the proof is divided into four steps. The first step will apply the three-series theorem to show that each series $\sum_{i=1}^{\infty} Z_i^j$, $j = 1, 2, 3$, converges a.s. In the second step, we will prove that $\sum_{i=1}^{\infty} Y_i^j$ also converges a.s. for $j = 1, 2, 3$. In the third step, we will prove that condition (1.3) is verified by $\sum_{i=1}^{\infty} Z_i^j$, $j = 1, 2, 3$. Finally, step four will prove the same thing for $\sum_{i=1}^{\infty} Y_i^j$, $j = 1, 2, 3$.

First step: almost sure convergence of $\sum_{i=1}^{\infty} Z_i^j$, $j = 1, 2, 3$.

Consider $Z^1 = \sum_{i=1}^{\infty} Z_i^1$. Fix $\lambda > 0$. We shall deal successively with the three series involved the three-series theorem.

First series: $S_1 = \sum_{i=1}^{\infty} \mathsf{P}(|Z_i^1| > \lambda)$. From Lemma 3.1 and (C4), one gets $S_1 < +\infty$.

Second series: $S_2^n = \sum_{i=1}^{n} \mathsf{E}(Z_i^1 \mathbb{1}\{|Z_i^1| \le \lambda\})$. One computes

$$\begin{aligned}\mathsf{E}(Z_i^1 \mathbb{1}\{|Z_i^1| \le \lambda\}) &= \mathsf{E}\big(\gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i) \mathbb{1}\big\{\big|a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i)\big| \le \lambda\big\}\big) \\ &= \mathsf{E}(\gamma_i)\mathsf{E}\big(a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i) \mathbb{1}\big\{\big|a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i)\big| \le \lambda\big\}\big) \\ &= 0,\end{aligned}$$

where we have used that γ_i is independent of (w_i, V_i) and $\mathsf{E}(\gamma_i) = 0$. Thus, $\lim_{n\to+\infty} S_2^n = 0$.

Third series: The final series we need to consider is $S_3 = \sum_{i=1}^{\infty} \mathsf{E}[(Z_i^1 \mathbb{1}\{|Z_i^1| \le \lambda\})^2]$. Choose $\lambda = 1$[1]. Let η be such that $d < \eta < 2$.

$$\begin{aligned}(Z_i^1 \mathbb{1}\{|Z_i^1| \le 1\})^2 &\le |Z_i^1|^\eta \mathbb{1}\{|Z_i^1| \le 1\}, \\ \mathsf{E}[(Z_i^1 \mathbb{1}\{|Z_i^1| \le 1\})^2] &\le \mathsf{E}[|Z_i^1|^\eta \mathbb{1}\{|Z_i^1| \le 1\}] \\ &= \int_0^{+\infty} \mathsf{P}(|Z_i^1|^\eta \mathbb{1}\{|Z_i^1| \le 1\} > x)\,\mathrm{d}x \\ &\le \int_0^1 \mathsf{P}(|Z_i^1|^\eta > x)\,\mathrm{d}x.\end{aligned}$$

[1] Recall that, in the three series theorem, for the series $\sum_{i=1}^{\infty} X_i$ to converge a.s., it is necessary that, for all $\lambda > 0$, the three series $\sum_{i=1}^{\infty} \mathsf{P}(|X_i| > \lambda)$, $\sum_{i=1}^{\infty} \mathsf{E}(X_i \mathbb{1}(|X_i| \le \lambda))$, and $\sum_{i=1}^{\infty} \mathrm{Var}(X_i \mathbb{1}(|X_i| \le \lambda))$ converge, and it is sufficient that they converge for *one* $\lambda > 0$, see, for example, [18], Theorem 6.1.

Now from Lemma 3.1 and (C4), there exists a positive finite constant $K \in (0, \infty)$ such that

$$S_3 \leq K \int_0^1 \frac{\mathrm{d}x}{\min(x^{c/\eta}, x^{d/\eta})} = K \int_0^1 \frac{\mathrm{d}x}{x^{d/\eta}} < +\infty.$$

The case of the $Z^2 = \sum_{i=1}^{\infty} Z_i^2$ is treated similarly, since the conditions required on (a', f) in the proof above are also satisfied by (a, f_u').

Consider finally $Z^3 = \sum_{i=1}^{\infty} Z_i^3$. Since the series $\sum_{i=1}^{\infty} \gamma_i(\Phi_i(v) - \Phi_i(u))$ converges a.s. (see, e.g., [15], page 132), the convergence of Z^1 and Z^2 imply the convergence of Z^3.

We have thus shown that the series Z^1, Z^2 and Z^3 are a.s. convergent.

Second step: almost sure convergence of $\sum_{i=1}^{\infty} Y_i^j$, $j = 1, 2, 3$.

To show that the series $\sum_{i=1}^{\infty} Y_i^j$, $j = 1, 2$ converge a.s., we will first prove that it is enough to show that $\sum_{i=1}^{\infty} Y_i^j \mathbf{1}_{\{1/2 \leq \Gamma_i/i \leq 2\} \cap \{|Y_i^j| \leq 1\}}$ converges a.s. for $j = 1, 2$. Indeed, we prove now that $\sum_{i=1}^{\infty} \mathsf{P}(\overline{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}} \cup \{|Y_i^j| > 1\}) < \infty$ for $j = 1, 2$, where $\overline{T}$ denotes the complementary set of the set T, and conclude with the Borel Cantelli lemma. The case of $\sum_{i=1}^{+\infty} Y_i^3$ is then treated as was the case of $\sum_{i=1}^{+\infty} Z_i^3$: we know that the series $\sum_{i=1}^{\infty} \gamma_i(\Psi_i(v) - \Psi_i(u))$ converges a.s. (see [15], page 132); the convergence of $\sum_{i=1}^{+\infty} Y_i^1$ and $\sum_{i=1}^{+\infty} Y_i^2$ will then entail the one of $\sum_{i=1}^{+\infty} Y_i^3$. Now:

$$\begin{aligned}
&\mathsf{P}\left(\overline{\left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}} \cup \{|Y_i^j| > 1\}\right) \\
&\quad = \mathsf{P}\left(\overline{\left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}} \cup \left[\{|Y_i^j| > 1\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right]\right) \\
&\quad \leq \mathsf{P}\left(\Gamma_i < \frac{i}{2}\right) + \mathsf{P}(\Gamma_i > 2i) + \mathsf{P}\left(\{|Y_i^j| > 1\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right).
\end{aligned}$$

Γ_i, as a sum of independent and identically distributed exponential random variables with mean 1, satisfy a Large Deviation Principle with rate function $\Lambda^*(x) = x - 1 - \log(x)$ for $x > 0$ and infinity for $x \leq 0$ (see [4], page 35), thus $\sum_{i \geq 1} \mathsf{P}(\Gamma_i < \frac{i}{2}) < +\infty$ and $\sum_{i \geq 1} \mathsf{P}(\Gamma_i > 2i) < +\infty$.

Consider now $\sum_{i \geq 1} \mathsf{P}(\{|Y_i^j| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\})$, for $j = 1, 2$.

Case $j = 1$:

Let $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$.

$$\begin{aligned}
&\mathsf{P}\left(\{|Y_i^1| > 1\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right) \\
&\quad = \mathsf{P}\left(\left\{\left|a'(x_i) i^{-1/\alpha(x_i)} f(v, x_i, V_i)\right| \left|\left(\frac{\Gamma_i}{i}\right)^{-1/\alpha(x_i)} - 1\right| > 1\right\} \cap B_i\right) \\
&\quad \leq \mathsf{P}\left(\left\{\left(2^{1/\alpha(x_i)} - 1\right)\left|a'(x_i) i^{-1/\alpha(x_i)} f(v, x_i, V_i)\right| > 1\right\} \cap B_i\right) \\
&\quad \leq \mathsf{P}\left(\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} > Ki\right),
\end{aligned}$$

where K is a positive constant. Thus $\sum_{i\geq 1} \mathsf{P}(\{|Y_i^1| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}) < +\infty$.

Case $j = 2$: Since the conditions needed on (a', f) in the proof above are satisfied by (a, f'_u), $\sum_{i\geq 1} \mathsf{P}(\{|Y_i^2| > 1\} \cap B_i) < +\infty$.

It remains to prove the a.s. convergence of $\sum_{i=1}^{\infty} Y_i^j \mathbf{1}_{\{1/2\leq\Gamma_i/i\leq 2\}\cap\{|Y_i^j|\leq 1\}}$, $j = 1, 2$. In that view, we use the following well-known lemma.

Lemma 3.2. *Let* $\{X_k, k \geq 1\}$ *be a sequence of random variables such that* $\sum_{n=1}^{+\infty} \mathsf{E}|X_n| < +\infty$, *then* $\sum_{n=1}^{+\infty} X_n$ *converges a.s.*

Let us show that $\sum_{i=1}^{\infty} \mathsf{E}[|Y_i^j| \mathbf{1}_{\{1/2\leq\Gamma_i/i\leq 2\}\cap\{|Y_i^j|\leq 1\}}] < +\infty$.

$$\begin{aligned}
\mathsf{E}\big[|Y_i^j| \mathbf{1}_{\{1/2\leq\Gamma_i/i\leq 2\}\cap\{|Y_i^j|\leq 1\}}\big] &= \int_0^{\infty} \mathsf{P}\left(\{1 \geq |Y_i^j| > x\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right) \mathrm{d}x \\
&\leq \int_0^{1} \mathsf{P}\left(\{|Y_i^j| > x\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right) \mathrm{d}x.
\end{aligned}$$

Let $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$.

Case $j = 1$:

Using the finite-increments formula applied to $y \mapsto y^{-1/\alpha(x_i)}$ on $[\frac{1}{2}, 2]$, one shows that

$$\begin{aligned}
\mathsf{P}(\{|Y_i^1| > x\} \cap B_i) &\leq \mathsf{P}\left(\left\{\left|a'(x_i) i^{-1/\alpha(x_i)} f(v, x_i, V_i)\right| \left|\frac{\Gamma_i}{i} - 1\right| > x \frac{c}{2^{1+1/c}}\right\} \cap B_i\right) \\
&\leq \mathsf{P}\left(\left\{|f(v, x_i, V_i)|^{\alpha(x_i)} \left|\frac{\Gamma_i}{i} - 1\right|^{\alpha(x_i)} > K_c i x^{\alpha(x_i)}\right\} \cap B_i\right)
\end{aligned}$$

with $K_c := \inf_{w\in B(u,\varepsilon)}[(\frac{c}{2^{1+1/c}|a'(w)|})^{\alpha(w)}] > 0$ by assumptions on a', α (we may assume again that $a' \not\equiv 0$, otherwise there is nothing to prove). Thus, for $x \in (0, 1)$,

$$\mathsf{P}(\{|Y_i^1| > x\} \cap B_i) \leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \left|\frac{\Gamma_i}{i} - 1\right|^{c} > K_c i x^{d}\right).$$

Case $d \geq 1$:

Fix $\zeta \in (d, 1 + \frac{c}{2})$ (since α is continuous and $d < 2$, by decreasing if necessary ε, one may ensure that $d < 1 + c/2$). By Markov and Hölder inequalities, and the independence of V_1, Γ_i,

$$\begin{aligned}
\mathsf{P}(\{|Y_i^1| > x\} \cap B_i) &\leq \mathsf{P}\left(\Big[\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)}\Big]^{1/\zeta} \left|\frac{\Gamma_i}{i} - 1\right|^{c/\zeta} > K_c^{1/\zeta} i^{1/\zeta} x^{d/\zeta}\right) \\
&\leq \frac{1}{(K_c i x^d)^{1/\zeta}} \left[\mathsf{E}\left|\frac{\Gamma_i}{i} - 1\right|^2\right]^{c/2\zeta} \left(\sup_{v\in B(u,\varepsilon)} \mathsf{E}\Big(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)}\Big)\right)^{1/\zeta} \\
&\leq \frac{K}{x^{d/\zeta}} \frac{1}{i^{1/\zeta + c/2\zeta}},
\end{aligned}$$

where we have used that the variance of Γ_i is equal to i, and K does not depend on v thanks to (C4). Thus, $\mathsf{E}[|Y_i^1|\mathbf{1}_{\{1/2\le\Gamma_i/i\le 2\}\cap\{|Y_i^1|\le 1\}}] \le \frac{K}{i^{1/\zeta+c/2\zeta}}$ where $\frac{1}{\zeta}+\frac{c}{2\zeta}>1$.

Case $d<1$:

$$\mathsf{P}(\{|Y_i^1|>x\}\cap B_i) \le \frac{1}{x^d K_c i}\mathsf{E}\Big(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}\Big)\mathsf{E}\left|\frac{\Gamma_i}{i}-1\right|^c$$

$$\le K\frac{1}{x^d}\frac{1}{i}\left(\mathsf{E}\left|\frac{\Gamma_i}{i}-1\right|^2\right)^{c/2} \le K\frac{1}{i^{1+c/2}}\frac{1}{x^d},$$

thus $\mathsf{E}[|Y_i^1|\mathbf{1}_{\{1/2\le\Gamma_i/i\le 2\}\cap\{|Y_i^1|\le 1\}}] \le \frac{K}{i^{1+c/2}}$ with $1+\frac{c}{2}>1$.

The case of $\sum_{i\ge 1}\mathsf{E}[|Y_i^2|\mathbf{1}_{\{B_i\cap\{|Y_i^2|\le 1\}}]$ is treated similarly, since the conditions required on (a',f) in the proof above are satisfied by (a,f_u'). For $j=3$, we have also $\sum_{i\ge 1}\mathsf{E}[|Y_i^3|\times \mathbf{1}_{\{B_i\cap\{|Y_i^3|\le 1\}}]<+\infty$, because there exists a positive finite constant $K\in(0,\infty)$ such that for $x\in(0,1)$, $\mathsf{P}(\{|Y_i^3|>x\}\cap B_i)\le \frac{K}{x^{d/\eta}}\frac{(\log i)^{d/\eta}}{i^{1/\eta+c/2\eta}}$ for $d\ge 1$ and $\mathsf{P}(\{|Y_i^3|>x\}\cap B_i)\le \frac{K}{x^d}\frac{(\log i)^d}{i^{1+c/2}}$ for $d<1$.

Thus, for $j=1,2,3$, $\sum_{i=1}^{+\infty}Y_i^j$ converges almost surely. We now move to the last two steps of the proof: to verify h-localisability, we need to check that for some η such that $h<\eta<1$, $\mathsf{P}(|\sum_{i=1}^{\infty}Z_i^j|\ge|v-u|^{\eta-1})$ and $\mathsf{P}(|\sum_{i=1}^{\infty}Y_i^j|\ge|v-u|^{\eta-1})$ tend to 0 when v tends to u, for $j=1,2,3$.

Third step: verification of (1.3) for $\sum_{i=1}^{\infty}Z_i^j$, $j=1,2,3$.

We need to estimate $\mathsf{P}(|\sum_{i=1}^{\infty}Z_i^j|\ge|v-u|^{\eta-1})$. Let $\eta\in(0,1)$ and $a\in(0,1-\eta)$.

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty}Z_i^j\right|>|v-u|^{\eta-1}\right)$$

$$\le \mathsf{P}\left(\left|\sum_{i=1}^{\infty}Z_i^j\mathbf{1}_{|Z_i^j|\le|v-u|^{-a}}\right|>\frac{|v-u|^{\eta-1}}{2}\right)$$

$$+\mathsf{P}\left(\left|\sum_{i=1}^{\infty}Z_i^j\mathbf{1}_{|Z_i^j|>|v-u|^{-a}}\right|>\frac{|v-u|^{\eta-1}}{2}\right).$$

Since γ_i is independent from γ_k, $i\ne k$ and $|Z_i^j|$ is independent of γ_i, Markov inequality yields

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty}Z_i^j\mathbf{1}_{|Z_i^j|\le|v-u|^{-a}}\right|>\frac{|v-u|^{\eta-1}}{2}\right)\le\frac{4}{|v-u|^{2(\eta-1)}}\sum_{i=1}^{\infty}\mathsf{E}\big[|Z_i^j|^2\mathbf{1}_{|Z_i^j|\le|v-u|^{-a}}\big].$$

Let $\gamma\in(d,2)$. For any $M>1$, we get:

$$\mathsf{E}\big[|Z_i^j|^2\mathbf{1}_{|Z_i^j|\le M}\big]=M^2\mathsf{E}\left[\frac{|Z_i^j|^2}{M^2}\mathbf{1}_{|Z_i^j|\le M}\right]\le M^2\int_0^1\mathsf{P}(|Z_i^j|>Mx^{1/\gamma})\,\mathrm{d}x.$$

For $j = 1$, we use again Lemma 3.1: there exists a positive constant K such that, for any $M > 1$

$$\sum_{i=1}^{\infty} \mathsf{E}\big[|Z_i^1|^2 \mathbf{1}_{|Z_i^j| \le M}\big] \le M^2 K \int_0^1 \frac{\mathrm{d}x}{\min(M^c x^{c/\gamma}, M^d x^{d/\gamma})}$$
$$\le M^{2-c} K \int_0^1 \frac{\mathrm{d}x}{\min(x^{c/\gamma}, x^{d/\gamma})}.$$

Thus, there exists a positive constant K such that

$$\sum_{i=1}^{\infty} \mathsf{E}\big[|Z_i^1|^2 \mathbf{1}_{|Z_i^1| \le M}\big] \le K M^{2-c}.$$

The same conclusion holds for $j = 2$: $\sum_{i=1}^{\infty} \mathsf{E}[|Z_i^2|^2 \mathbf{1}_{|Z_i^2| \le M}] \le K M^{2-c}$.

For $j = 3$, fix $\lambda > 0$.

$$\mathsf{P}(|Z_i^3| > \lambda) \le \mathsf{P}\bigg(|f(v, w_i, V_i)|^{\alpha(w_i)} > K'' \lambda^{\alpha(w_i)} \frac{i}{(\log i)^{\alpha(w_i)}}\bigg),$$

where $K'' := \inf_{w \in B(u,\varepsilon)}[(\frac{\alpha(w)^2}{|a(w)\alpha'(w)|})^{\alpha(w)}] > 0$ by assumptions on a, α and α' (we may assume without loss of generality that $a\alpha' \not\equiv 0$, otherwise there is nothing to prove). In the sequel, K will always denote a finite positive constant, that may however change from line to line.

Let $g_i(x) = \frac{x}{(\log x)^{\alpha(w_i)}}$ for $x > 1$ and $i \in \mathbb{N}^*$. For x large enough and for all i, g_i is strictly increasing and $\lim_{x \to +\infty} g_i(x) = +\infty$. For z large enough (independently of i),

$$g_i\big(2z(\log z)^{\alpha(w_i)}\big) = \frac{2z(\log 2z)^{\alpha(w_i)}}{(\log(2z) + \alpha(w_i)\log\log(2z))^{\alpha(w_i)}} \ge z.$$

Let $A \in \mathbf{R}$, $A > \mathrm{e}$ be such that $\forall z \ge A, \forall i \in \mathbb{N}^*, g_i^{-1}(z) \le 2z(\log z)^{\alpha(w_i)}$. A depends only on α. Let $U_i = |f(v, w_i, V_i)|^{\alpha(w_i)}$ and $i^* \in \mathbb{N}$, $i^* \ge 3$ depending only on α such that $\forall i \ge i^*$, $\frac{i}{(\log i)^{\alpha(w_i)}} \ge A$. Then:

$$\begin{aligned}
&\forall i \ge i^*, \\
&\quad \mathsf{P}(|Z_i^3| > \lambda) \\
&\quad \le \mathsf{P}\bigg(\frac{U_i}{K''\lambda^{\alpha(w_i)}} > \frac{i}{(\log i)^{\alpha(w_i)}}\bigg) \qquad (3.6)\\
&\quad \le \mathsf{P}\bigg(i \le \frac{K U_i}{K''\lambda^{\alpha(w_i)}}\bigg|\log\bigg(\frac{U_i}{K''\lambda^{\alpha(w_i)}}\bigg)\bigg|^{\alpha(w_i)}\bigg) \\
&\quad \le \mathsf{P}\bigg(i \le K\frac{U_i|\log U_i|^{\alpha(w_i)}}{\lambda^{\alpha(w_i)}} + K\frac{U_i}{\lambda^{\alpha(w_i)}} + K\frac{|\log\lambda|^{\alpha(w_i)}}{\lambda^{\alpha(w_i)}} U_i\bigg)
\end{aligned}$$

$$\leq \mathsf{P}\Big(\sup_{w\in B(u,\varepsilon)}\big[|f(v,w,V_1)\log|f(v,w,V_1)||^{\alpha(w)}\big] \geq Ki\min(\lambda^c,\lambda^d)\Big)$$

$$+\,\mathsf{P}\Big(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)} \geq Ki\min(\lambda^c,\lambda^d)\Big)$$

$$+\,\mathsf{P}\Big(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)} \geq Ki\min\Big(\Big(\frac{\lambda}{|\log\lambda|}\Big)^c,\Big(\frac{\lambda}{|\log\lambda|}\Big)^d\Big)\Big). \tag{3.7}$$

Finally, with (C4) and (C2), for $M > \mathrm{e}$,

$$\begin{aligned}
&\sum_{i=i^*}^{\infty}\int_0^1 \mathsf{P}(|Z_i^3| > Mx^{1/\gamma})\,\mathrm{d}x\\
&\quad\leq K\int_0^1 \frac{1}{\min(M^c x^{c/\gamma}, M^d x^{d/\gamma})}\,\mathrm{d}x\\
&\qquad + K\int_0^1 \frac{1}{\min(((Mx^{1/\gamma})/|\log Mx^{1/\gamma}|)^c, ((Mx^{1/\gamma})/|\log Mx^{1/\gamma}|)^d)}\,\mathrm{d}x\\
&\quad\leq K\frac{|\log(M)|^d}{M^c}.
\end{aligned}$$

We get then

$$\sum_{i=i^*}^{\infty}\mathsf{E}\big[|Z_i^3|^2\mathbf{1}_{|Z_i^3|\leq M}\big] \leq K|\log(M)|^d M^{2-c}.$$

Let $M = |v-u|^{-a}$. Using previously obtained inequalities, since $\sum_{i=1}^{i^*}\mathsf{E}[|Z_i^3|^2\mathbf{1}_{|Z_i^3|\leq M}] \leq i^*M^2$, we get, for $j = 1, 2, 3$:

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|\leq|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right) \leq K|v-u|^{2(1-\eta-a)}$$

and

$$\lim_{v\to u}\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|\leq|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right) = 0.$$

We consider now the second term $\mathsf{P}(|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2})$. Let $\bar{i} = \inf\{n \geq 1\colon\ i \geq n, |Z_i^j| \leq |v-u|^{-a}\}$. Since $\sum_{i\geq 1}\mathsf{P}(|Z_i^j| > |v-u|^{-a}) < +\infty$, the Borel–Cantelli lemma yields

$\mathsf{P}(\bar{i}=+\infty)=0$. As a consequence,

$$\begin{aligned}
&\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right)\\
&\quad=\sum_{n=1}^{\infty}\mathsf{P}\left(\left\{\left|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right\}\cap\{\bar{i}=n\}\right)\\
&\quad=\sum_{n=2}^{\infty}\mathsf{P}\left(\left\{\left|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right\}\cap\{\bar{i}=n\}\right)\\
&\quad\le\sum_{n=2}^{\infty}\mathsf{P}(\bar{i}=n).
\end{aligned}$$

For $n\ge 2$, $\mathsf{P}(\bar{i}=n)\le\mathsf{P}(|Z_{n-1}^j|>|v-u|^{-a})$.

For $j=1$, $\mathsf{P}(\bar{i}=n)\le\mathsf{P}(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}>|v-u|^{-ac}K(n-1))$, and thus

$$\begin{aligned}
\sum_{n=2}^{\infty}\mathsf{P}(\bar{i}=n)&\le K|v-u|^{ac}\mathsf{E}\Big(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}\Big)\\
&\le K|v-u|^{ac}\sup_{t\in B(u,\varepsilon)}\mathsf{E}\Big(\sup_{w\in B(u,\varepsilon)}|f(t,w,V_1)|^{\alpha(w)}\Big).
\end{aligned}$$

For $j=2$,

$$\sum_{n=2}^{\infty}\mathsf{P}(\bar{i}=n)\le K|v-u|^{ac}\sup_{t\in B(u,\varepsilon)}\mathsf{E}\Big(\sup_{w\in B(u,\varepsilon)}|f_u'(t,w,V_1)|^{\alpha(w)}\Big),$$

and for $j=3$,

$$\sum_{n=2}^{\infty}\mathsf{P}(\bar{i}=n)\le\sum_{i=1}^{i^*-1}\mathsf{P}(|Z_i^3|>|v-u|^{-a})+\sum_{i=i^*}^{\infty}\mathsf{P}(|Z_i^3|>|v-u|^{-a}).$$

The first term on the right-hand side is a finite sum and it is bounded from above by $K|v-u|^{ac}$, because there exists a positive constant K such that

$$\mathsf{P}(|Z_i^3|>|v-u|^{-a})\le\mathsf{P}\left(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}\ge\frac{Ki}{(\log i)^d}|v-u|^{-ac}\right).$$

The second term, using previous inequalities estimating $\mathsf{P}(|Z_i^3|>\lambda)$ for $i\ge i^*$, is bounded from above by $K|v-u|^{ac}|\log|v-u||^d$. Finally,

$$\lim_{v\to u}\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}\right| > \frac{|v-u|^{\eta-1}}{2}\right)=0.$$

Fourth step: verification of (1.3) for $\sum_{i=1}^{\infty} Y_i^j$, $j=1,2,3$.

We consider now $\mathsf{P}(|\sum_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1})$.

Note that, with $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$, we have $\sum_{i\geq 1} \mathsf{P}(\{|Y_i^3|>1\} \cap B_i) < +\infty$. Indeed, for $j=3$ and $i>1$,

$$\begin{aligned}
&\mathsf{P}(\{|Y_i^3|>1\} \cap B_i) \\
&\quad = \mathsf{P}\left(\left\{\left|a(x_i)\frac{\alpha'(x_i)}{\alpha(x_i)^2}\log(i) i^{-1/\alpha(x_i)} f(v,x_i,V_i)\right|\left|\frac{\log\Gamma_i}{\log i}\left(\frac{\Gamma_i}{i}\right)^{-1/\alpha(x_i)} - 1\right| > 1\right\} \cap B_i\right).
\end{aligned}$$

$(\frac{\log\Gamma_i}{\log i}(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)} - 1)_{(i>1)}$ is bounded on $\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$, thus there exists $K>0$ such that, for $i \geq 3$,

$$\mathsf{P}\left(\{|Y_i^3|>1\} \cap \left\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\right\}\right) \leq \mathsf{P}\left(|f(v,x_i,V_i)|^{\alpha(x_i)} > \frac{Ki}{(\log i)^{\alpha(x_i)}}\right).$$

Following the computations done after equation (3.6), we conclude that

$$\sum_{i\geq 1} \mathsf{P}\left(|f(v,x_i,V_i)|^{\alpha(x_i)} > \frac{Ki}{(\log i)^{\alpha(x_i)}}\right) < +\infty$$

and then $\sum_{i\geq 1} \mathsf{P}(\{|Y_i^3|>1\} \cap B_i) < +\infty$.

Let $i_0 = \inf\{n \geq 1\colon i \geq n, |Y_i^j| \leq 1 \text{ and } \frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$.

Since $\sum_{i\geq 1} \mathsf{P}(\{|Y_i^j|>1\} \cup \{\Gamma_i < \frac{i}{2}\} \cup \{\Gamma_i > 2i\}) < +\infty$, the Borel–Cantelli lemma yields $\mathsf{P}(i_0 = +\infty) = 0$. As a consequence,

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Y_i^j\right| \geq |v-u|^{\eta-1}\right) = \sum_{n\geq 1} \mathsf{P}\left(\left\{\left|\sum_{i=1}^{\infty} Y_i^j\right| \geq |v-u|^{\eta-1}\right\} \cap \{i_0 = n\}\right).$$

Let $b_n(v) = \mathsf{P}(\{|\sum_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1}\} \cap \{i_0 = n\})$. Our strategy is the following: we show that, for each fixed n, $b_n(v)$ tends to 0 when v tends to u. Then we prove that there exists a summable sequence $(c_n)_n$ such that, for all n and all v, $b_n(v) \leq c_n$. We conclude using the dominated convergence theorem that $\sum_{n\geq 1} b_n(v)$ tends to 0 when v tends to u. For all $n \geq 1$,

$$b_n(v) \leq \mathsf{P}\left(\left|\sum_{i=1}^{n-1} Y_i^j\right| \geq \frac{|v-u|^{\eta-1}}{2}\right) + \mathsf{P}\left(\left|\sum_{i=n}^{\infty} Y_i^j \mathbf{1}_{\{|Y_i^j|\leq 1\} \cap \{1/2 \leq \Gamma_i/i \leq 2\}}\right| \geq \frac{|v-u|^{\eta-1}}{2}\right).$$

For $n \geq 2$, consider $\mathsf{P}(|\sum_{i=1}^{n-1} Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2})$.

$$\mathsf{P}\left(\left|\sum_{i=1}^{n-1} Y_i^j\right| \geq \frac{|v-u|^{\eta-1}}{2}\right) \leq \sum_{i=1}^{n-1} \mathsf{P}\left(|Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right).$$

Let $p \in (0, \frac{c}{d})$. With K a positive constant depending on n but not on v, we have, for $j = 1$:

$$\begin{aligned}
&\mathsf{P}\biggl(|Y_i^1| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\biggr) \\
&\quad \leq \mathsf{P}\biggl(\Bigl(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)}\Bigr)^p \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/\alpha(x_i)} - 1\biggr|^{p\alpha(x_i)} \geq K|v-u|^{pc(\eta-1)}\biggr).
\end{aligned}$$

We use the following chain of inequalities:

$$\begin{aligned}
\biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/\alpha(x_i)} - 1\biggr|^{p\alpha(x_i)} &= \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/\alpha(x_i)} - 1\biggr|^{p\alpha(x_i)} \mathbf{1}_{\Gamma_i \geq i} + \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/\alpha(x_i)} - 1\biggr|^{p\alpha(x_i)} \mathbf{1}_{\Gamma_i < i} \\
&\leq 1 + \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/c} - 1\biggr|^{p\alpha(x_i)} \mathbf{1}_{\Gamma_i < i} \\
&\leq 1 + \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/c} - 1\biggr|^{pc} + \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/c} - 1\biggr|^{pd},
\end{aligned}$$

and obtain

$$\begin{aligned}
&\mathsf{P}\biggl(|Y_i^1| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\biggr) \\
&\quad \leq \mathsf{P}\biggl(\Bigl(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)}\Bigr)^p \geq \frac{K}{3}|v-u|^{pc(\eta-1)}\biggr) \\
&\qquad + \mathsf{P}\biggl(\Bigl(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)}\Bigr)^p \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/c} - 1\biggr|^{pc} \geq \frac{K}{3}|v-u|^{pc(\eta-1)}\biggr) \\
&\qquad + \mathsf{P}\biggl(\Bigl(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)}\Bigr)^p \biggl|\biggl(\frac{\Gamma_i}{i}\biggr)^{-1/c} - 1\biggr|^{pd} \geq \frac{K}{3}|v-u|^{pc(\eta-1)}\biggr).
\end{aligned}$$

Since $p < \frac{c}{d} < 1$, $\mathsf{E}(|(\frac{\Gamma_i}{i})^{-1/c} - 1|^{pc}) < +\infty$ and $\mathsf{E}(|(\frac{\Gamma_i}{i})^{-1/c} - 1|^{pd}) < +\infty$ (just compute these expectations using the density of Γ_i). By independence of V_i and Γ_i and Markov inequality,

$$\begin{aligned}
\mathsf{P}\biggl(|Y_i^1| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\biggr) &\leq K|v-u|^{pc(1-\eta)} \mathsf{E}\biggl(\Bigl(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)}\Bigr)^p\biggr) \\
&\leq K|v-u|^{pc(1-\eta)},
\end{aligned}$$

and

$$\lim_{v \to u} \mathsf{P}\biggl(|Y_i^1| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\biggr) = 0.$$

Since the conditions required on (a', f) are also satisfied by (a, f'_u),

$$\lim_{v\to u} \mathsf{P}\left(|Y_i^2| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right) = 0.$$

We consider now the case $j = 3$. When $i = 1$:

$$\begin{aligned}
&\mathsf{P}\left(|Y_1^3| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right)\\
&\quad = \mathsf{P}\left(\left|\log(\Gamma_1)\Gamma_1^{-1/\alpha(x_1)} f(v, x_1, V_1)\right| |a(x_1)\alpha'(x_1)| \geq \frac{\alpha(x_1)^2}{2(n-1)} |v-u|^{\eta-1}\right)\\
&\quad \leq K|v-u|^{pc(1-\eta)} \mathsf{E}\left(\left(\frac{|\log(\Gamma_1)|^c + |\log(\Gamma_1)|^d}{\Gamma_1}\right)^p\right)
\end{aligned}$$

(K depends on n but not on v). Since $p < 1$, $\mathsf{E}((\frac{|\log(\Gamma_1)|^c+|\log(\Gamma_1)|^d}{\Gamma_1})^p) < +\infty$, and

$$\lim_{v\to u} \mathsf{P}\left(|Y_1^3| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right) = 0.$$

For $i \geq 2$,

$$\begin{aligned}
&\mathsf{P}\left(|Y_i^3| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right)\\
&\quad = \mathsf{P}\left(\left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/\alpha(x_i)} - 1\right| |f(v, x_i, V_i)a(x_i)\alpha'(x_i)| \geq \frac{\alpha(x_i)^2 i^{1/\alpha(x_i)}|v-u|^{\eta-1}}{\log(i)2(n-1)}\right).
\end{aligned}$$

One has:

$$\begin{aligned}
&\left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/\alpha(x_i)} - 1\right|^{\alpha(x_i)p}\\
&\quad \leq \left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/c} - 1\right|^{cp} + \left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/c} - 1\right|^{dp}\\
&\qquad + \left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/d} - 1\right|^{cp} + \left|\frac{\log(\Gamma_i)}{\log(i)}\left(\frac{\Gamma_i}{i}\right)^{-1/d} - 1\right|^{dp}.
\end{aligned}$$

Since $p \in (0, \frac{c}{d})$, the four terms in the right-hand side of the above inequality have finite expectation (use again the density of Γ_i). Reasoning as in the case of Y_1^3, one gets:

$$\lim_{v\to u} \mathsf{P}\left(|Y_i^3| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right) = 0.$$

Finally, we have, for $j \in \{1,2,3\}$,

$$\lim_{v\to u} \mathsf{P}\left(\left|\sum_{i=1}^{n-1} Y_i^j\right| \geq \frac{|v-u|^{\eta-1}}{2}\right) = 0.$$

Let us now consider, for $n \geq 1$, $\mathsf{P}(|\sum_{i=n}^{\infty} Y_i^j \mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{1/2\leq\Gamma_i/i\leq 2\}}| \geq \frac{|v-u|^{\eta-1}}{2})$:

$$\begin{aligned}
&\mathsf{P}\left(\left|\sum_{i=n}^{\infty} Y_i^j \mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{1/2\leq\Gamma_i/i\leq 2\}}\right| \geq \frac{|v-u|^{\eta-1}}{2}\right) \\
&\quad\leq 2|v-u|^{1-\eta}\mathsf{E}\left[\left|\sum_{i=n}^{\infty} Y_i^j \mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{1/2\leq\Gamma_i/i\leq 2\}}\right|\right] \\
&\quad\leq K|v-u|^{1-\eta}
\end{aligned}$$

(recall that the constants K used in bounding the series $\mathsf{E}(|Y_i^j|\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{1/2\leq\Gamma_i/i\leq 2\}})$ do not depend on v). Thus $b_n(v) \to 0$ when $v \to u$ for each n. In view of using the dominated convergence theorem, we compute (recall that $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$):

$$\begin{aligned}
b_n(v) &\leq \mathsf{P}(\{i_0 = n\}) \leq \mathsf{P}(\{|Y_{n-1}^j| > 1\} \cup \overline{B_{n-1}}) \\
&\leq \mathsf{P}(\{|Y_{n-1}^j| > 1\} \cap B_{n-1}) + \mathsf{P}\left(\frac{\Gamma_{n-1}}{n-1} < \frac{1}{2}\right) + \mathsf{P}\left(\frac{\Gamma_{n-1}}{n-1} > 2\right).
\end{aligned}$$

For $j = 1$ and $d \geq 1$,

$$\mathsf{P}(\{|Y_{n-1}^1| > 1\} \cap B_{n-1}) \leq \frac{K}{(n-1)^{1/\zeta + c/2\zeta}}\left(\sup_{t\in B(u,\varepsilon)} \mathsf{E}\left(\sup_{w\in B(u,\varepsilon)} |f(t,w,V_1)|^{\alpha(w)}\right)\right)^{1/\zeta}$$

and if $d < 1$,

$$\mathsf{P}(\{|Y_{n-1}^1| > 1\} \cap B_{n-1}) \leq \frac{K}{(n-1)^{1+c/2}}\left(\sup_{t\in B(u,\varepsilon)} \mathsf{E}\left(\sup_{w\in B(u,\varepsilon)} |f(t,w,V_1)|^{\alpha(w)}\right)\right).$$

The same conclusion holds for $j = 2$. The case $j = 3$ is treated in a similar way (with an additional "log" term). □

4. A Ferguson–Klass–LePage representation of localisable processes in the σ-finite measure space case

When the space E has infinite measure, one cannot use the representation above. This is a drawback, because many applications deal with processes defined on the real line, that is, $E = \mathbb{R}$ and m is Lebesgue measure. In the σ-finite case, one may perform a change of measure that allows to reduce to the finite case (see [20], Proposition 3.11.3). Section 5 contains specific examples of changes of measure. In terms of localisability, this translates into adding a natural condition involving the kernel and the change of measure.

Theorem 4.1. *Let $(E,\mathcal{E},m)$ be a σ-finite measure space. Let $r\colon E\to\mathbb{R}_+$ be such that $\hat{m}(\mathrm{d}x)=\frac{1}{r(x)}m(\mathrm{d}x)$ is a probability measure. Let α be a C^1 function defined on $\mathbf{R}$ and ranging in $(0,2)$. Let b be a C^1 function defined on $\mathbf{R}$. Let $f(t,u,\cdot)$ be a family of functions such that, for all $(t,u)\in\mathbf{R}^2$, $f(t,u,\cdot)\in\mathcal{F}_{\alpha(u)}(E,\mathcal{E},m)$. Consider the following random field*:

$$X(t,u)=b(u)C_{\alpha(u)}^{1/\alpha(u)}\sum_{i=1}^{\infty}\gamma_i\Gamma_i^{-1/\alpha(u)}r(V_i)^{1/\alpha(u)}f(t,u,V_i). \tag{4.1}$$

Assume $X(\cdot,u)$ is localisable at u with exponent $h\in(0,1)$ and local form $X'_u(t,u)$ and that there exists $\epsilon>0$ such that:

(Cs1) *$v\to f(t,v,x)$ is differentiable on $B(u,\varepsilon)$ for any $t\in B(u,\varepsilon)$ and almost all $x\in E$ (the derivatives of f with respect to v are denoted by f'_v)*,

(Cs2)

$$\sup_{t\in B(u,\varepsilon)}\int_E\sup_{w\in B(u,\varepsilon)}\bigl[|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)}\bigr]m(\mathrm{d}x)<\infty, \tag{4.2}$$

(Cs3)

$$\sup_{t\in B(u,\varepsilon)}\int_E\sup_{w\in B(u,\varepsilon)}\bigl(|f'_u(t,w,x)|^{\alpha(w)}\bigr)m(\mathrm{d}x)<\infty, \tag{4.3}$$

(Cs4)

$$\sup_{t\in B(u,\varepsilon)}\int_E\sup_{w\in B(u,\varepsilon)}\bigl[|f(t,w,x)\log(r(x))|^{\alpha(w)}\bigr]m(\mathrm{d}x)<\infty. \tag{4.4}$$

Then $Y(t)\equiv X(t,t)$ is localisable at u with exponent h and local form $Y'_u(t)=X'_u(t,u)$.

Remark: from (4.1), it may seem as though the process Y depends on the particular change of measure used, for example, the choice of r. However, this is not case: see Proposition 6.1.

Proof of Theorem 4.1. We apply Theorem 3.2 with $g(t,w,x)=r(x)^{1/\alpha(w)}f(t,w,x)$ on $(E,\mathcal{E},\hat{m})$.

- By (Cs1), the family of functions $v\to f(t,v,x)$ is differentiable $\forall(v,t)$ in a neighbourhood of u and a.a. x in E thus $v\to g(t,v,x)$ is differentiable and (C1) holds.
- Choose $\varepsilon>0$ such that (Cs2) and (Cs4) hold.

$$\begin{aligned}&\int_{\mathbf{R}}\sup_{w\in B(u,\varepsilon)}\bigl[|g(t,w,x)\log|g(t,w,x)||^{\alpha(w)}\bigr]\,\hat{m}(\mathrm{d}x)\\&\quad=\int_{\mathbf{R}}\sup_{w\in B(u,\varepsilon)}\Bigl[\Bigl|f(t,w,x)\log\Bigl|r(x)^{1/\alpha(w)}f(t,w,x)\Bigr|\Bigr|^{\alpha(w)}\Bigr]m(\mathrm{d}x).\end{aligned}$$

Expanding the logarithm above and using the inequality $|a+b|^\delta \le \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$, valid for all real numbers a, b and all positive δ, one sees that (C2) holds.

- Choose $\varepsilon > 0$ such that (Cs3) and (Cs4) hold.

$$g'_u(t,w,x) = r(x)^{1/\alpha(w)}\left(f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)}\log(r(x))f(t,w,x)\right)$$

and

$$\begin{aligned}&\int_{\mathbf{R}} \sup_{w\in B(u,\varepsilon)} \big(|g'_u(t,w,x)|^{\alpha(w)}\big)\hat{m}(\mathrm{d}x)\\ &\quad= \int_{\mathbf{R}} \sup_{w\in B(u,\varepsilon)} \left[\left|f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)}\log(r(x))f(t,w,x)\right|^{\alpha(w)}\right] m(\mathrm{d}x).\end{aligned}$$

The inequality $|a+b|^\delta \le \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$ shows that (C3) holds. □

5. Examples of localisable processes

In this section, we apply the results above and obtain some localisable processes of interest, in particular "multistable versions" of several classical processes. Similar multistable extensions were considered in [8], to which the interested reader might refer for comparison. We first recall some definitions. In the sequel, M will denote a symmetric α-stable $(0 < \alpha < 2)$ random measure on $\mathbf{R}$ with control measure the Lebesgue measure $\mathcal{L}$. We write

$$L_\alpha(t) := \int_0^t M(\mathrm{d}z)$$

for α-stable Lévy motion. The *log-fractional stable motion* is defined as

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} \big(\log(|t-x|) - \log(|x|)\big)M(\mathrm{d}x) \qquad (t \in \mathbf{R}).$$

This process is well defined only for $\alpha \in (1,2]$ (the integrand does not belong to $\mathcal{F}_\alpha$ for $\alpha \le 1$). Both Lévy motion and log-fractional stable motion are $1/\alpha$-self-similar with stationary increments. The following process is called *linear fractional α-stable motion*:

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+,b^-,t,x)M(\mathrm{d}x),$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, and

$$f_{\alpha,H}(b^+,b^-,t,x) = b^+\big((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\big) + b^-\big((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\big).$$

L_{α,H,b^+,b^-} is an s.s.s.i. process. If $b^+ = b^- = 1$, this process is called well-balanced linear fractional α-stable motion, denoted $L_{\alpha,H}$. The reverse Ornstein–Uhlenbeck process is:

$$Y(t) = \int_t^\infty \exp\bigl(-\lambda(x-t)\bigr)M(\mathrm{d}x) \qquad (t \in \mathbf{R}, \lambda > 0).$$

The localisability of Lévy motion, log-fractional stable motion and linear fractional α-stable motion stems from the fact that they are s.s.s.i. The localisability of the reverse Ornstein–Uhlenbeck process is proved in [7]. We now define multistable versions of these processes.

Theorem 5.1 (Symmetric multistable Lévy motion, compact case). *Let $\alpha : [0,T] \to (1,2)$ and $b : [0,T] \to (0,+\infty)$ be continuously differentiable. Let $\hat{m}(\mathrm{d}x)$ be the uniform distribution on $[0,T]$. Define*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} T^{1/\alpha(t)} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \mathbf{1}_{[0,t]}(V_i) \qquad (t \in [0,T]). \tag{5.1}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in (0,T)$, with local form $Y'_u = b(u) L_{\alpha(u)}$.

The proof is a simple application of Theorem 3.2, and is omitted.

Theorem 5.2 (Symmetric multistable Lévy motion, noncompact case). *Let $\alpha : \mathbf{R} \to (1,2)$ and $b : \mathbf{R} \to (0,+\infty)$ be continuously differentiable. Let $\hat{m}(\mathrm{d}x) = \sum_{j=1}^{+\infty} 2^{-j} \mathbf{1}_{[j-1,j[}(x)\,\mathrm{d}x$ on $\mathbf{R}$. Define*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 2^{j/\alpha(t)} \mathbf{1}_{[0,t] \cap [j-1,j[}(V_i) \qquad (t \in \mathbf{R}_+). \tag{5.2}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in \mathbf{R}_+$, with local form $Y'_u = b(u) L_{\alpha(u)}$.

Again, the proof is a straightforward application of Theorem 4.1 with $m(\mathrm{d}x) = \mathrm{d}x$, $r(x) = \sum_{j=1}^{\infty} 2^j \mathbf{1}_{[j-1,j[}(x)$, $f(t,u,x) = \mathbf{1}_{[0,t]}(x)$, and is omitted.

Theorem 5.3 (Log-fractional multistable motion). *Let $\alpha : \mathbf{R} \to (1,2)$ and $b : \mathbf{R} \to (0,+\infty)$ be continuously differentiable. Let $\hat{m}(\mathrm{d}x) = \frac{3}{\pi^2} \sum_{j=1}^{+\infty} j^{-2} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)\,\mathrm{d}x$ on $\mathbf{R}$. Define*

$$\begin{aligned} Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} (\log|t - V_i| - \log|V_i|) \\ \times \frac{\pi^{2/\alpha(t)}}{3^{1/\alpha(t)}} j^{2/\alpha(t)} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i) \qquad (t \in \mathbf{R}). \end{aligned} \tag{5.3}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in \mathbf{R}$, with $Y'_u = b(u) \Lambda_{\alpha(u)}$.

Proof. We apply Theorem 4.1 with $m(\mathrm{d}x) = \mathrm{d}x$, $r(x) = \frac{\pi^2}{3}\sum_{j=1}^{\infty} j^2 \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$, $f(t,u,x) = \log(|t-x|) - \log(|x|)$ and the random field

$$X(t,u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} (\log|t-V_i| - \log|V_i|) \times \frac{\pi^{2/\alpha(u)}}{3^{1/\alpha(u)}} j^{2/\alpha(u)} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i).$$

$X(\cdot,u)$ is the symmetrical $\alpha(u)$-Log-fractional motion. It is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(\cdot,u) = b(u)\Lambda_{\alpha(u)}$ [20]. The remaining of the proof is easy and is left to the reader. □

Theorem 5.4 (Linear multistable multifractional motion). *Let $b:\mathbf{R}\to(0,+\infty)$, $\alpha:\mathbf{R}\to(0,2)$ and $h:\mathbf{R}\to(0,1)$ be continuously differentiable.*

Let $\hat{m}(\mathrm{d}x) = \frac{3}{\pi^2}\sum_{j=1}^{+\infty} j^{-2}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)\,\mathrm{d}x$ on $\mathbf{R}$. Define for $t \in \mathbf{R}$

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i,j=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \big(|t-V_i|^{h(t)-1/\alpha(t)} - |V_i|^{h(t)-1/\alpha(t)}\big) \times \left(\frac{\pi^2 j^2}{3}\right)^{1/\alpha(t)} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i). \tag{5.4}$$

The process Y is $h(u)$-localisable at all $u \in \mathbf{R}$, with $Y'_u = b(u)L_{\alpha(u),h(u)}$ (the well balanced linear fractional stable motion).

Proof. We apply Theorem 4.1 with $m(\mathrm{d}x) = \mathrm{d}x$, $r(x) = \frac{\pi^2}{3}\sum_{j=1}^{\infty} j^2 \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$, $f(t,u,x) = |t-x|^{h(u)-1/\alpha(u)} - |x|^{h(u)-1/\alpha(u)}$ and the random field

$$X(t,u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \left(\frac{\pi^2 j^2}{3}\right)^{1/\alpha(u)} \times \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \big(|t-V_i|^{h(u)-1/\alpha(u)} - |V_i|^{h(u)-1/\alpha(u)}\big) \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i).$$

$X(\cdot,u)$ is the $(\alpha(u),h(u))$-well balanced linear fractional stable motion and it is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(\cdot,u) = b(u)L_{\alpha(u),h(u)}$ [8]. (Cs1) is easy.

- (Cs3) First, we note (Cs5) the following condition:
  There exists $\varepsilon > 0$ such that:

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \big(|f(t,w,x)|^{\alpha(w)}\big) m(\mathrm{d}x) < \infty. \tag{5.5}$$

In [8], it is shown that, given $u \in \mathbf{R}$, one may choose $\varepsilon > 0$ small enough and numbers a, b, h_-, h_+ with $0 < a < \alpha(w) < b < 2$, $0 < h_- < h(w) < h_+ < 1$ and $\frac{1}{a} - \frac{1}{b} < h_- < h_+ < 1 - (\frac{1}{a} - \frac{1}{b})$ such that, for all t and w in $U := (u - \varepsilon, u + \varepsilon)$:

$$|f(t, w, x)|, |f'_u(t, w, x)| \le k_1(t, x) \qquad (x \in \mathbf{R}), \tag{5.6}$$

where

$$k_1(t, x) = \begin{cases} c_1 \max\{1, |t - x|^{h_- - 1/a} + |x|^{h_- - 1/a}\} & \left(|x| \le 1 + 2 \max_{t \in U} |t|\right), \\ c_2 |x|^{h_+ - 1/b - 1} & \left(|x| > 1 + 2 \max_{t \in U} |t|\right) \end{cases} \tag{5.7}$$

for appropriately chosen constants c_1 and c_2. The conditions on a, b, h_-, h_+ entail that $\sup_{t \in U} \|k_1(t, \cdot)\|_{a,b} < \infty$, where

$$\|k_1(t, \cdot)\|_{a,b} = \left(\int_{\mathbf{R}} |k_1(t, x)|^a \,\mathrm{d}x\right)^{1/a} + \left(\int_{\mathbf{R}} |k_1(t, x)|^b \,\mathrm{d}x\right)^{1/b}, \tag{5.8}$$

and (Cs5) holds for k_1. (Cs3) is obtained with (5.6).

- (Cs2) One computes:

$$\begin{aligned} &|f(t, w, x) \log(|f(t, w, x)|)|^{\alpha(w)} \\ &\quad \le |f(t, w, x)|^{\alpha(w)} + |f(t, w, x) \log(|f(t, w, x)|)|^{\alpha(w)} \mathbf{1}_{\{|f(t,w,x)| > \mathrm{e}\}} \\ &\qquad + |f(t, w, x) \log(|f(t, w, x)|)|^{\alpha(w)} \mathbf{1}_{\{|f(t,w,x)| < 1/\mathrm{e}\}}. \end{aligned}$$

Since $|f(t, w, x)| \le k_1(t, x)$, one gets

$$\begin{aligned} |f(t, w, x) \log(|f(t, w, x)|)| \mathbf{1}_{\{|f(t,w,x)| > \mathrm{e}\}} &\le k_1(t, x) \log(k_1(t, x)) \mathbf{1}_{\{|f(t,w,x)| > \mathrm{e}\}} \\ &\le |k_1(t, x) \log(k_1(t, x))|, \\ |f(t, w, x) \log(|f(t, w, x)|)|^{\alpha(w)} \mathbf{1}_{\{|f(t,w,x)| > \mathrm{e}\}} &\le |k_1(t, x) \log(k_1(t, x))|^{\alpha(w)}. \end{aligned}$$

Fix $\eta > 0$ such that $1 < \eta < a + \frac{a}{b} - a h_+$ and $\lambda > 0$ such that $\frac{1}{\eta} < \lambda < 1$. Then:

$$|f(t, w, x) \log(|f(t, w, x)|)|^{\alpha(w)} \mathbf{1}_{\{|f(t,w,x)| < 1/\mathrm{e}\}} \le K |f(t, w, x)|^{\lambda \alpha(w)} \le K |k_1(t, x)|^{\lambda \alpha(w)}$$

and thus, since $|k_1|^\lambda$ satisfies (Cs5) and k_1 satisfies (Cs2), (Cs2) holds for f.

- (Cs4) For j large enough ($j > j^*$),

$$\begin{aligned} |f(t, w, x) \log(r(x))|^{\alpha(w)} &\le K_1 |k_1(t, x)|^{\alpha(w)} \\ &\quad + K_2 \sum_{j=j^*}^{+\infty} |f(t, w, x)|^{\alpha(w)} (\log(j))^d \mathbf{1}_{[-j, -j+1[\cup [j-1, j[}(x). \end{aligned}$$

Also

$$|f(t,w,x)|^{\alpha(w)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x) \le K_3 \frac{1}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

Thus,

$$|f(t,w,x)\log(r(x))|^{\alpha(w)} \le K_1|k_1(t,x)|^{\alpha(w)} + K_4\sum_{j=j^*}^{+\infty}\frac{(\log(j))^d}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

To conclude, note that

$$\int_{\mathbf{R}}\frac{(\log(j))^d}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)\,\mathrm{d}x = 2(\log(j))^d\int_{j-1}^{j}\frac{\mathrm{d}x}{|x|^{a(1+1/b-h_+)}} \sim 2\frac{(\log(j))^d}{j^{a(1+1/b-h_+)}} \qquad \square$$

Theorem 5.5 (Multistable reverse Ornstein–Uhlenbeck process). *Let* $\lambda > 0$, $\alpha : \mathbf{R} \to (1,2)$ *and* $b : \mathbf{R} \to (0,+\infty)$ *be continuously differentiable.*

Let $\hat{m}(\mathrm{d}x) = \sum_{j=1}^{+\infty} 2^{-j-1}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)\,\mathrm{d}x$ *on* $\mathbf{R}$. *Define*

$$Y(t) = b(t)C_{\alpha(t)}^{1/\alpha(t)}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(t)}2^{(j+1)/\alpha(t)} \times \mathrm{e}^{-\lambda(V_i-t)}\mathbf{1}_{[t,+\infty)\cap([-j,-j+1)\cup[j-1,j))}(V_i) \qquad (t\in\mathbf{R}). \tag{5.9}$$

Then Y *is* $1/\alpha(u)$*-localisable at any* $u \in \mathbf{R}$, *with local form* $Y'_u = b(u)L_{\alpha(u)}$.

The proof is similar to previous ones and is omitted.

We provide now an example of a process whose kernel f does not satisfy the criteria of Theorem 5.2 in [8] for localisability. Thus, it is not possible to prove that the process associated to f through the Poisson sum representation is localisable. Indeed, as is readily verified, f is such that, for all open set $U \subset (0, 1/\mathrm{e})$, all $t \in U$ and all increasing function α, $\|f(t,u,\cdot)\|_{c,d} = +\infty$ (recall (5.8)) where $c = \inf_{v\in U}\alpha(v)$, $d = \sup_{v\in U}\alpha(v)$. However, it is possible to prove that the process defined by f in the Ferguson–Klass–LePage representation is localisable thanks to Theorem 3.2.

Theorem 5.6. *Let* $U = (0, \frac{1}{2\mathrm{e}})$, $\alpha : U \to \mathbf{R}$, $\alpha(t) = 1 + t$. *Let* $\hat{m}(\mathrm{d}x)$ *be the uniform distribution on* U. *Define*

$$Y(t) = C_{\alpha(t)}^{1/\alpha(t)}\frac{1}{(2\mathrm{e})^{1/\alpha(t)}}\sum_{i=1}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(t)}\frac{1}{V_i^{1/\alpha(t)}|\ln V_i|^{4/\alpha(t)}}\mathbf{1}_{[0,t]}(V_i) \qquad (t \in U). \tag{5.10}$$

Then Y *is* $1/\alpha(u)$*-localisable at any* $u \in U$, *with local form* $Y'_u = \frac{1}{u^{1/\alpha(u)}|\ln u|^{4/\alpha(u)}}L_{\alpha(u)}$.

Proof. Apply Theorem 4.1 with $m(\mathrm{d}x) = \mathrm{d}x$, $f(t,u,x) = \frac{1}{x^{1/\alpha(u)}|\ln x|^{4/\alpha(u)}}\mathbf{1}_{[0,t]}(x)$ and

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \frac{1}{(2\mathrm{e})^{1/\alpha(u)}} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \frac{1}{V_i^{1/\alpha(u)}|\ln V_i|^{4/\alpha(u)}} \mathbf{1}_{[0,t]}(V_i).$$

The computation of the characteristic function of $X(\cdot,u)$ (as in Section 6) allows to verify that $X(\cdot,u)$ is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(\cdot,u) = \frac{1}{u^{1/\alpha(u)}|\ln u|^{4/\alpha(u)}} L_{\alpha(u)}$. □

6. Finite dimensional distributions

In this section, we compute the finite dimensional distributions of the family of processes defined in Theorem 4.1, and compare the results with the ones in [8].

Proposition 6.1. *With notations as above, let $\{X(t,u), t,u \in \mathbf{R}\}$ be as in (4.1) and $Y(t) \equiv X(t,t)$. The finite dimensional distributions of the process Y are equal to*

$$\mathsf{E}\big(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j Y(t_j)}\big) = \exp\left(-2\int_E \int_0^{+\infty} \sin^2\left(\sum_{j=1}^m \theta_j b(t_j) \frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}} f(t_j,t_j,x)\right) \mathrm{d}y\, m(\mathrm{d}x)\right)$$

for $m \in \mathbb{N}$, $\boldsymbol{\theta} = (\theta_1,\dots,\theta_m) \in \mathbf{R}^m$, $\mathbf{t} = (t_1,\dots,t_m) \in \mathbf{R}^m$.

Proof. Let $m \in \mathbb{N}$ and write $\phi_t(\theta) = \mathsf{E}(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j Y(t_j)})$. We proceed as in [20], Theorem 1.4.2. Let $\{U_i\}_{i\in\mathbb{N}}$ be an i.i.d. sequence of uniform random variables on $(0,1)$, independent of the sequences $\{\gamma_i\}$ and $\{V_i\}$, and $g(t,u,x) = b(u) C_{\alpha(u)}^{1/\alpha(u)} r(x)^{1/\alpha(u)} f(t,u,x)$. For all $n \in \mathbb{N}$,

$$\begin{aligned} &\sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} \sum_{k=1}^n \gamma_k U_k^{-1/\alpha(t_j)} g(t_j,t_j,V_k) \\ &\quad \overset{d}{=} \sum_{j=1}^m \theta_j \left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha(t_j)} \sum_{k=1}^n \gamma_k \Gamma_k^{-1/\alpha(t_j)} g(t_j,t_j,V_k). \end{aligned} \tag{6.1}$$

The right-hand side of (6.1) converges a.s. to $\sum_{j=1}^m \theta_j Y(t_j)$ when n tends to infinity and thus

$$\phi_t(\theta) = \lim_{n\to+\infty} \mathsf{E}\big(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} \sum_{k=1}^n \gamma_k U_k^{-1/\alpha(t_j)} g(t_j,t_j,V_k)}\big).$$

Set $\phi_t^n(\theta) = \mathsf{E}(\mathrm{e}^{\mathrm{i}\sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} \sum_{k=1}^n \gamma_k U_k^{-1/\alpha(t_j)} g(t_j,t_j,V_k)})$. This function may be written as:

$$\begin{aligned} \phi_t^n(\theta) &= \mathsf{E}\left(\prod_{k=1}^n \mathrm{e}^{\mathrm{i}\gamma_k \sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} U_k^{-1/\alpha(t_j)} g(t_j,t_j,V_k)}\right) \\ &= \big(\mathsf{E}\big(\mathrm{e}^{\mathrm{i}\gamma_1 \sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1)}\big)\big)^n, \end{aligned}$$

since all the sequences $\{\gamma_k\}$, $\{U_k\}$, $\{V_k\}$ are i.i.d. We compute now the expectation using conditioning and independence of the sequences $\{\gamma_k\}$, $\{U_k\}$ and $\{V_k\}$.

$$\begin{aligned}
\mathsf{E}\left(\mathrm{e}^{\mathrm{i}\gamma_1 \sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1)}\right) &= \mathsf{E}\left(\mathsf{E}\left(\mathrm{e}^{\mathrm{i}\gamma_1 \sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1)} | U_1, V_1\right)\right) \\
&= \mathsf{E}\left(\cos\left(\sum_{j=1}^m \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right)\right) \\
&= \mathsf{E}\left(\frac{1}{n}\int_0^n \cos\left(\sum_{j=1}^m \theta_j y^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right) \mathrm{d}y\right) \\
&= 1 - \frac{2}{n}\int_0^n \mathsf{E}\left(\sin^2\left(\sum_{j=1}^m \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right)\right) \mathrm{d}y.
\end{aligned}$$

The function $\sin^2$ is positive and thus, when n tends to $+\infty$,

$$\begin{aligned}
&\int_0^n \mathsf{E}\left(\sin^2\left(\sum_{j=1}^m \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right)\right) \mathrm{d}y \\
&\quad \to \int_0^{+\infty} \mathsf{E}\left(\sin^2\left(\sum_{j=1}^m \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right)\right) \mathrm{d}y.
\end{aligned}$$

To conclude, note that

$$\mathsf{E}\left(\sin^2\left(\sum_{j=1}^m \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j, t_j, V_1)\right)\right) = \int_E \sin^2\left(\sum_{j=1}^m \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j, t_j, x)\right) \hat{m}(\mathrm{d}x). \qquad \square$$

Comparing with Proposition 8.2, Theorems 9.3, 9.4, 9.5 and 9.6 in [8], it is easy to prove the following corollary.

Corollary 6.1. *The linear multistable multifractional motion, multistable Lévy motion, log-fractional multistable motion and multistable reverse Ornstein–Uhlenbeck process defined in Section 5 have the same finite dimensional distributions as the corresponding processes of* [8].

7. Numerical experiments

We display in this section graphs of synthesized paths of some of the processes defined above. The idea is to picture how multistability translates on the behaviour of trajectories, and, in the case of linear multistable multifractional motion, to visualize the effect of both a varying H and a varying α, these two parameters corresponding to two different notions of irregularity. The synthesis method is described in [7], pages 17–18. Theoretical results concerning the convergence of this method will be presented elsewhere. The graphs on the first line of Figure 1 ((a) and (b)) dis-

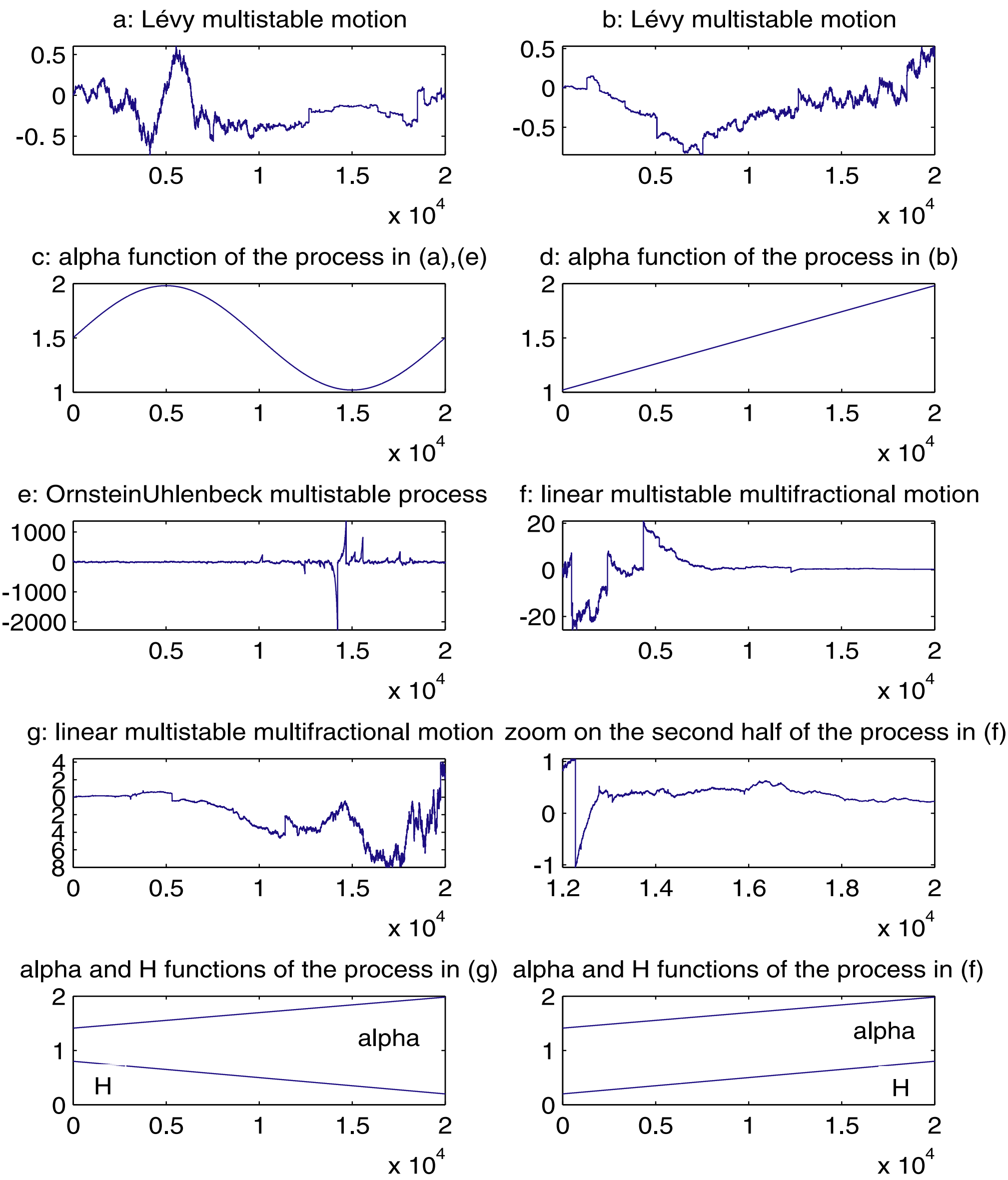

Figure 1. Paths of multistable processes. First line: Levy multistable motions with sine (a) and linear (b) α function. Second line: (c) α function for the process in (a), (d) α function for the process in (b). Third line: (e) multistable Ornstein–Uhlenbeck process with α function displayed in (c), and (f) linear multistable multifractional motion with increasing α and H functions. Fourth line: (g) linear multistable multifractional motion with increasing α function and decreasing H function, and zoom on the second part of the process in (f). Last line: α and H functions for the process in (g) (left), and in (f) (right).

play multistable Lévy motions, with respectively α increasing linearly from 1.02 to 1.98 (shown in (c)) and α a sine function ranging in the same interval (shown in (d)). The graph (e) displays an Ornstein–Uhlenbeck multistable process with same sine α function. A linear multistable multifractional motion with linearly increasing α and H functions is shown in (f). H increases from 0.2 to 0.8 and α from 1.41 to 1.98 (these functions are displayed on the right part of the bottom line). The graph in (g) is a linear multistable multifractional motion with linearly increasing α and linearly decreasing H. H decreases from 0.8 to 0.2 and α increases from 1.41 to 1.98 (these functions are displayed on the left part of the bottom line). Finally, a zoom on the second half of the process in (f) is shown, that allows to appreciate how the graph becomes smoother as H increases. In all the graphs, one sees how the variations of α translate in terms of the "intensity" of jumps. Additionally, in the case of linear multistable multifractional motions, the interplay between the smoothness governed by H and the jumps tuned by α indicate that such processes may prove useful in applications.

Acknowledgement

We thank the anonymous referee for many comments and corrections that allowed us to significantly improve our paper.

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Received January 2010 and revised March 2011