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Asymptotics of the maximal radius of an L^r-optimal sequence of quantizers

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Let P be a probability distribution on $\mathbb{R}^d$ (equipped with an Euclidean norm $|\cdot|$). Let $r > 0$ and let $(\alpha_n)_{n\ge1}$ be an (asymptotically) $L^r(P)$-optimal sequence of n-quantizers. We investigate the asymptotic behavior of the maximal radius sequence induced by the sequence $(\alpha_n)_{n\ge1}$ defined for every $n \ge 1$ by $\rho(\alpha_n) = \max\{|a|, a \in \alpha_n\}$. When card(supp($P$)) is infinite, the maximal radius sequence goes to $\sup\{|x|, x \in \operatorname{supp}(P)\}$ as n goes to infinity. We then give the exact rate of convergence for two classes of distributions with unbounded support: distributions with hyper-exponential tails and distributions with polynomial tails. In the one-dimensional setting, a sharp rate and constant are provided for distributions with hyper-exponential tails.

Keywords: distribution tail; function with regular variation; maximal radius of a quantizer; optimal quantization; Zador theorem

1. Introduction

The aim of this paper (which is a part of the second author's Ph.D. thesis [19]) is to provide some precise upper and lower bounds for the radius of a sequence of quantizers of an $\mathbb{R}^d$-valued random vector. Our motivation is that it is a first attempt toward the elucidation of the geometric structure of an optimal quantizer in higher dimension.

Quantization has become an important field of information theory since the early 1940's. Nowadays, it plays an important role in digital signal processing (DSP), the basis of many areas of technology, from mobile phones to modems and multimedia PCs. In DSP, vector quantization is the process of approximating a continuous range of values or a very large set of discrete values by a relatively small set of discrete values. A common use of quantization is the conversion of a continuous signal into a digital signal. This is performed in analog-to-digital converters with a given quantization level.

Recently, optimal vector quantization has become a promising tool in numerical probability: it is an efficient method to produce grids optimally fitted to the distribution of a random vector X. This leads to some cubature formulas that may approximate either expectations (see [14]) or, more significantly, conditional expectations (see [17]). This ability to approximate conditional expectations is the key property called upon in the quantization-based numerical schemes used to solve some problems arising in finance, including optimal stopping problems (pricing and hedging American-style options, see [1,2]), the pricing of swing options (see [3,4]), stochastic control problems (see [7,16]) for portfolio management and nonlinear filtering (see [15,18]).

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Other applications, like some new schemes for the discretization of Zakai and McKean–Vlasov equations, have also been investigated (see [10]).

At this stage, we need to recall some basic facts on optimal quantization. At this level of generality, we just assume that $\mathbb{R}^d$ is endowed with a norm $|\cdot|$, possibly not Euclidean.

Let $X \in L^r(\Omega, \mathcal{A}, \mathbb{P})$ be an $\mathbb{R}^d$-valued random vector with distribution $P = \mathbb{P}_X$. The $L^r(P)$-optimal quantization problem at level n for X consists in finding the best approximation of X by $q(X)$ for the $L^r(\mathbb{P})$-norm, where q is a Borel function taking at most n values. This leads to the following minimization problem:

$$\inf\{\|X - q(X)\|_r, q : \mathbb{R}^d \xrightarrow{\text{Borel}} \mathbb{R}^d, \operatorname{card}(q(\mathbb{R}^d)) \leq n\},$$

where $\operatorname{card}(\alpha)$ stands for the cardinality of α. The solution, $e_{n,r}(X)$, of the previous problem is called the L^r-optimal mean quantization error induced by X (at level n). Note that, in fact, $e_{n,r}(X)$ only depends on the distribution of X so that we will occasionally use the notation $e_{n,r}(P)$. However, for every Borel function $q : \mathbb{R}^d \to \alpha$, $\alpha \subset \mathbb{R}^d$, $\operatorname{card}(\alpha) \leq n$, we have

$$|X - q(X)| \geq d(X, \alpha) := \min_{a \in \alpha} |X - a| \qquad \mathbb{P}\text{-a.s.}$$

Consider $\alpha \subset \mathbb{R}^d$ with $\operatorname{card}(\alpha) \leq n$ (called an n-quantizer). Let $(C_a(\alpha))_{a\in\alpha}$ be a Voronoi partition of $\mathbb{R}^d$ (with respect to the norm $|\cdot|$), that is, a Borel partition of $\mathbb{R}^d$ satisfying for every $a \in \alpha$,

$$C_a(\alpha) \subset \left\{x \in \mathbb{R}^d \colon |x - a| = \min_{b\in\alpha} |x - b|\right\}$$

and let $\widehat{X}^\alpha = \sum_{a\in\alpha} a \mathbf{1}_{\{X \in C_a(\alpha)\}}$. Then $\widehat{X}^\alpha$ is a projection on α following the nearest neighbor rule and satisfying $|X - \widehat{X}^\alpha| = d(X, \alpha)$ so that one also has

$$\begin{aligned} e_{n,r}(X) &= \inf_{\substack{\alpha\subset\mathbb{R}^d \\ \operatorname{card}(\alpha)\leq n}} \left(\int_{\mathbb{R}^d} d(x, \alpha)^r P(\mathrm{d}x)\right)^{1/r} \\ &= \inf\{(\mathbb{E}|X - \widehat{X}^\alpha|^r)^{1/r}, \alpha \subset \mathbb{R}^d, \operatorname{card}(\alpha) \leq n\}. \end{aligned} \tag{1.1}$$

For every $n \geq 1$, the infimum in (1.1) holds as a (finite) minimum attained by (at least) one so-called *$L^r(P)$-optimal n-quantizer* $\alpha^\star$ (see, e.g., [14], Proposition 11 or [11], Theorem 4.1), also called, especially when dealing with numerical applications, *the optimal n-grid*. A sequence of n-quantizers $(\alpha_n)_{n\geq1}$ is $L^r(P)$-optimal if, for every $n \geq 1$, α_n is $L^r(P)$-optimal. A sequence $(\alpha_n)_{n\geq1}$ is *asymptotically $L^r(P)$-optimal* if

$$\int_{\mathbb{R}^d} d(x, \alpha_n)^r P(\mathrm{d}x) = e^r_{n,r}(X) + \mathrm{o}(e^r_{n,r}(X)) \qquad \text{as } n \to \infty$$

($f(x) = \mathrm{o}(g(x))$, as $x \to \infty$, if $f(x) = \epsilon(x)g(x)$ with $\lim_{x\to\infty} \epsilon(x) = 0$ for two $\mathbb{R}$-valued functions f and g). Moreover, the $L^r(P)$-optimal mean quantization error $e_{n,r}(X)$ decreases to 0 as n goes to infinity. As soon as X has a finite r'-moment for some $r' > r$, its rate of convergence to 0 is ruled by the so-called Zador theorem.

Theorem 1.1 (Zador theorem, see [6,11,21]). *Let $X \in L^{r'}(\mathbb{P})$ for an $r' > 0$, with distribution $P = f\lambda_d + P_s$ (where P_s denotes the singular part of P with respect to λ_d). Then,*

$$\forall r \in (0, r') \qquad \lim_n n^{r/d}(e_{n,r}(P))^r = Q_r(P), \tag{1.2}$$

where

$$Q_r(P) = J_{r,d}\left(\int_{\mathbb{R}^d} f^{d/(d+r)}\,\mathrm{d}\lambda_d\right)^{(d+r)/d} = J_{r,d}\|f\|_{d/(d+r)} \in [0, +\infty), \tag{1.3}$$

with

$$J_{r,d} = \inf_{n\geq 1} n^{r/d} e^r_{n,r}(U([0,1]^d)) \in (0, +\infty)$$

and $U([0,1]^d)$ stands for the uniform distribution on $[0,1]^d$.

Note that $\mathbb{E}|X|^{r'} < +\infty$ implies $\|f\|_{d/(d+r)} < +\infty$ and that $J_{r,d}$ depends upon the norm $|\cdot|$ on $\mathbb{R}^d$.

Let us come back to our topic of interest, that is, the asymptotic behavior of the radii of a sequence $(\alpha_n)_{n\geq 1}$ of L^r-optimal quantizers. The maximal radius (or simply radius) $\rho(\alpha)$ of a quantizer $\alpha \subset \mathbb{R}^d$ is defined by

$$\rho(\alpha) = \max\{|a|, a \in \alpha\}.$$

In a one-dimensional setting ($d = 1$), one can define the *one-sided* (right) radius of α by removing absolute values in the above definition. The one-sided left radius is defined as the opposite of the right radius of $-\alpha$ viewed as a quantizer of $-X$.

From now on, $|\cdot|$ will denote an Euclidean norm on $\mathbb{R}^d$, except where explicitly stated otherwise. Except in ambiguous cases, we will denote $(\rho_n)_{n\geq 1}$ for the sequence $(\rho(\alpha_n))_{n\geq 1}$ of radii of $(\alpha_n)_{n\geq 1}$.

We will first show that, if the support of P, denoted $\mathrm{supp}(P)$, is unbounded, then $\lim_{n\to+\infty} \rho_n = +\infty$ (when $d = 1$, the sequence of one-sided right radii goes to infinity as soon as $\sup \mathrm{supp}(P) = +\infty$). The key inequalities to get the upper and lower estimates of the maximal radius sequence are provided in Theorems 3.2 and 4.2. In these theorems, we point out the close connection between the asymptotics of ρ_n and the generalized survival function of X defined on $\mathbb{R}_+ := [0, +\infty)$ by $\bar{F}_r(\xi) = \mathbb{E}(|X|^r \mathbf{1}_{\{|X|>\xi\}})$. The regular variation index will play an important role since we elucidate the asymptotic behaviour of ρ_n (or $\log \rho_n$) from the asymptotic behavior of the function $-\log \bar{F}_r$ as a regularly varying function. We present below two typical results obtained for important families of (essentially radial) distributions: a sharp rate for $\log \rho_n$ for distributions with polynomial tails and an exact rate for ρ_n for distributions with hyper-exponential tails (also made sharp when $d = 1$ and $r \geq 1$).

Theorem 1.2. *Let $P = f\lambda_d$.*

(a) Polynomial tail. *If there exists* $K > 0$, $\beta \in \mathbb{R}$, $c > r + d$ *and a real number* $A > 0$ *such that*

$$\forall x \in \mathbb{R}^d \qquad |x| \geq A \quad \Longrightarrow \quad f(x) = K \frac{(\log|x|)^\beta}{|x|^c},$$

then

$$\lim_n \frac{\log \rho_n}{\log n} = \frac{1}{c - r - d} \frac{r + d}{d}.$$

(b) Hyper-exponential tail. *If there exists* $K > 0$, κ, $\vartheta > 0$, $c \in \mathbb{R}$ *and a real number* $A > 0$ *such that*

$$\forall x \in \mathbb{R}^d \qquad |x| \geq A \quad \Longrightarrow \quad f(x) = K|x|^c \mathrm{e}^{-\vartheta |x|^\kappa},$$

then

$$\vartheta^{-1/\kappa} \left(1 + \frac{r}{d}\right)^{1/\kappa} \leq \liminf_n \frac{\rho_n}{(\log n)^{1/\kappa}} \leq \limsup_n \frac{\rho_n}{(\log n)^{1/\kappa}} \leq 2\vartheta^{-1/\kappa} \left(1 + \frac{r}{d}\right)^{1/\kappa}.$$

Furthermore, *if* $d = 1$ *and* $r \geq 1$,

$$\lim_n \frac{\rho_n}{(\log n)^{1/\kappa}} = \vartheta^{-1/\kappa} (1 + r)^{1/\kappa}.$$

(c) *If* f *has a one-sided polynomial or hyper-exponential tail*, *say on* $\mathbb{R}_+$, *then the maximal radius sequence satisfies the above asymptotic bounds.*

Remarks.

• Note that the Euclidean norm appearing in the statement of the above theorem needs to be the one used to define the radius and the distance between the random vector and the quantizer. If X has a $\mathcal{N}(0, I_d)$ distribution, this norm is the canonical one. As concerns the $\mathcal{N}(0, \Sigma)$ distribution, the "reference" Euclidean norm is $|\cdot|_{\Sigma^{-1}}$ induced by the inverse Σ^{-1} ($|x|^2_{\Sigma^{-1}} := x'\Sigma^{-1}x$ for a (column) vector $x \in \mathbb{R}^d$ with x' standing for the transpose of x). To derive asymptotic bounds from such results for the radius *measured in the canonical Euclidean norm* one needs to use the strong equivalence of the norms, namely $\frac{1}{\lambda_{\Sigma,\max}}|\cdot| \leq |\cdot|_{\Sigma^{-1}} \leq \frac{1}{\lambda_{\Sigma,\min}}|\cdot|$, where $\lambda_{\Sigma,\max}$ and $\lambda_{\Sigma,\min}$ are the maximum and the minimum eigenvalues of Σ, respectively.

• Note that as concerns asymptotic lower estimates, we propose in Section 4.2 an alternative approach based on random quantization.

The paper is organized as follows. We first give, as a preliminary result, the limit of the maximal radius for distributions supported by a set of infinite cardinality. Section 2 is devoted to the upper estimate of the maximal radius based on the asymptotic estimates of survival functions of X. Section 3 is devoted to the lower limit where our results are obtained by two different methods – one still based on survival functions and one based on mean random quantization. In both cases, we strongly rely on recent results obtained in [12] about the L^s-behaviour of L^r-optimal quantizers when $r < s < r + d$.

Notation (additional). *For every $r \geq 0$, we define $L^{r+}(\mathbb{P}) = \bigcup_{\varepsilon>0} L^{r+\varepsilon}(\mathbb{P})$ and the generalized r-survival function $\bar{F}_r(\xi) = \mathbb{E}(|X|^r \mathbf{1}_{\{|X|>\xi\}})$ of a random vector $X \in L^r(\mathbb{P})$. Note that $\bar{F}_r$ is defined on $\mathbb{R}_+$ and takes values in $[0, \mathbb{E}|X|^r]$. $\bar{F}_0$ is the regular survival function denoted $\bar{F}$.*

Let $A \subset \mathbb{R}^d$. $\overline{A}$ will stand for its closure, ∂A for its boundary, Conv(A) *for its convex hull, $\mathring{A}$ for its interior and A^c for its complement. $[x]$ will denote the integer part of an $x \in \mathbb{R}$. $B(x,r)$, $r > 0$, will denote the open ball with center $x \in \mathbb{R}^d$ and radius $r \geq 0$ and $d(x, A)$ the distance of x to the set $A \subset \mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, $(x|y)$ will denote the inner product of x and y with respect to the specified Euclidean norm and for two real-valued functions f and g, $f(x) = \mathrm{O}(g(x))$ as $x \to \infty$ if there is a positive real constant C such that $|f(x)| \leq C|g(x)|$, for all large enough x.*

2. A first preliminary result

As a first necessary step we elucidate the connections between the asymptotics of the maximal radius sequence and the "supremum" of the support of the distribution P.

Proposition 2.1. (a) *Let $|\cdot|$ be an arbitrary norm on $\mathbb{R}^d$ and $X \in L^r(\mathbb{P})$. Let $(\alpha_n)_{n\geq 1}$ be a sequence of n-quantizers such that $\int_{\mathbb{R}^d} d(x,\alpha_n)^r P(\mathrm{d}x) \to 0$ as $n \to +\infty$. Then,*

$$\liminf_n \rho_n \geq \sup\{|x|, x \in \operatorname{supp}(P)\}. \tag{2.1}$$

(b) *Suppose that $|\cdot|$ is an Euclidean norm on $\mathbb{R}^d$. If* card(supp$(P)) = +\infty$, *then for any $L^r(P)$-optimal sequence of n-quantizers $(\alpha_n)_{n\geq 1}$*

$$\lim_n \rho_n = \sup_{n\geq 1} \rho_n = \sup\{|x|, x \in \operatorname{supp}(P)\}. \tag{2.2}$$

Proof. (a) Let $x \in \operatorname{supp}(P)$ and let $\varepsilon > 0$. For every $n \geq 1$,

$$\begin{aligned}
\|d(X,\alpha_n)\|_r &\geq \|d(X, B(0,\rho_n))\|_r \qquad \big(\text{since } \alpha_n \subset B(0,\rho_n)\big)\\
&\geq \big\|d(X, B(0,\rho_n))\mathbf{1}_{\{X\in B(x,\varepsilon)\}}\big\|_r\\
&\geq d(B(x,\varepsilon), B(0,\rho_n))\mathbb{P}\big(X \in B(x,\varepsilon)\big)^{1/r}.
\end{aligned}$$

Consequently, $d(B(x,2\varepsilon), B(0,\rho_n)) = 0$ for large enough n since $\|d(X,\alpha_n)\|_r \to 0$ so that $|x| - 2\varepsilon \leq \rho_n$, which eventually implies $\liminf_n \rho_n \geq |x|$.

(b) We will show first that if α is an L^r-optimal quantizer at level n and if card(supp$(P)) \geq n$, then

$$\alpha \subset \overline{\operatorname{Conv}(\operatorname{supp}(P))} \quad \text{and} \quad \rho_n \leq \sup\{|x|, x \in \operatorname{supp}(P)\}. \tag{2.3}$$

Note first that if α is L^r-optimal at level n, then card$(\alpha) = n$ since card(supp$(P)) \geq n$ (see [14], Proposition 11 or [11], Theorem 4.1). Now, suppose that there exists $a \in \alpha \cap (\overline{\operatorname{Conv}(\operatorname{supp}(P))})^c$ and set

$$\alpha' = (\alpha\backslash\{a\}) \cup \{\Pi(a)\},$$

where Π denotes the projection on the non-empty closed convex set $\overline{\mathrm{Conv}(\mathrm{supp}(P))}$. The projection is 1-Lipschitz (see, e.g., [13], Chapter III, page 116) and X is $\mathbb{P}$-a.s. $\mathrm{supp}(P)$-valued, hence

$$d(X, a) \geq d(\Pi(X), \Pi(a)) \overset{\mathbb{P}\text{-a.s.}}{=} d(X, \Pi(a)). \tag{2.4}$$

It follows that

$$d(X, \alpha) \geq d(X, \alpha') \qquad \mathbb{P}\text{-a.s.}$$

Since α is $L^r(P)$-optimal at level n and $\mathrm{card}(\alpha') \leq \mathrm{card}(\alpha) = n$,

$$\mathbb{E}(d(X, \alpha')^r) = \mathbb{E}(d(X, \alpha)^r)$$

so that the following two statements hold:

- $d(X, \alpha') = d(X, \alpha)$ $\mathbb{P}$-a.s.
- $\Pi(a) \notin \alpha \backslash \{a\}$ since α' is $L^r(P)$-optimal (which implies that $\mathrm{card}(\alpha') = n$).

On the other hand, it follows from equation (2.4) that

$$\big(a - \Pi(a) | X - \Pi(a)\big) \leq 0 \qquad \mathbb{P}\text{-a.s.}$$

Consequently

$$\begin{aligned} |X - a|^2 - |X - \Pi(a)|^2 &= 2\big(\Pi(a) - a | X - \Pi(a)\big) + |a - \Pi(a)|^2 \\ &\geq |a - \Pi(a)|^2 > 0 \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

since $a \notin \overline{\mathrm{Conv}(\mathrm{supp}(P))}$. As a consequence

$$d(X, \alpha') < d(X, \alpha) \qquad \mathbb{P}\text{-a.s. on } \big\{X \in \mathring{C}_{\Pi(a)}(\alpha')\big\},$$

where $\mathring{C}_{\Pi(a)}(\alpha') = \{\xi \in \mathbb{R}^d, d(\xi, \Pi(a)) < d(\xi, \alpha \backslash \{a\})\}$ since the norm is Euclidean.

This implies that $\mathbb{P}(X \in \mathring{C}_{\Pi(a)}(\alpha')) = 0$; if so, $\alpha' \setminus \{\Pi(a)\} = \alpha \setminus \{a\}$ would clearly be optimal at level n (since $d(X, \alpha) = d(X, \alpha \setminus \{a\})$ a.s.) with a cardinality equal to $n - 1$, which is impossible since $e_{n,r}(X)$ decreases (strictly) to 0 (see again [11,14]). Hence $\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))}$.

Now, let us prove that $\rho_n \leq \sup\{|x|, x \in \mathrm{supp}(P)\}$. Note first that this assertion is obvious if $\mathrm{supp}(P)$ is unbounded. Otherwise, if $\mathrm{supp}(P)$ is bounded, then it is compact and so is $\mathrm{Conv}(\mathrm{supp}(P))$. Let $x_0 \in \mathrm{Conv}(\mathrm{supp}(P))$ be such that $|x_0| = \sup\{|x|, x \in \mathrm{Conv}(\mathrm{supp}(P))\}$. Thus

$$x_0 = \lambda_0 \xi_1 + (1 - \lambda_0)\xi_2, \qquad \xi_1, \xi_2 \in \mathrm{supp}(P)$$

and $\lambda \mapsto |\lambda \xi_1 + (1 - \lambda)\xi_2|$ is convex so that it attains its maximum at $\lambda = 0$ or $\lambda = 1$. Consequently $x_0 \in \mathrm{supp}(P)$. Hence $\rho_n \leq \sup\{|x|, x \in \mathrm{supp}(P)\}$, which, combined with (2.1), yields the conclusion. □

Remark. Note that (b) follows from the fact that if α is an L^r-optimal quantizer at level n, then

$$\alpha \subset \overline{\mathrm{Conv}(\mathrm{supp}(P))} \tag{2.5}$$

as soon as card(supp(P)) $\geq n$. But this result holds true only for Euclidean norms on $\mathbb{R}^d$. For an arbitrary norm, this assertion may fail. A counterexample is given with the l_∞-norm in [11], page 25.

Before dealing with the general case we give two examples of distributions (exponential and Pareto) for which the sharp convergence rate of the maximal radius sequence can be easily derived from semi-closed forms established in [9] for their L^r-optimal quantizers.

$\triangleright$ *Exponential distribution.* Let $r > 0$ and let P be an exponential distribution with parameter $\lambda > 0$. Then

$$\rho_n = \frac{r+1}{\lambda}\log n + \frac{C_r}{\lambda} + \mathrm{O}\left(\frac{1}{n}\right), \tag{2.6}$$

where C_r is a real constant depending only on r.

$\triangleright$ *Pareto distribution.* Let $r > 0$ and let P be a Pareto distribution with index $\gamma > r$. Then,

$$\rho_n = K_r n^{(r+1)/(\gamma-r)}\left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right), \tag{2.7}$$

where K_r is a positive real constant depending only on r.

A short proof of these results is given in the Appendix. These rates will be useful to validate the asymptotic rates obtained by other approaches.

3. Asymptotic upper bounds for the radius

We investigate in this section the upper rate of convergence of (ρ_n) to infinity. We next give some definitions and some hypotheses that will be useful later on.

Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of quantizers at level n. For every $n \geq 1$, we denote by $M(\alpha_n)$ the set of points in α_n for which the maximal norm is reached, namely,

$$M(\alpha_n) = \left\{a \in \alpha_n \text{ such that } |a| = \max_{b\in\alpha_n}|b|\right\}.$$

We will need the following (light) assumption on the distribution P:

$$(\mathbf{H}) \equiv \exists x_0 \in \mathbb{R}^d, \exists \varepsilon_0 > 0, \exists r_0 > 0 \text{ such that } P(\mathrm{d}x) \geq \varepsilon_0 \mathbf{1}_{B(x_0,r_0)}(x)\lambda_d(\mathrm{d}x),$$

which means that P is locally lower bounded as a measure by the Lebesgue measure on a ball. This assumption holds as soon as P has a density f, bounded away from 0 on a non-empty open set.

In order to get a sharp estimate for ρ_n for one-dimensional distributions with hyper-exponential tails, we will need the following more technical assumption (for $r \in [1, +\infty)$):

$(\mathbf{G_r}) \equiv P = f \cdot \lambda_1$, where $f > 0$ is non-increasing to 0 on $[A, +\infty)$, non-decreasing from 0 on $(-\infty, -A]$ for some real constant $A \geq 0$ and

$$\lim_{|y| \to +\infty} \int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} \, \mathrm{d}u = 0. \tag{3.1}$$

Such an assumption is clearly satisfied by distributions with hyper-exponential tails, that is, of the form $f(x) = K|x|^c \mathrm{e}^{-\vartheta |x|^\kappa}$, $|x| > A > 0$, $\vartheta, \kappa > 0$, $c \in \mathbb{R}$. Indeed, such a density f is non-increasing outside a compact interval and we have

$$\int_1^{+\infty} (u-1)^{r-1} \frac{f(uy)}{f(y)} \, \mathrm{d}u = y^{-c} \int_1^{+\infty} (u-1)^{r-1} u^c \mathrm{e}^{-\vartheta y^\kappa (u^\kappa - 1)} \, \mathrm{d}u \overset{y \to +\infty}{\longrightarrow} 0$$

by the Lebesgue convergence theorem. A *one-sided version* of condition $(\mathbf{G_r})$ can be stated by restricting f on $[A, +\infty)$ or $(-\infty, -A]$ for some $A \geq 0$.

3.1. Main results on asymptotic upper bounds

The main result of this section, stated below, makes the connection between the asymptotic behaviour of ρ_n and that of its survival function (through some asymptotic "semi-inverse" of $-\log \bar{F}_r$ or $-\log \bar{F}_r(\mathrm{e}^{\bullet})$), where $\bar{F}_r(\xi) = \mathbb{E}(|X|^r \mathbf{1}_{\{|X| > \xi\}})$ denotes the generalized survival function.

First we need to briefly recall some background on inverse function and regular variations.

It is clear that the function $\bar{F}_r$ is non-increasing and goes to 0 as $\xi \to +\infty$ (provided $\mathbb{E}|X|^r < +\infty$). Consequently, $\xi \mapsto -\log \bar{F}_r(\xi)$ is monotone non-decreasing and goes to $+\infty$ as ξ goes to $+\infty$.

It is well known that if a function f defined on $(0, +\infty)$ is non-decreasing to $+\infty$, its generalized inverse function $f^{\leftarrow}$ defined for every $y > 0$ by

$$f^{\leftarrow}(y) = \inf\{\xi > 0, f(\xi) \geq y\} \tag{3.2}$$

is non-decreasing to $+\infty$. If, furthermore (see [5], Theorem 1.5.12.), f is regularly varying (at $+\infty$) with index $1/\delta$, $\delta > 0$ (i.e., for every $t > 0$, $\frac{f(t\xi)}{f(\xi)} \to t^{1/\delta}$ as $\xi \to +\infty$), then there exists a function ψ, regularly varying with index δ and satisfying

$$\lim_{\xi \to +\infty} \frac{\psi(f(\xi))}{\xi} = \lim_{y \to +\infty} \frac{f(\psi(y))}{y} = 1. \tag{3.3}$$

Such a function ψ is called an *asymptotic inverse* of f. It is neither necessarily increasing nor continuous. Moreover, ψ is unique up to asymptotic equivalence at $+\infty$ and $f^{\leftarrow}$ is one version of ψ. (By asymptotic equivalence (at $+\infty$), we mean $f \sim g$ if $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$.)

We show in the theorem below how to derive from the regularly varying property of a function ψ_r with upper bounds $(-\log \bar{F}_r)^{\leftarrow}$ or $(-\log \bar{F}_r(\mathrm{e}^{\bullet}))^{\leftarrow}$ an asymptotic upper estimate for ρ_n or $\log(\rho_n)$.

Theorem 3.1. *Let $r > 0$ and let $X \in L^r(\mathbb{P})$ with distribution P having an unbounded support and satisfying* (**H**). *Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *If ψ_r is a non-decreasing function, regularly varying with index δ and*

$$\lim_{\xi\to+\infty} \frac{\psi_r(-\log \bar{F}_r(\mathrm{e}^{\xi}))}{\xi} \geq 1, \tag{3.4}$$

then

$$\limsup_n \frac{\log \rho_n}{\psi_r(\log n)} \leq \left(1+\frac{r}{d}\right)^{\delta}. \tag{3.5}$$

If $-\log \bar{F}_r(\mathrm{e}^{\bullet})$ has regular variation of index $1/\delta$ then (3.5) *holds with $\psi_r = (-\log \bar{F}_r(\mathrm{e}^{\bullet}))^{\leftarrow}$.*

(b) *If ψ_r is a non-decreasing function, regularly varying with index δ and*

$$\lim_{\xi\to+\infty} \frac{\psi_r(-\log \bar{F}_r(\xi))}{\xi} \geq 1, \tag{3.6}$$

then

$$\limsup_n \frac{\rho_n}{\psi_r(\log n)} \leq c_{r,d}\left(1+\frac{r}{d}\right)^{\delta}, \tag{3.7}$$

where $c_{r,d} = 1$ if $d = 1, r \geq 1$ and ($\mathbf{G_r}$) *holds and $c_{r,d} = 2$ otherwise. In particular, if $-\log \bar{F}_r$ has regular variation with index $1/\delta$, then* (3.7) *holds with $\psi_r = (-\log \bar{F}_r)^{\leftarrow}$.*

Further comments on the choice of ψ_r. As we will show further on, claim (a) is devoted to distributions with polynomial tails whereas claim (b) will be applied to distributions with hyper-exponential tails. Note that for distributions with exponential tails, the function ψ_r in (b) can be chosen independently of r (see the proof of Corollary 3.1). Also note that if $-\log \bar{F}_r$ (resp., $-\log \bar{F}_r(\mathrm{e}^{\bullet})$) is measurable, locally bounded and regularly varying with index $1/\delta$, $\delta > 0$, then its generalized inverse function ϕ_r (resp., Φ_r) is measurable increasing to $+\infty$, regularly varying with index δ and $\phi_r(-\log \bar{F}_r(x)) = x + \mathrm{o}(x)$ (resp., $\Phi_r(-\log \bar{F}_r(\mathrm{e}^x)) = x + \mathrm{o}(x)$). Consequently, inequality (3.7) (resp., (3.5)) holds with ϕ_r (resp., Φ_r) in place of ψ_r. However, ϕ_r (resp., Φ_r) is, in general, not easy to compute and the examples below show that it is often easier to directly exhibit a function ψ_r satisfying the announced hypotheses without inducing any asymptotic loss of accuracy.

The above theorem is a consequence of the following more abstract result, which connects ρ_n and the generalized functions $\bar{F}_r$.

Theorem 3.2. *Let $r > 0$ and let $X \in L^r(\mathbb{P})$ with a distribution P having an unbounded support and satisfying* (**H**). *Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers. Then,*

$$\lim_{\varepsilon\downarrow 0} \liminf_n \left(n^{1+r/d}\,\bar{F}_r\left(\frac{\rho_n}{c_{r,d}+\varepsilon}\right)\right) \geq C_{r,d} \in (0,\infty), \tag{3.8}$$

where $c_{r,d}$ is defined in Theorem 3.1.

We will temporarily admit this result to prove Theorem 3.1.

Proof of Theorem 3.1. (a) It follows from (3.8) that, for every $\varepsilon > 0$, there is a positive real constant $C_{r,d,\varepsilon}$ such that $n^{-(d+r)/d}C_{r,d,\varepsilon} \le \bar{F}_r(\frac{\rho_n}{c_{r,d}+\varepsilon})$. Therefore, one has

$$\frac{r+d}{d}\log n - \log(C_{r,d,\varepsilon}) \ge -\log \bar{F}_r\left(\frac{\rho_n}{c_{r,d}+\varepsilon}\right).$$

Combining the fact that ψ_r is non-decreasing with assumption (3.4) yields

$$\begin{aligned}\psi_r\left(\frac{r+d}{d}\log n - \log(C_{r,d,\varepsilon})\right) &\ge \psi_r\left(-\log \bar{F}_r\left(\frac{\rho_n}{c_{r,d}+\varepsilon}\right)\right)\\ &\ge \log \rho_n - \log(c_{r,d}+\varepsilon) + \mathrm{o}(\log \rho_n).\end{aligned}$$

Moreover, dividing by $\psi_r(\log n)$ (which is positive for large enough n) yields

$$\frac{\log \rho_n}{\psi_r(\log n)} \le \left(1 - \frac{\log(c_{r,d}+\varepsilon)}{\log \rho_n} + \frac{\mathrm{o}(\log \rho_n)}{\log \rho_n}\right)^{-1} \frac{\psi_r(((r+d)/d)\log n - \log(C_{r,d,\varepsilon}))}{\psi_r(\log n)}.$$

Owing to the regularly varying hypothesis on ψ_r and the fact that $\lim_n \rho_n = +\infty$ (which follows from Proposition 2.1), we have

$$\limsup_n \frac{\log \rho_n}{\psi_r(\log n)} \le \left(1 + \frac{r}{d}\right)^{\delta}.$$

(b) As previously, one derives from (3.6) and from the non-decreasing hypothesis on ψ_r that

$$\begin{aligned}\psi_r\left(\frac{r+d}{d}\log n - \log(C_{r,d,\varepsilon})\right) &\ge \psi_r\left(-\log \bar{F}_r\left(\frac{\rho_n}{c_{r,d}+\varepsilon}\right)\right)\\ &\ge \frac{\rho_n}{c_{r,d}+\varepsilon} + \mathrm{o}(\rho_n).\end{aligned}$$

It follows that

$$\frac{\rho_n}{\psi_r(\log n)} \le (c_{r,d}+\varepsilon)\left(1 + \frac{\mathrm{o}(\rho_n)}{\rho_n}\right)^{-1} \frac{\psi_r(((r+d)/d)\log n - \log(C_{r,d,\varepsilon}))}{\psi_r(\log n)}.$$

The regularly varying hypothesis on ψ_r and the fact that $\lim_n \rho_n = +\infty$ yields

$$\forall \varepsilon > 0 \qquad \limsup_n \frac{\rho_n}{\psi_r(\log n)} \le (c_{r,d}+\varepsilon)\left(\frac{r+d}{d}\right)^{\delta}.$$

The result follows by letting $\varepsilon \to 0$. $\square$

Now we pass to the proof of Theorem 3.2, which is based on the following two lemmas.

Lemma 3.1. *Let $r > 0$ and let $X \in L^r(\mathbb{P})$ with a distribution P on $\mathbb{R}^d$ having an unbounded support. Let $(\alpha_n)_{n\geq 1}$ be a sequence of n-quantizers, such that $\mathbb{E}d(X,\alpha_n)^r \to 0$. Then,*

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ such that } \forall n \geq n_\varepsilon, \forall a \in M(\alpha_n), \forall y \in C_a(\alpha_n) \qquad |y| \geq \frac{\rho_n}{c_{r,d}+\varepsilon}, \tag{3.9}$$

where $c_{r,d}$ is defined in Theorem 3.1.

Proof. *Step* 1. Let $r > 0$ and let $d \geq 1$. Since $\mathbb{E}d(X,\alpha_n)^r \to 0$ as $n \to +\infty$, the following asymptotic density property of (α_n) in the support of P holds:

$$\forall \varepsilon > 0, \forall x \in \operatorname{supp}(P), \exists n_{\varepsilon,x} \in \mathbb{N}, \forall n \geq n_{\varepsilon,x} \qquad B(x,\varepsilon) \cap \alpha_n \neq \varnothing. \tag{3.10}$$

Otherwise, there exists $x \in \operatorname{supp}(P)$, $\varepsilon > 0$ and a subsequence $(\alpha_{n_k})_{k\geq 1}$ so that $\forall k \geq 1$, $B(x,\varepsilon) \cap \alpha_{n_k} = \varnothing$. Then, for every $k \geq 1$,

$$\|d(X,\alpha_{n_k})\|_r \geq \big\|d(X,\alpha_{n_k})\mathbf{1}_{X\in B(x,\varepsilon/2)}\big\|_r \geq \frac{\varepsilon}{2} P\big(B(x,\varepsilon/2)\big)^{1/r} > 0,$$

which contradicts the fact that $\|d(X,\alpha_n)\|_r \to 0$ as $n \to +\infty$.

Assume first that $0 \in \operatorname{supp}(P)$. Let $\varepsilon > 0$ and $a \in M(\alpha_n)$. There exists an $N_1 \in \mathbb{N}$ such that $B(0,\varepsilon) \cap \alpha_n \neq \varnothing$ for every $n \geq N_1$. Now $\rho_n \to +\infty$ implies the existence of $N_1' \in \mathbb{N}$, $N_1' \geq N_1$ such that $B(0,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n)) \neq \varnothing$ for $n \geq N_1'$.

Let $n \geq N_1'$ and let $b \in B(0,\varepsilon) \cap (\alpha_n \backslash M(\alpha_n))$. For every $y \in C_a(\alpha_n)$, we have $|y-b|^2 \geq |y-a|^2$, so that

$$2(y|a-b) \geq |a|^2 - |b|^2 = \rho_n^2 - |b|^2 \geq 0.$$

Now, if $|y||a-b| \geq (y|a-b)$, then,

$$|y||a-b| \geq \frac{(\rho_n+|b|)(\rho_n-|b|)}{2}.$$

Moreover, $0 < |a-b| \leq |a| + |b| = \rho_n + |b|$. One finally gets

$$|y| \geq \frac{\rho_n - |b|}{2} \geq \frac{\rho_n - \varepsilon}{2}.$$

Since $\rho_n \to +\infty$, then $|y| \geq \frac{\rho_n}{2+\varepsilon}$ as soon as $n \geq \max(N_1', N_2)$, with N_2 such that $\rho_{N_2} \geq 2+\varepsilon$.

If $0 \notin \operatorname{supp}(P)$, we show likewise that $|y| \geq \frac{\rho_n - |x_0| - \varepsilon}{2}$, where $x_0 \in \operatorname{supp}(P)$ is fixed. This implies the announced result since $\rho_n \to +\infty$.

Step 2. Suppose that $d = 1$, $r \geq 1$ and $(\mathbf{G_r})$ holds. First, we use the well-known fact (see, e.g., [11], Lemma 4.10 or [14], Proposition 9) that the L^r-distortion function

$$\alpha = (\alpha_1,\dots,\alpha_n) \longmapsto D^X_{n,r}(\alpha) = \mathbb{E}\Big(\min_{i=1,\dots,n} |X-\alpha_i|^r\Big)$$

is differentiable at any codebook $\alpha \in (\mathbb{R}^d)^n$ having pairwise distinct components and that

$$\nabla D^X_{n,r}(\alpha) = r\left(\int_{C_i(\alpha)} (\alpha_i - u)|u - \alpha_i|^{r-2} f(u)\,\mathrm{d}u\right)_{1\le i\le n}. \tag{3.11}$$

An optimal L^r-quantizer $\alpha = \{\alpha_1, \dots, \alpha_n\}$ at level n for $P = f\lambda_1$ has full size n so that

$$\nabla D^X_{n,r}(\alpha) = 0. \tag{3.12}$$

Note that for any (ordered) quantizer $\alpha_n = \{x_1^{(n)}, \dots, x_n^{(n)}\}$, $x_1^{(n)} < \dots < x_n^{(n)}$ at level n, its Voronoi partition is given by

$$C_1(\alpha_n) = \big(-\infty, x^{(n)}_{1/2}\big], \qquad C_n(\alpha_n) = \big(x^{(n)}_{n-1/2}, +\infty\big),$$
$$C_i(\alpha_n) = \big(x^{(n)}_{i-1/2}, x^{(n)}_{i+1/2}\big], \qquad i = 2, \dots, n-1,$$

with $x^{(n)}_{i\pm 1/2} = \frac{x_i^{(n)} + x_{i\pm1}^{(n)}}{2}$. We will focus on the one-sided setting by considering

$$\rho_n = \rho_n^+ := \max\{x, x \in \alpha\}.$$

All results on $\rho_n^- := \max\{-x, x \in \alpha\}$ follow by considering $-X$ instead of X. Finally, one will conclude by noting that the bi-sided radius is given by $\rho_n = \max(\rho_n^+, \rho_n^-)$.

Let $\alpha_n = \{x_1^{(n)}, \dots, x_n^{(n)}\}$ with $x_1^{(n)} < \dots < x_n^{(n)}$ and suppose that (up to a subsequence) $\frac{x^{(n)}_{n-1}}{x^{(n)}_n} \to \rho < 1$.

Let $\varepsilon > 0$ such that $\rho + \varepsilon < 1$. We have for large enough n, $\frac{x^{(n)}_{n-1}}{x^{(n)}_n} < \rho + \varepsilon < 1$ or, equivalently,

$$\frac{x^{(n)}_{n-1} + x^{(n)}_n}{2} < x^{(n)}_n \frac{1 + \rho + \varepsilon}{2}. \tag{3.13}$$

Let ρ' be such that $0 < \rho' < \frac{1-(\rho+\varepsilon)}{2}$, that is, $\frac{1+\rho+\varepsilon}{2} < 1 - \rho' < 1$. It follows from (3.13) that

$$\begin{aligned}
\int_{(x^{(n)}_{n-1}+x^{(n)}_n)/2}^{x^{(n)}_n} \left(1 - \frac{u}{x^{(n)}_n}\right)^{r-1} f(u)\,\mathrm{d}u &\ge \int_{x^{(n)}_n(1+\rho+\varepsilon)/2}^{x^{(n)}_n(1-\rho')} \left(1 - \frac{u}{x^{(n)}_n}\right)^{r-1} f(u)\,\mathrm{d}u \\
&\ge (\rho')^{r-1} \int_{x^{(n)}_n(1+\rho+\varepsilon)/2}^{x^{(n)}_n(1-\rho')} f(u)\,\mathrm{d}u \\
&\ge \rho'' x^{(n)}_n f(c_n)
\end{aligned} \tag{3.14}$$

with $\rho'' = (\rho')^{r-1}(\frac{1}{2} - \rho' - \frac{\rho+\varepsilon}{2}) > 0$ and $c_n \in (x^{(n)}_n(1+\rho+\varepsilon)/2, x^{(n)}_n(1-\rho'))$. On the other hand, since we have

$$\frac{1}{x^{(n)}_n f(x^{(n)}_n)} \int_{x^{(n)}_n}^{+\infty} \left(\frac{u}{x^{(n)}_n} - 1\right)^{r-1} f(u)\,\mathrm{d}u = \int_1^{+\infty} (u-1)^{r-1} \frac{f(ux^{(n)}_n)}{f(x^{(n)}_n)}\,\mathrm{d}u,$$

it follows from assumption ($\mathbf{G_r}$) that

$$\lim_n \frac{1}{x_n^{(n)} f(x_n^{(n)})} \int_{x_n^{(n)}}^{+\infty} \left(\frac{u}{x_n^{(n)}} - 1\right)^{r-1} f(u)\,\mathrm{d}u = 0.$$

Consequently, for large enough n,

$$\frac{1}{x_n^{(n)} f(x_n^{(n)})} \int_{x_n^{(n)}}^{+\infty} \left(\frac{u}{x_n^{(n)}} - 1\right)^{r-1} f(u)\,\mathrm{d}u < \rho''$$

so that using (3.14) and the fact that f is non-increasing in $[A, +\infty)$ and $A < c_n < x_n^{(n)}$ for large enough n, one gets

$$\begin{aligned}\int_{x_n^{(n)}}^{+\infty} \left(\frac{u}{x_n^{(n)}} - 1\right)^{r-1} f(u)\,\mathrm{d}u &< \rho'' x_n^{(n)} f\big(x_n^{(n)}\big)\\ &\le \rho'' x_n^{(n)} f(c_n) \le \int_{(x_{n-1}^{(n)}+x_n^{(n)})/2}^{x_n^{(n)}} \left(1 - \frac{u}{x_n^{(n)}}\right)^{r-1} f(u)\,\mathrm{d}u.\end{aligned}$$

This leads to a contradiction since the L^r-stationary equation (3.12) implies in particular

$$\int_{(x_{n-1}^{(n)}+x_n^{(n)})/2}^{x_n^{(n)}} \left(1 - \frac{u}{x_n^{(n)}}\right)^{r-1} f(u)\,\mathrm{d}u = \int_{x_n^{(n)}}^{+\infty} \left(\frac{u}{x_n^{(n)}} - 1\right)^{r-1} f(u)\,\mathrm{d}u.$$

We therefore have shown that $\lim_n \frac{x_n^{(n)}}{x_{n-1}^{(n)}} = 1$. It follows that

$$\forall \varepsilon > 0, \exists n_\varepsilon \text{ such that } \forall n \ge n_\varepsilon \qquad x_n^{(n)} < (1+\varepsilon) x_{n-1}^{(n)}.$$

Thus, one completes the proof by noting that

$$\forall y \in C_a(\alpha_n), a \in M(\alpha_n) \qquad \rho_n = x_n^{(n)} < (1+\varepsilon) x_{n-1}^{(n)} < (1+\varepsilon) y. \qquad \square$$

Lemma 3.2. *Let $r > 0$ and let $X \in L^r(\mathbb{P})$ with distribution P satisfying* (**H**). *Let $(\alpha_n)_{n\ge1}$ be a sequence of L^r-optimal n-quantizers of the distribution P. Then for large enough n,*

$$e_{n,r}^r(X) - e_{n+1,r}^r(X) \ge C_{r,d}\, n^{-(r+d)/d}, \tag{3.15}$$

with

$$C_{r,d} = \frac{r}{2^{(r+d)}(d+r)} \left(\frac{d}{d+r}\right)^{d/r} \frac{\varepsilon_0}{1+\varepsilon_0} Q_{d+r}\big(U\big(\bar B(x_0, r_0/2)\big)\big), \tag{3.16}$$

where $U(\bar B(x_0, \frac{r_0}{2}))$ stands for the uniform distribution on the closed ball $\bar B(x_0, \frac{r_0}{2})$, the constants ε_0, x_0, r_0 come from assumption (**H**) *and Q_{d+r} is defined by* (1.3) *in Zador's theorem.*

Proof. *Step* 1. Let $y \in \mathbb{R}^d$. We temporarily set $\delta_n = d(y, \alpha_n)$ and may assume $\delta_n > 0$. Following the lines of the proof of Theorem 2 in [12], we have for every $x \in B(y, \delta_n/2)$ and $a \in \alpha_n$,

$$|x - a| \geq |y - a| - |x - a| \geq \delta_n/2$$

and hence

$$d(x, \alpha_n) \geq \delta_n/2 \geq |x - y|, \qquad x \in B(y, \delta_n/2).$$

It follows, by setting $\beta_n = \alpha_n \cup \{y\}$, that $d(x, \alpha_n) \geq d(x, \beta_n)$ and $d(x, \beta_n) = |x - y|, x \in B(y, \delta_n/2)$. Consequently for every $b \in (0, 1/2)$,

$$\begin{aligned} e_{n,r}^r(X) - e_{n+1,r}^r(X) &\geq \int_{B(y,\delta_n b)} \big(d(x, \alpha_n)^r - d(x, \beta_n)^r\big) P(\mathrm{d}x) \\ &= \int_{B(y,\delta_n b)} \big(d(x, \alpha_n)^r - |x - y|^r\big) P(\mathrm{d}x) \\ &\geq \int_{B(y,\delta_n b)} \big((\delta_n/2)^r - (\delta_n b)^r\big) P(\mathrm{d}x) \\ &= (2^{-r} - b^r)\delta_n^r P(B(y, \delta_n b)). \end{aligned}$$

Step 2. This step is the core of our proof. Let x_0 and r_0 be as in (**H**). For every $y \in \bar{B}(x_0, \frac{r_0}{2})$,

$$\begin{aligned} e_{n,r}^r(X) - e_{n+1,r}^r(X) &\geq (2^{-r} - b^r)\delta_n^r P\left(B\left(y, \min\left(b\delta_n, \frac{r_0}{2}\right)\right)\right) \\ &\geq (2^{-r} - b^r)\delta_n^r \varepsilon_0 \min\left((b\delta_n)^d, \left(\frac{r_0}{2}\right)^d\right). \end{aligned}$$

We know from [8] that, as soon as $d(x, \alpha_n) \to 0$ as $n \to \infty$ in $L^r(P)$, the convergence will hold uniformly on compact sets as well. In particular, we have

$$\sup_{y \in \bar{B}(x_0, r_0/2)} d(y, \alpha_n) \to 0$$

so that there exists $N(x_0, r_0) \in \mathbb{N}$ such that for every $n \geq N(x_0, r_0)$,

$$\sup_{y \in \bar{B}(x_0, r_0/2)} d(y, \alpha_n) \leq \frac{r_0}{2}.$$

Consequently

$$e_{n,r}^r(X) - e_{n+1,r}^r(X) \geq (2^{-r} - b^r) b^d d(y, \alpha_n)^{d+r} \varepsilon_0 \mathbf{1}_{\{y \in \bar{B}(x_0, r_0/2)\}}.$$

It follows that

$$\begin{aligned} e_{n,r}^r(X) - e_{n+1,r}^r(X) &\geq (2^{-r} - b^r)\varepsilon_0 b^d \int_{\bar{B}(x_0, r_0/2)} d(y, \alpha_n)^{d+r} \frac{\lambda_d(\mathrm{d}y)}{\lambda_d(\bar{B}(x_0, r_0/2))} \\ &\geq (2^{-r} - b^r) b^d \varepsilon_0 \lambda_d\big(\bar{B}(x_0, r_0/2)\big) e_{n,r+d}^{r+d}\big(U\big(\bar{B}(x_0, r_0/2)\big)\big), \end{aligned}$$

where we used in the last inequality the fact that α_n is suboptimal for the uniform distribution over $\bar{B}(x_0, \frac{r_0}{2})$. As a consequence,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq (2^{-r} - b^r)b^d \varepsilon_0 e^{r+d}_{n,r+d}\big(U\big(\bar{B}(x_0, r_0/2)\big)\big).$$

Finally, one completes the proof by noting that, for large enough $n \geq N(x_0, r_0)$,

$$e^r_{n,r}(X) - e^r_{n+1,r}(X) \geq \sup_{b\in(0,1/2)} \big((2^{-r} - b^r)b^d\big)\frac{\varepsilon_0}{1+\varepsilon_0} Q_{d+r}\big(U\big(\bar{B}(x_0, r_0/2)\big)\big)n^{-(d+r)/d}. \qquad \square$$

Now we are in position to complete the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $a \in M(\alpha_n)$ and $\varepsilon > 0$. We have,

$$e^r_{n-1,r}(X) = \mathbb{E}|X - \widehat{X}^{\alpha_{n-1}}|^r \leq \mathbb{E}\big|X - \widehat{X}^{\alpha_n\setminus\{a\}}\big|^r$$

since α_{n-1} is L^r-optimal at level $n-1$. Hence

$$\mathbb{E}\big|X - \widehat{X}^{\alpha_n\setminus\{a\}}\big|^r = \mathbb{E}\big(|X - \widehat{X}^{\alpha_n}|^r \mathbf{1}_{\{X\in C^c_a(\alpha_n)\}}\big) + \mathbb{E}\Big(\min_{b\in\alpha_n\setminus\{a\}} |X-b|^r \mathbf{1}_{\{X\in C_a(\alpha_n)\}}\Big)$$

$$\leq e^r_{n,r}(X) + \mathbb{E}\Big(\min_{b\in\alpha_n\setminus\{a\}} (|X| + |b|)^r \mathbf{1}_{\{X\in C_a(\alpha_n)\}}\Big).$$

It follows from Lemma 3.1 that, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$, $|X| > \frac{\rho_n}{c_{r,d}+\varepsilon}$, on the event $\{X \in C_a(\alpha_n)\}$. Consequently, for all $b \in \alpha_n\setminus\{a\}$, $|b| \leq |a| = \rho_n < (c_{r,d}+\varepsilon)|X|$. Hence,

$$e^r_{n-1,r}(X) - e^r_{n,r}(X) \leq (c_{r,d} + 1 + \varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>\rho_n/(c_{r,d}+\varepsilon)\}}\big).$$

Lemma 3.2 yields for large enough n (since $(n-1)^{-(r+d)/d} \sim n^{-(r+d)/d}$ as $n \to +\infty$),

$$(1+\varepsilon)^{-1} C_{r,d} n^{-(r+d)/d} \leq (c_{r,d} + 1 + \varepsilon)^r \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>\rho_n/(c_{r,d}+\varepsilon)\}}\big)$$

so that for every $\varepsilon > 0$,

$$\liminf_n \left(n^{(r+d)/d} \bar{F}_r\left(\frac{\rho_n}{c_{r,d}+\varepsilon}\right)\right) \geq \frac{C_{r,d}}{(c_{r,d}+1+\varepsilon)^r(1+\varepsilon)}.$$

Letting $\varepsilon \to 0$ yields the statement (3.8). $\square$

3.2. Applications to distributions with polynomial and hyper-exponential tails

We next give an explicit asymptotic upper bound for the convergence rate of the maximal radius sequence by making the function ψ_r explicit. These bounds are derived in terms of the rate of decay of the generalized survival function $\bar{F}_r$.

Proposition 3.1. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ with distribution P having an unbounded support and satisfying* (**H**). *Suppose that $(\alpha_n)_{n\geq 1}$ is an L^r-optimal sequence of n-quantizers for X.*

(a) Polynomial tail. *Set*

$$\zeta^\star = \sup\Big\{\zeta > 0, \limsup_{\xi\to+\infty} \xi^{\zeta-r}\bar{F}_r(\xi) < +\infty\Big\} = \sup\{\zeta > r, \mathbb{E}|X|^\zeta < +\infty\}. \tag{3.17}$$

Then $\zeta^\star \in (r, +\infty]$ and

$$\limsup_n \frac{\log \rho_n}{\log n} \leq \frac{1}{\zeta^\star - r}\frac{r+d}{d}. \tag{3.18}$$

(b) Hyper-exponential tail. *Assume there exists $\kappa > 0$ such that $\mathrm{e}^{|X|^\kappa} \in L^{0+}(\mathbb{P})$. Set*

$$\theta^\star = \sup\Big\{\theta > 0, \limsup_{\xi\to+\infty} \mathrm{e}^{\theta\xi^\kappa}\bar{F}_r(\xi) < +\infty\Big\} = \sup\big\{\theta > 0, \mathbb{E}\mathrm{e}^{\theta|X|^\kappa} < +\infty\big\}. \tag{3.19}$$

Then $\theta^\star \in (0, +\infty]$ and

$$\limsup_n \frac{\rho_n}{(\log n)^{1/\kappa}} \leq c_{r,d}\left(\frac{r+d}{d\theta^\star}\right)^{1/\kappa}. \tag{3.20}$$

Remark 3.1. If $X \in \bigcap_{r>0} L^r(\mathbb{P})$, then $\zeta^\star = +\infty$ and, consequently, $\lim_{n\to+\infty} \frac{\log\rho_n}{\log n} = 0$. This confirms that this asymptotics is not the significant one for distributions with hyper-exponential tails.

Proof of Proposition 3.1. The equalities in (3.19) and (3.17) are elementary.

(a) Let $\zeta \in (r, \zeta^\star)$. We have

$$\begin{aligned}\mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|>\xi\}}\big) &= \mathbb{E}\big(|X|^r \mathbf{1}_{\{1<\xi^{-\zeta+r}|X|^{\zeta-r}\}}\big)\\ &\leq \xi^{-\zeta+r}\mathbb{E}|X|^\zeta.\end{aligned}$$

Then $-\log \bar{F}_r(\xi) \geq (\zeta - r)\log\xi + C$, $C \in \mathbb{R}$, so that by setting $\psi_r(\xi) = \frac{\xi}{\zeta-r}$, it follows from Theorem 3.1(a) that

$$\limsup_n \frac{\log\rho_n}{\log n} \leq \frac{1}{\zeta - r}\frac{r+d}{d}.$$

Letting ζ go to $\zeta^\star$ yields the assertion (3.18).

(b) Let $\theta \in (0, \theta^\star)$. We have

$$\mathbb{E}\big(|X|^r\mathbf{1}_{\{|X|>\xi\}}\big) = \mathbb{E}\big(|X|^r\mathbf{1}_{\{\mathrm{e}^{\theta|X|^\kappa}>\mathrm{e}^{\theta\xi^\kappa}\}}\big) \leq \mathrm{e}^{-\theta\xi^\kappa}\mathbb{E}\big(|X|^r\mathrm{e}^{\theta|X|^\kappa}\big).$$

Now, the right-hand side of this last inequality is finite because if $\theta' \in (\theta, \theta^\star)$, there exists a positive constant $C_{\theta,\theta'}$ such that, for every $\xi \in \mathbb{R}^d$, $|\xi|^r\mathrm{e}^{\theta|\xi|^\kappa} \leq 1 + C_{\theta,\theta'}\mathrm{e}^{\theta'|\xi|^\kappa}$. As a consequence,

$$-\log\bar{F}_r(\xi) \geq \theta\xi^\kappa + C_{\theta,X}, \qquad C_{\theta,X} \in \mathbb{R}.$$

Let $\psi_\theta(y) = (\frac{y}{\theta})^{1/\kappa}$. As a function of y, ψ_θ is continuous increasing to $+\infty$, regularly varying with index $\delta = \frac{1}{\kappa}$ and we have

$$\psi_\theta(-\log \bar{F}_r(\xi)) \geq \left(\xi^\kappa + \frac{C_X}{\theta}\right)^{1/\kappa} = \xi + \mathrm{o}(\xi) \qquad \text{as } \xi \to +\infty.$$

It follows from Theorem 3.1(b) that, for every $\theta \in (0, \theta^\star)$,

$$\limsup_n \frac{\rho_n}{(\log n)^{1/\kappa}} \leq c_{r,d}\left(\frac{d+r}{d\theta}\right)^{1/\kappa}.$$

Letting $\theta \to \theta^\star$ completes the proof. □

We now give more explicit results for two wide classes of density functions in $\mathbb{R}^d$: the distributions with polynomial tails and hyper-exponential tails which, among others, include the Pareto, Gaussian, Weibull, gamma and double-sided gamma distributions, respectively.

Corollary 3.1. (a) *If the density f of X satisfies*

$$\limsup_{|x|\to+\infty} |x|^c f(x) < +\infty \qquad \textit{for some } c > r + d, \tag{3.21}$$

then $X \in L^{r+}(\mathbb{P})$ and

$$\zeta^\star \geq c - d \quad \textit{and} \quad \limsup_n \frac{\log \rho_n}{\log n} \leq \frac{1}{c-d-r}\frac{r+d}{d}. \tag{3.22}$$

(b) *If the density of X satisfies*

$$\limsup_{|x|\to+\infty} \frac{\log f(x)}{|x|^\kappa} = -\vartheta < 0 \qquad \textit{for some } \kappa > 0, \tag{3.23}$$

then $X \in L^{r+}(\mathbb{P})$ and

$$\theta^\star \geq \vartheta \quad \textit{and} \quad \limsup_n \frac{\rho_n}{(\log n)^{1/\kappa}} \leq \frac{c_{r,d}}{\vartheta^{1/\kappa}}\left(1 + \frac{r}{d}\right)^{1/\kappa}. \tag{3.24}$$

Proof. (a) Let $A, B > 0$ such that for every x with $|x| \geq B$, $f(x) \leq \frac{A}{|x|^c}$. Then, as soon as $\xi \geq B$,

$$\bar{F}_r(\xi) = \mathbb{E}\big(|X|^r \mathbf{1}_{\{|X|\geq\xi\}}\big) \leq A \int_{\{|x|\geq\xi\}} |x|^r \frac{\mathrm{d}x}{|x|^c} = A d V_d \det(S) \frac{\xi^{r+d-c}}{r+d-c},$$

where V_d denotes the hyper-volume of the unit Euclidean ball of $\mathbb{R}^d$ and $|x|^2 = {}^t xSx$. As a consequence, for any $\zeta < c - d$ and any $\xi \geq B$,

$$\xi^{\zeta-r}\overline{F}_r(\xi) \leq A d V_d \det(S) \frac{\xi^{r+d-c}}{r+d-c}$$

so that $\overline{\lim}_{\xi\to\infty}\xi^{\zeta-r}\bar{F}_r(\xi)=0$, that is, $\zeta^\star \geq c-d$ by Proposition 3.1(a).

(b) It follows from the assumption that, for every $\eta\in(0,\vartheta/3)$, there exists $B>0$ such that, for every x with $|x|\geq B$, $f(x)\leq \mathrm{e}^{-(\vartheta-\eta)|x|^\kappa}$. Hence, as soon as $\xi\geq B$,

$$\begin{aligned}\bar{F}_r(\xi) &= \mathbb{E}\big(|X|^r\mathbf{1}_{\{|X|\geq\xi\}}\big)\\ &\leq \int_{\{|x|\geq\xi\}}|x|^r\mathrm{e}^{-(\vartheta-\eta)|x|^\kappa}\,\mathrm{d}x\\ &= d V_d\det(S)\int_{\{u\geq\xi\}}u^{r+d-1}\mathrm{e}^{-(\vartheta-\eta)u^\kappa}\,\mathrm{d}u\end{aligned}$$

so that

$$\mathrm{e}^{(\vartheta-3\eta)\xi^\kappa}\bar{F}_r(\xi)\leq dV_d\det(S)\mathrm{e}^{-\eta\xi^\kappa}\int_{\{u\geq B\}}u^{r+d-1}\mathrm{e}^{-\eta u^\kappa}\,\mathrm{d}u.$$

Consequently, $\theta^\star\geq\vartheta-3\eta$ and letting η go to 0 shows that $\theta^\star\geq\vartheta$, which completes the proof. □

4. Lower estimate and asymptotic rates

In this section we study the asymptotic lower estimate of the maximal radius sequence $(\rho_n)_{n\geq1}$ induced by an L^r-optimal sequence of n-quantizers. First we introduce the family of the (r,s)-distributions, which will play a crucial role to obtain the sharp lower estimate of the maximal radius sequence.

Let $r>0$ and let $s>r$. Since the L^r-norm is increasing, it is clear that, for every $s\leq r$, any L^r-optimal sequence of quantizers $(\alpha_n)_{n\geq1}$ is L^s*-rate optimal*, that is, $\limsup_n n^{1/d}\|X-\widehat{X}^{\alpha_n}\|_s<+\infty$.

But if $s>r$ (and $X\in L^s(\mathbb{P})$), this asymptotic rate optimality usually fails. This is always the case when $s>r+d$ and X has a probability distribution f satisfying $\lambda_d(f>0)=+\infty$, as pointed out in [12], Corollaries 3 and 4. It is established in [20] that some linear transformation of the L^r-optimal quantizers (α_n) makes it possible to overcome the critical exponent $r+d$; that is, one can always construct an L^s-rate-optimal sequence of quantizers up to an affine transformation of the L^r-optimal sequence of quantizers (α_n).

However, there are many (usual) distributions for which L^s-rate optimality does hold for every $s\in[r,r+d)$. This leads to the following definition.

Definition 4.1. *Let $r>0$ and $\nu\in(0,d)$. A random vector $X\in L^{r+}(\mathbb{P})$ has an $(r,r+\nu)$-distribution if any L^r-optimal sequence $(\alpha_n)_{n\geq1}$ is $L^{r+\nu}$-rate optimal, that is,*

$$\limsup_n n^{1/d}\|X-\widehat{X}^{\alpha_n}\|_{r+\nu}<+\infty.$$

Note that if X has an $(r, r+\nu)$-distribution, then $X \in L^{r+\nu}(\mathbb{P})$. A necessary condition for a distribution P with density f to have an $(r, r+\nu)$-distribution is (see [12]):

$$\int_{\mathbb{R}^d} f(x)^{-(r+\nu)/(d+r)} P(\mathrm{d}x) < +\infty. \tag{4.1}$$

For $\nu \in (0, d)$, criterions that imply that X has an $(r, r+\nu)$-distribution have been provided in [12]. We mention two of them below.

Proposition 4.1 (Radial tail). *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ with distribution $P = f\lambda_d$ having an unbounded support that is the intersection of finitely many half-spaces.*

(a) *Suppose f has a radial tail, that is, there exists a norm $N(\cdot)$ on $\mathbb{R}^d$ and $R_0 \in \mathbb{R}_+$ such that*

$$f = h(N(\cdot)) \text{ on } B_{N(\cdot)}(0, R_0)^c, \text{ where } h : [R_0, +\infty) \to \mathbb{R}_+ \text{ is a decreasing function.} \tag{4.2}$$

Let $\nu \in (0, d)$. If

$$\int_{\mathbb{R}^d} f(\rho x)^{-(r+\nu)/(r+d)} P(\mathrm{d}x) < +\infty \qquad \text{for some } \rho > 1, \tag{4.3}$$

then X has an $(r, r+\nu)$-distribution.

(b) *Assume $d = 1$. If $\operatorname{supp}(P) \subset [A_0, +\infty)$ for some $A \in \mathbb{R}$, $f_{|(R_0,+\infty)}$ is decreasing for $R_0 \geq A_0$ and, if, furthermore, assumption (4.3) holds for some $\rho > 1$, then X has an $(r, r+\nu)$-distribution.*

The following proposition works for distributions with non-radial tails.

Proposition 4.2. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ with distribution $P = f\lambda_d$ having a convex (unbounded) support. Assume that f satisfies the following local decay control assumption: There exist real numbers $\varepsilon \geq 0$, $\eta \in (0, 1)$, M, $K > 0$ such that*

$$\forall x, y \in \operatorname{supp}(P), |x| \geq M, |y - x| \leq \eta|x| \quad \Longrightarrow \quad f(y) \geq K f(x)^{1+\varepsilon}. \tag{4.4}$$

Let $\nu \in (0, d)$. If

$$\int_{\mathbb{R}^d} f(x)^{-(r+\nu)(1+\varepsilon)/(r+d)} P(\mathrm{d}x) < +\infty, \tag{4.5}$$

then X has an $(r, r+\nu)$-distribution.

It follows from Proposition 4.1 that the Gaussian, Weibull and gamma distributions are all $(r, r+\nu)$-distributions for every $\nu \in (0, d)$. The Pareto distribution with index $\gamma > r$ has an $(r, r+\nu)$-distribution if and only if $\nu \in (0, \frac{\gamma-r}{\gamma+1})$.

More generally, if a distribution $P = f\lambda_d$ is supported by a convex subset C of $\mathbb{R}^d$ such that

$$f(x) = \mathrm{e}^{-g(x)^\kappa}, \qquad g : C \to \mathbb{R}_+, \text{ Lipschitz continuous}, \kappa > 0,$$

or

$$f(x) = \frac{1}{g(x)^c}, \qquad g : C \to \mathbb{R}_+, \text{ Lipschitz continuous, } g \geq \varepsilon_0 \text{ on } B(0, M)^c, c > d,$$

then P satisfies the local decay control criterion (4.4) of Proposition 4.2 for arbitrarily small positive ε and $\varepsilon = 0$, respectively.

Now, suppose that X has an $(r, r+\nu)$-distribution for some $\nu \in (0, d)$ and set

$$\nu_X^\star := \sup\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} \in [0, d].$$

Note that $X \in L^{r+\nu}(\mathbb{P})$ for every $\nu \in (0, \nu_X^\star)$ and that

$$\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = (0, \nu_X^\star) \text{ or } (0, \nu_X^\star].$$

When $\{\nu > 0 \text{ s.t. } X \text{ has an } (r, r+\nu)\text{-distribution}\} = \varnothing$, we set

$$\nu_X^\star = 0^+ \qquad \text{with the convention } [0, 0^+) = \{0\}.$$

This convention is consistent with the Zador theorem satisfied by $X \in L^{r+}(\mathbb{P})$. Note that $\nu_X^\star$ may be lower than d, as is the case for the Pareto distribution.

We present below two different approaches to derive the asymptotic lower bound. The first one is based on tail estimates and involves the generalized survival functions $\bar{F}_r$ like for the upper estimate. The second one is based on a new connection with mean random quantization.

4.1. Distribution tail approach

4.1.1. *General results on asymptotic lower bounds*

The main result of this section is the theorem below, which connects the asymptotic lower estimate for ρ_n with the regularly varying property of "the" asymptotic inverse of $-\log \bar{F}_r$ (or one of its lower bound).

Theorem 4.1. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ be an $\mathbb{R}^d$-valued random variable with distribution P having an unbounded support. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers.*

(a) *Let $\nu \in [0, \nu_X^\star)$. If there is a non-decreasing function $\psi_{r,\nu}$ going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\limsup_{\xi \to +\infty} \frac{\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(e^\xi))}{\xi} \leq 1, \tag{4.6}$$

then

$$\liminf_n \frac{\log \rho_n}{\psi_{r,\nu}(\log n)} \geq \left(\frac{r+\nu}{d}\right)^\delta. \tag{4.7}$$

In particular, if $-\log \bar{F}_{r+\nu}(e^x)$ has regular variation with index $1/\delta$, then (4.7) holds with $\psi_{r,\nu}(x) = (-\log \bar{F}_{r+\nu}(e^x))^{\leftarrow}$.

(b) *If ψ is a non-decreasing function going to $+\infty$ as $x \to +\infty$, regularly varying with index δ and satisfying*

$$\limsup_{\xi \to +\infty} \frac{\psi(-\log \bar{F}(\xi))}{\xi} \leq 1, \tag{4.8}$$

then

$$\liminf_n \frac{\rho_n}{\psi(\log n)} \geq \left(\frac{r + \nu_X^\star}{d}\right)^\delta. \tag{4.9}$$

If $-\log \bar{F}$ has regular variation of index $1/\delta$, then (4.9) holds with $\psi = (-\log \bar{F})^{\leftarrow}$.

Similar to the upper limit, one may note that for distribution with exponential tails, the function ψ does not depend on r and ν even if in assumption (4.8) we take the generalized survival function $\bar{F}_{r+\nu}$ instead of the regular survival function $\bar{F}$. However, for distributions with polynomial tails like the Pareto distribution, the function $\psi_{r,\nu}$ in (4.6) may depend on r and consideration of the standard survival function $\bar{F}$ in place of $\bar{F}_{r+\nu}$ would lead to a less accurate lower bound.

As for the upper limit, this result essentially relies on a more abstract result that connects ρ_n and the (generalized) survival functions $\bar{F}_r$.

Theorem 4.2. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ be an $\mathbb{R}^d$-valued random variable with distribution P. Let $(\alpha_n)_{n\geq 1}$ be an $L^r(P)$-optimal sequence of n-quantizers. For every $\nu \in [0, \nu_X^\star)$, the following statements hold*:

$$\text{(a)} \quad \limsup_n \sup_{c>0} \bigl(c^{r+\nu} n^{(r+\nu)/d} \bar{F}(\rho_n + c)\bigr) < +\infty. \tag{4.10}$$

$$\text{(b)} \quad \limsup_n \sup_{u>1} \bigl((1 - 1/u)^{r+\nu} n^{(r+\nu)/d} \bar{F}_{r+\nu}(u\rho_n)\bigr) < +\infty. \tag{4.11}$$

We temporarily admit this theorem to prove Theorem 4.1.

Proof of Theorem 4.1. Let us focus on (b) (claim (a) is proved in a similar manner by considering $\bar{F}_{r+\nu}$ instead of $\bar{F}$, for $\nu \in [0, \nu_X^\star)$). Let $\nu \in [0, \nu_X^\star)$. It follows from (4.10) that for large enough n,

$$-\log \bar{F}(\rho_n + c) \geq -\log(C_{\nu,c}) + \frac{r + \nu}{d} \log n,$$

where $C_{\nu,c}$ is a positive real constant depending on the indexing parameters. We derive from the fact that ψ is non-decreasing and goes to $+\infty$ and from assumption (4.8) that

$$\frac{\rho_n}{\psi(\log n)} \geq \left(1 + \frac{c}{\rho_n} + \frac{\mathrm{o}(\rho_n)}{\rho_n}\right)^{-1} \frac{\psi((r+\nu)/d \log n - \log(C_{\nu,c}))}{\psi(\log n)}.$$

Since ψ is regularly varying with index δ we have

$$\forall \nu \in [0, \nu_X^\star) \qquad \liminf_n \frac{\rho_n}{\psi(\log n)} \geq \left(\frac{r + \nu}{d}\right)^\delta.$$

When $\nu_X^\star > 0$, letting $\nu \to \nu_X^\star$ yields the announced result. $\square$

Proof of Theorem 4.2. (a) Let $n \ge 1$, let $c > 0$ and let $\nu \in [0, \nu_X^\star)$. Then

$$\mathbb{E}|X - \widehat{X}^{\alpha_n}|^{r+\nu} \ge \mathbb{E}\Big(\min_{a\in\alpha_n} |X - a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\Big).$$

In the event $\{|X| > \rho_n + c\}$, we have $|X| > \rho_n + c > \rho_n \ge |a|$ for every $a \in \alpha_n$. Then

$$\begin{aligned} n^{(r+\nu)/d}\mathbb{E}|X - \widehat{X}^{\alpha_n}|^{r+\nu} &\ge n^{(r+\nu)/d}\mathbb{E}\Big(\min_{a\in\alpha_n} |X - a|^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\Big) \\ &\ge n^{(r+\nu)/d}\mathbb{E}\Big(\min_{a\in\alpha_n} (|X| - |a|)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\Big) \\ &\ge n^{(r+\nu)/d}\mathbb{E}\big((|X| - \rho_n)^{r+\nu}\mathbf{1}_{\{|X|>\rho_n+c\}}\big) \\ &\ge c^{r+\nu} n^{(r+\nu)/d}\mathbb{P}(|X| > \rho_n + c). \end{aligned} \tag{4.12}$$

Taking the supremum over $c > 0$ and using that X has an $(r, r+\nu)$-distribution, we complete the proof.

(b) is proved like (a). Inequality (4.12) has the following counterpart: For every $u > 1$,

$$\mathbb{E}|X - \widehat{X}^{\alpha_n}|^{r+\nu} \ge \mathbb{E}\big((|X| - \rho_n)^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\big) \ge \mathbb{E}\big(|X|^{r+\nu}(1 - 1/u)^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\big).$$

Inequality (4.11) follows from

$$n^{(r+\nu)/d}\mathbb{E}|X - \widehat{X}^{\alpha_n}|^{r+\nu} \ge \sup_{u>1}\big[(1 - 1/u)^{r+\nu} n^{(r+\nu)/d}\mathbb{E}\big(|X|^{r+\nu}\mathbf{1}_{\{|X|>u\rho_n\}}\big)\big]. \qquad \square$$

4.1.2. *Application to distributions with polynomial or hyper-exponential tails*

The next proposition is the counterpart of Proposition 3.1 devoted to the asymptotic lower bound.

Proposition 4.3. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ be an $\mathbb{R}^d$-valued random variable having an unbounded support.*

(a) Polynomial tail. *Set*

$$\zeta_\star = \inf\Big\{\zeta > 0, \forall \nu \in [0, \nu_X^\star), \liminf_{\xi\to+\infty} \xi^{\zeta - r - \nu}\bar{F}_{r+\nu}(\xi) > 0\Big\} \in [r + \nu_X^\star, +\infty]. \tag{4.13}$$

Then

$$\liminf_n \frac{\log \rho_n}{\log n} \ge \frac{1}{\zeta_\star - r - \nu_X^\star}\,\frac{r + \nu_X^\star}{d}. \tag{4.14}$$

(b) Hyper-exponential tail. *Set*

$$\theta_\star = \inf\Big\{\theta > 0, \liminf_{\xi\to+\infty} \mathrm{e}^{\theta\xi^\kappa}\mathbb{P}(|X| > \xi) > 0\Big\} \in [0, +\infty]. \tag{4.15}$$

Then, $\theta^\star \le \theta_\star$ *and*

$$\liminf_n \frac{\rho_n}{(\log n)^{1/\kappa}} \ge \left(\frac{r+\nu_X^\star}{d\theta_\star}\right)^{1/\kappa}. \tag{4.16}$$

Proof. (a) Let $\zeta \in (0, \zeta_\star)$. For every $\nu \in (0, \nu_X^\star)$, there exists a positive real constant C_ν such that $\bar{F}_{r+\nu}(\xi) \ge C_\nu \xi^{-\zeta+r+\nu}$ for large enough ξ. Setting $\psi_{r,\nu}(y) = \frac{y}{\zeta-r-\nu}$ yields $\psi_{r,\nu}(-\log \bar{F}_{r+\nu}(\xi)) \le \log\xi + \mathrm{o}(\log\xi)$. It follows from Theorem 4.1(a) that

$$\liminf_n \frac{\log \rho_n}{\log n} \ge \frac{1}{\zeta-r-\nu}\frac{r+\nu}{d}.$$

Letting ν and ζ go to $\nu_X^\star$ and $\zeta_\star$ yields the announced result.

(b) Let $\theta \in (\theta_\star, +\infty)$. Then, there exists a positive real constant C such that $\bar{F}(\xi) \ge C\mathrm{e}^{-\theta\xi^\kappa}$ for large enough x. Therefore $-\log \bar{F}(\xi) \le \theta\xi^\kappa(1-\xi^{-\kappa}\log(C))$ so that, by setting $\psi_\theta(y) = (y/\theta)^{1/\kappa}$, we have

$$\psi_\theta(-\log \bar{F}(\xi)) \le \xi + \mathrm{o}(\xi).$$

It follows from Theorem 4.1(b) that

$$\liminf_n \frac{\rho_n}{(\log n)^{1/\kappa}} \ge \left(\frac{r+\nu_X^\star}{\theta d}\right)^{1/\kappa}.$$

Letting θ go to $\theta_\star$ completes the proof. Finally, the inequality between $\theta_\star$ and $\theta^\star$ is an easy consequence of the fact that $\bar{F}_r(\xi) \ge \xi^r \bar{F}(\xi)$. □

Now we give explicit bounds and rates for several families of distribution tails (which include most usual distributions). To do so, we combine asymptotic upper bound results from Section 3.2 with asymptotic lower bound results obtained in this section. The results below are fully explicit in that we make no *a priori assumptions* on $\nu_X^\star$.

Corollary 4.1. *Let* $r > 0$ *and let* $X \in L^{r+}(\mathbb{P})$ *be an* $\mathbb{R}^d$*-valued random variable*, *with probability density* f, *having an unbounded convex support*.

(a) Polynomial tail. *If there exists* $c' \ge c > r+d$ *such that*

$$0 < \liminf_{|x|\to+\infty} |x|^{c'} f(x) \quad \textit{and} \quad \limsup_{|x|\to+\infty} |x|^{c} f(x) < +\infty,$$

then f *satisfies* (4.4),

$$\begin{aligned} &d\left(1-\frac{d+r}{c'}\right) - (r+d)\left(1-\frac{c}{c'}\right) \le \nu_X^\star \le d\left(1-\frac{d+r}{c'}\right), \\ &c-d \le \zeta^\star, \qquad \zeta_\star \le c'-d \end{aligned} \tag{4.17}$$

and

$$\frac{1}{c'-r-d}\left(1+\frac{r}{d}\right) \le \liminf_n \frac{\log \rho_n}{\log n} \le \limsup_n \frac{\log \rho_n}{\log n} \le \frac{1}{c-r-d}\left(1+\frac{r}{d}\right).$$

Finally, if $c = c'$, then

$$\nu_X^\star = d\left(1-\frac{d+r}{c'}\right), \qquad \zeta_\star = \zeta^\star = c-d \quad \textit{and} \quad \lim_n \frac{\log \rho_n}{\log n} = \frac{1}{c-r-d}\left(1+\frac{r}{d}\right). \tag{4.18}$$

(b) Hyper-exponential tail. *If there exists $\kappa > 0$ such that*

$$\lim_{|x|\to+\infty} \frac{\log f(x)}{|x|^\kappa} = -\vartheta \in (-\infty, 0), \tag{4.19}$$

then

$$\nu_X^\star = d \quad \textit{and} \quad \theta_\star = \theta^\star = \vartheta$$

so that

$$\frac{1}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa} \le \liminf_n \frac{\rho_n}{(\log n)^{1/\kappa}} \le \limsup_n \frac{\rho_n}{(\log n)^{1/\kappa}} \le \frac{2}{\vartheta^{1/\kappa}}\left(1+\frac{r}{d}\right)^{1/\kappa}. \tag{4.20}$$

When $d = 1$, $r \ge 1$, then the following sharp rate holds

$$\lim_n \frac{\rho_n}{(\log n)^{1/\kappa}} = \left(\frac{r+1}{\vartheta}\right)^{1/\kappa}. \tag{4.21}$$

Remark. When $d = 1$, a one-sided result follows by considering "$x \to +\infty$" instead of "$|x| \to \infty$".

Proof of Corollary 4.1. (a) First we need to check that f satisfies the control criterion (4.4) from Proposition 4.2: Let A, A' and B be such that $A'|x|^{-c'} \le f(x) \le A|x|^{-c}$ for every $x \in \mathbb{R}^d$, $|x| \ge B$. Then, if $\eta \in (0, 1)$, one checks that the criterion is satisfied with $M = \frac{B}{1-\eta}$, $K = \frac{A'}{A^{c'/c}}(1+\eta)^{-c'}$ and $\varepsilon = \frac{c'-c}{c} \ge 0$.

Using that $A'|x|^{-c'} \le f(x)$ and that f is a probability density (so that f^a is locally integrable if $a \in (0, 1]$) yields by checking (4.1) the upper bound for $\nu_X^\star$. Checking now the integral criterion (4.5) yields the lower bound.

The lower bound for $\zeta^\star$ is established in Corollary 3.1. The upper-bound is obtained by similar computations that show that, if $\zeta > c' - d$, then for ξ large enough, $\xi^{\zeta-r}\bar{F}_r(\xi) \ge AdV_d\xi^{\zeta-(c'-d)}$ for some real constant $A > 0$. This shows that $\zeta^\star \le c' - d$. The bounds for $\zeta_\star$ are obtained by similar computations.

As concerns the lower bound for the radius, one concludes by plugging all these estimates into (4.14). Combining this with Corollary 3.1(a) completes this part of the proof.

(b) First we need to check that f satisfies the control criterion (4.4). We know from assumption (4.19) that for every $\bar{\eta} \in (0, \vartheta)$, there exists $B_{\bar{\eta}} > 0$ such that $e^{-(\vartheta+\bar{\eta})|x|^\kappa} \le f(x) \le e^{(-\vartheta+\bar{\eta})|x|^\kappa}$, as soon as $|x| \geq B_{\bar{\eta}}$. Then, one shows that the criterion is satisfied with $M = \frac{B_{\bar{\eta}}}{1-\eta}$, $K = 1$, $\varepsilon = \frac{\vartheta+\bar{\eta}}{\vartheta-\bar{\eta}}(1+\eta)^\kappa - 1$. Then, one checks that $\nu_X^\star \geq d - (r+d)\frac{\varepsilon}{1+\varepsilon}$ since $\int_{\{|x|\geq B\}} \exp(-\mu|x|^\kappa)\,\mathrm{d}x < +\infty$ for every B, $\mu > 0$. Letting η and $\bar{\eta} \to 0$ yields $\nu_X^\star = d$.

To compute $\theta_\star$, one first notes that, as soon as $\xi \geq B_{\bar{\eta}}$,

$$\begin{aligned}\mathbb{P}(|X| > \xi) &\geq d V_d \int_{\{u>\xi\}} \mathrm{e}^{-(\vartheta+\bar{\eta})u^\kappa} u^{d-1}\,\mathrm{d}u \\ &= \mathrm{O}\big(\mathrm{e}^{-(\vartheta+\bar{\eta})\xi^\kappa} \xi^{d-\kappa}\big),\end{aligned}$$

where the equality follows by a standard argument based on an integration by parts and a comparison theorem for integrals. As a consequence $\theta_\star \leq \vartheta + \bar{\eta}$, which finally implies $\theta_\star \leq \vartheta$. Combining this with Corollary 3.1(b) and Proposition 4.3(b) yields $\theta_\star = \theta^\star = \vartheta$. □

Proof of Theorem 1.2. Claim (a) follows from the former Corollary 4.1(a) once it is noted that for every $\varepsilon \in (0, c)$, $\liminf_{|x|\to\infty} |x|^{c+\varepsilon} f(x) > 0$ and $\limsup_{|x|\to\infty} |x|^{c-\varepsilon} f(x) < +\infty$. Claim (b) directly follows from (b) in the above corollary. □

4.2. An alternative approach based on random quantization

Random quantization is another tool to compute the lower estimate of the maximal radius sequence of a random vector X with distribution P. It makes a connection between ρ_n and the maximum of an i.i.d. sequence of random variables with distribution P.

Theorem 4.3. *Let $r > 0$ and let $X \in L^{r+}(\mathbb{P})$ be a random variable taking values in $\mathbb{R}^d$ with an absolutely continuous distribution P. Assume $(\alpha_n)_{n\geq 1}$ is a sequence of $L^r(P)$-optimal n-quantizers. Let $(X_k)_{k\geq 1}$ be an i.i.d. sequence of $\mathbb{R}^d$-valued copies of X. For every $\nu \in [0, \nu_X^\star)$ such that $r + \nu \geq 1$, there exists a real constant $C_{r,\nu} \in (0, \infty)$ such that*

$$\liminf_n \Big(\rho_n - \mathbb{E}\Big(\max_{k\leq[n^{(r+\nu)/d}]} |X_k|\Big)\Big) \geq -C_{r,\nu}. \tag{4.22}$$

Proof. Let $\nu \in [0, \nu_X^\star)$ and set $\widehat{X}_k^{\alpha_n} = \sum_{a\in\alpha_n} a\mathbf{1}_{\{X_k\in C_a(\alpha_n)\}}$. We have, for integer $m \geq 1$,

$$\begin{aligned}\rho_n &\geq \max_{k\leq m} |\widehat{X}_k^{\alpha_n}| \\ &\geq \sum_{k=1}^m \max_{l\leq m} |\widehat{X}_l^{\alpha_n}| \mathbf{1}_{\{|X_k|>\max\{|X_i|, i\in\{1,\ldots,m\}, i\neq k\}\}} \\ &\geq \sum_{k=1}^m |\widehat{X}_k^{\alpha_n}| \mathbf{1}_{\{|X_k|>\max_{i\neq k}|X_i|\}} \\ &\geq \sum_{k=1}^m (|X_k| - |X_k - \widehat{X}_k^{\alpha_n}|)\mathbf{1}_{\{|X_k|>\max_{i\neq k}|X_i|\}}.\end{aligned}$$

Taking the expectation of both sides of the previous inequality yields

$$\rho_n \ge \mathbb{E}\max_{k\le m}|X_k| - \sum_{k=1}^{m}\mathbb{E}\big(|X_k - \widehat{X}_k^{\alpha_n}|\mathbf{1}_{\{|X_k|>\max_{i\neq k}|X_i|\}}\big).$$

Furthermore, $\forall k \ge 1$, $|X_k - \widehat{X}_k^{\alpha_n}|\mathbf{1}_{\{|X_k|>\max_{i\neq k}|X_i|\}}$ and $|X_1 - \widehat{X}_1^{\alpha_n}|\mathbf{1}_{\{|X_1|>\max_{i\neq 1}|X_i|\}}$ have the same distribution. Hence,

$$\begin{aligned}\rho_n &\ge \mathbb{E}\max_{k\le m}|X_k| - m\mathbb{E}\big(|X_1 - \widehat{X}_1^{\alpha_n}|\mathbf{1}_{\{|X_1|>\max_{i\neq 1}|X_i|\}}\big)\\ &\ge \mathbb{E}\max_{k\le m}|X_k| - m\|X_1 - \widehat{X}_1^{\alpha_n}\|_{r+\nu}\Big(\mathbb{P}\Big(|X_1| > \max_{i\neq 1}|X_i|\Big)\Big)^{1-1/(r+\nu)}\end{aligned}$$

owing to the Hölder inequality. Since the events $\{|X_k| > \max_{i\neq k}|X_i|\}$, $k = 1,\ldots,m$, are pairwise disjoint with the same probability, we have $\mathbb{P}(|X_1| > \max_{i\neq 1}|X_i|) \le \frac{1}{m}$. Finally,

$$\rho_n \ge \mathbb{E}\max_{k\le m}|X_k| - m^{1/(r+\nu)}\|X - \widehat{X}^{\alpha_n}\|_{r+\nu}.$$

It follows, by setting $m = [n^{(r+\nu)/d}]$, that

$$\liminf_n\Big(\rho_n - \mathbb{E}\Big(\max_{k\le[n^{(r+\nu)/d}]}|X_k|\Big)\Big) \ge -\limsup_n n^{1/d}\|X - \widehat{X}^{\alpha_n}\|_{r+\nu}.$$

The upper limit on the right-hand side is finite since X has an $(r, r+\nu)$-distribution. □

Example 4.1 (Exponential distribution). Let $r > 0$ and let X be an exponentially distributed random variable with parameter $\lambda > 0$. If $(\alpha_n)_{n\ge 1}$ is an L^r-optimal sequence of n-quantizers for X, then Theorem 4.3 implies

$$\liminf_n \frac{\rho_n}{\log n} \ge \frac{r+1}{\lambda}, \tag{4.23}$$

which corresponds to the sharp rates given by (2.6) and (4.18), respectively.

Indeed, let $\nu \in (0, \nu_X^\star)$ and let $(X_i)_{i=1,\ldots,[n^{r+\nu}]}$, be an i.i.d. exponentially distributed sequence of random variables with parameter λ. We have for every $u \ge 0$,

$$\mathbb{P}\Big(\max_{i\le[n^{r+\nu}]} X_i \ge u\Big) = 1 - \mathbb{P}(X \le u)^{[n^{r+\nu}]} = 1 - F(u)^{[n^{r+\nu}]},$$

where F is the distribution function of X (we will denote by f its density function). Then

$$\begin{aligned}\mathbb{E}\Big(\max_{i\le[n^{r+\nu}]} X_i\Big) &= \int_0^{+\infty}\mathbb{P}\Big(\max_{i\le[n^{r+\nu}]} X_i \ge u\Big)\,\mathrm{d}u = \int_0^{+\infty}\Big(1-(1-\mathrm{e}^{-\lambda u})^{[n^{r+\nu}]}\Big)\,\mathrm{d}u\\ &= \int_0^{+\infty}\big(1 + F(u) + \cdots + F(u)^{[n^{r+\nu}]-1}\big)\frac{f(u)}{\lambda}\,\mathrm{d}u\end{aligned}$$

$$= \frac{1}{\lambda}\left(1 + \frac{1}{2} + \cdots + \frac{1}{[n^{r+\nu}]}\right)$$
$$\geq \frac{1}{\lambda}\log(1 + [n^{r+\nu}]) \geq \frac{r+\nu}{\lambda}\log n.$$

Consequently, it follows from the super-additivity of the liminf that for every $\nu \in (0,1)$,

$$\liminf_n \frac{\rho_n}{\log n} \geq \liminf_n \frac{\rho_n - \mathbb{E}(\max_{i\leq[n^{r+\nu}]} X_i)}{\log n} + \liminf_n \frac{\mathbb{E}(\max_{i\leq[n^{r+\nu}]} X_i)}{\log n} \geq \frac{r+\nu}{\lambda}.$$

The result follows by letting ν go to $\nu_X^\star = 1$.

In fact, one may easily extend this example to a more general framework, although, overall, the connection made in Theorem 4.3 seems less straightforward in terms of deriving explicit asymptotic lower bounds than the former approach based on more geometric arguments.

Example 4.2 (Radial distribution with exponential tails). Let X be an $\mathbb{R}^d$-valued random vector with an unbounded support having an absolutely continuous distribution with a radial probability density $f(x) = g(|x|_S)$ with respect to an Euclidean norm $|\cdot|_S$ so that $\bar{F}(\xi) = K_{d,S}\int_\xi^{+\infty} u^{d-1}g(u)\,\mathrm{d}u$, $\xi > 0$, with $K_{d,S} = dV_d(\det(S))^{-1/2} > 0$. Assume that $\bar{F}(\xi) \geq cf(\xi)$ for $\xi \geq A > 0$ for some real constant $c > 0$. Then

$$\liminf_n \frac{\rho_n}{\log n} \geq c(r + \nu_X^\star).$$

Example 4.3 (Pareto distribution). Let X be a random variable having a Pareto distribution with index $\gamma > 0$. If $(\alpha_n)_{n\geq1}$ is an asymptotically L^r-optimal sequence of n-quantizers for X, $r \in (0,\gamma)$, then Theorem 4.3 yields

$$\liminf_n \frac{\log \rho_n}{\log n} \geq \frac{r+1}{\gamma+1},$$

which is not the sharp rate given by (2.7).

Notice that if $\gamma > r$, then $X \in L^{r+\eta}(\mathbb{P})$ for $\eta \in (0, \gamma - r)$. Now, to prove this result, let $\nu \in (0, \nu_X^\star)$ and let $(X_i)_{i\geq1}$ be an i.i.d. sequence of Pareto-distributed random variables (with index γ). We have

$$\forall m \geq 1, \forall u \geq 1 \qquad \mathbb{P}\Big(\max_{i\leq m} X_i \leq u\Big) = (1 - u^{-\gamma})^m.$$

Then, the density function of $\max_{1\leq i\leq m} X_i$ is $m\gamma u^{-(\gamma+1)}(1-u^{-\gamma})^{m-1}$. Hence,

$$\mathbb{E}\Big(\max_{1\leq i\leq m} X_i\Big) = m\gamma\int_1^{+\infty} x^{-\gamma}(1 - x^{-\gamma})^{m-1}\,\mathrm{d}x = mB\left(1 - \frac{1}{\gamma}, m\right)$$
$$= \frac{\Gamma(1-1/\gamma)\Gamma(m+1)}{\Gamma(m+1-1/\gamma)} \sim \Gamma\left(1 - \frac{1}{\gamma}\right)m^{1/\gamma} \qquad \text{as } m \to +\infty,$$

where we used Stirling's formula for the last statement ($B(\cdot,\cdot)$ denotes the beta function of the first kind). We finally set $m = [n^{r+\nu}]$ to get

$$\mathbb{E}\Big(\max_{1\le i\le [n^{r+\nu}]} X_i\Big) \sim \Gamma\Big(1-\frac{1}{\gamma}\Big) n^{(r+\nu)/\gamma}.$$

It follows from (4.22) that for every $\varepsilon \in (0,1)$, $\rho_n - (1-\varepsilon)\Gamma(1-\frac{1}{\gamma})n^{(r+\nu)/\gamma} \ge -(\mathrm{C}_{r,\nu}+\varepsilon)$. Dividing both sides of the inequality by $n^{(r+\nu)/\gamma}$ and taking the logarithm yields

$$\log\rho_n - \frac{r+\nu}{\gamma}\log n \ge \log\bigg((1-\varepsilon)\Gamma\Big(1-\frac{1}{\gamma}\Big) - (\varepsilon + \mathrm{C}_{r,\nu})n^{-(r+\nu)/\gamma}\bigg).$$

Consequently $\liminf_{n\to+\infty}\frac{\log\rho_n}{\log n} \ge \frac{r+\nu}{\gamma}$ for every $\nu \in (0,\nu_X^\star)$. One concludes by letting ν go to $\nu_X^\star = \frac{\gamma-r}{\gamma+1}$.

Comment. *Let ϕ be the inverse (if any) function of $-\log\bar F$. Notice that in both examples above we have*

$$\lim_n \frac{\mathbb{E}(\max_{k\le [n^{r+\nu_X^\star}]}|X_k|)}{\phi((r+\nu_X^\star)\log n)} = 1, \tag{4.24}$$

which, for distributions with hyper-exponential tails, leads to the asymptotic lower bound (4.9) for the sequence $(\rho_n)_{n\ge1}$. As concerns Pareto distribution, using the approximation (4.24) to compute the asymptotic lower estimate of the maximal radius sequence induces the loss of the "$-r$" term in the exact asymptotics. To recover this remaining term we have simply to consider the inverse function of $-\log\bar F_{r+\nu_X^\star}$ (as done in the previous section) instead of $-\log\bar F$, and the random quantization approach clearly does not allow us to do so.

4.2.1. *A conjecture about the sharp rate*

The previous results related to distributions with hyper-exponential tails strongly suggest the following conjecture: Suppose X is a distribution with hyper-exponential tail in the sense of statement (4.19). Then, for every $r > 0$ and $d \ge 1$,

$$\lim_n \frac{\rho_n}{(\log n)^{1/\kappa}} = \Big(\frac{r+d}{d\theta^\star}\Big)^{1/\kappa}.$$

This conjecture is proved for $d = 1$ and $r \ge 1$. To be satisfied for higher dimensions we need to prove that the geometric statement (3.9) of Lemma 3.1 holds true with $c_{r,d} = 1$ for every $r > 0$, $d \ge 1$.

Appendix

$\triangleright$ *Exponential distribution.* $\rho_n = \frac{r+1}{\lambda}\log n + \frac{C_r}{\lambda} + \mathrm{O}(\frac{1}{n})$, we use the following result (see [9]): If X is exponentially distributed with parameter $\lambda > 0$, then, for any $n \ge 1$, the L^r-optimal quantizer $\alpha_n = (\alpha_{n,1}, \dots, \alpha_{n,n})$ is unique and given by

$$\alpha_{n,k} = \frac{1}{\lambda}\left(\frac{a_n}{2} + \sum_{i=n+1-k}^{n-1} a_i\right), \qquad 1 \le k \le n, \tag{A.1}$$

where $(a_k)_{k\ge 1}$ is an $\mathbb{R}_+$-valued sequence recursively defined by the following implicit equation: $a_0 := +\infty$, $\phi_r(-a_{k+1}) := \phi_r(a_k)$, $k \ge 0$, with $\phi_r(x) := \int_0^{x/2} |u|^{r-1}\,\mathrm{sign}(u)\mathrm{e}^{-u}\,\mathrm{d}u$ (convention: $0^0 = 1$). Furthermore, the sequence $(a_k)_{k\ge 1}$ decreases to zero and for every $k \ge 1$, $a_k = \frac{r+1}{k}(1 + \frac{c_r}{k} + \mathrm{O}(\frac{1}{k^2}))$ for some positive real constant c_r. Then it follows that $\lambda\rho_n = \frac{a_n}{2} + \sum_{i=1}^{n-1} a_i$ so that

$$\lambda\rho_n = \frac{a_n}{2} + (r+1)\sum_{i=1}^{n-1}\frac{1}{i} + c_r\sum_{i=1}^{n-1}\frac{1}{i^2} + \sum_{i=1}^{n-1}\mathrm{O}(1/i^3) = (r+1)\log n + C_r + \mathrm{O}\left(\frac{1}{n}\right).$$

$\triangleright$ *Pareto distribution.* The proof is similar after noting that $\rho_n = \frac{1}{1+a_n}\prod_{i=1}^{n-1}(1+a_i)$, where $(a_n)_{n\ge 1}$ is an $\mathbb{R}_+$-valued sequence, decreasing to zero and satisfying, for every $n \ge 1$, $a_n = \frac{r+1}{(\gamma-r)n}(1 + c_r/n + \mathrm{O}(1/n^2))$ for some real constant c_r. Hence, if $i_0 := \max\{i \mid |a_i| \ge 1\}$,

$$\log(\rho_n) = -\log(1+a_n) + C_{i_0} + \sum_{i=i_0+1}^{n-1}\left(a_i - \frac{a_i^2}{2} + \mathrm{O}(a_i^3)\right) = \frac{r+1}{\gamma - r}\log n + C_r + \mathrm{O}\left(\frac{1}{n}\right),$$

where we used that $\sum_{i=1}^{\infty} a_i^2 < \infty$ and $\sum_{i=1}^{\infty}\mathrm{O}(a_i^3) < \infty$.

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