A unified minimax result for restricted parameter spaces

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We provide a development that unifies, simplifies and extends considerably a number of minimax results in the restricted parameter space literature. Various applications follow, such as that of estimating location or scale parameters under a lower (or upper) bound restriction, location parameter vectors restricted to a polyhedral cone, scale parameters subject to restricted ratios or products, linear combinations of restricted location parameters, location parameters bounded to an interval with unknown scale, quantiles for location-scale families with parametric restrictions and restricted covariance matrices.

Keywords: covariance matrices; linear combinations; location parameters; minimax; polyhedral cones; quantiles; restricted parameters; scale parameters

1. Introduction

We provide a development that unifies, simplifies and extends considerably a number of minimax results in the restricted parameter space estimation literature. As illustrated with a series of examples, the unified minimax result has wide applicability with respect to the nature of the constraint, the underlying probability model and the loss function utilized.

To further put into context the findings of this paper, consider a basic situation where $X \sim N(\theta, 1)$, with $\theta \geq a$ ($a > -\infty$ known), and where θ is estimated under squared error loss $(d - \theta)^2$. Katz [10] established that the Bayes estimator δ_U with respect to the flat prior on (a, ∞) dominates the minimum risk equivariant (MRE) estimator $\delta_0(X) = X$. However, δ_0 remains a useful benchmark estimator with its constant risk matching the minimax risk, and with any improvement, such as δ_U, being necessarily minimax as well. In a technical sense and roughly speaking, the form (and unboundedness) of the restricted parameter space $[a, \infty)$ preserves a common structure with the unrestricted parameter space $\Re$, and the constructions of the least favourable sequence of priors for both problems are isomorphic, leading to the same minimax values. In contrast, the restriction to a compact interval $\theta \in [a, b]$ is quite different and lowers the minimax risk (see example (C) in Section 3).

Now, the above phenomenon is not only more general (e.g., general location families with absolutely continuous Lebesgue densities and strictly convex loss [5]), but similar results have been established in various other situations, beginning with Blumenthal and Cohen [2] in the context of ordered location parameters. Many other such contributions will be referred to below, but at this point we refer to the monograph of van Eeden [22], as well as the review paper by

Marchand and Strawderman [17], which contain a substantial amount of material and references relating to such problems.

In this paper, we provide a unified framework for the above-mentioned problems, as well as many others either for more general loss and/or model, or for new situations such as estimating quantiles or covariance matrices under parametric restrictions. While results for certain of the problems (e.g., (A) and (B)) are not new (although we generalize some to more general loss functions) and certain others have been studied for squared error loss, we greatly expand the set of loss functions for which minimaxity is established (e.g., (C)–(F)), certain of the problems (e.g., (G) and (H) and Remark 2) have not been extensively studied and thus our problems and results are mostly new. In Section 2, we formalize the general argument, relying on the existence of a least favourable sequence (Proposition 1), setting up *conditions* on the restricted parameter space that facilitate a correspondence with the above sequence (Theorem 1) and inferring (Corollary 1) that a minimax MRE estimator remains minimax with the introduction of a restriction on the parameter space under given *conditions*. Detailed examples follow in Section 3. These include the estimation of location or scale parameters under a lower (or upper) bound restriction, location parameter vectors restricted to a polyhedral cone, scale parameters subject to restricted ratios or products, linear combinations of restricted location parameters, location parameters bounded to an interval with unknown scale, quantiles for location-scale families with parametric restrictions and covariance matrices with restricted traces or determinants.

2. Main result

We begin with the following fact concerning minimax problems, presented as a synthesized version of parts of the Appendix of [3], pages 254–268.

Proposition 1. *Let $R < \infty$ be the minimax value in a problem with sample space X and parameter space Ω, both Euclidean. Suppose the probability measures are absolutely continuous with respect to a σ-finite measure, that the loss $L(\theta, \cdot)$ is lower semicontinuous on the action space and that $L(\theta, a) \to b(\theta) = \sup L(\theta, \cdot)$ as $\|a\| \to \infty$ for all θ. Then there exists a sequence of prior distributions with finite support and with Bayes risks equal to r_n, such that r_n approaches R as $n \to \infty$, and there also exists a minimax procedure.*

We make use of a classical framework for invariant statistical problems. This includes a group of transformations G with an invariant family of probability measures $\{P_\theta \colon \theta \in \Omega\}$, where $X \sim P_\theta$ and $X' = g(X)$ implies $X' \sim P_{\theta'}$ with $\theta' = \bar{g}\theta$ and $\bar{G} = \{\bar{g} \colon g \in G\}$ forming a corresponding group of actions on Ω. As well, for estimating a parametric function $\tau(\theta)$ with loss L, additional assumptions include the condition that $\tau(\bar{g}\theta)$ depends on θ only through $\tau(\theta)$, and that the group action on the decision space D satisfies the condition $L(\bar{g}\theta, g^*d) = L(\theta, d)$ for all θ, d (e.g., [16], Section 3.2). With the help of Proposition 1, we obtain the following result.

Theorem 1. *Let a problem satisfying the conditions of Proposition 1 be invariant under a group G and let $\delta_0(X)$ be minimax for a full parameter space Ω. Suppose now that the parameter space is restricted to a subset Ω^*; that there exist sequences $\bar{g}_n \in \bar{G}$ and $B_n \subseteq \Omega^*$, such that $\bar{g}_n B_n \subset$*

$\bar{g}_{n+1}B_{n+1}$; *and that* $\bigcup_n \bar{g}_n B_n = \Omega$. *Then* $\delta_0(X)$ *remains minimax in the restricted parameter space problem.*

Proof. Let π_n, S_n and r_n be, respectively, Proposition 1's sequence of priors, sequence of corresponding finite supports and Bayes risks, with $r_n \to R$ as $n \to \infty$. Choose $m(n)$ sufficiently large so that $m \geq m(n)$ implies $\bar{g}_m B_m \supset S_n$. As we show below, the prior distribution with finite support $S_n^* = \bar{g}_{m(n)}^{-1}(S_n) \subset \Omega^*$, given by $\pi_n^*(\theta) = \pi_n(\bar{g}_{m(n)}^{-1}\theta)$, has Bayes risk $r_n^* = r_n$. This implies directly that $\delta_0(X)$ is minimax, since $r_n^* = r_n \to R$, as $n \to \infty$ by Theorem 5.18 of [1]. It remains to show that the Bayes risks of π_n and π_n^* coincide, and a standard argument is as follows. Let $\delta(X)$ be any estimator. Then, for its risk, we have:

$$R(\theta, \delta) = E_\theta L(\theta, \delta(X)) = E_{\bar{g}\theta} L(\theta, \delta(g^{-1}X)) = E_{\bar{g}\theta} L(\bar{g}\theta, g^*\delta(g^{-1}X)) = R(\bar{g}\theta, g^*\delta(g^{-1}X)),$$

by invariance. It follows that, if we set $\tilde{\theta} = \bar{g}_{m(n)}\theta$, then

$$r_n = E^\theta[R(\theta, \delta_n(X))] = E^\theta\big[R\big(\bar{g}_{m(n)}^{-1}\theta, g_{m(n)}^{*-1}\delta_n\big(g_{m(n)}X\big)\big)\big]$$

$$= E^{\tilde{\theta}}\big[R\big(\tilde{\theta}, g_{m(n)}^{*-1}\delta_n\big(g_{m(n)}(X)\big)\big)\big] = r_n^*,$$

where $\delta_n(X)$ is the Bayes estimator corresponding to π_n and hence $g_{m(n)}^{*-1}\delta_n(g_{m(n)}(X))$ is the Bayes estimator corresponding to π_n^*. $\square$

For applications, we will take B_n of Theorem 1 to match Ω^*, but it is potentially more convenient to take B_n as a sequence of open neighborhoods. Now, since the best equivariant estimators are often minimax, we deduce the following widely applicable result.

Corollary 1. *If an MRE estimator in a given problem satisfying the conditions of Theorem 1 is minimax, then it remains minimax in the restricted problem provided the restricted parameter space Ω^* satisfies the conditions of Theorem 1.*

For the sake of clarity, we do not assume that the action space and the image of the restricted parameter space coincide. Hence, minimax estimators that can be derived from Theorem 1 or Corollary 1 are not forced to take values in Ω^*. The main motivation resides in the benchmarking (i.e., dominating estimators that take values in Ω^* are necessarily minimax) and preservation of minimaxity (the minimax risks on Ω and Ω^* are equivalent). An important class of further applications of Corollary 1 will arise in cases where δ_{MRE} is minimax for the unrestricted problem Ω and the parameter space Ω^* and loss $L(\theta, \cdot)$ are convex, in which case the projection of δ_{MRE} onto Ω^* will dominate δ_{MRE} and hence be minimax.

Remark 1. Notwithstanding the conditions required on the restricted parameter space Ω^*, the applicability of Corollary 1 hinges on the minimaxity of the best equivariant estimator, in particular for unrestricted parameter space versions. As studied and established by several authors, it turns out that it is frequently the case that a minimax equivariant rule exists. We refer to [1], Section 6.7, [20], Section 9.5 and [16], note 9.3, pages 421–422, for general expositions and many

useful references. In particular, the Hunt–Stein theorem gives, for invariant problems, conditions on the group (amenability) that guarantee the existence of a minimax equivariant estimator whenever a minimax procedure exists. [11] is a key reference. All of the examples below relate to amenable groups, such as the additive and multiplicative groups, the group of location-scale transformations and the group of lower triangular $p \times p$ non-singular matrices with positive diagonal elements.

3. Examples

We focus here on various applications, illustrating how the results of Section 2 apply to both existing and new results. We accompany this with further observations and remarks. As previously mentioned, such applications are quite varied with respect to model, loss and shape of the restricted parameter space. At the expense of some redundancy, some particular cases are singled out (e.g., (A) is a particular case of (D), while (B) is a particular case of (E)) for their practical or historical importance. However, we do not focus here on specific determinations of the MRE estimators, but do refer to textbooks that treat in detail such topics (e.g., [16]). Throughout, we consider loss functions that satisfy the conditions of Proposition 1, and our findings relate to univariate and multivariate continuous probability models with absolutely continuous Lebesgue densities.

(A) (A single location parameter.) Consider a location model with $X \sim f_0(x_1 - \theta, \ldots, x_n - \theta)$, with $\Omega = \Re$, known f_0 and invariant loss $\rho(d - \theta)$. Consider further a lower (or upper) bounded parameter space Ω^* (i.e., $\Omega^* = [a, \infty)$ or $\Omega^* = (-\infty, a]$). On one hand, Ω^* satisfies the conditions of Theorem 1 with the choices $B_n = \Omega^*$, $\bar{g}_n = -n$ for $\Omega^* = [a, \infty)$ (and $B_n = \Omega^*$, $\bar{g}_n = n$ for $\Omega^* = (-\infty, a]$). On the other hand, following [11] or [7] for squared error loss, the MRE or Pitman estimator is minimax (and also Bayes with respect to the flat prior for θ on $\Re$). Thus Corollary 1 applies and the MRE estimator is minimax as well for the restricted parameter space Ω^*. The result is not new (see, e.g., [10], for a normal model and squared error loss; [5], for strictly convex ρ; [18], for strict bowl-shaped losses). Finally, we mention the implication that the minimaxity property is hence shared by any dominator of the MRE estimator, which includes quite generally the Bayes estimator of μ associated with the flat prior on Ω^* (e.g., [5,18]).

(B) (A single scale parameter.) Analogously, consider scale families with densities $\frac{1}{\sigma^n} f_1(\frac{x_1}{\sigma}, \ldots, \frac{x_n}{\sigma})$, with natural parameter space $\Omega = \Re^+$, known f_1, invariant loss $\rho(d/\sigma)$ and restricted parameter spaces $\Omega^* = [a, \infty)$ or $\Omega^* = (0, a]$ (with $a > 0$ known). With the multiplicative group on $\Re^+$, these restricted parameter spaces satisfy the conditions of Theorem 1 with $B_n = \Omega^*$ and the choices $\bar{g}_n = \frac{1}{n}$ and $\bar{g}_n = n$ for $\Omega^* = [a, \infty)$ and $\Omega^* = (0, a]$, respectively. From [11], whenever a minimax estimator exists for the unconstrained case $\sigma > 0$, it is necessarily given by the MRE estimator or equivalently by the Bayes estimator with respect to the non-informative prior $\pi(\sigma) = \frac{1}{\sigma} I_{(0,\infty)}(\sigma)$. Thus Theorem 1 and Corollary 1 apply and such MRE estimators remain minimax for constrained parameter spaces $[a, \infty)$ and $(0, a]$; and this quite generally with respect to model f_1 and loss ρ.

A version of the above minimaxity result for strict bowl-shaped losses was obtained by Marchand and Strawderman [19]. Kubokawa [12] provided the result for entropy loss

(i.e., $\rho(z) = z - \log z - 1$), while van Eeden [22] provided the result (actually more general, which relates to a vector of scale parameters as in (E) below) for scale invariant squared error loss (i.e., $\rho(z) = z^2$). Also, we refer to the three last references for earlier results obtained for specific models f_1, namely gamma and Fisher models. Finally, we also point out that the above development applies to estimating powers σ^r of σ by the transformation $x_i \to x_i^r$ (e.g., [19], for more details).

(C) (Location-scale families with the location parameter restricted to an interval (possibly compact).) For location-scale families with observables $X_1, \ldots, X_n$ having joint density $\frac{1}{\sigma^n} f_2(\frac{x_1-\mu}{\sigma}, \ldots, \frac{x_n-\mu}{\sigma})$, consider estimating μ with $\sigma > 0$ (unknown) under either: (i) the compact interval restriction $\mu \in [a, b]$, or (ii) $\mu \in [a, \infty)$; f_2 known, invariant loss $\rho(\frac{d-\mu}{\sigma})$. For (i), Theorem 1 applies with $B_n = \Omega^*$, $\bar{g}_n = (-\frac{n(a+b)}{2}, n)$, and $\bar{g}_n \Omega^* = \{(\mu, \sigma) \in \Re \times \Re^+ : \mu \in [-\frac{n(b-a)}{2}, \frac{n(b-a)}{2}], \sigma > 0\}$. As well, Kiefer [11] tells us that the MRE estimator of μ or, equivalently, Bayes with respect to the Haar right invariant prior $\pi(\mu, \sigma) = \frac{1}{\sigma} 1_{(0,\infty)}(\sigma)$, is minimax for the unrestricted problem with $\Omega = \Re \times \Re^+$ (subject to existence). The conclusion derived from Corollary 1 is that δ_{MRE} is also minimax for the restricted parameter space with $\mu \in [a, b], \sigma > 0$, while a similar development and conclusion applies for (ii) with $B_n = \Omega^*$ and $\bar{g}_n = (-n, 1)$, a result of which also follows from (G) below. The result for compact interval restriction (i) generalizes the result previously obtained for scaled squared error loss (i.e., $\rho(z) = z^2$) by Kubokawa [13].

Finally, we point out that a compact interval restriction on μ with known σ typically leads to a different conclusion, with a corresponding MRE estimator that is not minimax. A somewhat familiar justification for this (e.g., see [16], page 327 for a normal mean μ and squared error ρ) is as follows. Consider ρ to be strictly bowled-shaped in the sense that $\rho'(\cdot)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Denote V_0 and δ_{TMRE} as the constant risk of δ_{MRE} and the truncation of δ_{MRE} onto the parameter space $[a, b]$, respectively. Observe that $V_0 = R(\mu, \delta_{\text{MRE}}) > R(\mu, \delta_{\text{TMRE}})$ for all $\mu \in [a, b]$, and that the compactness of the parameter space coupled with the continuity of the risk $R(\mu, \delta_{\text{TMRE}})$ imply that $\sup_{\mu \in [a,b]} R(\mu, \delta_{\text{TMRE}}) < V_0$ and that, consequently, δ_{MRE} is not minimax.

(D) (Location parameters restricted to a polyhedral cone.) Consider independently generated copies of $X \sim f_0(x_1 - \mu_1, \ldots, x_p - \mu_p)$, with f_0 known, and $\mu = (\mu_1, \ldots, \mu_p)'$ restricted to a Polyhedral cone

$$\Omega_C^* = \{\mu \in \Re^p : C\mu \geq 0\}, \tag{1}$$

where $C (q \times p)$ $(q \leq p)$ is of full rank (and the 0 is a $q \times 1$ vector of 0's). Such restricted parameter spaces include:

(i) orthant restrictions where some or all of the μ_i's are bounded below by 0;

(ii) order restrictions of the type $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r$ with $r \leq p$;

(iii) tree order restrictions of the type $\mu_1 \leq \mu_i$ for some or all of the μ_i's;

(iv) umbrella order restrictions of the type $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m \geq \cdots \geq \mu_p$ (m known).

With $B_n = \Omega_C^*$ and $\bar{g}_n \in \Re^p$ as the additive group elements such that $C\bar{g}_n = -n(1, \ldots, 1)'$, we obtain $\bar{g}_n \Omega_C^* = \{\mu \in \Re^p : C\mu \geq -n(1, \ldots, 1)'\}$ and choices that satisfy Theorem 1. Furthermore, for invariant losses $\rho(\|d - \mu\|)$, the results of Kiefer [11] tell us that, subject to risk finiteness, the MRE or Bayes estimator for μ with a flat prior on

$\mathfrak{R}^p$ is minimax for the unconstrained problem $\mu \in \mathfrak{R}^p$. We infer by Corollary 1 that the same estimator is minimax for any polyhedral cone Ω_C^* as in (1).

Other than problems in (A), the above unifies and extends several previously established results, beginning with the Blumenthal and Cohen [2] case of order constraints and squared error ρ, and including more recent findings by Tsukuma and Kubokawa [21] for multivariate normal models, the general constraint in (1) and squared error ρ (also see [15] and [22], for further results and references). A much-studied and important case is the normal model with $X \sim N_p(\mu, I_p)$, $\mu \in \Omega_C^*$ and loss $\|d - \mu\|^2$, for which the above results apply with $\delta_{\mathrm{MRE}}(X) = X$. As an interesting corollary of a result by Hartigan [8] and of the above, it follows that the Bayes estimator δ_U of μ with respect to a flat prior on Ω_C^*, which Hartigan showed dominates X, is minimax for Ω_C^*. To conclude, we point out that a particular case of Hartigan's result was obtained by Blumenthal and Cohen [2] for ordered location parameters in (ii) with $r = p = 2$. They actually provide a class of model densities f_0, including normal, uniform and gamma densities, through conditions that ensure that δ_U (also referred to as the Pitman estimator by the authors) is minimax under squared error loss. They also report on numerical evidence indicating that δ_U is not minimax in general with respect to f_0.

Remark 2. As an extension of the above, a similar development holds with the introduction of an unknown scale parameter σ ($\sigma > 0$). Indeed for (at least two) independent copies from density $\frac{1}{\sigma^p} f_2(\frac{x_1 - \mu_1}{\sigma}, \ldots, \frac{x_p - \mu_p}{\sigma})$, invariant loss $\rho(\frac{\|d - \mu\|}{\sigma})$ and restricted parameter space $\mu \in \Omega_C^*$, $\sigma > 0$, Theorem 1 and Corollary 1 apply as above, but with the MRE estimator of μ now being generalized Bayes with respect to the prior measure $\pi(\mu, \sigma) = \frac{1}{\sigma} 1_{(0,\infty)}(\sigma) 1_{\mathfrak{R}^p}(\mu)$. Moreover, if estimating an unconstrained σ (or σ^r) is the objective, the MRE estimator of σ can be shown to be minimax as well with the parametric restrictions (subject to risk finiteness). This means that any minimax estimator of σ^r for an unconstrained problem remains minimax even when $\mu \in \Omega_C^*$.

(E) (Ratios or products of scale parameters.) For independently generated copies of $X \sim (\prod_i \sigma_i)^{-1} f_1(\frac{x_1}{\sigma_1}, \ldots, \frac{x_p}{\sigma_p})$ with f_1 a known Lebesgue density (on $(\mathfrak{R}^+)^p$), $\Omega = (\mathfrak{R}^+)^p$, consider the restriction $\tau = \prod_i (\sigma_i)^{r_i} \geq c > 0$, with the r_i's known and estimating τ under invariant loss $\rho(\frac{d}{\tau})$. The parametric function τ includes interesting cases of ratios $\frac{\sigma_i}{\sigma_j}$ and products $\sigma_i \sigma_j$ (with or without nuisance parameters σ_k, $k \neq i, j$), and the constraint on τ represents a natural scale parameter analog of (1) with $q = 1$. With $B_n = \Omega^*$, and $\bar{g}_n \in (\mathfrak{R}^+)^p$ the multiplicative group element given by $\bar{g}_n = (n^{-1/r_1}, \ldots, n^{-1/r_p})$, we obtain $\bar{g}_n \Omega^* = \{(\sigma_1, \ldots, \sigma_p) \in (\mathfrak{R}^+)^p : \prod_i (\sigma_i)^{r_i} \geq \frac{c}{n^p}\}$. Thus, the conditions of Theorem 1 are satisfied, and Theorem 1 applies. Corollary 1 applies as well, by virtue of Kiefer [11], indicating that the MRE estimator (if it exists), or equivalently Bayes with respect to the prior measure $\prod_i \frac{1}{\sigma_i} 1_{(0,\infty)}(\sigma_i)$, remains minimax for estimating τ under the lower bound constraint above. We refer to [16], Chapter 3, problems 3.34–3.37 for examples. Finally, with the minimax result here being quite general with respect to the loss ρ (as well as with respect to the type of constraint and the model), we point out that the particular case of scale-invariant squared error loss (i.e., $\rho(z) = (z - 1)^2$) is covered by van Eeden [22], Lemma 4.5.

(F) (Linear combinations of restricted location parameters.) Consider location models with $X = (X_1, \ldots, X_k)' \sim \prod_i f_i(x_i - \mu_i)$, known f_i's, where we wish to estimate $\theta = \sum_{i=1}^k a_i \mu_i = a'\mu$, under loss $\rho(d - \theta)$ and the restriction $\mu \in \Omega^* = \{\mu \in \Re^n \colon \mu_i \geq 0$ for $i = 1, \ldots, k\}$. For the unconstrained version with $\mu \in \Omega = \Re^k$, the MRE estimator (also Bayes with respect to the flat prior on $\Re^k$) is minimax [11], subject to existence and risk finiteness. Hence, Corollary 1 (or Theorem 1) applies with $B_n = \Omega^*$, $\bar{g}_n = (-n, \ldots, -n)$ indicating that an MRE estimator remains minimax in the constrained problem $\mu \in \Omega^*$ for estimating θ. Kubokawa [14] has recently established the above for squared error loss, where the MRE estimator, whenever it exists, is the unbiased estimator $\sum_{i=1}^k a_i (X_i - b_i)$ with $E(X_i - \mu_i) = b_i$. The result is extended here with respect to ρ, and achieved with a different and more general proof.

(G) (Quantiles with parameter space restrictions.) Consider location-scale models with $(X_1, \ldots, X_m)' \sim \frac{1}{\sigma^m} \prod_i f_0(\frac{x_i - \mu}{\sigma})$; $m \geq 2$, f_0 known, $\mu \in \Re$, $\sigma > 0$; with the objective of estimating a quantile parameter $\mu + \eta\sigma$; (of known order $\int_{-\infty}^\eta f_0(z)\,\mathrm{d}z$) under invariant loss $\rho(\frac{d - \mu - \eta\sigma}{\sigma})$. Now, consider restricted parameter spaces such as:

$$\Omega_1^* = \{(\mu, \sigma) \in \Re \times \Re^+ \colon \mu + \eta\sigma \geq 0\}$$

and

$$\Omega_2^* = \{(\mu, \sigma) \in \Re \times \Re^+ \colon \mu \geq a, \sigma \geq b \geq 0\}.$$

Taking $B_n = \Omega_1^*$ and $\bar{g}_n = (-n, \frac{1}{n})$ such that $\bar{g}_n \Omega_1^* = \{(\mu, \sigma) \in \Re \times \Re^+ \colon \mu + \eta\sigma \geq -n\}$ and $\bar{g}_n \Omega_2^* = \{(\mu, \sigma) \in \Re \times \Re^+ \colon \mu \geq \frac{a}{n} - n, \sigma \geq \frac{b}{n}\}$, we see that the conditions of Theorem 1 are satisfied. Moreover, subject to existence or risk finiteness, the results of Kiefer [11] tell us the MRE estimator is minimax for the unrestricted parameter space $\Omega = \Re \times \Re^+$. Hence, Corollary 1 applies and tells us that such MRE estimators are minimax for restricted parameter spaces Ω_1^* and Ω_2^*. Previously studied models, for which the above results apply, include exponential and normal f_0's. For instance, consider a standard normal f_0 and squared error ρ, where equivariant estimators are of the form $\bar{X} + \eta c S$, $\delta_{\mathrm{MRE}}(X_1, \ldots, X_m) = \bar{X} + \eta c_m S$, with constant and minimax risk $1 + \eta^2(1 - (m-1)c_m^2)$, and where $\bar{X} = \frac{1}{m} \sum_{i=1}^m X_i$, $S^2 = \sum_{i=1}^m (X_i - \bar{X})^2$ and $c_m = \frac{\Gamma(m/2)}{\sqrt{2}\Gamma((m+1)/2)}$ ([6], page 182). The general result above tells us the δ_{MRE} remains minimax for parameter spaces Ω_1^* and Ω_2^* under squared error loss.

Observe also that the above development relative to Ω_1^* is still valid whenever $\eta = 0$, which relates to the problem of estimating a median or mean for symmetric f_0's, with the corresponding minimaxity result previously obtained by Kubokawa [12] for scale-invariant squared error loss (i.e., $\rho(z) = z^2$ above). Similar results follow with an upper bound of 0 for Ω_1^*, as well as an upper bound for μ and/or an upper bound for σ in the case of Ω_2^*. Finally, we point out that the minimaxity result and development above follow without emendation for the general case of non-independent components with joint density $\frac{1}{\sigma^m} f(\frac{x_1 - \mu}{\sigma}, \ldots, \frac{x_m - \mu}{\sigma})$.

(H) (Restricted covariance matrices.) Consider a summary statistic $S \sim \mathrm{Wishart}(\Sigma, p, m)$ with $m \geq p$ and Σ positive definite. Moreover, suppose that we wish to estimate Σ

with invariant loss (under the general linear group) $L(\Sigma, \delta) = \psi(\Sigma^{-1}\delta)$, with $\psi(y) =$ tr$(y) - \log|y| - p$ and $\psi(y) = \text{tr}(y - I_p)^2$ as interesting examples. A standard method to derive a minimax estimator here (e.g., [4], Section 6.2) is to consider the best equivariant estimator under the subgroup G_T^+ of lower triangular matrices with positive diagonal elements. Such equivariant estimators can be shown to have constant risk, be of the form $\delta_A(S) = (S^{1/2})A(S^{1/2})'$ with A symmetric and $S^{1/2}$ the unique square root of S element belonging to G_T^+ and with the optimal choice (MRE) being minimax. For instance, under loss tr$(\Sigma^{-1}\delta - I_p)^2$, the BEE is minimax and given by δ_{A_0}, with A_0 the diagonal matrix with elements $(m + p - 2i + 1)^{-1}$; $i = 1, \ldots, p$; [9].

Now, consider restrictions on Σ of the type $\Omega^* = \{\Sigma > 0: |\Sigma| \geq c_1 > 0\}$ or $\Omega^* = \{\Sigma > 0: \text{tr}(\Sigma) \geq c_2 > 0\}$. It is easy to see in both cases that the conditions of Theorem 1 apply with $\bar{g}_n = \frac{1}{n}I_p$ and $B_n = \Omega^*$. Hence, the above MRE estimators remain minimax under the above restrictions by virtue of Corollary 1.

Concluding remarks

We have provided in this paper a rich and vast collection of novel minimax findings for restricted parameter spaces. Furthermore, we have established a unified framework not only applicable to many new situations, but also covering many generalizations of existing minimax results with respect to model and loss. For the sake of clarity and in a summary attempt to draw a sharper distinction between existing and new results to the best of our knowledge, we point out or reiterate that:

- Results in (A) and (B) are not new except for the slight generalization on the loss with our results here applicable to losses that are not necessarily strictly bowl-shaped.
- Situations (C)–(F) have been studied by others with existing minimax results for squared error ρ. Our results cover more general losses ρ in all these cases.
- Remark 2, situations (G) and (H), correspond for the most part to new problems and the given results are novel.

Acknowledgements

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