

Limit experiments of GARCH

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GARCH is one of the most prominent nonlinear time series models, both widely applied and thoroughly studied. Recently, it has been shown that the COGARCH model (which was introduced a few years ago by Klüppelberg, Lindner and Maller) and Nelson's diffusion limit are the only functional continuous-time limits of GARCH in distribution. In contrast to Nelson's diffusion limit, COGARCH reproduces most of the *stylized facts* of financial time series. Since it has been proven that Nelson's diffusion is not asymptotically equivalent to GARCH in deficiency, in the present paper, we investigate the relation between GARCH and COGARCH in Le Cam's framework of statistical equivalence. We show that GARCH converges generically to COGARCH, even in deficiency, provided that the volatility processes are observed. Hence, from a theoretical point of view, COGARCH can indeed be considered as a continuous-time equivalent to GARCH. Otherwise, when the observations are incomplete, GARCH still has a limiting experiment, which we call MCOGARCH, which is not equivalent, but nevertheless quite similar, to COGARCH. In the COGARCH model, the jump times can be more random than for the MCOGARCH, a fact practitioners may see as an advantage of COGARCH.

Keywords: COGARCH; Le Cam's deficiency distance; random thinning; statistical equivalence; time series

1. Introduction

Since the seminal papers of Engle [10] and Bollerslev [4] the discrete-time GARCH methodology has become a widely applied tool in the modeling of heteroscedasticity in financial times series. On the other hand, continuous-time models are very useful, for instance, in option pricing, as shown by Black and Scholes [3] and Merton [21], in the analysis of tick-by-tick data and for modeling irregularly spaced time series.

In the 1990s, researchers tried to bridge the gap between continuous and discrete time. Nelson [23] showed that an appropriately parametrized GARCH can be seen as a discrete-time approximation of a bivariate diffusion model on an approximating time grid. However, this diffusion model does not capture most of the so-called *stylized facts* reflecting empirical findings in financial time series: for example, volatility exhibits heavy tails, jumps upward and clusters on high levels. To overcome the shortcomings of the diffusion model, Klüppelberg *et al.* [17] introduced a new continuous-time GARCH model which they called COGARCH. In contrast to the bivariate diffusion, this model exhibits many of the stylized facts. We refer the reader to Fasen *et al.* [12] for an extensive discussion of the stylized facts and the various competing volatility models proposed in the literature.

Recently, Kallsen and Vesenmayer [16] and Maller *et al.* [20] have identified COGARCH as a functional limit of GARCH in distribution. Most notably, Kallsen and Vesenmayer [16] have argued that Nelson's diffusion and COGARCH are the only possible limits of GARCH in distribution in a semimartingale setting.

The passage from discrete to continuous time has an obviously appealing practical purpose: we can estimate the underlying continuous-time model parameters by a time series formulation and plug them into the continuous-time limit for other purposes. As argued by Wang [28], such a passage is, in general, only justified if the corresponding statistical experiments converge in Le Cam's framework of deficiency (see Le Cam [18], Le Cam and Young [19] and Strasser [27]).

In particular, Wang [28] (see also Brown *et al.* [6]) showed, assuming independent Gaussian innovations, that Nelson's diffusion approximation of GARCH is not valid in deficiency: the innovations encounter the two models in an intrinsically different way. Whereas GARCH is driven by one-dimensional innovations, its diffusion limit is driven by planar Brownian motion.

In contrast to Nelson's approximation, COGARCH is driven by a Lévy process of only one dimension, thereby mimicking one of the key features of GARCH. Naturally, the following questions arise: Are the approximations of COGARCH by GARCH, as proposed by Kallsen and Vesenmayer [16] and Maller *et al.* [20], also valid in deficiency? Does the limiting model depend on the underlying sampling scheme?

Dealing with Le Cam's distance in deficiency is a challenging task. In particular, asymptotic equivalence results for dependent data are very scarce; see Dalalyan and Reiß [8] for an overview. Further obstacles arise from the intrinsic heteroscedasticity of GARCH. Therefore, in this paper, we restrict ourselves to compound Poisson processes as driving Lévy process and assume that the innovations are randomly thinned. This approximation scheme also occurs in the papers [16] and [20]. Random thinning is a standard limiting procedure in many other areas of probability theory and statistics. In particular, we mention the peak-over-threshold method in extreme value theory (see Remark 2.1(ii)). In contrast to our approximation scheme, most papers on statistical equivalence deal with aggregated innovations where the experiments are compared to Gaussian shift experiments (see Brown and Low [5], Nussbaum [24], Grama and Neumann [13] and Carter [7] and references therein; see also Milstein and Nussbaum [22] with potential applications to time series analysis). We point out once again that for the GARCH, aggregated innovations lead to the diffusion limit investigated by Nelson [23] and Wang [28].

The paper is organized as follows. Section 2 contains our main results. We introduce the experiments and sampling schemes in Section 2.1. In Section 2.2, we construct a limiting experiment for randomly thinned GARCH with unobserved conditional variances. As shown in Section 2.3, using both theoretical and numerical methods, this experiment is generically not equivalent to COGARCH. However, if the conditional variances are observable in full, then all experiments are generically (asymptotically) equivalent to COGARCH – this is shown in Section 2.4. We conclude in Section 3. In Sections 4–7 we give the proofs of all theorems and propositions from Section 2. Section 4 contains the proof of Theorem 2.1, Section 5 the proof of Theorem 2.2 and Section 6 the proof of Theorem 2.3. The proofs of all propositions in Section 2.4 are reported in Section 7. In the [Appendix](#), we review some of the basic notions of Le Cam's convergence in deficiency.

2. Main results

2.1. GARCH-type experiments in discrete and continuous time

For all $n \in \mathbb{N}$ we consider an n -dimensional vector $Z_n = (Z_{n,k})_{1 \leq k \leq n}$ with distribution

$$\mathcal{L}(Z_n) = ((1 - p_n)\varepsilon_0 + p_n Q_n)^{\otimes n}, \quad (2.1)$$

where, for all $n \in \mathbb{N}$, $p_n \in (0, 1)$ and Q_n is a probability measure on the Borel field $\mathcal{B}(\mathbb{R})$. Here, ε_0 denotes the Dirac measure with total mass in zero.

The parameter p_n modulates our random thinning. In accordance with the law of rare events, we assume that the following limit exists in $(0, \infty)$:

$$\gamma = \lim_{n \rightarrow \infty} np_n \in (0, \infty). \quad (2.2)$$

In the sequel, we will encounter several GARCH-type processes, all indexed by $\theta \in [0, \infty)^4$. In discrete time, processes will be further indexed by $n \in \mathbb{N}$ and a suitable parametrization. Throughout this paper, a parametrization is a pair $(\Theta, (H_n)_{n \in \mathbb{N}})$, where Θ is a non-empty subset of $[0, \infty)^4$, and for all $n \in \mathbb{N}$, H_n is a mapping $H_n = (h_{0,n}, \beta_n, \alpha_n, \lambda_n): \Theta \rightarrow [0, \infty)^4$. Here, h_0 ($h_{0,n}(\theta)$) denotes the unknown initial value of the volatility h_0 , which is treated as an additional unknown parameter in this paper. For the corresponding continuous-time limits, α is the mean reversion parameter of the volatility processes and β/α the asymptotically stable fixed point of the (unperturbed) volatility SDE (ODE). λ is a scaling parameter for the corresponding jumps of the volatility processes.

For a parametrization $(\Theta, (H_n)_{n \in \mathbb{N}})$, we consider the sequence of partial sums corresponding to a *randomly thinned* GARCH model indexed by $\theta \in \Theta$ and $n \in \mathbb{N}$, defined by

$$\begin{aligned} G_n(k) &= G_n(k-1) + h_n^{1/2}(k-1)Z_{n,k}, & G_n(0) &= 0, \\ h_n(k) &= \beta_n(\theta) + \alpha_n(\theta)h_n(k-1) + \lambda_n(\theta)h_n(k-1)Z_{n,k}^2, & (2.3) \\ h_n(0) &= h_{0,n}(\theta), & 1 \leq k \leq n, \theta \in \Theta, \end{aligned}$$

where $H_n(\theta) = (h_{0,n}(\theta), \beta_n(\theta), \alpha_n(\theta), \lambda_n(\theta))$ for all $\theta \in \Theta$. Note that this specification of a GARCH does not quite follow the traditional one, but enumerating the indices generates the same processes. Also, observe that the definition of (G_n, h_n) in (2.3) depends on the choice of $(\Theta, (H_n)_{n \in \mathbb{N}})$.

Provided that Q_n converges weakly to some probability measure Q , the limit in (2.2) sets up convergence in distribution of $\sum_{k=1}^{[nt]} Z_{n,k}$ to a compound Poisson process with rate γ and jump distribution Q as $n \rightarrow \infty$. For a choice of $(\Theta, (H_n)_{n \in \mathbb{N}})$ it is thus natural to ask whether the limit of $(G_n([nt]), h_n([nt]))_{0 \leq t \leq 1}$ in distribution exists along $H_n(\theta)$ as $n \rightarrow \infty$ for fixed $\theta \in \Theta$. In [16] and [20], such parametrizations have been successfully constructed. Moreover, the corresponding continuous-time limit equals COGARCH driven by a compound Poisson process.

COGARCH is a process $(G, h) = (G(t), h(t))_{0 \leq t \leq 1}$ that is indexed by $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$ and determined as the unique pathwise solution of the following system of integral

equations:

$$\begin{aligned} G(t) &= \int_{[0,t] \times \mathbb{R}} h^{1/2}(s-)zN(ds, dz), \\ h(t) &= h_0 + \int_{[0,t]} \beta - \alpha h(s-) ds + \lambda \int_{[0,t] \times \mathbb{R}} h(s-)z^2N(ds, dz), \end{aligned} \tag{2.4}$$

where N is a Poisson point measure on $[0, 1] \times \mathbb{R}$ with intensity $\gamma \ell \otimes Q$.

In the sequel, we restrict our analysis to the two following sampling schemes:

- *Incomplete observations*: only G and G_n ($n \in \mathbb{N}$) are observable in full, whereas the corresponding volatility processes, h and h_n ($n \in \mathbb{N}$), are unobservable.
- *Complete observations*: both processes (G, h) and (G_n, h_n) are observable in full.

We will deal with these two sampling schemes in Sections 2.2–2.3 and Section 2.4, respectively. Not surprisingly, a simpler theory is in place in the case of complete observations. In a more realistic scenario, where observations of the volatility processes are not available, results are more difficult due to the nonlinearity of (CO)GARCH.

Throughout the paper, the space of right-continuous functions $g : [0, 1] \rightarrow \mathbb{R}^d$ with left limits on $[0, 1]$ is denoted by D_d . We endow D_d with the σ -algebra \mathcal{D}_d , generated by point evaluations (see Billingsley [2]). Furthermore, let \mathbb{M}_d be the space of all non-negative point measures on $[0, 1] \times \mathbb{R}^d$ with finite support. We equip this space with the σ -algebra \mathcal{M}_d generated by the point evaluations $A \mapsto \mu(A)$, $A \in \mathcal{B}([0, 1] \times \mathbb{R}^d)$, $\mu \in \mathbb{M}_0$ (see Reiss [26], pages 5–6).

The trace of the Borel field in $\overline{\mathbb{R}}^d = (\mathbb{R} \cup \{-\infty, \infty\})^d$ with respect to $A \subseteq \overline{\mathbb{R}}^d$ is denoted by $\mathcal{B}(A)$. The Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and the Dirac measure with total mass in x are denoted by ℓ and ϵ_x , respectively. If (E, \mathcal{A}) is a measurable space and X is a random element taking values in (E, \mathcal{A}) , then its distribution is denoted by $\mathcal{L}(X)$. Whenever this distribution depends on a parameter θ , we employ the notation $\mathcal{L}_\theta(X)$. If (E_i, \mathcal{A}_i) , $i = 1, 2$, are measurable spaces and $X : E_1 \rightarrow E_2$ is $\mathcal{A}_1/\mathcal{A}_2$ measurable, then μ^X denotes the image of a measure μ under X .

We refer to the Appendix and [27] for unexplained notation relating to convergence in deficiency.

2.2. Limit experiments of GARCH (incomplete observations)

In this subsection, we assume that the volatility processes are unobservable. To pursue our program, we introduce another class of processes. Let $(\widehat{G}, \widehat{h}) = (\widehat{G}(t), \widehat{h}(t))_{0 \leq t \leq 1}$ be the unique pathwise solution of the following system of integral equations:

$$\begin{aligned} \widehat{G}(t) &= \int_{[0,t] \times \mathbb{R}} \widehat{h}^{1/2}(s-)zN(ds, dz), \\ \widehat{h}(t) &= h_0 + \int_{[0,t]} \beta - \alpha \widehat{h}(s-) dT_N(s) + \lambda \int_{[0,t] \times \mathbb{R}} \widehat{h}(s-)z^2N(ds, dz), \end{aligned} \tag{2.5}$$

where $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$. Here, $T : \mathbb{M}_1 \rightarrow D_1$, $\sigma \mapsto T_\sigma$ is defined as follows: If, for some $m \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_m < 1$ and $x_1, \dots, x_m \in \mathbb{R}$, $\sigma \in \mathbb{M}$ admits a representation of

the form $\sigma = \sum_{k=1}^m \varepsilon_{(t_k, x_k)}$, where $0 = t_0 < t_1 < \dots < t_m < 1$, then we set

$$\begin{aligned} T_\sigma(t) &= \frac{t - t_k}{m(t_k - t_{k-1})} + \frac{k}{m}, & t \in [t_{k-1}, t_k), 1 \leq k \leq m, \\ T_\sigma(t) &= \frac{t - t_m}{m(t_m - t_{m-1})} + 1, & t \in [t_m, 1]. \end{aligned} \quad (2.6)$$

If such a representation does not exist, then we set $T_\sigma(t) = t$ for all $t \in [0, 1]$.

We call $(\widehat{G}, \widehat{h})$ the MCOGARCH, an acronym referring to *modified* COGARCH. To illustrate the difference between COGARCH and MCOGARCH, we next consider a simpler representation of \widehat{G} (we will return to (2.5) in our analysis in Section 2.4).

To this end, let $\nu = (\nu(t))_{0 \leq t \leq 1}$ be a Poisson process with rate γ and $(Z_k)_{k \in \mathbb{N}}$ be a sequence of independent random variables, independent of ν . By solving the integral equations for \widehat{h} in (2.5), we observe that

$$\mathcal{L}_\theta(\widehat{G}) = \mathcal{L}_\theta \left(\sum_{k=1}^{\nu(\cdot)} \widehat{h}_{\nu(1), k, \theta}^{1/2} Z_k \right), \quad (2.7)$$

where, for $k, m \in \mathbb{N}$, $k \geq 2$, we set

$$\begin{aligned} \widehat{h}_{m, k, \theta} &= \frac{\beta}{\alpha} (1 - e^{-\alpha/m}) + e^{-\alpha/m} \widehat{h}_{m, k-1, \theta} [1 + \lambda Z_{k-1}^2], \\ \widehat{h}_{m, 1, \theta} &= \frac{\beta}{\alpha} (1 - e^{-\alpha/m}) + e^{-\alpha/m} h_0 \end{aligned} \quad (2.8)$$

for $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, $\alpha > 0$, with the convention that $\sum_{\emptyset} = 0$. Here, we extend the definition of $\widehat{h}_{m, k, \theta}$ to $\theta = (h_0, \beta, 0, \lambda) \in [0, \infty)^4$ by taking $\alpha \downarrow 0$ in (2.8).

In view of (2.8), note that the magnitudes of the jumps of \widehat{G} (in space) depend on their multiplicities and the sizes of innovations, but not on their arrival times. This attribute is not shared by COGARCH. Therefore, it is, to some extent, justified to speak of \widehat{G} and G as experiments driven by two and three sources of randomness, respectively: the number of jumps, the innovations and the arrival times.

As no information about the volatility processes is assumed in this subsection, we consider the following experiment of MCOGARCH type:

$$\widehat{\mathcal{E}} = (D_1, \mathcal{D}_1, (\mathcal{L}_\theta(\widehat{G}))_{\theta \in [0, \infty)^4}). \quad (2.9)$$

For a parametrization $(\Theta, (H_n)_{n \in \mathbb{N}})$, we consider the corresponding GARCH experiments in discrete time,

$$\mathcal{E}_{n, H_n}(\Theta) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), (\mathcal{L}_\theta(G_n))_{\theta \in \Theta}), \quad n \in \mathbb{N}, \quad (2.10)$$

where, for $n \in \mathbb{N}$, $G_n = (G_n(k))_{1 \leq k \leq n}$ is defined by (2.3) via the parametrization $(\Theta, (H_n)_{n \in \mathbb{N}})$. We write $\mathcal{E}_{n, H_n} = \mathcal{E}_{n, H_n}(\Theta)$, provided that we have $\Theta = [0, \infty)^4$ in (2.10).

Next, we give a GARCH parametrization such that the randomly thinned GARCH converges strongly to the MCOGARCH experiment $\widehat{\mathcal{E}}$ in deficiency; therefore, we pick $\theta = (h_0, \beta, \alpha, \lambda) \in$

Θ and $n \in \mathbb{N}$. If $\alpha > 0$, then we set

$$\begin{aligned} h_{0,n}^{(0)}(\theta) &= h_0 e^{-\alpha/n} + \frac{\beta}{\alpha}(1 - e^{-\alpha/n}), & \beta_n^{(0)}(\theta) &= \frac{\beta}{\alpha}(1 - e^{-\alpha/n}), \\ \alpha_n^{(0)}(\theta) &= e^{-\alpha/n}, & \lambda_n^{(0)}(\theta) &= \lambda e^{-\alpha/n} \end{aligned} \quad (2.11)$$

and, otherwise, if $\alpha = 0$, then we set

$$h_{0,n}^{(0)}(\theta) = h_0 + \frac{\beta}{n}, \quad \beta_n^{(0)}(\theta) = \frac{\beta}{n}, \quad \alpha_n^{(0)}(\theta) = 1, \quad \lambda_n^{(0)}(\theta) = \lambda. \quad (2.12)$$

Let $([0, \infty)^4, (H_n^{(0)}))$ be the corresponding parametrization and $G_n^{(0)}$ the corresponding partial sum processes of GARCH in (2.3).

Although the parametrization in (2.11) and (2.12) is quite elaborate, we show that the corresponding GARCH experiment converges to the experiment of MCOGARCH-type with no restrictions on the limiting probability measure Q assumed (see Section 4 for a proof).

Theorem 2.1. *Let (2.2) be satisfied for some $\gamma \in (0, \infty)$ and $p_n \in (0, 1)$, $n \in \mathbb{N}$. If Q_n tends to a probability measure Q in total variation as $n \rightarrow \infty$, then $\mathcal{E}_{n, H_n^{(0)}}$ converges strongly to $\widehat{\mathcal{E}}$ in deficiency as $n \rightarrow \infty$.*

If Q is absolutely continuous with respect to the Lebesgue measure, then Theorem 2.1 partially extends to other GARCH parametrizations (see Section 5 for a proof of the following theorem).

Theorem 2.2. *Let (2.2) be satisfied for some $\gamma \in (0, \infty)$ and $p_n \in (0, 1)$, $n \in \mathbb{N}$. Suppose both that Q_n tends to a probability measure Q in total variation as $n \rightarrow \infty$ and that $Q \ll \ell$.*

Let $\Theta \neq \emptyset$ with compact closure $\overline{\Theta}$ in $(0, \infty) \times [0, \infty)^3$. For $n \in \mathbb{N}$, let $H_n = (h_{0,n}, \beta_n, \alpha_n, \lambda_n) : \Theta \rightarrow [0, \infty)^4$ be a GARCH parametrization and G_n the corresponding GARCH model in (2.3).

If there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that, for all $n \geq n_0$, both

$$\sup_{\theta=(h_0, \beta, \alpha, \lambda) \in \Theta} \max\{|h_{0,n}(\theta) - h_0|, |\lambda_n(\theta) - \lambda|\} \leq \frac{C}{n} \quad (2.13)$$

and

$$\sup_{\theta=(h_0, \beta, \alpha, \lambda) \in \Theta} \max\{|n\beta_n(\theta) - \beta|, |n(\alpha_n(\theta) - 1) + \alpha|\} \leq C, \quad (2.14)$$

then

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \|\mathcal{L}_\theta(G_n) - \mathcal{L}_\theta(G_n^{(0)})\| = 0 \quad (2.15)$$

and $\mathcal{E}_{n, H_n}(\Theta)$ converges strongly to $\widehat{\mathcal{E}}(\Theta)$ in deficiency as $n \rightarrow \infty$.

Remark 2.1. (i) Let $Q = Q_n$ for all $n \in \mathbb{N}$. In Kallsen and Vesenmayer [16] and Maller *et al.* [20], the GARCH parametrizations $(\Theta, (H_n^{(KV)})_{n \in \mathbb{N}})$ and $(\Theta, (H_n^{(M)})_{n \in \mathbb{N}})$ have been considered where, for $\theta = (h_0, \beta, \alpha, \lambda) \in (0, \infty)^3 \times [0, \infty)$, using the obvious notation, $\Theta = (0, \infty)^3 \times [0, \infty)$ and

$$\begin{aligned} h_{0,n}^{(KV)}(\theta) &= h_{0,n}^{(M)}(\theta) = h_0, \\ \beta_n^{(KV)}(\theta) &= \beta_n^{(M)}(\theta) = \frac{\beta}{n}, \\ \alpha_n^{(KV)}(\theta) &= \alpha_n^{(M)}(\theta) = e^{-\alpha/n}, \\ \lambda_n^{(KV)}(\theta) &= \lambda, \quad \lambda_n^{(M)}(\theta) = e^{-\alpha/n} \lambda. \end{aligned} \tag{2.16}$$

Kallsen and Vesenmayer [16] have shown that $(G_n[n \cdot], h_n[n \cdot])$, as defined in (2.3) by $H_n(\theta) = H_n^{(KV)}(\theta)$, converge to COGARCH with parameter θ in (2.4) in law, with respect to the Skorokhod topology, as $n \rightarrow \infty$, for all $\theta \in \Theta$.

Maller *et al.* [20] have encountered a slightly different scenario. For $\theta \in \Theta$, they have embedded a sequence of GARCH models into a given COGARCH and obtained the convergence with respect to the same topology, now driven by a general Lévy process, even in probability. If the driving process is a compound Poisson process with rate γ and jump size distribution Q , then their analysis relates to a situation where the corresponding partial sums have the same law as $(G_n[n \cdot], h_n[n \cdot])$ under the parametrization $H_n^{(M)}(\theta)$, $\theta \in \Theta$, $n \in \mathbb{N}$.

In short, it follows from the analyses in [16] that, for fixed $\theta \in \Theta$, the partial sum processes of GARCH with parametrization $H_n^{(KV)}(\theta)$ converge to COGARCH in law, as $n \rightarrow \infty$, with a similar result being true for the parametrization $H_n^{(M)}(\theta)$ in [20]. On the other hand, both parametrizations fall into the framework of Theorem 2.2. Hence, if the distribution of the innovations admits a Lebesgue density, then the limiting experiment is given by MCOGARCH $\widehat{\mathcal{E}}(\Theta)$ rather than COGARCH $\mathcal{E}(\Theta)$.

(ii) In part (i), $Q = Q_n$ does not depend on n . Potential applications where Q_n depends on n arise in the peak-over-threshold method in extreme value theory; see, for instance, Embrechts *et al.* [9], Resnick [25] and Falk *et al.* [11]. Here, Q_n equals the laws of rescaled innovations, conditioned on the event that they exceed a given threshold. Under reasonable assumptions, Q_n converge weakly to a generalized Pareto distribution Q as $n \rightarrow \infty$. Also, the corresponding GARCH models converge in distribution in law to a COGARCH driven by a compound Poisson process with jump distribution Q . In this sense, COGARCH serves as a good approximation of GARCH *in law* if we are interested in the extreme parts of the innovations. On the other hand, if Q_n converges to Q , even in total variation norm, then it follows from Theorem 2.1 that the corresponding limiting experiment must be statistically equivalent to MCOGARCH.

2.3. COGARCH vs. MCOGARCH (incomplete observations)

In this subsection, we investigate whether the experiments induced by COGARCH and MCOGARCH are of the same type. Here, we again assume that the volatility processes are unobservable.

Therefore, we recall (2.4) and consider the experiment

$$\mathcal{E} = (D_1, \mathcal{D}_1, (\mathcal{L}_\theta(G))_{\theta \in [0, \infty)^4}). \quad (2.17)$$

Note that both experiments \mathcal{E} and $\widehat{\mathcal{E}}$ depend on the intensity measure $\gamma \ell \otimes Q$ which enters (2.4) and (2.5) via N . In this subsection, we include this dependence in our notation by writing $\mathcal{E}_{\gamma, Q}$ and $\widehat{\mathcal{E}}_{\gamma, Q}$ instead of \mathcal{E} and $\widehat{\mathcal{E}}$, respectively.

Let $f : \mathbb{R} \rightarrow (0, \infty]$ be a strictly positive probability density with respect to Lebesgue measure and set

$$g_{f, \zeta}(h) := h^\zeta \int_{\mathbb{R}} f(hz)^\zeta f(z)^{1-\zeta} dz, \quad h > 0, \zeta \in (0, 1). \quad (2.18)$$

By Hölder's inequality, $g_{f, \zeta}$ defines a function $g_{f, \zeta} : (0, \infty) \rightarrow (0, 1]$ with $g_{f, \zeta}(1) = 1$. Note that $g_{f, \zeta}$ satisfies both a scaling and a reflection property: for all $0 < \zeta < 1$, $a, h > 0$,

$$g_{af(a), \zeta}(h) = g_{f, \zeta}(h), \quad g_{f, \zeta}(h) = g_{f, 1-\zeta}(1/h). \quad (2.19)$$

Next, we investigate how COGARCH relates to MCOGARCH in deficiency (see Section 6 for a proof).

Theorem 2.3. *Let $\emptyset \neq \Theta \subseteq (0, \infty) \times [0, \infty)^3$. Assume that Q admits a strictly positive Lebesgue density f such that for some $\zeta_0 \in (0, 1)$, $g_{f, \zeta_0} : (0, \infty) \rightarrow [0, 1]$ is strictly increasing on $(0, 1]$.*

Let $(\gamma_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ be a sequence such that $\gamma = \lim_{n \rightarrow \infty} \gamma_n$ exists in $[0, \infty)$ and $\gamma_n \neq \gamma$ for all $n \in \mathbb{N}$.

If $\mathcal{E}_{\gamma_n, Q}(\Theta)$ is equivalent to $\widehat{\mathcal{E}}_{\gamma_n, Q}(\Theta)$ for all $n \in \mathbb{N}$, then we have:

- (i) *if $(h_{0,1}, \beta, \alpha, \lambda), (h_{0,2}, \beta, \alpha, \lambda) \in \Theta$ and $\beta > 0$, then $h_{0,1} = h_{0,2}$;*
- (ii) *if $(h_0, \beta_1, \alpha, \lambda), (h_0, \beta_2, \alpha, \lambda) \in \Theta$, then $\beta_1 = \beta_2$;*
- (iii) *if $(h_0, \beta, \alpha_1, \lambda), (h_0, \beta, \alpha_2, \lambda) \in \Theta$ and $\beta = 0$, then $\alpha_1 = \alpha_2$;*
- (iv) *if $(h_0, \beta, \alpha_1, \lambda), (h_0, \beta, \alpha_2, \lambda) \in \Theta$ and $\alpha_1 = 0$, then $\alpha_2 = 0$;*
- (v) *if $(h_0, \beta, \alpha_1, \lambda), (h_0, \beta, \alpha_2, \lambda) \in \Theta$ and $\alpha_1 < \alpha_2$, then $h_0 > \beta/\alpha_1$.*

Theorem 2.3 indicates that equivalence of MCOGARCH and COGARCH is restricted to parameter sets that are of considerably lower dimension and which have non-empty interiors. Hence, we do not have equivalence in deficiency.

Observe that $\zeta \mapsto g_{f, \zeta}(h)$ occurs as the Hellinger transformation of the scaling experiment $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \{\mathcal{L}(Z), \mathcal{L}(Z/h)\})$, where Z is a random variable with Lebesgue density f . Next, we verify the monotonicity property of $g_{f, \zeta}(h)$ in a number of examples.

Generalized symmetric gamma distribution. Let $a, b, c > 0$ and Γ be Euler's gamma function. Assume that $f : \mathbb{R} \rightarrow (0, \infty]$ has the following form:

$$f(z) = \frac{1}{2} \frac{a^{c/b}}{\Gamma(c/b)} e^{-a|z|^b} |z|^{c-1}, \quad z \in \mathbb{R}.$$

This class of distributions covers important special cases such as the normal distribution with zero mean and the Laplace distribution. It follows straightforwardly that

$$g_{f,\zeta}(h) = \left(\frac{h^{b\zeta}}{h^{b\zeta} + (1-\zeta)} \right)^{c/b}, \quad 0 < \zeta < 1, h > 0.$$

Observe that $g_{f,\zeta} : (0, \infty) \rightarrow [0, 1]$ is strictly increasing on $(0, 1]$ for all $0 < \zeta < 1$ and thus f satisfies the monotonicity assumption of Theorem 2.3.

Centered Cauchy distribution. Let $a > 0$ and $f_a(z) = \frac{a}{\pi} \frac{1}{1+(az)^2}$ be the density of the centered Cauchy distribution $\text{Cauchy}(0, a)$ with scaling parameter a . By the scaling property in (2.19), we have $g_{f_a,1/2}(h) = g_{f_1,1/2}(h)$ for all $h > 0$. By differentiating this under the integral sign, we obtain

$$\frac{d}{dh} g_{f_a,1/2}(h) = \frac{1-h^2}{2\pi h^2} \int_0^\infty \frac{\sqrt{x} dx}{(1+(1+h^2)x/h+x^2)^{3/2}} > 0, \quad a > 0, 0 < h \leq 1.$$

Consequently, the centered Cauchy distribution satisfies the monotonicity assumption of Theorem 2.3.

Next, we present a simulation-based approach to assess non-equivalence. This approach can be used in cases not covered by Theorem 2.3 (or when it is not clear whether the assumption of Theorem 2.3 is satisfied). Recall that statistical equivalence of the experiments \mathcal{E} and $\widehat{\mathcal{E}}$ is implied (see [27], Theorem 53.10) when, for all finite subsets $\Theta \subseteq [0, \infty)^4$ and all $\theta_0 \in \Theta$, we have

$$\mathcal{L}_{\theta_0} \left(\left(\frac{d\mathcal{L}_\theta(G)}{d\mathcal{L}_{\theta_0}(G)} \right)_{\theta \in \Theta} \right) = \mathcal{L}_{\theta_0} \left(\left(\frac{d\mathcal{L}_\theta(\widehat{G})}{d\mathcal{L}_{\theta_0}(\widehat{G})} \right)_{\theta \in \Theta} \right). \quad (2.20)$$

We generated samples from these two distributions according to the recursion (6.1) in the proof of Theorem 2.3 in Section 6. To this end, we first restricted the parameter space to a set with two elements, θ_0 and θ . While fixing θ_0 to $(2, 1, 1, 0.1)$, we have chosen eight vectors θ_{ij} , $i = 1, \dots, 4$, $j = 1, 2$, for the parameter vector θ , which differ from θ_0 in only one component; see Table 1. Second, we checked the distributional equality (2.20) for three different jump dis-

Table 1. Choices of θ_0 and $\theta = \theta_{ij}$ in equation (2.20)

θ_0	2	1	1	0.1
θ_{11}	0.4	1	1	0.1
θ_{12}	10	1	1	0.1
θ_{21}	2	0.2	1	0.1
θ_{22}	2	5	1	0.1
θ_{31}	2	1	0.2	0.1
θ_{32}	2	1	5	0.1
θ_{41}	2	1	1	0.02
θ_{42}	2	1	1	0.5

Table 2. Estimated 25% quantiles, medians and 75% quantiles for the distributions in (2.20)

Jumps Quantiles	COGARCH MCOGARCH								
	$N(0, 1)$			Cauchy(0, 1)			Mixed N		
	25%	Median	75%	25%	Median	75%	25%	Median	75%
θ_{11}	0.1081	0.5560	1.3888	0.5521	0.7775	1.1767	0.0909	0.5329	1.3918
	0.1785	0.6977	1.3495	0.5884	0.8226	1.1811	0.1558	0.6743	1.3543
θ_{12}	0.1505	0.3152	0.6449	0.4173	0.8127	1.4573	0.1436	0.3008	0.6136
	0.1637	0.3377	0.6768	0.4412	0.8335	1.4505	0.1575	0.3264	0.6559
θ_{21}	0.8326	1.0168	1.1711	0.9273	0.9761	1.0393	0.8307	1.0201	1.1766
	0.7605	1.0114	1.2459	0.9051	0.9566	1.0539	0.7560	1.0155	1.2512
θ_{22}	0.4883	0.7071	1.0086	0.7765	1.0229	1.2130	0.4797	0.6956	1.0000
	0.4201	0.6077	1.0000	0.7010	1.0247	1.2676	0.4100	0.5988	0.9798
θ_{31}	0.6928	0.8543	1.0621	0.8497	1.0000	1.1506	0.6863	0.8476	1.0530
	0.6304	0.7841	1.0629	0.8029	1.0000	1.1881	0.6248	0.7757	1.0524
θ_{32}	0.0053	0.1702	1.1056	0.3853	0.6449	1.1172	0.0028	0.1392	1.0856
	0.0010	0.0590	0.9129	0.3093	0.5650	1.1090	0.0005	0.0437	0.8703
θ_{41}	0.9864	1.0104	1.0735	0.8265	1.0000	1.0798	0.9863	1.0114	1.0762
	0.9884	1.0100	1.0693	0.8357	1.0000	1.0779	0.9884	1.0109	1.0722
θ_{42}	0.6851	0.8870	1.0000	0.6217	0.9328	1.0418	0.6750	0.8802	1.0000
	0.6963	0.8942	1.0000	0.6281	0.9360	1.0388	0.6865	0.8874	1.0000

tributions: the standard normal, the standard Cauchy distribution Cauchy(0, 1) (for comparison, note that both of these are covered by Theorem 2.3) and the normal mixture distribution

$$\frac{1}{2}N(-0.5, 0.75) + \frac{1}{2}N(0.5, 0.75),$$

which has mean 0 and variance 1. The intensity γ was always fixed to 4.

For each of the eight pairs (θ_0, θ_{ij}) and each of the three jump distributions, we generated 10^6 samples of the two distributions referring to the COGARCH and MCOGARCH in equation (2.20). The left-hand column of Table 2 reports the choice of θ_{ij} , whereas the other nine columns report, for each of the three jump distributions, the 25% quantile, median and 75% quantile of the distribution in equation (2.20).

Next, we applied the Wilcoxon rank sum test (also known as Mann–Whitney test) to investigate the null hypothesis *the median of the likelihood ratio for the COGARCH experiment equals the median of the likelihood ratio for the MCOGARCH experiment*. Table 3 reports the values of the Wilcoxon test statistic W , together with the corresponding p -values. For each jump distribution, the first column corresponds to a sample size of 10^4 , the second to 10^5 and the third to a sample size of 10^6 per experiment. Obviously, the p -values tend to 0 as the sample size increases. Based on 10^6 samples, the null hypothesis is most significantly rejected, for all three jump distributions and all eight parameter vectors θ_{ij} . In other words, there is strong evidence that

Table 3. Wilcoxon rank sum test: Values of Wilcoxon test statistic W and corresponding p -values

Jumps Sample size	W statistic p -value								
	$N(0, 1)$			Cauchy(0, 1)			Mixed N		
	10^4	10^5	10^6	10^4	10^5	10^6	10^4	10^5	10^6
θ_{11}	-7.10 0.0000	-24.12 0.0000	-73.91 0.0000	-8.82 0.0000	-25.46 0.0000	-73.81 0.0000	-8.11 0.0000	-24.01 0.0000	-71.91 0.0000
θ_{12}	-3.04 0.0024	-14.90 0.0000	-47.21 0.0000	-0.35 0.7245	-2.73 0.0064	-9.90 0.0000	-6.04 0.0000	-15.37 0.0000	-48.40 0.0000
θ_{21}	-1.56 0.1189	-3.20 0.0014	-12.52 0.0000	8.71 0.0000	31.90 0.0000	98.90 0.0000	-0.45 0.6545	-3.45 0.0006	-14.05 0.0000
θ_{22}	12.10 0.0000	44.13 0.0000	136.09 0.0000	-1.92 0.0546	-2.69 0.0070	-8.28 0.0000	14.17 0.0000	45.16 0.0000	141.15 0.0000
θ_{31}	12.38 0.0000	37.96 0.0000	116.09 0.0000	1.76 0.0788	2.16 0.0311	10.30 0.0000	12.07 0.0000	38.89 0.0000	119.95 0.0000
θ_{32}	11.34 0.0000	39.48 0.0000	126.96 0.0000	11.63 0.0000	33.04 0.0000	100.34 0.0000	13.59 0.0000	42.66 0.0000	131.75 0.0000
θ_{41}	0.83 0.4054	1.41 0.1572	5.79 0.0000	-1.52 0.1280	-2.85 0.0044	-4.98 0.0000	2.83 0.0047	3.64 0.0003	5.35 0.0000
θ_{42}	-2.96 0.0031	-3.71 0.0002	-13.73 0.0000	-1.29 0.1963	-2.35 0.0189	-2.94 0.0032	-1.63 0.1041	-4.81 0.0000	-15.26 0.0000

in the case of incomplete observations, the randomly thinned GARCH and COGARCH experiments are not statistically equivalent for these jump distributions. This confirms our conjecture, that Theorem 2.3 holds in a much more general formulation for quite arbitrary jump distributions.

2.4. Complete observations

In the previous subsections, we investigated both convergence and equivalence in deficiency of a variety of GARCH-type experiments under the assumption that their volatility processes h_n , h and \hat{h} are unobservable. In this subsection, we deal with the situation where the corresponding volatility processes are observable in full. Of course, this situation is mainly of theoretical interest and will primarily help us to learn about the structural connections between GARCH and COGARCH. However, we want to briefly mention some modern approaches by which the unobservability of the volatility process can be dealt with in practice. For example, there are several modern ways to estimate the local volatility directly; see, for example, Aït-Sahalia, Mykland and Zhang [1] and references therein or Jacod, Klüppelberg and Müller [15], who use local volatility estimates also in a COGARCH context, and many others. The paper by Hubalek and Posedel [14] contains another very interesting idea. They use martingale estimating functions to estimate the parameters in the Barndorff-Nielsen–Shephard model, which is composed of a stochastic differential equation (SDE) for the log prices and another SDE for the variance.

However, the martingale estimating functions approach requires that both processes can be observed. Hence, Hubalek and Posedel [14] reinterpret the volatility equation as an equation for some other observable measure of trading intensity (such as trading volume or the number of trades), assuming that the instantaneous variance process behaves (up to a time-independent constant) exactly as the observable trading volume (or the number of trades). As they show in their real-data example, this approach leads to quite satisfying results. The same idea could be used, of course, for the COGARCH model, to bypass problems with the unobservability of the volatility process in practice.

Returning to theoretical matters, we now consider the following GARCH-type experiments in continuous time with fully observed volatilities, denoted by

$$\mathcal{E}_h = (D_2, \mathcal{D}_2, (\mathcal{L}_\theta(G, h))_{\theta \in [0, \infty)^4}), \quad \widehat{\mathcal{E}}_h = (D_2, \mathcal{D}_2, (\mathcal{L}_\theta(\widehat{G}, \widehat{h}))_{\theta \in [0, \infty)^4}),$$

where \widehat{h} is defined by the specification in (2.5) and (2.6). Similarly to Sections 2.1 and 2.2, where we dealt with continuous time, both experiments \mathcal{E}_h and $\widehat{\mathcal{E}}_h$ depend on Q and $\gamma > 0$ as well. In this subsection, we will suppress this dependence in our notation.

We need to specify a set $\Theta_e \subseteq [0, \infty)^4$ of *exceptional points* in the parameter space $[0, \infty)^4$ as follows:

$$\Theta_e = \{\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4 : h_0\alpha = \beta\}. \tag{2.21}$$

Observe that Θ_e is closely connected to the fixed point of the affine differential equation $h'(t) = \beta - \alpha h(t)$. Indeed, if $\theta = (h_0, \beta, \alpha, \lambda) \in \Theta_e$, then we have $h(t) = \widehat{h}(t) \equiv h_0$ for all $t \in [0, T)$, where T is the first jump of (M)COGARCH. It is impossible to recover the parameters β, α, λ in full within the time horizon $[0, T)$. Otherwise, if h_0 is not the fixed point of this differential equation, then it is always possible to recover parts of θ by taking appropriate derivatives. In the next proposition, we formalize this idea and show that both \mathcal{E}_h and $\widehat{\mathcal{E}}_h$ are equivalent to a simple reference experiment (see Section 7.1 for a proof).

Proposition 2.1. *If $Q(\{0\}) = 0$, then both \mathcal{E}_h and $\widehat{\mathcal{E}}_h$ are equivalent to $\mathcal{F} = ([0, \infty]^4, \mathcal{B}([0, \infty]^4), (Q_\theta)_{\theta \in [0, \infty)^4})$ where, for $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, $\gamma > 0$, we set*

$$Q_\theta = \begin{cases} e^{-\gamma} \varepsilon_{(h_0, \beta, \alpha, \infty)} + (1 - e^{-\gamma}) \varepsilon_\theta, & \theta \notin \Theta_e, \\ e^{-\gamma} \varepsilon_{(h_0, \infty, \infty, \infty)} + (1 - e^{-\gamma}) \varepsilon_\theta, & \theta \in \Theta_e, h_0 > 0, \lambda > 0, \\ e^{-\gamma} \varepsilon_{(h_0, \infty, \infty, \infty)} + (1 - e^{-\gamma}) \varepsilon_{(h_0, \infty, \infty, 0)}, & \theta \in \Theta_e, h_0 > 0, \lambda = 0, \\ \varepsilon_{(0, \infty, \infty, \infty)}, & \theta \in \Theta_e, h_0 = 0, \end{cases} \tag{2.22}$$

and Θ_e is the set defined as in (2.21).

Remark 2.2. In the situation of Proposition 2.1, we require Q to satisfy $Q(\{0\}) = 0$. Indeed, if $Q = \varepsilon_0$, then it is easy to see that both \mathcal{E}_h and $\widehat{\mathcal{E}}_h$ are equivalent to \mathcal{F} , where we formally set $\gamma = 0$ in (2.22). Otherwise, if $Q(\{0\}) \in [0, 1)$, then we may adjust the intensity measures of the driving Poisson measure accordingly, to see that both \mathcal{E}_h and $\widehat{\mathcal{E}}_h$ are equivalent to \mathcal{F} , but with γ replaced by $\gamma Q(\mathbb{R} \setminus \{0\})$ in the definition of Q_θ . Analogously, one can adjust the discrete-time experiments that we consider in Proposition 2.2. We leave the details to the reader.

Next, we investigate the discrete-time experiments. Note that the initial value of h is observable in continuous time. As a result, it is always possible to recover the parameter h_0 in full. To account for this phenomenon in discrete time, we introduce the following sequence of experiments, \mathcal{E}_{h,n,H_n} , indexed by $n \in \mathbb{N}$, where

$$\mathcal{E}_{h,n,H_n} = ([\mathbb{R}^{n+1}]^2, \mathcal{B}([\mathbb{R}^{n+1}]^2), (\mathcal{L}_\theta(G_n, h_n))_{\theta \in [0, \infty)^4}), \quad n \in \mathbb{N}. \quad (2.23)$$

Here, $([0, \infty)^4, (H_n))$ is a parametrization of the full parameter space $[0, \infty)^4$; both $G_n = (G_{n,k})_{0 \leq k \leq n}$ and $h_n = (h_{n,k})_{0 \leq k \leq n}$ are defined by (2.3) via $H_n(\theta) = (h_{0,n}(\theta), \beta_n(\theta), \alpha_n(\theta), \lambda_n(\theta))$ for $n \in \mathbb{N}$ and $\theta \in [0, \infty)^4$ (by a slight abuse of previous notation). We are now in a position to state a discrete-time analog of Proposition 2.1 (see Section 7.2 for a proof).

Proposition 2.2. *Suppose that (2.2) is satisfied for some $\gamma \in (0, \infty)$ and $p_n \in (0, 1)$, $n \in \mathbb{N}$. Let $([0, \infty)^4, H_n)_{n \in \mathbb{N}}$ be the parametrization in (2.11) and (2.12). Also, let $([0, \infty)^4, H_n^{(\text{KV})})_{n \in \mathbb{N}}$ and $([0, \infty)^4, H_n^{(\text{M})})_{n \in \mathbb{N}}$ be the parametrizations in (2.16), respectively.*

If $Q(\{0\}) = Q_n(\{0\}) = 0$ for all $n \in \mathbb{N}$, then the following assertions hold as $n \rightarrow \infty$, both in deficiency:

- (i) \mathcal{E}_{h,n,H_n} converges strongly to \mathcal{F} ;
- (ii) both $\mathcal{E}_{h,n,H_n^{(\text{KV})}}$ and $\mathcal{E}_{h,n,H_n^{(\text{M})}}$ are asymptotically equivalent to

$$\mathcal{F}_n = ([0, \infty]^4, \mathcal{B}([0, \infty]^4), (Q_{\theta,n})_{\theta \in [0, \infty)^4}),$$

where, for $n \in \mathbb{N}$ and $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, we define $Q_{\theta,n}$ as Q_θ in (2.22), but with Θ_e replaced by

$$\Theta_{e,n} = \{\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4 : h_0 n (1 - e^{-\alpha/n}) = \beta\}.$$

Finally, we are concerned with the relationships between the experiments \mathcal{F} and \mathcal{F}_n , $n \in \mathbb{N} \cup \{\infty\}$ (see Section 7.3 for a proof).

Proposition 2.3. *Let $\gamma > 0$ and $\emptyset \neq \Theta \subseteq [0, \infty)^4$. Let $\mathcal{F}, \mathcal{F}_n$, $n \in \mathbb{N}$, be the experiments in Propositions 2.1 and 2.2, respectively.*

Let $\widehat{\mathcal{F}} = ([0, \infty]^4, \mathcal{B}([0, \infty]^4), (\widehat{Q}_\theta)_{\theta \in [0, \infty)^4})$ be the experiment where, for $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, we define \widehat{Q}_θ as Q_θ in (2.22), but with Θ_e replaced by

$$\widehat{\Theta}_e = \{0\}^2 \times (0, \infty) \times [0, \infty) \cup [0, \infty) \times \{0\}^2 \times [0, \infty).$$

The following assertions then hold:

- (i) $\delta(\widehat{\mathcal{F}}(\Theta), \mathcal{F}(\Theta)) = \delta(\widehat{\mathcal{F}}(\Theta), \mathcal{F}_n(\Theta)) = 0$ for all $n \in \mathbb{N}$;
- (ii) $\delta(\mathcal{F}(\Theta), \widehat{\mathcal{F}}(\Theta)) = 0$ if and only if, for all $h_0 > 0$,

$$\begin{aligned} & \{(\beta, \alpha, \lambda) \in [0, \infty)^3 : (h_0, \beta, \alpha, \lambda) \in \Theta \cap \Theta_e \cap \widehat{\Theta}_e^c\} \neq \emptyset \\ & \Rightarrow \#\{(\beta, \alpha) \in [0, \infty)^2 : \exists \lambda \geq 0 (h_0, \beta, \alpha, \lambda) \in \Theta_e \cap \Theta\} = 1; \end{aligned} \quad (2.24)$$

(iii) $\lim_{n \rightarrow \infty} \delta(\mathcal{F}_n(\Theta), \widehat{\mathcal{F}}(\Theta)) = 0$ if and only if there exists some n_0 such that for all $n \geq n_0$ and $h_0 > 0$, (2.24) holds, but with Θ_e replaced by $\Theta_{e,n}$. In particular, \mathcal{F}_n converges weakly to $\widehat{\mathcal{F}}$ as $n \rightarrow \infty$ in deficiency.

Let us rephrase our results in terms of the GARCH experiments, with the volatility processes fully observed in both continuous and discrete time. In contrast to the situation in Theorem 2.3, it follows from Proposition 2.1 that the continuous-time experiments induced by (M)COGARCH are mutually equivalent in deficiency. Depending on the parametrization, (M)COGARCH also occurs as the limit in deficiency of discrete-time GARCH; in particular, this is the case for the parametrization in Proposition 2.2. In contrast to Theorem 2.3, for a large class of parameter sets Θ , all of these discrete-time experiments, that is, $\mathcal{E}_{h,n,H_n^{(0)}}(\Theta)$, $\mathcal{E}_{h,n,H_n^{(KV)}}(\Theta)$, $\mathcal{E}_{h,n,H_n^{(M)}}(\Theta)$, are asymptotically equivalent to (M)COGARCH $\mathcal{E}_h(\Theta)$ and $\widehat{\mathcal{E}}_h(\Theta)$, in deficiency, as $n \rightarrow \infty$. This happens, for instance, if $\Theta \subseteq [0, \infty)^4$ does not contain an open neighborhood of Θ_e . Since the set Θ_e is of lower dimension than $[0, \infty)^4$ it is thus justified to say that the randomly thinned GARCH is *generically* equivalent to COGARCH in deficiency as $n \rightarrow \infty$.

3. Conclusion

In Le Cam's framework, Wang [28] and Brown *et al.* [6] investigated GARCH and Nelson's diffusion limit. These authors dealt with aggregated Gaussian innovations. For a suitable parametrization, Maller *et al.* [20] and Kallsen and Vesenmayer [16] showed that the GARCH model converges to the COGARCH model in probability and in distribution, respectively, when the innovations are randomly thinned. These papers dealt with a general Lévy process as the driving process of the COGARCH. In this paper, we have studied an important special case in Le Cam's framework of statistical experiments, namely, we have assumed that the driving process of COGARCH is a compound Poisson process. GARCH then converges generically to COGARCH, even in deficiency, provided that the volatility processes are observed. Hence, from a theoretical point of view, COGARCH can indeed be considered as a continuous-time equivalent to GARCH. Otherwise, when the observations are incomplete, GARCH still has a limiting experiment, which we call MCOGARCH, but this will usually not be equivalent to COGARCH in deficiency. Nevertheless, this limiting experiment is, from a statistical point of view, quite similar to COGARCH since the only difference is the exact localization of the jump times. For COGARCH, the jump times can be more random than for MCOGARCH, but practitioners may see this as an additional advantage of COGARCH.

It would be interesting to extend the analysis to more general Lévy processes, rather than Brownian motion and compound Poisson processes. However, this would first require substantial investigations of the approximation and randomizations of Lévy processes themselves and, therefore, seems out of reach at the present stage of research.

4. Proof of Theorem 2.1

For the reader's convenience, we first provide a brief roadmap for the proof of Theorem 2.1. The proof is split into two parts, which appear in Sections 4.1 and 4.3, respectively. The second

part uses a lemma which we formulate and prove in Section 4.2. To prove that $\mathcal{E}_{n, H_n^{(0)}} \rightarrow \hat{\mathcal{E}}$ in deficiency, we will introduce intermediate experiments $\mathcal{E}_{1,n}^*$ and $\mathcal{E}_{2,n}^*$. The first of these two experiments corresponds to a deterministic time grid, the latter to a randomized time grid. First, we will show that $\mathcal{E}_{n, H_n^{(0)}}$ is equivalent to $\mathcal{E}_{1,n}^*$ in deficiency and then, using Lemma 4.1 from Section 4.2, that $\mathcal{E}_{2,n}^*$ converges strongly to $\hat{\mathcal{E}}$. Finally, we will prove that $\mathcal{E}_{1,n}^*$ and $\mathcal{E}_{2,n}^*$ are equivalent.

4.1. Proof of Theorem 2.1 (part I)

For $n \in \mathbb{N}$, define a point measure $N_{1,n}$ on $[0, 1] \times \mathbb{R}$ by

$$N_{1,n} = \sum_{k=1}^n 1_{Z_{n,k} \neq 0} \mathcal{E}(k/n, Z_{n,k}), \quad n \in \mathbb{N}. \quad (4.1)$$

Using $N_{1,n}$, we pass from discrete to continuous time. For $n \in \mathbb{N}$, define

$$\mathcal{E}_{1,n}^* = (D_1, \mathcal{D}_1, (\mathcal{L}_\theta(G_{1,n}))_{\theta \in [0, \infty)^4}),$$

where, for all $0 \leq t \leq 1$, $n \in \mathbb{N}$ and $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, $(G_{1,n}, h_{1,n})$ is the unique pathwise solution of the following system of integral equations ($t \in [0, 1]$):

$$\begin{aligned} G_{1,n}(t) &= \int_{[0,t] \times \mathbb{R}} h_{1,n}^{1/2}(s-) z N_{1,n}(ds, dz), \\ h_{1,n}(t) &= h_0 + \int_{[0,t]} \beta - \alpha h_{1,n}(s-) ds + \lambda \int_{[0,t] \times \mathbb{R}} h_{1,n}(s-) z^2 N_{1,n}(ds, dz). \end{aligned} \quad (4.2)$$

Fix $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$ with $\alpha \neq 0$. By solving the linear ODE for $h_{1,n}$ in (4.2), observe that

$$h_{1,n}(t) = \frac{\beta}{\alpha} [1 - e^{-\alpha[t - (k-1)/n]}] + e^{-\alpha[t - (k-1)/n]} h_{1,n}\left(\frac{k-1}{n}\right) \quad (4.3)$$

for $(k-1)/n \leq t < k/n$, $1 \leq k \leq n$ and $n \in \mathbb{N}$. It thus follows from (2.11) and (4.3) that for all $n \in \mathbb{N}$,

$$\begin{aligned} h_{1,n}(1/n-) &= h_0 e^{-\alpha/n} + \frac{\beta}{\alpha} [1 - e^{-\alpha/n}] = h_{0,n}(\theta), \\ h_{1,n}\left(\frac{k}{n}-\right) &= \beta_n(\theta) + h_{1,n}\left(\frac{k-1}{n}-\right) [\alpha_n(\theta) + \lambda_n(\theta) Z_{n,k-1}^2], \quad 2 \leq k \leq n. \end{aligned}$$

In view of (2.3) and the identities in the last display, we thus have $h_n(k) = h_{1,n}(((k+1)/n)-)$ for all $n \in \mathbb{N}$, $0 \leq k \leq n-1$ and $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$ with $\alpha > 0$. A similar argument is applicable to (2.12) and $\theta = (h_0, \beta, 0, \lambda) \in [0, \infty)^4$. It thus follows from (2.3) and (4.2) that

$$\mathcal{L}_\theta((G_{1,n}(k/n))_{1 \leq k \leq n}) = \mathcal{L}_\theta((G_n(k))_{1 \leq k \leq n}), \quad n \in \mathbb{N}, \theta \in [0, \infty)^4.$$

Note that $G_{1,n}$ is constant on $[(k-1)/n, k/n)$, $1 \leq k \leq n$ and $n \in \mathbb{N}$. Hence, $\mathcal{E}_{n, H_n^{(0)}}$ is equivalent to $\mathcal{E}_{1,n}^*$ in deficiency for all $n \in \mathbb{N}$ by (A.2) and the monotonicity theorem for Markov kernels (see [26], Lemma 1.4.2(i)).

Next, we randomize the deterministic time grid. Therefore, let $(U_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence of random variables independent of the vector Z_n , where U_k is uniformly distributed on $[0, 1]$. Set

$$V_{n,k} = ((k-1) + U_k)/n, \quad 1 \leq k \leq n, \quad (4.4)$$

and define a point process $N_{2,n}$ by

$$N_{2,n} = \sum_{k=1}^n 1_{Z_{n,k} \neq 0} \varepsilon_{(V_{n,k}, Z_{n,k})}, \quad n \in \mathbb{N}. \quad (4.5)$$

Let T be as in (2.6). For $n \in \mathbb{N}$, let $\mathcal{E}_{2,n}^* = (D_1, \mathcal{D}_1, (\mathcal{L}_\theta(G_{2,n}))_{\theta \in [0, \infty)^4})$, where for all $0 \leq t \leq 1$, $n \in \mathbb{N}$ and $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty)^4$, $(G_{2,n}, h_{2,n})$ is the pathwise unique solution of the following system of integral equations:

$$\begin{aligned} G_{2,n}(t) &= \int_{[0,t] \times \mathbb{R}} h_{2,n}^{1/2}(s-) z N_{2,n}(ds, dz), \\ h_{2,n}(t) &= h_0 + \int_{[0,t]} \beta - \alpha h_{2,n}(s-) dT_{N_{2,n}}(s) + \lambda \int_{[0,t] \times \mathbb{R}} h_{2,n}(s-) z^2 N_{2,n}(ds, dz). \end{aligned} \quad (4.6)$$

To proceed with the proof of Theorem 2.1, we need the following lemma.

4.2. Statement and proof of Lemma 4.1

Lemma 4.1. *Let N be a Poisson measure with intensity measure $\gamma \ell \otimes Q$ and $N_{2,n}$ as in (4.5). Suppose that (2.2) holds. If Q_n tends to Q in total variation as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \|\mathcal{L}(N_{2,n}) - \mathcal{L}(N)\| = 0$.*

Proof. Suppose that (2.2) is satisfied for $n \in \mathbb{N}$, $p_n \in (0, 1)$ and $\gamma \in (0, \infty)$. Let $B_{n,1}, \dots, B_{n,n}$ be independent Bernoulli variables with parameter p_n . Suppose that $(U_k, \zeta_{n,k})_{k \in \mathbb{N}}$ is an i.i.d. sequence of random vectors with independent components, where U_k is uniformly distributed on $(0, 1)$ and $\mathcal{L}(\zeta_{n,k}) = Q_n$. Suppose that $B_{n,1}, \dots, B_{n,n}$ and $(U_k, \zeta_{n,k})_{k \in \mathbb{N}}$ are independent. Observe that

$$\mathcal{L}(N_{2,n}) = \mathcal{L}\left(\sum_{k=1}^n B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}\right)$$

with $V_{n,k} = (k-1 + U_k)/n$ for all $n \in \mathbb{N}$ and $1 \leq k \leq n$.

Let \widehat{N}_n be a Poisson measure on $[0, 1] \otimes \mathbb{R}$ with intensity measure $np_n \ell \otimes Q_n$ and define

$$\widehat{N}_{n,k}(B) = \widehat{N}_n\left(B \cap \left(\left(\frac{k-1}{n}, \frac{k}{n}\right] \times \mathbb{R}\right)\right), \quad B \in \mathcal{B}([0, 1] \times \mathbb{R}).$$

$\widehat{N}_{n,1}, \dots, \widehat{N}_{n,n}$ are then independent Poisson point processes, where for all $n \in \mathbb{N}$, $1 \leq k \leq n$, $\widehat{N}_{n,k}$ has intensity measure

$$np_n[\ell \otimes Q_n] \left(B \cap \left(\left(\frac{k-1}{n}, \frac{k}{n} \right] \times \mathbb{R} \right) \right), \quad B \in \mathcal{B}([0, 1] \times \mathbb{R}).$$

By the monotonicity theorem for Markov kernels (see [26], Lemma 1.4.2(i)), observe that for all $n \in \mathbb{N}$,

$$\|\mathcal{L}(N_{2,n}) - \mathcal{L}(\widehat{N}_n)\| \leq \left\| \bigotimes_{k=1}^n \mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}) - \bigotimes_{k=1}^n \mathcal{L}(\widehat{N}_{n,k}) \right\|. \quad (4.7)$$

Denote the Hellinger distance between two probability measures P_1 and P_2 by $H(P_1, P_2)$. This gives us the following upper bound (see [26], Section 1.3, equation (1.23), and Section 1.3, equation (1.25)):

$$\begin{aligned} & \left\| \bigotimes_{k=1}^n \mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}) - \bigotimes_{k=1}^n \mathcal{L}(\widehat{N}_{n,k}) \right\| \\ & \leq H \left(\bigotimes_{k=1}^n \mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}), \bigotimes_{k=1}^n \mathcal{L}(\widehat{N}_{n,k}) \right) \\ & \leq \left(\sum_{k=1}^n H^2(\mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}), \mathcal{L}(\widehat{N}_{n,k})) \right)^{1/2}. \end{aligned} \quad (4.8)$$

Fix $n \in \mathbb{N}$ and $1 \leq k \leq n$. Let $(V_{n,k,l}, \zeta_{n,k,l})_{l \in \mathbb{N}}$ be an i.i.d. sequence of random vectors with $\mathcal{L}(V_{n,k,l}, \zeta_{n,k,l}) = \mathcal{L}(V_{n,k}) \otimes Q_n$, $l \in \mathbb{N}$. Suppose that $(V_{n,k,l})$ is independent of $B_{n,k}$ and $\tau_{n,k}$, where $\tau_{n,k}$ is a Poisson variable with parameter p_n . We then have the following identities:

$$\mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, \zeta_{n,k})}) = \mathcal{L} \left(\sum_{l=1}^{B_{n,k}} \varepsilon_{(V_{n,k,l}, \zeta_{n,k,l})} \right), \quad \mathcal{L}(\widehat{N}_{n,k}) = \mathcal{L} \left(\sum_{l=1}^{\tau_{n,k}} \varepsilon_{(V_{n,k,l}, \zeta_{n,k,l})} \right).$$

By [26], Lemma 1.4.2(ii), for $n \in \mathbb{N}$ and $1 \leq k \leq n$, we must have

$$H(\mathcal{L}(B_{n,k} \varepsilon_{(V_{n,k}, Z_{n,k})}), \mathcal{L}(\widehat{N}_{n,k})) \leq H(\mathcal{L}(B_{n,k}), \mathcal{L}(\tau_{n,k})). \quad (4.9)$$

As $H(\mathcal{L}(B_{n,k}), \mathcal{L}(\tau_{n,k})) \leq 3^{1/2} p_n$ (see [26], Theorem 1.3.1(ii)), it follows from (4.7)–(4.9) and (2.2) that

$$\limsup_{n \rightarrow \infty} \|\mathcal{L}(\tilde{N}_n) - \mathcal{L}(\widehat{N}_n)\| \leq \limsup_{n \rightarrow \infty} (3np_n^2)^{1/2} = 0. \quad (4.10)$$

In view of a well-known upper bound of the laws of Poisson point measures in terms of the corresponding intensity measures (see [26], Section 3.2, equation (3.8)), it follows from (2.2)

and $\|\ell \otimes Q\| = 1$ that

$$\begin{aligned} \|\mathcal{L}(\widehat{N}_n) - \mathcal{L}(N)\| &\leq 3\|\gamma\ell \otimes Q - np_n\ell \otimes Q_n\| \\ &\leq 3|np_n - \gamma| + 3np_n\|Q - Q_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By means of (4.10) and (4.11), this completes the proof of the lemma. \square

4.3. Proof of Theorem 2.1 (part II)

Let N be a Poisson measure with intensity measure $\gamma\ell \otimes Q$. It follows from (2.5) and (4.6) that there exists a family of deterministic Markov kernels $K_\theta: \mathbb{M}_1 \times \mathcal{D}_1 \rightarrow [0, 1]$, indexed by $\theta \in [0, \infty)^4$, such that both $\mathcal{L}_\theta(G_{2,n}) = K_\theta\mathcal{L}(N_{2,n})$ and $\mathcal{L}_\theta(\widehat{G}) = K_\theta\mathcal{L}(N)$ for all $n \in \mathbb{N}$ and $\theta \in \Theta$. Since we have assumed (2.2), the assertion of Lemma 4.1 holds and we thus get from (A.4) and the monotonicity theorem for Markov kernels (see [26], Lemma 1.4.2(i)) that as $n \rightarrow \infty$,

$$\Delta(\widehat{\mathcal{E}}, \mathcal{E}_{2,n}^*) \leq \sup_{\theta \in [0, \infty)^4} \|\mathcal{L}_\theta(\widehat{G}) - \mathcal{L}_\theta(G_{2,n})\| \leq \|\mathcal{L}(N) - \mathcal{L}(N_{2,n})\| \rightarrow 0.$$

Consequently, $\mathcal{E}_{2,n}^*$ converges (strongly) to $\widehat{\mathcal{E}}$ in deficiency as $n \rightarrow \infty$. Recall that $\mathcal{E}_{n, H_n^{(0)}}$ is equivalent to $\mathcal{E}_{1,n}^*$ in deficiency for all $n \in \mathbb{N}$. To complete the proof of the theorem, it thus suffices to show that $\mathcal{E}_{1,n}^*$ is equivalent to $\mathcal{E}_{2,n}^*$.

Therefore, let \mathbb{M}_0 denote the space of all non-negative point measures on $[0, 1]$ with finite support. We equip this space with the σ -algebra \mathcal{M}_0 generated by point evaluations (see Reiss [26], pages 5–6). Let $\mathbb{M}_{0,1} \subseteq \mathbb{M}_0$ be the subset of point measures $\sigma \in \mathbb{M}_0$ such that there exist $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_m < 1$ with $\sigma = \sum_{k=1}^m \varepsilon_{t_k}$. For $\sigma \in \mathbb{M}_0$, we define mappings $T_{1,\sigma}, T_{2,\sigma}: [0, 1] \rightarrow [0, \infty)$ and $T_{3,\sigma}, T_{4,\sigma}: [0, 1] \times \mathbb{R} \rightarrow [0, \infty) \times \mathbb{R}$ as follows: if $\sigma \in \mathbb{M}_0 \setminus \mathbb{M}_{0,1}$, then for all $t \in [0, 1]$ and $x \in \mathbb{R}$, we set $T_{1,\sigma}(t) = T_{2,\sigma}(t) = t$ and $T_{3,\sigma}(t, x) = T_{4,\sigma}(t, x) = (t, x)$. Otherwise, if $\sigma \in \mathbb{M}_{0,1}$, then there exist $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_m < 1$ with $\sigma = \sum_{k=1}^m \varepsilon_{t_k}$ and we set

$$\begin{aligned} T_{1,\sigma}(t) &= \frac{t - t_k}{m(t_k - t_{k-1})} + \frac{k}{m}, & t \in [t_{k-1}, t_k], \quad 1 \leq k \leq m, \\ T_{1,\sigma}(t) &= \frac{t - t_m}{m(t_m - t_{m-1})} + 1, & t \in [t_m, 1]. \end{aligned}$$

In this case, we define $T_{4,\sigma}: [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ by $T_{4,\sigma} = (T_{1,\sigma}(t), x)$. Then, $T_{1,\sigma}: [0, t_m] \rightarrow [0, 1]$ and $T_{4,\sigma}: [0, t_m] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ are bijections and we let $T_{2,\sigma}: [0, 1] \rightarrow [0, t_m]$ and $T_{3,\sigma}: [0, 1] \times \mathbb{R} \rightarrow [0, t_m] \times \mathbb{R}$ be their corresponding inverses.

Let $n \in \mathbb{N}$. Recall (4.4) and set

$$\begin{aligned} M_{1,n} &= \sum_{k=1}^n \varepsilon_{V_{n,k}} 1_{G_{1,n}(k/n) - G_{1,n}((k-1)/n) \neq 0}, \\ M_{2,n} &= \sum_{0 \leq t \leq 1} \varepsilon_{(\lfloor tn \rfloor + 1)/n} 1_{G_{2,n}(t) - G_{2,n}(t-) \neq 0}. \end{aligned}$$

For $n \in \mathbb{N}$ and $i = 1, 2$, it follows from the transformation theorem that

$$\begin{aligned} G_{i,n} \circ T_{i,M_{i,n}}(t) &= \int_{[0,t] \times \mathbb{R}} (h_{n,i} \circ T_{M_{i,n}})^{1/2}(s-) z N_{i,n}^{T_{i+2}, M_{i,n}}(ds, dz), \\ h_{i,n} \circ T_{i,M_{i,n}}(t) &= h_0 + \int_{[0,t]} \beta - \alpha(h_{i,n} \circ T_{i,M_{i,n}})(s-) dT_{i,M_{i,n}}(s) \\ &\quad + \lambda \int_{[0,t] \times \mathbb{R}} (h_{i,n} \circ T_{i,M_{i,n}})(s-) z^2 N_n^{T_{i+2}, M_{i,n}}(ds, dz) \end{aligned} \quad (4.11)$$

for all $t \in [0, 1]$ and $\theta = (h_0, \beta, \alpha, \gamma) \in [0, \infty)^4$.

Let $\theta = (h_0, \beta, \alpha, \gamma) \in [0, \infty)^4$. If $h_0 = \beta = 0$, then it follows from (4.2), (4.6) and (4.11) that $h_{i,n} = h_{i,n} \circ T_{i,M_{i,n}} \equiv 0$, $i = 1, 2$, a.s. and thus

$$\mathcal{L}_\theta(G_{i,n}) = \mathcal{L}_\theta(G_{i,n} \circ T_{i,M_{i,n}}) = \varepsilon_0, \quad n \in \mathbb{N}, i = 1, 2.$$

Otherwise, if $h_0 + \beta > 0$, it follows from (4.2) and (4.6) that $h_{i,n}(t) > 0$ for all $t \in (0, 1]$ a.s., $i = 1, 2$. In this case, we have $M_{1,n} = N_{2,n}$, $M_{2,n} = N_{1,n}$, $N_{1,n}^{T_{3}, M_{1,n}} = N_{n,2}$ and $N_{2,n}^{T_{4}, M_{2,n}} = N_{n,1}$ and thus we get from (4.11) that both

$$\mathcal{L}_\theta(G_{1,n}) = \mathcal{L}_\theta(G_{2,n} \circ T_{2,M_{2,n}}) \quad \text{and} \quad \mathcal{L}_\theta(G_{2,n}) = \mathcal{L}_\theta(G_{1,n} \circ T_{1,M_{1,n}})$$

for $n \in \mathbb{N}$. In other words, for all $n \in \mathbb{N}$, there are Markov kernels $K_{1,2,n} : D_1 \times \mathcal{D}_1 \rightarrow [0, 1]$ and $K_{2,1,n} : D_1 \times \mathcal{D}_1 \rightarrow [0, 1]$, not depending on $\theta \in [0, \infty)^4$, such that $K_{1,2,n} \mathcal{L}_\theta(G_{2,n}) = \mathcal{L}_\theta(G_{1,n})$ and $K_{2,1,n} \mathcal{L}_\theta(G_{1,n}) = \mathcal{L}_\theta(G_{2,n})$ for all $\theta \in [0, \infty)^4$. Hence, $\mathcal{E}_{1,n}^*$ is equivalent to $\mathcal{E}_{2,n}^*$ in deficiency by (A.2) for all $n \in \mathbb{N}$. This completes the proof of the theorem.

5. Proof of Theorem 2.2

The proof of Theorem 2.2 is split into two parts, reported in Sections 5.1 and 5.4, respectively. We will need two additional results, which appear as Lemmas 5.1 and 5.2, together with their respective proofs in Sections 5.2 and 5.3.

5.1. Proof of Theorem 2.2 (part I)

Recall that Le Cam's distance is a pseudo-metric. In view of (A.4) and Theorem 2.1, it thus suffices to show (2.15). For $n \in \mathbb{N}$, let $Z_n = (Z_{n,k})_{1 \leq k \leq n}$ be a random vector with a distribution as in (2.1).

First, we assume that

$$Q_n = Q, \quad n \in \mathbb{N}. \quad (5.1)$$

At the end of the proof we will relax this condition to $\|Q_n - Q\| \rightarrow 0$ as $n \rightarrow \infty$.

Let N_n be as in (4.1) and set $\|N_n\| = N_n([0, 1] \times \mathbb{R})$, $n \in \mathbb{N}$. Let Θ be as in the assertion of the theorem. Suppose that $H_{1,n} = H_n = (h_{0,1,n}, \beta_{1,n}, \alpha_{1,n}, \lambda_{1,n}) : \Theta \rightarrow [0, \infty)^4$ satisfies the assumptions of the theorem. Further, let $H_{2,n} = (h_{0,2,n}, \beta_{2,n}, \alpha_{2,n}, \lambda_{2,n}) = H_n^{(0)} : \Theta \rightarrow [0, \infty)^4$ be defined by the identities in (2.11) and (2.12).

For $\theta \in \Theta$ and $i = 1, 2$, we define $X_{i,n} = (X_{i,n}(k))_{1 \leq k \leq n}$ by

$$\begin{aligned} X_{i,n}(k) &= h_{i,n}^{1/2}(k-1)Z_{n,k}, & X_{i,n}(0) &= 0, \\ h_{i,n}(k) &= \beta_{i,n}(\theta) + h_{i,n}(k-1)[\alpha_{i,n}(\theta) + \lambda_{i,n}(\theta)Z_{n,k}^2], \\ h_{i,n}(0) &= h_{0,i,n}(\theta), & n \in \mathbb{N}, 1 \leq k \leq n. \end{aligned} \quad (5.2)$$

Hence, $X_{1,n}$ corresponds to the GARCH processes G_n as in the theorem, and $X_{2,n}$ to the GARCH processes $G_n^{(0)}$ defined directly after (2.12). Let

$$M_{n,k} = \left\{ \sigma = (\sigma_l)_{1 \leq l \leq k} \in \mathbb{N}^k : \sum_{l=1}^k \sigma_l \leq n \right\}, \quad 1 \leq k \leq n, n \in \mathbb{N}. \quad (5.3)$$

By employing the conventions $0^0 = 1$ and $\sum_{l=k}^m = 0$ for $m < k$, we set

$$\begin{aligned} \eta_{i,n,1,l,\sigma}(\theta) &= \beta_{i,n}(\theta) \sum_{m=0}^{\sigma_{l+1}-1} [\alpha_{i,n}(\theta)]^m, \\ \eta_{i,n,2,l,\sigma}(\theta) &= [\alpha_{i,n}(\theta)]^{\sigma_{l+1}}, \\ \eta_{i,n,3,l,\sigma}(\theta) &= \lambda_{i,n}(\theta)[\alpha_{i,n}(\theta)]^{\sigma_{l+1}-1} \end{aligned} \quad (5.4)$$

for $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $i = 1, 2$ and $n \in \mathbb{N}$.

Also, we recursively define functions from $\mathbb{R}^k \rightarrow \mathbb{R}$ by setting

$$\begin{aligned} \hat{g}_{i,n,0,\sigma,\theta} &\equiv h_{0,i,n}(\theta)\alpha_{i,n}(\theta)^{\sigma_1-1} + \beta_{i,n}(\theta) \sum_{m=0}^{\sigma_1-2} \alpha_{i,n}^m(\theta), \\ \hat{g}_{i,n,l,\sigma,\theta}(y) &= \eta_{i,n,1,l,\sigma}(\theta) + \eta_{i,n,2,l,\sigma}(\theta)\hat{g}_{i,n,l-1,\sigma,\theta}(y) + \eta_{i,n,3,l,\sigma}(\theta)y_l^2 \end{aligned} \quad (5.5)$$

for $y \in \mathbb{R}^k$, $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $i = 1, 2$ and $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. On $\{\|N_n\| = k\}$, we consider the following stopping times:

$$\tau_0 = 0, \quad \tau_m = \min\{v \in \{\tau_{m-1} + 1, \dots, n\} : Z_{n,v} \neq 0\}, \quad 1 \leq m \leq k.$$

Using these stopping times, let $\Delta\tau = ((\Delta\tau_m)_{1 \leq m \leq k}) \in M_{n,k}$ be the random vector defined componentwise by $\Delta\tau_m = \tau_m - \tau_{m-1}$ for $1 \leq m \leq k$.

Let $i = 1, 2$, $n \in \mathbb{N}$, $1 \leq k \leq n$ and $\theta \in \Theta$. On $\{\|N_n\| = 0\}$, set $Y_{i,n} = 0$, and otherwise,

$$Y_{i,n} = (Y_{i,n}(l))_{1 \leq l \leq \|N_n\|} = (X_{i,n}(\tau_l))_{1 \leq l \leq \|N_n\|}. \quad (5.6)$$

In the notation of (5.4) and (5.5), $Y_{i,n}$ satisfies the following recursion on $\{\|N_n\| = k\}$:

$$\begin{aligned} Y_{i,n}(l) &= g_{i,n}^{1/2}(l-1)Z_{n,\tau_i}, & Y_{i,n}(0) &= 0, & 1 \leq l \leq k, \\ g_{i,n}(l) &= \eta_{i,n,1,l,\Delta\tau}(\theta) + g_{i,n}(l-1)\eta_{i,n,2,l,\Delta\tau}(\theta) \\ &\quad + \eta_{i,n,3,l,\Delta\tau}(\theta)g_{i,n}(l-1)Z_{n,\tau_i}^2, & 1 \leq l \leq k-1, \\ g_{i,n}(0) &= \hat{g}_{i,n,0,\Delta\tau,\theta}. \end{aligned} \tag{5.7}$$

Recall (5.3). For all $n \in \mathbb{N}$, $1 \leq k \leq n$ and $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, let

$$A_{n,k,\sigma} = \{\|N_n\| = k, \Delta\tau = \sigma\}. \tag{5.8}$$

For future purposes, we collect some useful inequalities in the next lemma.

5.2. Statement and proof of Lemma 5.1

Lemma 5.1. *Suppose that $(\Theta, (H_n)_{n \in \mathbb{N}})$ satisfies the assumptions of Theorem 2.2. Let $S \in (0, \infty)$ and suppose that $Q([-S, S]) = 1$.*

There then exist some $C = C(S, \Theta) \in (1, \infty)$ and $n_0 = n_0(S, \Theta) \in \mathbb{N}$ such that the following three inequalities hold:

$$|\hat{g}_{1,n,0,\sigma,\theta} - \hat{g}_{2,n,0,\sigma,\theta}| \leq \frac{C}{n}, \tag{5.9}$$

$$\hat{g}_{i,n,l,\sigma,\theta}(y) \geq C^{-1}, \tag{5.10}$$

$$E_\theta[|\hat{g}_{1,n,l,\sigma,\theta}(Y_{1,n}) - \hat{g}_{2,n,l,\sigma,\theta}(Y_{1,n})| | A_{n,k,\sigma}] \leq \frac{C^k}{n} \tag{5.11}$$

for all $n \geq n_0$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $\sigma \in M_{n,k}$, $i = 1, 2$, $\theta \in \Theta$, $y \in \mathbb{R}^k$ and $i = 1, 2$.

Proof. Let $(\Theta, (H_n)_{n \in \mathbb{N}})$ be as in Theorem 2.2. First, note that $(\Theta, (H_{1,n})_{n \in \mathbb{N}}) = (\Theta, (H_n)_{n \in \mathbb{N}})$ satisfies the assumptions of Theorem 2.2. Also, recall that $(\Theta, (H_{2,n})_{n \in \mathbb{N}}) = (\Theta, (H_n^{(0)})_{n \in \mathbb{N}})$ is defined in (2.11) and (2.12). In particular, observe that $\alpha_{i,n}(\theta) \rightarrow 1$ uniformly for all $\theta = (h_0, \beta, \alpha, \lambda) \in \Theta$ as $n \rightarrow \infty$, $i = 1, 2$, and thus there is an $n_1 = n_1(\Theta) \in \mathbb{N}$ satisfying

$$\frac{e^{-1}}{2} \leq [\alpha_{i,n}(\theta)]^n = \exp(n \log[n + n(\alpha_{i,n}(\theta) - 1)] - n \log n) \leq 2e \tag{5.12}$$

for all $n \geq n_1$, $i = 1, 2$ and $\theta = (h_0, \beta, \alpha, \lambda) \in \Theta$.

It follows from our assumptions on $(\Theta, (H_n)_{n \in \mathbb{N}})$ that there exist $n_0 = n_0(\Theta) \geq n_1$ and $C_1 = C_1(\Theta) \in (1, \infty)$ such that

$$\begin{aligned} \hat{g}_{i,n,l,\sigma,\theta}(y) &\geq h_{0,i,n}(\theta)[\alpha_{i,n}(\theta)]^{-1 + \sum_{m=1}^{l+1} \sigma_k} \geq \frac{e^{-1}}{2} \frac{h_{0,i,n}(\theta)}{\alpha_{i,n}(\theta)} \\ &\geq \frac{e^{-1}}{4} \inf_{(h_0, \beta, \alpha, \lambda) \in \Theta} h_0 \geq C_1^{-1} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \max \left\{ h_{0,i,n}(\theta), \beta_{i,n}(\theta), [\alpha_{i,n}(\theta)]^n, \frac{h_{0,i,n}(\theta)}{\alpha_{i,n}(\theta)} \right\} &\leq C_1, \\ \max \{ |h_{0,1,n}(\theta) - h_{0,2,n}(\theta)|, |\beta_{1,n}(\theta) - \beta_{2,n}(\theta)|, \\ |\alpha_{1,n}(\theta) - \alpha_{2,n}(\theta)|, |\lambda_{1,n}(\theta) - \lambda_{2,n}(\theta)| \} &\leq \frac{C_1}{n} \end{aligned} \quad (5.14)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, $i = 1, 2$, $\theta \in \Theta$ and $y \in \mathbb{R}^k$.

Recall (5.4) and (5.5). It follows from (5.14) that we have

$$\begin{aligned} \max \{ \eta_{i,n,2,l,\sigma}(\theta), \eta_{i,n,3,l,\sigma}(\theta) \} &\leq C_1^2, \\ \max \{ \eta_{i,n,1,l,\sigma}(\theta), \hat{g}_{i,n,0,\sigma,\theta} \} &\leq (k+1)C_1^2, \\ \max \{ |[\alpha_{1,n}(\theta)]^m - [\alpha_{1,n}(\theta)]^m| \} &\leq \frac{C_1^2 m}{n}, \\ \max \{ |\eta_{1,n,j,l,\sigma}(\theta) - \eta_{2,n,j,l,\sigma}(\theta)| : j = 1, 2, 3 \} &\leq \frac{(2e^2 C_1^3)^k}{n}, \\ |\hat{g}_{1,n,0,\sigma,\theta} - \hat{g}_{2,n,0,\sigma,\theta}| &\leq \frac{(4e^2 C_1^3)^k}{n} \end{aligned} \quad (5.15)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, $i = 1, 2$, $m \in \mathbb{N}_0$ and $\theta \in \Theta$.

Recall (5.7). Let $S > 1$ be such that $Q([-S, S]) = 1$ and set $C_2 = C_2(S, \theta) = e^2(1+S)^2 C_1^4$ and $C_3 = C_3(S, \theta) = S^2 C_2$. It follows from an induction and the inequalities in (5.15) that

$$\begin{aligned} E_\theta [g_{i,n}(l) | A_{n,k,\sigma}] &\leq C_1^2(k+1) + C_1^2(1+S^2)E_\theta [g_{i,n}(l-1) | A_{n,k,\sigma}] \\ &\leq (k+1) \sum_{m=0}^l (1+S^2)^m C_1^{2(1+m)} \leq C_2^k \end{aligned} \quad (5.16)$$

and thus that

$$E_\theta [Y_{i,n}^2(l) | A_{n,k,\sigma}] \leq C_3^k \quad (5.17)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$, $i = 1, 2$ and $\theta \in \Theta$.

Finally, let $C = C(S, \theta) = 12e^3 C_1^3 C_3$. By an induction, it follows from (5.15)–(5.17) that

$$\begin{aligned} E_\theta [|\hat{g}_{1,n,l,\sigma,\theta}(Y_{1,n}) - \hat{g}_{2,n,l,\sigma,\theta}(Y_{1,n})| | A_{n,k,\sigma}] \\ \leq C_3^k \sum_{j=1}^3 |\eta_{1,n,j,l}(\theta) - \eta_{2,n,j,l}(\theta)| + E_\theta [|\hat{g}_{1,n,l-1,\sigma,\theta}(Y_{1,n}) - \hat{g}_{2,n,l-1,\sigma,\theta}(Y_{1,n})| | A_{n,k,\sigma}] \\ \leq |\hat{g}_{1,n,0,\sigma,\theta} - \hat{g}_{2,n,0,\sigma,\theta}| + C_4^k \sum_{l=1}^{k-1} \sum_{j=1}^3 |\eta_{1,n,j,l}(\theta) - \eta_{2,n,j,l}(\theta)| \leq \frac{C^k}{n} \end{aligned}$$

for all $n \geq n_0$, $1 \leq k \leq n$, $0 \leq l \leq k-1$, $\sigma = (\sigma_l)_{1 \leq l \leq k} \in M_{n,k}$ and $\theta \in \Theta$. This completes the proof in view of (5.13) and (5.15). \square

5.3. Statement and proof of Lemma 5.2

We now provide an upper bound for conditional laws and their total variation norms in the next lemma.

Lemma 5.2. *Suppose that Q admits a Lebesgue density f , where f is globally Lipschitz and has a compact support $\{f > 0\}$.*

If $(\Theta, (H_n)_{n \in \mathbb{N}})$ satisfies the assumptions of Theorem 2.2, then there exist $n_0 = n_0(f, \Theta) \in \mathbb{N}$ and $C = C(f, \Theta) \in (0, \infty)$ such that

$$\|\mathcal{L}_\theta(Y_{1,n}|A_{n,k,\sigma}) - \mathcal{L}_\theta(Y_{2,n}|A_{n,k,\sigma})\| \leq \frac{C^k}{n} \quad (5.18)$$

for all $\theta \in \Theta$, $n \geq n_0$, $1 \leq k \leq n$ and $\sigma \in M_{n,k}$.

Proof. By assumption, we have $f(x) = 0$ for all $|x| \geq S$ and some $S > 0$. Hence, there are $n_0 = n_0(f, \theta) \in \mathbb{N}$ and $C_1 = C_1(f, \theta) \in (1, \infty)$ such that for C replaced by C_1 , the assertion of Lemma 5.1 holds.

Let $n \geq n_0$, $i = 1, 2$, $\theta \in \Theta$, $1 \leq k \leq n$ and $\sigma \in M_{n,k}$. Recall (5.5). In view of (5.10), $\Psi_{i,n,\theta}: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a well defined C^∞ diffeomorphism, where $\Psi_{i,n,\sigma,\theta} = (\psi_{i,n,l,\sigma,\theta})_{1 \leq l \leq k}: \mathbb{R}^k \rightarrow \mathbb{R}^k$ is defined by

$$\psi_{i,n,l,\sigma,\theta}(y) = \frac{y_l}{\hat{g}_{i,n,l-1,\sigma,\theta}^{1/2}(y)} \quad (5.19)$$

for $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ and $1 \leq l \leq k$. For all $n \geq n_0$, $\theta \in \Theta$, $n \geq n_0$, $1 \leq k \leq n$ and $\sigma \in M_{n,k}$, we define

$$\tilde{f}_{i,n,k,\sigma,\theta}(y) = \prod_{l=1}^k \frac{f(\psi_{i,n,l,\sigma,\theta}(y))}{\hat{g}_{i,n,l-1,\sigma,\theta}^{1/2}(y)}, \quad y \in \mathbb{R}^k, i = 1, 2.$$

It follows from (5.5), (5.7) and (5.19) that $\tilde{f}_{i,n,k,\sigma,\theta}$ is a density of the probability measure $\mathcal{L}_\theta(Y_{i,n}|A_{n,k,\sigma})$ with respect to the Lebesgue measure $\ell^{\otimes k}$ on $\mathcal{B}(\mathbb{R}^k)$. In particular, we must have

$$\|\mathcal{L}_\theta(Y_{1,n}|A_{n,k,\sigma}) - \mathcal{L}_\theta(Y_{2,n}|A_{n,k,\sigma})\| = \frac{1}{2} \int_{\mathbb{R}^k} |\tilde{f}_{1,n,k,\sigma,\theta}(y) - \tilde{f}_{2,n,k,\sigma,\theta}(y)| dy \quad (5.20)$$

for all $\theta \in \Theta$, $n \geq n_0$, $1 \leq k \leq n$ and $\sigma \in M_{n,k}$.

Suppose that $C_f \in (0, \infty)$ is a global Lipschitz constant of f . By means of simple substitutions, for all $\epsilon > 0$ and $w, v \geq \epsilon$, we can observe that

$$\frac{1}{2} \int \left| \frac{f(x/v)}{v} - \frac{f(x/w)}{w} \right| dx \leq \frac{1}{\epsilon} (S^2 C_f + 1) |v - w|.$$

Consequently, for all $\epsilon > 0$, we find a $\kappa_1 = \kappa_1(f, \epsilon) \in (1, \infty)$ such that

$$\frac{1}{2} \int \left| \frac{f(x/v)}{v} - \frac{f(x/w)}{w} \right| dx \leq \kappa_1(\epsilon) |v - w|, \quad v, w \geq \epsilon.$$

In view of (5.10), there thus exists some $\kappa_2 = \kappa_2(f, \Theta) \in (1, \infty)$ such that

$$\begin{aligned} & \frac{1}{2} \int \left| \frac{f(y_l / \hat{g}_{1,n,l-1,\sigma,\theta}^{1/2}(y))}{\hat{g}_{1,n,l-1,\sigma,\theta}^{1/2}(y)} - \frac{f(y_l / \hat{g}_{2,n,l-1,\sigma,\theta}^{1/2}(y))}{\hat{g}_{2,n,l-1,\sigma,\theta}^{1/2}(y)} \right| dy_l \\ & \leq \kappa_2 |\hat{g}_{1,n,l-1,\sigma,\theta}(y) - \hat{g}_{2,n,l-1,\sigma,\theta}(y)| \end{aligned} \quad (5.21)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $1 \leq l \leq k$, $\sigma \in M_{n,k}$, $y \in \mathbb{R}^k$ and $\theta \in \Theta$. By integrating over y_k , we get from (5.21) that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^k} |\tilde{f}_{1,n,k,\sigma,\theta}(y) - \tilde{f}_{2,n,k,\sigma,\theta}(y)| dy \\ & \leq \kappa_2 \int_{\mathbb{R}^{k-1}} \prod_{l=1}^{k-1} \frac{f(\psi_{1,n,l,\sigma,\theta}(y))}{\hat{g}_{1,n,l-1,\sigma,\theta}^{1/2}(y)} |\hat{g}_{1,n,k-1,\sigma,\theta}(y) - \hat{g}_{2,n,k-1,\sigma,\theta}(y)| dy \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^{k-1}} \left| \prod_{l=1}^{k-1} \frac{f(\psi_{1,n,l,\sigma,\theta}(y))}{\hat{g}_{1,n,l-1,\sigma,\theta}^{1/2}(y)} - \prod_{l=1}^{k-1} \frac{f(\psi_{2,n,l,\sigma,\theta}(y))}{\hat{g}_{2,n,l-1,\sigma,\theta}^{1/2}(y)} \right| dy \end{aligned} \quad (5.22)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $\sigma \in M_{n,k}$ and $\theta \in \Theta$. It follows from (5.11) that

$$\begin{aligned} & \int_{\mathbb{R}^{k-1}} \prod_{l=1}^{k-1} \frac{f(\psi_{1,n,l,\sigma,\theta}(y))}{\hat{g}_{1,n,l-1,\sigma,\theta}^{1/2}(y)} |\hat{g}_{1,n,k-1,\sigma,\theta}(y) - \hat{g}_{2,n,k-1,\sigma,\theta}(y)| dy \\ & = E_\theta[|\hat{g}_{1,n,k-1,\theta}(Y_{1,n}) - \hat{g}_{2,n,k-1,\theta}(Y_{1,n})| | A_{n,k,\sigma}] \leq \frac{C_1^k}{n} \end{aligned} \quad (5.23)$$

for all $n \geq n_0$, $1 \leq k \leq n$, $\sigma \in M_{n,k}$ and $\theta \in \Theta$.

Let $C = \epsilon \kappa_2 C_1$. By induction, we thus get from (5.9) and (5.22)–(5.23) that

$$\| \mathcal{L}_\theta(Y_{1,n} | A_{n,k,\sigma}) - \mathcal{L}_\theta(Y_{2,n} | A_{n,k,\sigma}) \| \leq \frac{C^k}{n},$$

uniformly for all $n \geq n_0$, $1 \leq k \leq n$, $\sigma \in M_{n,k}$ and $\theta \in \Theta$. This completes the proof of the lemma. \square

5.4. Proof of Theorem 2.2 (part II)

Let f be a Lebesgue density of Q and Θ be as in Theorem 2.2. We denote the positive part of a function $g: \mathbb{R} \rightarrow \mathbb{R}$ by g^+ . Let C_C^∞ be the space of infinitely often continuously differentiable

functions $g : \mathbb{R} \rightarrow \mathbb{R}$ with compact support $\overline{\{g > 0\}}$. As C_C^∞ is dense in L^1 , we find a sequence of $g_m \in C_C^\infty$, $m \in \mathbb{N}$, such that $\int |g_m - f| d\ell \rightarrow 0$ as $m \rightarrow \infty$. It is immediate that both $\int |g_m^+ - f| d\ell \rightarrow 0$ and $\int g_m^+ d\ell \rightarrow 1$ as $m \rightarrow \infty$. Without loss of generality, we may thus assume that $\int g_m^+ d\ell > 0$ for all $m \in \mathbb{N}$. Then, $h_m := g_m^+ / \int g_m^+ d\ell$ defines a sequence of globally Lipschitz continuous probability densities with a compact support $\overline{\{h_m > 0\}}$ such that $\int |h_m - f| d\ell \rightarrow 0$.

For $m \in \mathbb{N}$, let $Z_n^{(m)} = (Z_{n,k}^{(m)})_{1 \leq k \leq n}$ be a random vector with distribution

$$\mathcal{L}(Z_n^{(m)})(B) = \left((1 - p_n)\varepsilon_0(B) + p_n \int_B h_m d\ell \right)^{\otimes n},$$

with $B \in \mathcal{B}(\mathbb{R}^n)$, $m, n \in \mathbb{N}$, $1 \leq k \leq n$. If we replace $Z_{n,k}$ by $Z_{n,k}^{(m)}$ in (5.2), then we get yet another family of GARCH models, $X_{i,n}^{(m)} = (X_{i,n}^{(m)}(k))_{1 \leq k \leq n}$, say, indexed by $\theta \in \Theta$, $i = 1, 2$ and $m, n \in \mathbb{N}$.

It follows from the monotonicity theorem for Markov kernels and a well-known upper bound for product measures (see [26], Lemma 1.4.2(i) and page 23) that for all $i = 1, 2$,

$$\begin{aligned} \sup_{\theta \in \Theta} \|\mathcal{L}_\theta(X_{i,n}) - \mathcal{L}_\theta(X_{i,n}^{(m)})\| &\leq \|\mathcal{L}(Z_n) - \mathcal{L}(Z_n^{(m)})\| \\ &\leq n \|\mathcal{L}(Z_{n,1}) - \mathcal{L}(Z_{n,1}^{(m)})\| = \frac{np_n}{2} \int |h_m - f| d\ell. \end{aligned} \quad (5.24)$$

As h_m is globally Lipschitz with a compact support $\overline{\{h_m > 0\}}$ for all $m \in \mathbb{N}_0$, the assumptions of Lemma 5.2 hold. For all $m \in \mathbb{N}$, there thus exist $n_m \in \mathbb{N}$ and $C_m = C(h_m, \Theta) \in (0, \infty)$ such that for all $n \geq n_m$, we get, by conditioning and the monotonicity theorem for Markov kernels, that

$$\sup_{\theta \in \Theta} \|\mathcal{L}_\theta(X_{1,n}^{(m)}) - \mathcal{L}_\theta(X_{2,n}^{(m)})\| \leq \frac{1}{n} E[C_m^{\|N_n\|}] \quad (5.25)$$

for N_n as defined in (4.1). Recall (5.2). By combining (5.24) and (5.25), we get, from the triangle inequality, that

$$\sup_{\theta \in \Theta} \|\mathcal{L}_\theta(G_n) - \mathcal{L}_\theta(G_n^{(0)})\| \leq np_n \int |h_m - f| d\ell + \frac{1}{n} E[C_m^{\|N_n\|}]$$

for all $m \in \mathbb{N}$ and $n \geq n_m$. As (2.2) holds, we have $\lim_{n \rightarrow \infty} E C_m^{\|N_n\|} = e^{\lambda(C_m - 1)}$ and thus

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \|\mathcal{L}_\theta(G_n) - \mathcal{L}_\theta(G_n^{(0)})\| \leq \lambda \limsup_{m \rightarrow \infty} \int |h_m - f| d\ell = 0,$$

giving (2.15). This completes the proof of Theorem 2.2 in the case where $Q_n = Q$ for all $n \in \mathbb{N}$ (see (5.1)).

Now, assume that $Q_n \rightarrow Q$ in total variation norm as $n \rightarrow \infty$. For $m \in \mathbb{N}$, let $\widehat{Z}_n = (\widehat{Z}_{n,k})_{1 \leq k \leq n}$ be a random vector with distribution

$$\mathcal{L}(\widehat{Z}_n) = ((1 - p_n)\varepsilon_0 + p_n Q_n)^{\otimes n}, \quad n \in \mathbb{N}.$$

If we replace $Z_{n,k}$ by $\widehat{Z}_{n,k}$ in (5.2), then we get the GARCH models in the assertion of the theorem. We denote them by $\widehat{X}_{i,n}$, $n \in \mathbb{N}$, $i = 1, 2$. By the same argument as in (5.24), we must have, for all $i = 1, 2$ and $n \in \mathbb{N}$,

$$\sup_{\theta \in \Theta_0} \|\mathcal{L}_\theta(\widehat{X}_{i,n}) - \mathcal{L}_\theta(X_{i,n})\| \leq np_n \|Q_n - Q\|.$$

As the right-hand side tends to zero, this completes the proof of the theorem.

6. Proof of Theorem 2.3

We first need to make some preparations. Let $Z = (Z_n)_{n \in \mathbb{N}}$ and $U = (U_n)_{n \in \mathbb{N}}$ be independent sequences of i.i.d. random variables such that $\mathcal{L}(Z_1) = Q$ with Lebesgue density f and U_1 is uniformly distributed on $(0, 1)$. For $d \in \mathbb{N}$, we denote the order statistics of $0, U_1, \dots, U_d$ by $0 =: U_{d,0} < U_{d,1} \leq \dots \leq U_{d,d}$. For each $n \in \mathbb{N}$, let v_n be a Poisson random variable with parameter $\gamma_n > 0$, independent of Z and U .

In both (2.4) and (2.5), N admits a representation $N = \sum_{k=1}^{v_n} \varepsilon_{(U_{v_n,k}, Z_k)}$ since N is a Poisson measure with intensity $\gamma_n \ell \otimes Q$. On $\{v_n = 0\}$, we let $\Delta U_{v_n} = \Delta G_{v_n} = \Delta \widehat{G}_{v_n} = 0$, whereas on $\{v_n > 0\}$, we set

$$\begin{aligned} \Delta U_{v_n} &= (U_{v_n,k} - U_{v_n,k-1})_{1 \leq k \leq v_n}, \\ \Delta G_{v_n} &= (G(U_{v_n,k}) - G(U_{v_n,k-}))_{1 \leq k \leq v_n}, \\ \Delta \widehat{G}_{v_n} &= (\widehat{G}(U_{v_n,k}) - \widehat{G}(U_{v_n,k-}))_{1 \leq k \leq v_n}. \end{aligned}$$

Let $S_0 = \mathbb{R}^0 = \{0\}$ and $\widetilde{\mathbb{R}} = \bigcup_{d=0}^{\infty} \{d\} \times S_d \times \mathbb{R}^d$, where for $d \in \mathbb{N}$, S_d equals the set of all $w = (w_1, \dots, w_d)' \in (0, 1)^d$ such that $\sum_{i=1}^d w_i \leq 1$. We endow S_d and $\widetilde{\mathbb{R}}$ with the Borel trace field $\mathcal{B}(S_d)$ ($d \geq 0$) and the σ -algebra $\widetilde{\mathcal{B}}$, respectively, where $\widetilde{\mathcal{B}}$ is the set of all $B \subseteq \widetilde{\mathbb{R}}$ such that $B \cap (\{d\} \times S_d \times \mathbb{R}^d) \in \{\emptyset, \{d\}\} \otimes \mathcal{B}(S_d) \otimes \mathcal{B}(\mathbb{R}^d)$ for all $d \in \mathbb{N}_0$.

Since we have assumed that $\Theta \subseteq (0, \infty) \times [0, \infty)^3$, and since G and \widehat{G} always jump at the same time as N does, all arrival times are observed in full and thus $\mathcal{E}_{\gamma_n, Q}(\Theta)$ and $\widehat{\mathcal{E}}_{\gamma_n, Q}(\Theta)$ are equivalent to \mathcal{F}_n and $\widehat{\mathcal{F}}_n$ in deficiency, respectively, in view of (A.2), where for all $n \in \mathbb{N}$, we set

$$\begin{aligned} \mathcal{F}_n &= (\widetilde{\mathbb{R}}, \widetilde{\mathcal{B}}, (\mathcal{L}_\theta(v_n, \Delta U_{v_n}, \Delta G_{v_n}))_{\theta \in \Theta}), \\ \widehat{\mathcal{F}}_n &= (\widetilde{\mathbb{R}}, \widetilde{\mathcal{B}}, (\mathcal{L}_\theta(v_n, \Delta U_{v_n}, \Delta \widehat{G}_{v_n}))_{\theta \in \Theta}). \end{aligned}$$

Let $\widehat{w}_0 = 0$ and for $d > 0$, set $\widehat{w}_d = (1/d, \dots, 1/d) \in \mathbb{R}^d$. Recall that $\Theta \subseteq (0, \infty) \times [0, \infty)^3$ and pick $d \in \mathbb{N}_0$, $\theta = (h_0, \beta, \alpha, \lambda) \in \Theta$, $w = (w_1, \dots, w_d) \in S_d \cup \{\widehat{w}_d\}$. We define a diffeomorphism $\Psi_{d,w,\theta} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as follows: if $d = 0$, then let $\Psi_{d,w,\theta} = 0$; otherwise, if $d > 0$, let

$$\Psi_{d,w,\theta}(z) = (h_{d,w,\theta,k}^{1/2}(z) z_k)_{1 \leq k \leq d}, \quad z = (z_1, \dots, z_d) \in \mathbb{R}^d,$$

where for $2 \leq k \leq d$, we recursively define

$$\begin{aligned} h_{d,w,\theta,k}(z) &= \frac{\beta}{\alpha}(1 - e^{-\alpha w_k}) + e^{-\alpha w_k}(1 + \lambda z_{k-1}^2)h_{d,w,\theta,k-1}(z), \\ h_{d,w,\theta,1}(z) &\equiv h_{d,w,\theta,1} = \frac{\beta}{\alpha}(1 - e^{-\alpha w_1}) + e^{-\alpha w_1}h_0, \end{aligned} \quad (6.1)$$

provided $\alpha > 0$ and otherwise, if $\alpha = 0$, we set

$$\begin{aligned} h_{d,\theta,w,k}(z) &= \beta w_k + h_{d,\theta,w,k-1}(z)(1 + \lambda z_{k-1}^2), \\ h_{d,\theta,w,1}(z) &\equiv h_{d,w,\theta,1} = \beta w_1 + h_0. \end{aligned} \quad (6.2)$$

Let f be a strictly positive Lebesgue density of Q and set

$$\begin{aligned} \mathcal{H}_{d,\theta_1,\theta_2,w}(\zeta) &= \int_{\mathbb{R}^d} (|J_{\Psi_{d,w,\theta_1}^{-1}}(x)|f^{\otimes d}(\Psi_{d,w,\theta_1}^{-1}(x)))^\zeta (|J_{\Psi_{d,w,\theta_2}^{-1}}(x)|f^{\otimes d}(\Psi_{d,w,\theta_2}^{-1}(x)))^{1-\zeta} dx \end{aligned}$$

for all $\theta_1, \theta_2 \in \Theta$, $0 < \zeta < 1$, $w \in S_d \cup \{\widehat{w}_d\}$.

To summarize, we have thus far shown that for all $n \in \mathbb{N}$, equivalence of $\mathcal{E}_{\gamma_n, Q}(\Theta)$ and $\widehat{\mathcal{E}}_{\gamma_n, Q}(\Theta)$ in deficiency is equivalent to equivalence of \mathcal{F}_n and $\widehat{\mathcal{F}}_n$ in deficiency. For the remaining part, recall that the two experiments are equivalent in deficiency if and only if their corresponding Hellinger transformations are equal (see [27], Corollary 53.8). By solving the differential equations in (2.4) and (2.5), we thus arrive at the following identity:

$$\sum_{d=1}^{\infty} \frac{\gamma_n^d e^{-\gamma_n}}{d!} \mathcal{H}_{d,\theta_1,\theta_2,\widehat{w}_d}(\zeta) = \sum_{d=1}^{\infty} \frac{\gamma_n^d e^{-\gamma_n}}{d!} \int_{S_d} \mathcal{H}_{d,\theta_1,\theta_2,w}(\zeta) \frac{dw}{\ell^{\otimes d}(S_d)}$$

for all $\theta_1, \theta_2 \in \Theta$, $0 < \zeta < 1$, $n \in \mathbb{N}$.

In the last display, the functions are analytical in γ_n ; consequently, for all $d \in \mathbb{N}$, $\theta_1, \theta_2 \in \Theta$, $0 < \zeta < 1$, we must have

$$\mathcal{H}_{d,\theta_1,\theta_2,\widehat{w}_d}(\zeta) = \int_{S_d} \mathcal{H}_{d,\theta_1,\theta_2,w}(\zeta) \frac{dw}{\ell^{\otimes d}(S_d)}. \quad (6.3)$$

Next, we return to the proof of the theorem. By our assumption, there exists $\zeta_0 \in (0, 1)$ such that, with $g_{f,\zeta_0}: (0, \infty) \rightarrow [0, 1]$ as in (2.18), g_{f,ζ_0} is strictly increasing on $(0, 1]$. As a result, $h \mapsto g_{f,\zeta_0}(\sqrt{h})$ is strictly increasing on $(0, 1]$.

For all $\theta_1, \theta_2 \in \Theta$, define $H_{\theta_1,\theta_2}: (0, 1] \rightarrow (0, \infty)$ by $H_{\theta_1,\theta_2}(w) := h_{1,w,\theta_2,1}(1)/h_{1,w,\theta_1,1}(1)$ for $0 < w \leq 1$. In particular, taking $d = 1$ and $\zeta = \zeta_0$ in (6.3), we must have

$$g_{f,\zeta_0}\{\sqrt{H_{\theta_1,\theta_2}(1)}\} = \int_{(0,1)} g_{f,\zeta_0}\{\sqrt{H_{\theta_1,\theta_2}(w)}\} dw \quad (6.4)$$

for all $\theta_1, \theta_2 \in \Theta$.

(i) and (ii) For $i = 1, 2$, let $\theta_i = (h_{0,i}, \beta_i, \alpha, \lambda) \in \Theta$. Then,

$$h_{1,w,\theta_{i,1}}^2(1)e^{\alpha w} \frac{d}{dw} H_{\theta_1,\theta_2}(w) = \beta_2 h_{0,1} - \beta_1 h_{0,2}, \quad 0 < w \leq 1.$$

(Note that this formula extends to $\alpha = 0$.) If $\beta_1 = \beta_2 > 0$ and $h_{0,1} > h_{0,2}$, then H_{θ_1,θ_2} is strictly increasing with $H_{\theta_1,\theta_2}(1) \leq 1$, contradicting (6.4) since $h \mapsto g_{f,\zeta_0}(\sqrt{h})$ is strictly increasing on $(0, 1]$. If $h_{0,1} = h_{0,2}$ and $\beta_2 < \beta_1$, then H_{θ_1,θ_2} is strictly decreasing with $H_{\theta_1,\theta_2}(0+) = 1$, contradicting (6.4) since $h \mapsto g_{f,\zeta_0}(\sqrt{h})$ is strictly increasing on $(0, 1]$. Reversing the roles of parameters by replacing H_{θ_1,θ_2} with H_{θ_2,θ_1} , the previous reasoning extends to the remaining cases where either $\beta_1 = \beta_2 > 0$ and $h_{0,1} < h_{0,2}$, or $h_{0,1} = h_{0,2}$ and $\beta_2 > \beta_1$. This completes the proof of (i) and (ii).

(iii) If $(h_0, \beta, \alpha_1, \lambda), (h_0, \beta, \alpha_2, \lambda) \in \Theta$ and $\beta = 0$, then we have $H_{\theta_1,\theta_2}(w) = e^{(\alpha_1 - \alpha_2)w}$ for all $w \in (0, 1]$. (Note that this formula extends to $\alpha_1 = 0$ or $\alpha_2 = 0$.) By the same arguments as in parts (i) and (ii), we get from (6.4) that $\alpha_1 = \alpha_2$.

(iv) In view of (iii), we may assume that $\beta > 0$. Contradicting the hypothesis, we assume that $\alpha_2 > 0$. It follows from the strict inequality $e^x - 1 > x, x > 0$, that

$$\begin{aligned} (h_0 + \beta w)^2 \frac{d}{dw} H_{\theta_1,\theta_2}(w) &= e^{-\alpha_2 w} \left\{ w(\beta^2 - \alpha_2 \beta h_0) - h_0^2 \alpha_2 - \frac{\beta^2}{\alpha_2} (e^{\alpha_2 w} - 1) \right\} \\ &< -\alpha_2 h_0 e^{-\alpha_2 w} (h_0 + w\beta) < 0 \end{aligned}$$

for all $w \in (0, 1]$. Thus, $w \mapsto H_{\theta_1,\theta_2}(w)$ is strictly decreasing on $(0, 1]$ with $H_{\theta_1,\theta_2}(0+) = 1$, contradicting (6.4). Thus, we must have $\alpha_2 = 0$.

(v) Let $(h_0, \beta, \alpha_1, \lambda), (h_0, \beta, \alpha_2, \lambda) \in \Theta$ with $\alpha_2 > \alpha_1$. Without loss of generality, we may assume that $\beta > 0$. First, assume that $\beta/\alpha_2 \leq h_0 \leq \beta/\alpha_1$. Then, $\beta - \alpha_1 h_0 \geq 0$ and $\beta - \alpha_2 h_0 \leq 0$. Note that we cannot simultaneously have that $\beta - \alpha_1 h_0 = \beta - \alpha_2 h_0 = 0$ such that

$$h_{1,w,\theta_{1,1}}^2 \frac{d}{dw} H_{\theta_1,\theta_2}(w) = (\beta - \alpha_2 h_0) e^{-\alpha_2 w} h_{1,w,\theta_{1,1}} - (\beta - \alpha_1 h_0) e^{-\alpha_1 w} h_{1,w,\theta_{2,1}} < 0$$

for all $0 < w \leq 1$. Consequently, H_{θ_1,θ_2} is strictly decreasing with $H_{\theta_1,\theta_2}(0+) = 1$, contradicting (6.4). Second, let $h_0 < \beta/\alpha_2$ and set

$$\psi(w) := (\beta - \alpha_2 h_0) h_{1,w,\theta_{1,1}} - (\beta - \alpha_1 h_0) e^{-(\alpha_1 - \alpha_2)w} h_{1,w,\theta_{2,1}}, \quad 0 < w \leq 1.$$

As we have $\alpha_2 > \alpha_1$ and $h_0 < \beta/\alpha_2$, we must have that $\beta - \alpha_1 h_0 > \beta - \alpha_2 h_0 > 0$ such that

$$\psi'(w) = (\alpha_1 - \alpha_2)(\beta - \alpha_1 h_0) e^{-(\alpha_1 - \alpha_2)w} h_{d,w,\theta_{2,1}} < 0, \quad 0 < w \leq 1.$$

Note that $\psi(0+) = (\alpha_1 - \alpha_2) h_0^2 < 0$ and thus $\psi(w) < 0$ for all $0 < w \leq 1$. Since $e^{\alpha_2 w} h_{d,w,\theta_{1,1}}^2 \frac{d}{dw} H_{\theta_1,\theta_2}(w) = \psi(w) < 0$ for all $0 < w \leq 1$, H_{θ_1,θ_2} is strictly decreasing with $H_{\theta_1,\theta_2}(0+) = 1$, contradicting (6.4). This completes the proof of (v).

7. Proofs of the results in Section 2.4

This section contains the proofs of Propositions 2.1–2.3.

7.1. Proof of Proposition 2.1

For $f \in D_d[0, 1]$, we write $\Delta f = f(t) - f(t-)$, $0 \leq t \leq 1$, with the convention that $\Delta f(0) = 0$. With the usual convention $\inf \emptyset = \infty$, we define $T(f) = \inf\{t \in [0, 1] : \Delta f_1(t) \neq 0\} \wedge 1$ for all $f = (f_1, f_2) \in D_2$. Let S be the set of all functions $f \in D_2$ with $T(f) \in (0, 1)$. Let $D''_0 \subseteq D_2[0, 1]$ be the set of all functions $f = (f_1, f_2)$ such that the right-hand derivatives $f'_1(0+)$ and $f''_2(0+)$ exist in \mathbb{R} . Further, let $D''_{0,T} \subseteq S \cap D''_0$ be the set of all functions $f = (f_1, f_2)$ such that the right-hand derivatives $f'_2(T(f)+)$ and $f''_2(T(f)+)$ exist in \mathbb{R} .

Let $f \in D_2$ with $T = T(f)$. If $f \in D''_{0,T}$ and $f'_2(0+) \neq 0$, then we set

$$X(f) = \left(|f_2(0)|, \left| \frac{(f'_2(0+))^2 - f_2(0)f''_2(0+)}{f'_2(0+)} \right|, \left| \frac{f''_2(0+)}{f'_2(0+)} \right|, \frac{|\Delta f_2(T)|}{(\Delta f_1(T))^2} \right).$$

If $f \in D''_{0,T}$, $f'_2(0+) = 0$ and $f'_2(T+) \neq 0$, then we set

$$X(f) = \left(|f_2(0)|, \left| \frac{(f'_2(T+))^2 - f_2(T)f''_2(T+)}{f'_2(T+)} \right|, \left| \frac{f''_2(T+)}{f'_2(T+)} \right|, \frac{|\Delta f_2(T)|}{(\Delta f_1(T))^2} \right).$$

If $f \in D''_{0,T}$, $f'(0+) = f'_2(T+) = 0$ and $\Delta f_2(T) \neq 0$, then we set

$$X(f) = \left(|f_2(0)|, 0, 0, \frac{|\Delta f_2(T)|}{(\Delta f_1(T))^2} \right).$$

If $f \in D''_{0,T}$, $f'(0+) = f'_2(T+) = 0$ and $\Delta f_2(T) = 0$, then we set $X(f) = (|f_2(0)|, \infty, \infty, 0)$. If $f \in D''_0 \setminus S$ and $f'_2(0+) \neq 0$, then we define

$$X(f) = \left(|f_2(0)|, \left| \frac{(f'_2(0+))^2 - f_2(0)f''_2(0+)}{f'_2(0+)} \right|, \left| \frac{f''_2(0+)}{f'_2(0+)} \right|, \infty \right).$$

For the remaining cases, we set $X(f) = (|f_2(0)|, \infty, \infty, \infty)$. Then, $X : D_2 \rightarrow [0, \infty]^4$ is a D_2 - $\mathcal{B}([0, \infty]^4)$ -measurable mapping. Since $Q(\{0\}) = 0$, it follows from (2.4) that $\mathcal{L}_\theta^X((G, h)) = Q_\theta$ for all $\theta \in [0, \infty]^4$ and thus $\delta(\mathcal{E}_h, \mathcal{F}) = 0$ by (A.2), where \mathcal{F} is the experiment as defined in the assertion of the proposition.

Next, we show that $\delta(\mathcal{F}, \mathcal{E}_h) = 0$. To this end, we define $\xi = (\xi_1, \dots, \xi_3) : [0, \infty]^4 \rightarrow [0, \infty]^3$ as follows. Let $\omega = (\omega_1, \dots, \omega_4) \in [0, \infty]^4$. If $(\omega_1, \dots, \omega_3) \in [0, \infty]^3$, then we set $\xi(\omega) = (\omega_1, \omega_2, \omega_3)$; if $\omega_1 \in [0, \infty)$ and either $\omega_2 = \infty$ or $\omega_3 = \infty$, then we set $\xi(\omega) = (\omega_1, 0, 0)$; otherwise, we set $\xi(\omega) = 0$.

In the notation of the [Introduction](#), we define $\Psi : [0, \infty]^3 \times \mathbb{M}_2 \rightarrow D_2$, where, for $0 \leq t \leq 1$, $\omega = (\omega_1, \omega_2, \omega_3) \in [0, \infty]^3$ and $\sigma \in \mathbb{M}_2$, $(f_1(t), f_2(t)) = \Psi[\omega, \sigma](t)$ is defined to be the unique

solution of the system of the following integral equations:

$$\begin{aligned} f_1(t) &= \int_{[0,t] \times \mathbb{R}^2} f_2^{1/2}(s-) z_1 \sigma(ds, dz_1, dz_2), \\ f_2(t) &= \omega_1 + \int_{[0,t]} (\omega_2 - \omega_3 f_2(s-)) ds + \int_{[0,t] \times \mathbb{R} \times (0, \infty)} f_2(s-) z_2^2 \sigma(ds, dz_1, dz_2). \end{aligned} \tag{7.1}$$

Clearly, Ψ is $(\mathcal{B}([0, \infty)^3) \otimes \mathcal{M}_2) / \mathcal{D}_2$ -measurable and thus defines a deterministic Markov kernel $K_2 : ([0, \infty)^3 \times \mathbb{M}_2) \times \mathcal{D}_2 \rightarrow [0, 1]$.

Let ν_0 be the zero measure on $\mathcal{B}([0, 1] \times \mathbb{R}^2)$. For $\lambda \geq 0$, let M_λ be a Poisson measure on $[0, 1] \times \mathbb{R}^2$ with the intensity measure $\gamma \ell \otimes \mathcal{L}(Z, \lambda^{1/2} Z)$, where $\mathcal{L}(Z) = Q$ and $\gamma > 0$ is the intensity parameter of N in (2.4). Consider the Markov kernel $K_1 : [0, \infty]^4 \times (\mathcal{B}([0, \infty)^3) \otimes \mathcal{M}_2) \rightarrow [0, 1]$ defined by

$$K_1[(\omega_1, \omega_2, \omega_3, \omega_4), \cdot] = \varepsilon_{\hat{\xi}(\omega)} \otimes \begin{cases} \varepsilon_{\nu_0}, & \omega_4 = \infty, \\ \mathcal{L}(M_{\omega_4} | M_{\omega_4} \neq \nu_0), & \omega_4 < \infty. \end{cases}$$

Observe that $K_2 K_1 Q_\theta = \mathcal{L}_\theta(G, h)$ for all $\theta \in [0, \infty)^4$, in view of (2.4). Hence, $\delta(\mathcal{F}, \mathcal{E}_h) = 0$, by (A.2).

To summarize, we have shown that \mathcal{E}_h is equivalent to \mathcal{F} in deficiency. By means of similar arguments, we can show that $\Delta(\mathcal{F}, \hat{\mathcal{E}}_h) = 0$.

7.2. Proof of Proposition 2.2

(i) Let $H_n = H_n^{(0)} : [0, \infty)^4 \rightarrow [0, \infty)^4$ be as defined in (2.11)–(2.12) and define $\bar{H}_n : [0, \infty)^3 \rightarrow M := \{(x_1, x_2, x_3) \in [0, \infty)^2 \times (0, 1] : x_1 \geq x_2\}$ by

$$\bar{H}_n(h_0, \beta, \alpha) = (h_{0,n}(h_0, \beta, \alpha, 0), \beta_n(h_0, \beta, \alpha, 0), \alpha_n(h_0, \beta, \alpha, 0)),$$

$h_0, \beta, \alpha \in [0, \infty)$. Then, $H_n : [0, \infty)^3 \rightarrow M \times [0, \infty)$ and $\bar{H}_n : [0, \infty)^3 \rightarrow M$ are both bijections with inverse functions $H_n^{-1} : M \times [0, \infty) \rightarrow [0, \infty)^4$ and $\bar{H}_n^{-1} : M \rightarrow [0, \infty)^3$, respectively. Define $\tilde{H}_n : \mathbb{R}^3 \rightarrow [0, \infty)^3$ and $\hat{H}_n : \mathbb{R}^4 \rightarrow [0, \infty)^4$ by $\tilde{H}_n(x_1, x_2, x_3) = \bar{H}_n^{-1}(|x_1| \vee |x_2|, |x_2|, |x_3| \wedge 1)$ and $\hat{H}_n(x_1, x_2, x_3, x_4) = H_n^{-1}(|x_1| \vee |x_2|, |x_2|, |x_3| \wedge 1, |x_4|)$ for $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with $x_3 \neq 0$.

In the sequel, we write $x = (x(k))_{0 \leq k \leq n}$ for a generic element of \mathbb{R}^{n+1} . Fix $n \geq 5$. Let $M_{0,n} \subseteq [\mathbb{R}^{n+1}]^2$ be the set of all (x, y) such that both $y(0) \neq y(1)$ and $y(1) \neq y(2)$. By employing the convention $\inf \emptyset = \infty$, we define $T_n : [\mathbb{R}^{n+1}]^2 \rightarrow \{1, \dots, n+1\}$ by

$$T_n(x, y) = \inf\{1 \leq k \leq n : x(k) \neq x(k-1)\} \wedge 1, \quad (x, y) \in [\mathbb{R}^{n+1}]^2.$$

Let S_n be the set of all $(x, y) \in [\mathbb{R}^{n+1}]^2$ with $3 \leq T(x, y) \leq n-2$ such that $x(T) = x(T+1) = x(T+2)$. Consider the subset $M_{T,n} \subseteq S_n$ of all $(x, y) \in [\mathbb{R}^{n+1}]^2$ such that both $y(T) \neq y(T+1)$ and $y(T+1) \neq y(T+2)$ are satisfied.

For all $n \geq 5$, we define a mapping $X_n : [\mathbb{R}^{n+1}]^2 \rightarrow [0, \infty]^4$ as follows: fix $(x, y) \in [\mathbb{R}^{n+1}]^2$ and set $T = T_n(x, y)$. If $(x, y) \in S_n \cap M_{0,n}$, then set

$$X_n(x, y) = \widehat{H}_n \left(y(0), \frac{y(1)^2 - y(0)y(2)}{y(1) - y(0)}, \frac{y(2) - y(1)}{y(1) - y(0)}, \frac{y(T)}{[x(T) - x(T-1)]^2} - \frac{y(1)^2 - y(0)y(2) + y(T-1)[y(2) - y(1)]}{[y(1) - y(0)][x(T) - x(T-1)]^2} \right).$$

If $(x, y) \in M_{T,n} \setminus M_{0,n}$, then set

$$\begin{aligned} X_n(x, y) &= \widehat{H}_n \left(y(0), \frac{y(T+1)^2 - y(T)y(T+2)}{y(T+1) - y(T)}, \frac{y(T+2) - y(T+1)}{y(T+1) - y(T)}, \frac{y(T)}{[x(T) - x(T-1)]^2} \right. \\ &\quad \left. - \frac{y(T+1)^2 - y(T)y(T+2) + y(T-1)[y(T+2) - y(T+1)]}{[y(T+1) - y(T)][x(T) - x(T-1)]^2} \right). \end{aligned}$$

If $(x, y) \in S_n \setminus (M_{0,n} \cup M_{T,n})$ and $y(T) \neq y(T-1)$, then set

$$X_n(x, y) = \left(y(0), 0, 0, \frac{|y(T) - y(T-1)|}{(x(T) - x(T-1))^2} \right).$$

If $(x, y) \in S_n \setminus (M_{0,n} \cup M_{T,n})$ and $y(T) = y(T-1)$, then set $X_n(x, y) = (|y(0)|, \infty, \infty, 0)$. If $(x, y) \in M_{0,n} \setminus S_n$ and $T = n+1$, then set

$$X_n(x, y) = \left(\widetilde{H}_n \left[y(0), \frac{y(1)^2 - y(0)y(2)}{y(1) - y(0)}, \frac{y(2) - y(1)}{y(1) - y(0)} \right], \infty \right).$$

Otherwise, set $X_n(x, y) = (|y(0)|, \infty, \infty, \infty)$.

Recall that both $G_n = (G_{n,k})_{0 \leq k \leq n}$ and $h_n = (h_{n,k})_{0 \leq k \leq n}$ are defined by (2.3) via (2.11) and (2.12). For $n \geq 5$, the mapping $X_n : [\mathbb{R}^{n+1}]^2 \rightarrow [0, \infty]^4$ is well defined and $\mathcal{B}([\mathbb{R}^{n+1}]^2) / \mathcal{B}([0, \infty]^4)$ -measurable. Recall that $Q_n(\{0\}) = 0$ for all $n \in \mathbb{N}$ and thus

$$\mathcal{L}_\theta^{X_n}(G_n, h_n) = \begin{cases} q_{1,n} \varepsilon_{(h_0, \beta, \alpha, \infty)} + q_{2,n} \varepsilon_\theta & \theta \notin \Theta_e, \\ + (1 - q_{1,n} - q_{2,n}) \varepsilon_{(h_0, n(\theta), \infty, \infty, \infty)}, & \theta \in \Theta_e, h_0 > 0, \lambda > 0, \\ (1 - q_{2,n}) \varepsilon_{(h_0, \infty, \infty, \infty)} + q_{2,n} \varepsilon_\theta, & \theta \in \Theta_e, h_0 > 0, \lambda = 0, \\ (1 - q_{2,n}) \varepsilon_{(h_0, \infty, \infty, \infty)} + q_{2,n} \varepsilon_{(h_0, \infty, \infty, 0)}, & \theta \in \Theta_e, h_0 = 0 \\ \varepsilon_{(0, \infty, \infty, \infty)}, & \theta \in \Theta_e, h_0 = 0 \end{cases}$$

for all $n \geq 5$, $\theta = (h_0, \beta, \alpha, \lambda) \in [0, \infty]^4$, where we set $q_{1,n} = (1 - p_n)^n$ and $q_{2,n} = (1 - p_n)^2 [1 - p_n - p_n(1 - p_n)] [1 - (1 - p_n)^{n-4}]$.

On the other hand, define a mapping $\xi_n = (\xi_{1,n}, \dots, \xi_{4,n}) : [0, \infty]^4 \rightarrow [0, \infty]^4$ as follows. Let $\omega = (\omega_1, \dots, \omega_4) \in [0, \infty]^4$. If $\omega \in [0, \infty]^4$, then set $\xi_n(\omega) = H_n(\omega)$. If $\omega \in [0, \infty]^3 \times \{\infty\}$, then set $\xi_n(\omega) = (\widetilde{H}_n(\omega_1, \omega_2, \omega_3), 0)$. If $\omega \in [0, \infty) \times (\{\infty\} \times [0, \infty] \cup [0, \infty] \times \{\infty\}) \times [0, \infty)$, then set $\xi_n(\omega) = (\omega_1, 0, 1, \omega_4)$. If $\omega \in [0, \infty) \times (\{\infty\} \times [0, \infty] \cup [0, \infty] \times \{\infty\}) \times \{\infty\}$, then

set $\xi_n(\omega) = (\omega_1, 0, 1, 0)$. Otherwise, set $\xi(\omega) = 0$. Define a Markov kernel $K_{1,n} : [0, \infty]^4 \times \mathcal{B}([0, \infty)^3 \times [\mathbb{R}^n]^2) \rightarrow [0, 1]$ by

$$K_{1,n}[\omega, \cdot] = \varepsilon_{(\xi_{n,1}(\omega), \xi_{n,2}(\omega), \xi_{n,3}(\omega))} \otimes \begin{cases} \varepsilon_0, & \omega_4 = \infty, \\ \mathcal{L}((Z_{n,k})_k, (\xi_4(\omega) Z_{n,k}^2)_k | (Z_{n,k})_k \neq 0), & \omega_4 < \infty \end{cases}$$

for $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in [0, \infty]^4$, where $Z_n = (Z_{n,k})_k$ is the random vector with the distribution as specified by (2.1).

Also, let $K_{2,n} : [0, \infty]^3 \times [\mathbb{R}^n]^2 \times \mathcal{B}([\mathbb{R}^{n+1}]^2) \rightarrow [0, 1]$ be the Markov kernel defined by the deterministic mapping $(\xi_1, \xi_2, \xi_3, z_1, z_2) \mapsto (x, y)$, where we recursively set $x(0) = 0$ and $y(0) = \xi_1$, and for $1 \leq k \leq n$,

$$x(k) = x(k-1) + y^{1/2}(k-1)z_1(k), \quad y(k) = \xi_2 + y(k-1)(\xi_3 + z_2(k)).$$

For $n \geq 5$, let $\mathcal{F}_n = ([0, \infty]^4, \mathcal{B}([0, \infty]^4), (\mathcal{L}_\theta^{X_n}(G_n, h_n))_{\theta \in [0, \infty)^4})$. By construction, we have $\delta(\mathcal{E}_{h,n}, \mathcal{F}_n) = 0$, by means of (A.2). For all $n \geq 5$, observe that

$$\begin{aligned} \delta(\mathcal{F}_n, \mathcal{E}_{h,n}) &\leq \sup_{\theta \in [0, \infty)^3} \|\mathcal{L}_\theta(G_n, h_n) - K_{2,n} K_{1,n} \mathcal{L}_\theta^{X_n}(G_n, h_n)\| \\ &\leq |1 - q_{1,n} - q_{2,n}| + |1 - (1 - p_n)^n - q_{2,n}| \end{aligned}$$

and thus $\mathcal{E}_{h,n}$ is strongly asymptotically equivalent to \mathcal{F}_n as $n \rightarrow \infty$, by means of (A.2) and (2.2). By (A.4), \mathcal{F}_n converges strongly to the experiment \mathcal{F} in the assertion of Proposition 2.1, completing the proof of (i).

(ii) This follows from the same arguments as in (i).

7.3. Proof of Proposition 2.3

(i) Define $X, X_n : [0, \infty]^4 \rightarrow [0, \infty]^4$ as follows. If $\omega = (\omega_1, \dots, \omega_4) \in [0, \infty)^3 \times \{\infty\}$ such that $\omega_1 \omega_3 = \omega_2$, then set $X(\omega) = (\omega_1, \infty, \infty, \infty)$; otherwise, set $X(\omega) = \omega$. If $\omega = (\omega_1, \dots, \omega_4) \in [0, \infty)^3 \times \{\infty\}$ such that $\omega_1 n(1 - e^{-\omega_3/n}) = \omega_2$, then set $X_n(\omega) = (\omega_1, \infty, \infty, \infty)$; otherwise, set $X_n(\omega) = \omega, n \in \mathbb{N}$.

By definition, the deficiency is non-decreasing in the parameter set with respect to set inclusions. Further, we have $\widehat{Q}_\theta^X = Q_\theta$ and $\widehat{Q}_\theta^{X_n} = Q_{\theta,n}$ for all $n \in \mathbb{N}$ and thus, by (A.2), $\delta(\widehat{\mathcal{F}}(\Theta), \mathcal{F}(\Theta)) \leq \delta(\widehat{\mathcal{F}}, \mathcal{F}) = 0$ and $\delta(\widehat{\mathcal{F}}(\Theta), \mathcal{F}_n(\Theta)) \leq \delta(\widehat{\mathcal{F}}, \mathcal{F}_n) = 0$ for all $n \in \mathbb{N}$, completing the proof of (i).

(ii) First, assume that Θ satisfies (2.24) for all $x > 0$. Without loss of generality, we may assume that $\Theta \subseteq [0, \infty)^4$ is a finite set (see [27], Theorem 51.4). Define Ω_Θ to be the set of all $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in (0, \infty) \times \{\infty\}^2 \times \{0, \infty\}$ such that $(\omega_1, \beta, \alpha, \lambda) \in (\Theta \cap \Theta_e) \setminus \widehat{\Theta}_e$ for some $(\beta, \alpha, \lambda) \in [0, \infty)^3$. If $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_\Theta$, then it follows from (2.24) that the corresponding pair $(\beta, \alpha) = (\beta(\omega_1), \alpha(\omega_1)) \in [0, \infty)^2$ is uniquely determined by ω_1 . Hence, we may define a mapping $Y : [0, \infty]^4 \rightarrow [0, \infty]^4$ as follows: if $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_\Theta$, then we

set $Y(\omega) = (\omega_1, \beta(\omega_1), \alpha(\omega_1), \omega_4)$; otherwise, if $\omega \in [0, \infty]^4 \setminus \Omega_\Theta$, then we set $Y(\omega) = \omega$. As Θ , and thus Ω_Θ , is a finite set, the mapping Y is $\mathcal{B}([0, \infty]^4)/\mathcal{B}([0, \infty]^4)$ -measurable. In view of (2.24), note that $Q_\theta^Y = \widehat{Q}_\theta$ for all $\theta \in \Theta$ and thus $\delta(\mathcal{F}(\Theta), \widehat{\mathcal{F}}(\Theta)) = 0$, by (A.2).

Second, assume that (2.24) is violated. There then exist some $h_0 > 0$, $\theta_1 = (h_0, \beta_1, \alpha_2, \lambda_1) \in \Theta \cap \Theta_e \cap \widehat{\Theta}_e^C$ and $\theta_2 = (h_0, \beta_2, \alpha_2, \lambda_2) \in \Theta \cap \Theta_e$ such that $(\beta_1, \alpha_1) \neq (\beta_2, \alpha_2)$.

Consider $\Theta_0 = \{\theta_1, \theta_2\}$ and the decision space $D = \{(\beta_1, \alpha_1), (\beta_2, \alpha_2)\}$, endowed with the discrete topology. For $\theta = (h_0, \beta, \alpha, \lambda) \in \Theta$, consider (continuous and bounded) loss functions $W_\theta : D \rightarrow \mathbb{R}$, where for $x = (x_1, \dots, x_4) \in [0, \infty]^4$, we set $W_\theta(x) = 1 - 1_{\{(\beta, \alpha)\}}(x_2, x_3)$. Further, we define a Markov kernel $\widehat{\rho} : [0, \infty]^4 \times \mathcal{B}(D) \rightarrow [0, 1]$, where for $x \in [0, \infty]^4$ and $B \in \mathcal{B}(D)$, we set

$$\widehat{\rho}(x, B) = \begin{cases} \epsilon_{(\beta_1, \alpha_1)}(B), & \text{if } x \in (0, \infty) \times \{\beta_1\} \times \{\alpha_1\} \times [0, \infty), \\ \epsilon_{(\beta_2, \alpha_2)}(B), & \text{otherwise.} \end{cases}$$

We then have $\int W_{\theta_i}(x) \widehat{\rho}(\omega, dx) \widehat{Q}_{\theta_i}(d\omega) = 0$ for $i = 1, 2$. On the other hand, any Markov kernel $\rho : [0, \infty]^4 \times \mathcal{B}(D) \rightarrow [0, 1]$ is of the form $\rho(\omega, B) = p(\omega) \epsilon_{(\beta_1, \alpha_1)}(B) + (1 - p(\omega)) \epsilon_{(\beta_2, \alpha_2)}(B)$, where $p : [0, \infty]^4 \rightarrow [0, 1]$ is Borel, $\omega \in [0, \infty]^4$ and $B \in \mathcal{B}(D)$. It is easy to see that for such a Markov kernel ρ , there exists a Markov kernel $\bar{\rho} : [0, \infty]^4 \times \mathcal{B}(D) \rightarrow [0, 1]$ such that both

$$\int W_{\theta_1}(x) \rho(\omega, dx) Q_{\theta_1}(d\omega) \geq e^{-\gamma} (1 - p(h_0, \infty, \infty, \infty))$$

and

$$\int W_{\theta_2}(x) \rho(\omega, dx) Q_{\theta_2}(d\omega) \geq e^{-\gamma} p(h_0, \infty, \infty, \infty).$$

In view of (A.1), we thus have $\delta(\mathcal{F}(\Theta), \widehat{\mathcal{F}}(\Theta)) \geq \delta(\mathcal{F}(\Theta_0), \mathcal{F}(\Theta_0)) \geq e^{-\gamma}/2$, which completes the proof of (ii).

(iii) This follows by the same arguments as in (ii).

Appendix

Here, we collect necessary facts regarding Le Cam's distance in deficiency. The reader is referred to Le Cam [18], Le Cam and Young [19] and Strasser's monograph [27] for unexplained notation encountered in this section. Let Θ be a non-empty set, (E, \mathcal{A}) be a measurable space and $(P_\theta)_{\theta \in \Theta}$ be a family of probability measures on \mathcal{A} . The triplet $\mathcal{E} = (E, \mathcal{A}, (P_\theta)_{\theta \in \Theta})$ is then called a (statistical) experiment. Consider two experiments $\mathcal{E}_i = (E_i, \mathcal{A}_i, (P_{i,\theta})_{\theta \in \Theta})$, $i = 1, 2$, indexed by Θ . A decision problem is a triple (Θ, D, W) , where D is a topological space and $W = (W_\theta)_{\theta \in \Theta}$ is a loss function $W_\theta : D \rightarrow \mathbb{R}$, $\theta \in \Theta$. Let $\|W_\theta\|_\infty = \sup_{d \in D} |W_\theta(d)|$. Also, let $\epsilon \geq 0$. Then, \mathcal{E}_1 is called ϵ -deficient with respect to \mathcal{E}_2 , notated as $\mathcal{E}_1 \supseteq_\epsilon \mathcal{E}_2$, if and only if for all decision problems (Θ, D, W) with W continuous and bounded, and all $\beta_2 \in \mathcal{B}(\mathcal{E}_2, D)$, there exists some $\beta_1 \in \mathcal{B}(\mathcal{E}_1, D)$ such that

$$\beta_1(W_\theta, P_{1,\theta}) \leq \beta_2(W_\theta, P_{2,\theta}) + \epsilon \|W_\theta\|_\infty, \quad \theta \in \Theta,$$

where $\mathcal{B}(\mathcal{E}_i, D)$ ($i = 1, 2$) is the space of generalized decision functions (see [27], Definition 42.2). The deficiency of \mathcal{E}_1 with respect to \mathcal{E}_2 is the number

$$\delta(\mathcal{E}_1, \mathcal{E}_2) = \inf\{\epsilon > 0 : \mathcal{E}_1 \supseteq_{\epsilon} \mathcal{E}_2\}. \quad (\text{A.1})$$

The relation $\mathcal{E}_1 \supseteq_{\epsilon} \mathcal{E}_2$ is interpreted in the following sense: we have $\mathcal{E}_1 \supseteq_{\epsilon} \mathcal{E}_2$ if \mathcal{E}_1 is more informative than \mathcal{E}_2 uniformly over all decision problems with continuous and bounded loss functions up to some error ϵ . Two experiments \mathcal{E}_1 and \mathcal{E}_2 are called *equivalent in deficiency* if and only if $\mathcal{E}_1 \supseteq_0 \mathcal{E}_2$ and $\mathcal{E}_2 \supseteq_0 \mathcal{E}_1$.

Recall that (see [27], Lemma 55.4 and Remark 55.6(2))

$$\delta(\mathcal{E}_1, \mathcal{E}_2) = \inf_K \sup_{\theta \in \Theta} \|P_{2,\theta} - K P_{1,\theta}\| \quad (\text{A.2})$$

with infimum now taken over all Markov kernels $K : E_1 \times \mathcal{E}_2 \rightarrow [0, 1]$.

Le Cam's distance between \mathcal{E}_1 and \mathcal{E}_2 is a pseudo-metric on the space of all experiments indexed by Θ (see [27], Corollary 59.6), defined by setting

$$\Delta(\mathcal{E}_1, \mathcal{E}_2) = \max\{\delta(\mathcal{E}_1, \mathcal{E}_2), \delta(\mathcal{E}_2, \mathcal{E}_1)\}. \quad (\text{A.3})$$

If $(E_1, \mathcal{A}_1) = (E_2, \mathcal{A}_2)$, then we have (see [27], Corollary 59.6)

$$\Delta(\mathcal{E}_1, \mathcal{E}_2) \leq \sup_{\theta \in \Theta} \|P_{1,\theta} - P_{2,\theta}\|. \quad (\text{A.4})$$

Clearly, if \mathcal{E}_1 and \mathcal{E}_2 are two experiments indexed by the same Θ , then \mathcal{E}_1 is equivalent to \mathcal{E}_2 in deficiency if and only if $\Delta(\mathcal{E}_1, \mathcal{E}_2) = 0$. Let $\mathcal{E}, \mathcal{E}_n, \mathcal{F}_n, n \in \mathbb{N}$, be experiments, all indexed by Θ . We then say that \mathcal{E}_n converges (strongly) in deficiency, or that \mathcal{E}_n and \mathcal{F}_n are (strongly) asymptotically equivalent in deficiency, if and only if $\Delta(\mathcal{E}_n, \mathcal{E}) \rightarrow 0$ and $\Delta(\mathcal{E}_n, \mathcal{F}_n) \rightarrow 0$, respectively, as $n \rightarrow \infty$.

For $\emptyset \neq \Theta_0 \subseteq \Theta$, we employ the notation $\mathcal{E}(\Theta_0) = (E, \mathcal{A}, (P_{\theta})_{\theta \in \Theta_0})$ for corresponding subexperiments of $\mathcal{E} = (E, \mathcal{A}, (P_{\theta})_{\theta \in \Theta})$. We refer to weak convergence and weak asymptotic equivalence in deficiency if and only if, for all non-empty and finite $\Theta_0 \subseteq \Theta$, the corresponding subexperiments converge strongly and are strongly asymptotically equivalent in deficiency, respectively.

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