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Integral representations and properties of operator fractional Brownian motions

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Operator fractional Brownian motions (OFBMs) are (i) Gaussian, (ii) operator self-similar and (iii) stationary increment processes. They are the natural multivariate generalizations of the well-studied fractional Brownian motions. Because of the possible lack of time-reversibility, the defining properties (i)–(iii) do not, in general, characterize the covariance structure of OFBMs. To circumvent this problem, the class of OFBMs is characterized here by means of their integral representations in the spectral and time domains. For the spectral domain representations, this involves showing how the operator self-similarity shapes the spectral density in the general representation of stationary increment processes. The time domain representations are derived by using primary matrix functions and taking the Fourier transforms of the deterministic spectral domain kernels. Necessary and sufficient conditions for OFBMs to be time-reversible are established in terms of their spectral and time domain representations. It is also shown that the spectral density of the stationary increments of an OFBM has a rigid structure, here called the *dichotomy principle*. The notion of operator Brownian motions is also explored.

Keywords: dichotomy principle; integral representations; long-range dependence; multivariate Brownian motion; operator fractional Brownian motion; operator self-similarity; time-reversibility

1. Introduction

Fractional Brownian motion (FBM), denoted $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ with $H \in (0, 1)$, is a stochastic process characterized by the following three properties:

(i) Gaussianity;
(ii) self-similarity with parameter H;
(iii) stationarity of the increments.

By self-similarity, it is meant that the law of B_H scales as

$$\{B_H(ct)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{c^H B_H(t)\}_{t\in\mathbb{R}}, \qquad c > 0, \tag{1.1}$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality of finite-dimensional distributions. By stationarity of the increments, it is meant that the process

$$\{B_H(t + h) - B_H(h)\}_{t\in\mathbb{R}}$$

has the same distribution for any time-shift $h \in \mathbb{R}$. It may be shown that these three properties actually characterize FBM in the sense that it is the *unique* (up to a constant) such process for a given $H \in (0, 1)$. FBM plays an important role in both theory and applications, especially in connection with long-range dependence [11,12].

We are interested here in the multivariate counterparts of FBM, called operator fractional Brownian motions (OFBMs). In the multivariate context, an OFBM $B_H = (B_{1,H}, \dots, B_{n,H})^* = \{(B_{1,H}(t), \dots, B_{n,H}(t))^* \in \mathbb{R}^n, t \in \mathbb{R}\}$ is a collection of random vectors, where the symbol $*$ denotes transposition. It is also Gaussian and has stationary increments. Moreover, as is standard for the multivariate context, in this paper, we assume that OFBMs are *proper*, that is, for each t, the distribution of $B_H(t)$ is not contained in a proper subspace of $\mathbb{R}^n$. However, self-similarity is now replaced by

(ii′) operator self-similarity.

A proper multivariate process B_H is called (strictly) operator self-similar (o.s.s.) if it is continuous in law for all t and the expression (1.1) holds for some matrix H. Here, the expression c^H is defined by means of the convergent series

$$c^H = \exp(\log(c)H) = \sum_{k=0}^{\infty} (\log c)^k \frac{H^k}{k!}, \qquad c > 0.$$

Operator self-similarity extends the usual notion of self-similarity and was first studied thoroughly in [19,22]; see also Section 11 in [28] and Chapter 9 in [12]. The theory of operator self-similarity runs somewhat parallel to that of operator stable measures (see [20,28]) and is also related to that of operator scaling random fields (see, e.g., [3]).

OFBMs are of interest in several areas and for reasons similar to those in the univariate case. For example, OFBMs arise and are used in the context of multivariate time series and long-range dependence (see, e.g., [4–6,9,26,30]). Another context is that of queueing systems, where reflected OFBMs model the size of multiple queues in particular classes of queueing models and are studied in problems related to, for example, large deviations (see [7,21,24,25]). Partly motivated by this interest in OFBMs, several authors consider constructions and properties of OFBMs. Maejima and Mason [23], in particular, construct examples of OFBMs through time domain integral representations. Mason and Xiao [27] study sample path properties of OFBMs. Bahadoran, Benassi and Dębicki [1] provide wavelet decompositions of OFBMs, study their sample path properties and consider questions of identification. Becker-Kern and Pap [2] consider estimation of the real spectrum of the self-similarity exponent. A number of other works on operator self-similarity are naturally related to OFBMs; see, for example, [28], Section 11, and references therein.

To the reader less familiar with OFBMs, we should note that the multivariate case is quite different from the univariate case. For example, consider an OFBM B_H whose exponent H has characteristic roots h_k with positive real parts. By using operator self-similarity and stationarity of increments, one can argue, as in the univariate case, that

$$\begin{aligned} &EB_H(t)B_H(s)^* + EB_H(s)B_H(t)^* \\ &\quad = EB_H(t)B_H(t)^* + EB_H(s)B_H(s)^* - E\big(B_H(t) - B_H(s)\big)\big(B_H(t) - B_H(s)\big)^* \qquad (1.2) \\ &\quad = |t|^H \Gamma(1,1)|t|^{H^*} + |s|^H \Gamma(1,1)|s|^{H^*} - |t-s|^H \Gamma(1,1)|t-s|^{H^*}, \end{aligned}$$

where $\Gamma(1,1) = EB_H(1)B_H(1)^*$ and the symbol $*$ denotes the adjoint operator. However, in contrast with the univariate case, it is *not* generally true that

$$EB_H(t)B_H(s)^* = EB_H(s)B_H(t)^* \tag{1.3}$$

and hence the OFBM is not characterized by H and a matrix $\Gamma(1,1)$. Another important difference is that the self-similarity exponent of an operator self-similar process is generally not unique. The latter fact has been well known since the fundamental work of Hudson and Mason [19]. We briefly recall it, together with some related results, in Section 2.2.

In this work, we address several new and, in our view, important questions about OFBMs. In view of (1.2) and (1.3), since the covariance structure of an OFBM cannot be determined in general, we pursue the characterization of OFBMs in terms of their integral representations (Section 3). In the spectral domain, under the mild and natural assumption that the characteristic roots of H satisfy

$$0 < \mathrm{Re}(h_k) < 1, \qquad k = 1, \dots, n, \tag{1.4}$$

we show that an OFBM admits the integral representation

$$\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} \big(x_+^{-(H-(1/2)I)} A + x_-^{-(H-(1/2)I)} \overline{A}\big) \widetilde{B}(\mathrm{d}x), \tag{1.5}$$

where A is a matrix with complex-valued entries, $\overline{A}$ denotes its complex conjugate, $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and $\widetilde{B}(\mathrm{d}x)$ is a suitable multivariate complex-valued Gaussian measure. In the time domain and when, in addition to (1.4), we have

$$\mathrm{Re}(h_k) \neq \frac{1}{2}, \qquad k = 1, \dots, n, \tag{1.6}$$

the OFBM admits the integral representation

$$\int_{\mathbb{R}} \Big(\big((t-u)_+^{H-(1/2)I} - (-u)_+^{H-(1/2)I}\big)M_+ + \big((t-u)_-^{H-(1/2)I} - (-u)_-^{H-(1/2)I}\big)M_-\Big) B(\mathrm{d}u), \tag{1.7}$$

where M_+, M_- are matrices with real-valued entries and $B(\mathrm{d}u)$ is a suitable multivariate real-valued Gaussian measure. The representation (1.7) is obtained from (1.5) by taking the Fourier transform of the deterministic kernel in (1.5). We shall provide rigorous arguments for this step by using primary matrix functions. (Even in the univariate case, very often this step is unjustifiably taken as more or less evident.) On a related note, but from a different angle, the representations (1.5) and (1.7) always define Gaussian processes with stationary increments that satisfy (1.1) for a matrix H. We shall provide sufficient condition for these processes to be proper (see Section 4) and, hence, to be OFBMs.

Subclasses of the representations (1.5) and (1.7) were considered in the works referenced above. Maejima and Mason [23] consider OFBMs given by the representation (1.7) with $M_+ = M_- = I$. Mason and Xiao [27] take (1.5) with $A = I$. Bahadoran *et al.* [1] consider (1.5) with A having full rank and real-valued entries. (Such OFBMs, for example, are necessarily time-reversible; see Theorem 5.1 and also Remark 3.1.) We would again like to emphasize that, in contrast with these works, the representations (1.5) and (1.7) characterize *all* OFBMs (under the mild and natural conditions (1.4) and (1.6)).

In particular, the representations (1.5) and (1.7) provide a natural framework for the study of many properties of OFBMs. In this paper, we provide conditions in terms of A in (1.5) (or M_+, M_- in (1.7)) for OFBMs to be time-reversible (see Section 5). Time-reversibility is shown to be equivalent to the condition (1.3) and hence, in view of (1.2), corresponds to the situation where the covariance structure of the OFBM is given by

$$EB_H(t)B_H(s)^* = \frac{1}{2}\bigl(|t|^H\Gamma(1,1)|t|^{H^*} + |s|^H\Gamma(1,1)|s|^{H^*} - |t-s|^H\Gamma(1,1)|t-s|^{H^*}\bigr). \quad (1.8)$$

Another interesting and little-explored direction of study of OFBMs is their uniqueness (identification). This encompasses the characterization of the different parameterizations for any given OFBM and, in particular, of the aforementioned non-uniqueness of the self-similarity exponents. Uniqueness questions in the context of OFBMs are the focus of Didier and Pipiras [8], where they are explored starting with the representation (1.5), and will be largely absent from this paper.

Furthermore, in this paper, we also discuss some additional properties of OFBMs which are of independent interest. First, we prove that OFBMs have a rigid dependence structure among components, which we call the *dichotomy principle* (Section 6). More precisely, under long-range dependence (in the sense considered in Section 6), we show that the components of the increments of an OFBM are either independent or long-range dependent, that is, they cannot be short-range dependent in a non-trivial way. Since, in the univariate case, the increments of FBM are often considered representative of all long-range dependent series, this result raises the question of whether OFBMs are flexible enough to capture multivariate long-range dependence structures. Second, we also discuss the notion of operator Brownian motions (OBMs) and related questions (Section 7). OBMs are defined as having independent increments and are known to admit $H = (1/2)I$ as an exponent. We also show, in particular, that an OFBM with $H = (1/2)I$ does not necessarily have independent increments and hence is not necessarily an OBM. (In contrast, in the univariate case, $H = 1/2$ necessarily implies Brownian motion.)

In summary, the structure of the paper is as follows. In Section 2, we provide the necessary background for the paper and some definitions. In Section 3, we construct integral representations for OFBMs in the spectral and time domains. Section 4 furnishes sufficient conditions for properness. Section 5 is dedicated to time-reversibility. The dichotomy principle is established in Section 6 and the properties of OBMs are studied in Section 7. Appendices A–D contain several important technical results used throughout the paper, as well as some proofs.

2. Preliminaries

We begin by introducing some notation and considering some preliminaries on the exponential map and operator self-similarity that are used throughout the paper.

2.1. Some notation

In this paper, the notation and terminology for finite-dimensional operator theory will be preferred over their matrix analogs. However, whenever convenient, the latter will be used.

All with respect to the field $\mathbb{R}$, $M(n)$ or $M(n,\mathbb{R})$ is the vector space of all $n \times n$ operators (endomorphisms), $GL(n)$ or $GL(n,\mathbb{R})$ is the general linear group (invertible operators, or automorphisms), $O(n)$ is the orthogonal group of operators O such that $OO^* = I = O^*O$ (i.e., the adjoint operator is the inverse), $SO(n) \subseteq O(n)$ is the special orthogonal group of operators (rotations) with determinant equal to 1 and $so(n)$ is the vector space of skew-symmetric operators (i.e., $A^* = -A$).

The notation will indicate a change to the field $\mathbb{C}$. For instance, $M(n,\mathbb{C})$ is the vector space of complex endomorphisms. Whenever it is said that $A \in M(n)$ has a complex eigenvalue or eigenspace, one is considering the operator embedding $M(n) \hookrightarrow M(n,\mathbb{C})$. The notation $\overline{A}$ indicates the operator whose matrix representation is entrywise equal to the complex conjugates of those of A. We will say that two endomorphisms $A, B \in M(n)$ are *conjugate* (or similar) when there exists $P \in GL(n)$ such that $A = PBP^{-1}$. In this case, P is called a *conjugacy*. The expression $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ denotes the operator whose matrix expression has the values $\lambda_1, \dots, \lambda_n$ on the diagonal and zeros elsewhere. The expression $\operatorname{tr}(A)$ denotes the trace of an operator $A \in M(n,\mathbb{C})$. We write $f \in L^2(\mathbb{R}, M(n,\mathbb{C}))$ for a matrix-valued function f when $\operatorname{tr}\{\int_{\mathbb{R}} f(u)^* f(u)\,\mathrm{d}u\} < +\infty$.

Throughout the paper, we set

$$D = H - (1/2)I \tag{2.1}$$

for an operator exponent H. The characteristic roots of H and D are denoted

$$h_k, d_k, \tag{2.2}$$

respectively. Here,

$$k = 1, \dots, N \text{ or } n, \tag{2.3}$$

where $N \le n$ is the number of different characteristic roots of H.

For notational simplicity when constructing the spectral and time domain filters, we will adopt the convention that $z^D = 0 \in M(n,\mathbb{R})$ when $z = 0$.

2.2. Operator self-similar processes

Operator self-similar (o.s.s.) processes were defined in Section 1. Any matrix H for which (1.1) holds is called an *exponent* of the o.s.s. process X. The set of all such H for X is denoted by $\mathcal{E}(X)$, which, in general, contains more than one exponent. The non-uniqueness of the exponent H depends on the symmetry group G_1 of X, which is defined as follows.

Definition 2.1. *The symmetry group of an o.s.s. process X is the set G_1 of matrices $A \in GL(n)$ such that*

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{AX(t)\}_{t\in\mathbb{R}}. \tag{2.4}$$

It turns out that the symmetry group G_1 is always compact, which implies that there exists a closed subgroup $\mathcal{O}_0$ of $O(n)$ such that $G_1 = W\mathcal{O}_0W^{-1}$, where W is a positive definite matrix (see, e.g., [20], Corollary 2.4.2, page 61). A process X that has maximal symmetry, that is, such that $G_1 = WO(n)W^{-1}$, is called *elliptically symmetric*.

Let G be a closed (sub)group of operators. The tangent space $T(G)$ of G is the set of $A \in M(n)$ such that

$$A = \lim_{n\to\infty} \frac{G_n - I}{d_n} \qquad \text{for some } \{G_n\} \subseteq G \text{ and some } 0 < d_n \to 0.$$

In this sense, $T(G)$ is, in fact, a linearization of G in a neighborhood of I. Hudson and Mason [19], Theorem 2, shows that for any given o.s.s. process X with exponent H, the set of exponents $\mathcal{E}(X)$ has the form $\mathcal{E}(X) = H + T(G_1)$, where $T(G_1) = W\mathcal{L}_0 W^{-1}$ for the positive definite conjugacy matrix W associated with G_1 and some subspace $\mathcal{L}_0$ of $so(n)$. Consequently, X has a unique exponent if and only if G_1 is finite.

3. Integral representations of OFBMs

Representations of OFBMs in the spectral domain are derived in Section 3.1. The corresponding representations in the time domain are given in Section 3.2. The derivation of these representations is quite different from that in the univariate case. In the latter case, it is enough to "guess" the form of the spectral representation and to verify that it gives self-similarity and stationarity of the increments (and hence, immediately, FBM). In the multivariate case, these representations actually have to be derived from the properties of OFBMs, without any guessing involved.

3.1. Spectral domain representations

In Theorem 3.1, we establish integral representations of OFBMs in the spectral domain.

Theorem 3.1. *Let $H \in M(n,\mathbb{R})$ with characteristic roots h_k satisfying*

$$0 < \mathrm{Re}(h_k) < 1, \qquad k = 1, \dots, n. \tag{3.1}$$

Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM with exponent H. Then $\{B_H(t)\}_{t\in\mathbb{R}}$ admits the integral representation

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \left\{ \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} (x_+^{-D} A + x_-^{-D} \overline{A}) \widetilde{B}(\mathrm{d}x) \right\}_{t\in\mathbb{R}} \tag{3.2}$$

for some $A \in M(n,\mathbb{C})$. Here, D is as in (2.1),

$$\widetilde{B}(x) := \widetilde{B}_1(x) + \mathrm{i}\widetilde{B}_2(x) \tag{3.3}$$

denotes a complex-valued multivariate Brownian motion such that $\widetilde{B}_1(-x) = \widetilde{B}_1(x)$ and $\widetilde{B}_2(-x) = -\widetilde{B}_2(x)$, $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent and the induced random measure $\widetilde{B}(\mathrm{d}x)$ satisfies $E\widetilde{B}(\mathrm{d}x)\widetilde{B}(\mathrm{d}x)^* = \mathrm{d}x$.

Proof. For notational simplicity, set $X = B_H$. Since X has stationary increments, we have

$$X(t) - X(s) = \int_{\mathbb{R}} \frac{e^{itx} - e^{isx}}{ix} \widetilde{Y}(dx), \tag{3.4}$$

where $\widetilde{Y}(dx)$ is an orthogonal-increment random measure in $\mathbb{C}^n$. The relation (3.4) can be proven following the approach for the univariate case found in [10], page 550, under the assumption that $E|X(t+h) - X(t)|^2 \to 0$ as $h \to 0$, that is, X is L^2-continuous at every t (see also [35], page 409, and [34], Theorem 7). The latter assumption is satisfied in our context because of the following. Property 2.1 in [23] states that, for an o.s.s. process Z with exponent H, if $\inf\{\mathrm{Re}(h_k); k = 1, \dots, n\} > 0$, then $Z(0) = 0$ a.s. Thus, in view of (3.1), $X(0) = 0$ a.s. So, by stationarity of the increments and continuity in law,

$$X(t+h) - X(t) \overset{\mathcal{L}}{=} X(h) \overset{\mathcal{L}}{\to} X(0) = 0, \qquad h \to 0. \tag{3.5}$$

Therefore, by relation (3.4) and again by Property 2.1 in [23],

$$X(t) = \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} \widetilde{Y}(dx). \tag{3.6}$$

Let $F_X(dx) = E\widetilde{Y}(dx)\widetilde{Y}(dx)^*$ be the multivariate spectral distribution of $\widetilde{Y}(dx)$. The remainder of the proof involves three steps:

(i) showing the existence of a spectral density function $f_X(x) = F_X(dx)/dx$;
(ii) decorrelating the measure $\widetilde{Y}(dx)$ componentwise by finding a filter based upon the spectral density function;
(iii) developing the form of the filter.

Step (i). Since X is o.s.s. with exponent H,

$$X(ct) \overset{\mathcal{L}}{=} c^H \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} \widetilde{Y}(dx) \tag{3.7}$$

for $c > 0$. On the other hand, through a change of variables $x = c^{-1}v$,

$$X(ct) \overset{\mathcal{L}}{=} \int_{\mathbb{R}} \frac{e^{itv} - 1}{iv} c\widetilde{Y}(c^{-1}\, dv). \tag{3.8}$$

The relations (3.7) and (3.8) provide two spectral representations for the process $\{X(ct)\}_{t\in\mathbb{R}}$. As a consequence of the uniqueness of the spectral distribution function of the stationary process $\{X(t) - X(t-1)\}_{t\in\mathbb{R}}$ and of the fact that $|\frac{e^{ix}-1}{ix}|^2 > 0$, $x \in \mathbb{R}\backslash\{2\pi k, k \in \mathbb{Z}\}$, we obtain that

$$c^2 F_X(c^{-1}\, dx) = c^H F_X(dx) c^{H^*}, \qquad c > 0.$$

Equivalently, by a simple change of variables, $F_X(c\, dx) = c^{I-H} F_X(dx) c^{(I-H)^*}$. Thus, for $c > 0$,

$$\int_{(0,1]} F_X(c\, dx) = F_X(0, c] = c^{I-H} F_X(0, 1] c^{(I-H)^*}, \tag{3.9}$$

$$\int_{(-1,0]} F_X(c\,\mathrm{d}x) = F_X(-c,0] = c^{I-H} F_X(-1,0] c^{(I-H)^*}. \tag{3.10}$$

By the explicit formula for c^{I-H} in Appendix D, each individual entry $F_X(0,c]_{ij}$, $i,j=1,\dots,n$, in the expression on the right-hand side of (3.9) is either a linear combination (with complex weights) of terms of the form

$$\frac{(\log(c))^l}{l!} c^{1-h_q} \frac{(\log(c))^m}{m!} c^{1-\overline{h}_k}, \qquad q,k=1,\dots,n,\ l,m=0,\dots,n-1, \tag{3.11}$$

or is identically zero for $c>0$. Thus, $F_X(c)$ is differentiable in c over $(0,\infty)$ since $F_X(0,c]_{ij} = F_X(c)_{ij} - F_X(0)_{ij}$. The differentiability of F_X on $(-\infty,0)$ follows from (3.10) and an analogous argument.

To finish the proof of the absolute continuity of F_X, it suffices to show that F_X is continuous at zero. Note that

$$F_X(-c,c] = c^{I-H} F_X(-1,1] c^{(I-H)^*} \to 0$$

as $c\to 0^+$. The limit holds because $\|c^{I-H}\| \to 0$ as $c \to 0^+$, where $\|\cdot\|$ is the matrix norm, which, in turn, follows from [23], Proposition 2.1(ii), under the assumption that $\mathrm{Re}(h_k) < 1$, $k=1,\dots,n$.

Step (ii). Denote the spectral density of X by f_X. Since $|\frac{1-\mathrm{e}^{-\mathrm{i}x}}{\mathrm{i}x}|^2 f_X(x)$ is the spectral density of the stationary process $\{X(t)-X(t-1)\}_{t\in\mathbb{R}}$, $f_X(x)$ is a positive semidefinite Hermitian symmetric matrix $\mathrm{d}x$-a.e. ([15], Theorem 1, page 34). The spectral theorem yields a (unique) positive semidefinite square root $\widehat{a}(x)$ of $f_X(x)$. Let $\widetilde{B}(x)$ be a complex-valued multivariate Brownian motion, as in the statement of the theorem. X can then also be represented as

$$X(t) \stackrel{\mathcal{L}}{=} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x} \widehat{a}(x) \widetilde{B}(\mathrm{d}x) \tag{3.12}$$

because

$$E(\widehat{a}(x)\widetilde{B}(\mathrm{d}x)\widetilde{B}(\mathrm{d}x)^*\widehat{a}(x)^*) = \widehat{a}(x)^2\,\mathrm{d}x = f_X(x)\,\mathrm{d}x = F_X(\mathrm{d}x)$$

and the processes on both sides of (3.12) are Gaussian and real-valued.

Step (iii). By using operator self-similarity and arguing as in step (i), the relation (3.12) implies that, for every $c>0$,

$$\widehat{a}(x)\widehat{a}(x)^* = c^{-D}\widehat{a}\biggl(\frac{x}{c}\biggr)\widehat{a}\biggl(\frac{x}{c}\biggr)^* c^{-D^*} \qquad \mathrm{d}x\text{-a.e.} \tag{3.13}$$

By Fubini's theorem, the relation (3.13) also holds $\mathrm{d}x\,\mathrm{d}c$-a.e.

Consider $x>0$. A change of variables leads to

$$\widehat{a}(x)\widehat{a}(x)^* = x^{-D} v^{D} \widehat{a}(v)\widehat{a}(v)^* v^{D^*} x^{-D^*} \qquad \mathrm{d}x\,\mathrm{d}v\text{-a.e.}$$

Thus, one can choose $v_+>0$ such that

$$\widehat{a}(x)\widehat{a}(x)^* = x^{-D} v_+^{D} \widehat{a}(v_+)\widehat{a}(v_+)^* v_+^{D^*} x^{-D^*} \qquad \mathrm{d}x\text{-a.e.} \tag{3.14}$$

This means, in particular, that if we set

$$\widehat{\alpha}_+(x) = x^{-D} v_+^D \widehat{a}(v_+)$$

for dx-a.e. $x > 0$, then $\widehat{\alpha}_+(x)\widehat{\alpha}_+(x)^* = f_X(x)$ on the same domain.

By again considering the stationary process $\{X(t) - X(t-1)\}_{t\in\mathbb{R}}$ and applying [15], Theorem 3, page 41, one can show that f_X is a Hermitian function. Thus,

$$\widehat{a}(-x)\widehat{a}(-x)^* = f_X(-x) = \overline{f_X(x)} = x^{-D} v_+^D \overline{\widehat{a}(v_+)\widehat{a}(v_+)^*} v_+^{D^*} x^{-D^*} \qquad \mathrm{d}x\text{-a.e.}$$

Hence, for $x < 0$, we can set

$$\widehat{\alpha}_-(x) = (-x)_+^{-D} v_+^D \overline{\widehat{a}(v_+)}$$

and, for $x \in \mathbb{R}$, we have

$$\widehat{\alpha}(x) = x_+^{-D} v_+^D \widehat{a}(v_+) + x_-^{-D} v_+^D \overline{\widehat{a}(v_+)} \qquad \mathrm{d}x\text{-a.e.},$$

where $\widehat{\alpha}(x)\widehat{\alpha}(x)^* = f_X(x)$ dx-a.e. Therefore, we can use $\widehat{\alpha}$ in place of $\widehat{a}$ in the spectral representation of X, which establishes relation (3.2). □

Remark 3.1. The invertibility of A in relation (3.2) is not a requirement for the process to be proper (compare with [1], page 9). In the Gaussian case, properness is equivalent to $EX(t)X(t)^*$ being a full rank matrix for all $t \neq 0$.

A simple example would be that of a bivariate OFBM whose spectral representation has matrix parameters $D = dI$, $0 < d < 1/2$, and A set to the (unique) non-negative square root of

$$A^2 = \begin{pmatrix} 1 & i \\ \bar{i} & 1 \end{pmatrix},$$

which is rank-deficient. Let

$$g(t) = \int_0^\infty \left| \frac{\mathrm{e}^{itx} - 1}{\mathrm{i}x} \right|^2 |x|^{-2d} \, \mathrm{d}x,$$

which is strictly positive for all $t \neq 0$. In this case,

$$\begin{aligned} EX(t)X(t)^* &= \int_{\mathbb{R}} \left| \frac{\mathrm{e}^{itx} - 1}{\mathrm{i}x} \right|^2 |x|^{-2d} \bigl(A^2 1_{\{x\geq 0\}} + \overline{A^2} 1_{\{x<0\}}\bigr) \, \mathrm{d}x \\ &= g(t) \begin{pmatrix} 2 & i + \bar{i} \\ i + \bar{i} & 2 \end{pmatrix} = 2g(t)I. \end{aligned}$$

Theorem 3.1 shows that an OFBM is characterized by a (potentially non-unique) o.s.s. exponent H and a matrix A. For the sake of simplicity, we will continue to use the notation B_H instead of the (more correct) notation $B_{H,A}$.

Remark 3.2. As a consequence of Maejima and Mason [23], Corollary 2.1, the characteristic roots h_k of the exponent H of an OFBM must satisfy $\mathrm{Re}(h_k) \le 1$, $k = 1, \dots, n$. However, the extension of the definition of OFBMs to the case of H with at least one characteristic root h_k satisfying $\mathrm{Re}(h_k) = 1$ can be subtle. In Proposition C.1, it is shown that there does not exist an OFBM with exponent

$$H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

whose characteristic roots are $h_1 = h_2 = 1$. (More precisely, it is shown that a Gaussian, H-o.s.s. process $X = (X_1, X_2)^*$ with stationary increments is necessarily such that $X_1(t) = 0$ and $X_2(t) = tY$ a.s. for a Gaussian variable Y and hence that it cannot be proper.)

3.2. Time domain representations

Our next goal is to provide integral representations of OFBMs in the time domain, which is done in Theorem 3.2. The key technical step in the proof is the calculation of the (entrywise) Fourier transform of the kernels

$$(t-u)_{\pm}^{D} - (-u)_{\pm}^{D} = \exp\bigl(\log(t-u)_{\pm} D\bigr) - \exp\bigl(\log(-u)_{\pm} D\bigr), \tag{3.15}$$

which are the multivariate analogs of the corresponding univariate FBM time domain kernels. It is natural and convenient to carry out this step in the framework of the so-called primary matrix functions. The latter allows one to naturally define matrix analogs $f(D)$, $D \in M(n, \mathbb{R})$, of univariate functions $f(d)$, $d \in \mathbb{R}$, and to say when two such matrix-valued functions are equal based on their univariate counterparts.

For the reader's convenience, we recall the definition of primary matrix functions (more details and properties can be found in [17], Sections 6.1 and 6.2). Let $\Lambda \in M(n, \mathbb{C})$ with minimal polynomial

$$q_\Lambda(z) = (z - \lambda_1)^{r_1} \cdots (z - \lambda_N)^{r_N}, \tag{3.16}$$

where $\lambda_1, \dots, \lambda_N$ are pairwise distinct and $r_k \ge 1$ for $k = 1, \dots, N$, $N \le n$. We denote by $\Lambda = PJP^{-1}$ the Jordan decomposition of Λ, where J is in Jordan canonical form with the Jordan blocks $J_{\lambda_1}, \dots, J_{\lambda_N}$ on the diagonal.

Let $U \subseteq \mathbb{C}$ be an open set. Given a function $h : U \to \mathbb{C}$ and some $\Lambda \in M(n, \mathbb{C})$ as above, consider the following conditions: (M1) $\lambda_k \in U$, $k = 1, \dots, N$; (M2) if $r_k > 1$, then $h(z)$ is analytic in a neighborhood $U_k \ni \lambda_k$, where $U_k \subseteq U$. Let $\mathcal{M}_h = \{\Lambda \in M(n, \mathbb{C})$; conditions (M1) and (M2) hold at the characteristic roots $\lambda_1, \dots, \lambda_N$ of $\Lambda\}$. We now define the primary matrix function $h(\Lambda)$ associated with the scalar-valued stem function $h(z)$.

Definition 3.1. *The primary matrix function $h : \mathcal{M}_h \to M(n, \mathbb{C})$ is defined as*

$$h(\Lambda) = Ph(J)P^{-1} = P \begin{pmatrix} h(J_{\lambda_1}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & h(J_{\lambda_N}) \end{pmatrix} P^{-1},$$

where

$$h(J_{\lambda_k}) = \begin{pmatrix} h(\lambda_k) & 0 & \dots & 0 \\ h'(\lambda_k) & h(\lambda_k) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{h^{(r_k-1)}(\lambda_k)}{(r_k-1)!} & \dots & h'(\lambda_k) & h(\lambda_k) \end{pmatrix}.$$

The following technical result is proved in Appendix A. The functions $(t-u)_{\pm}^{D}$, $\Gamma(D+I)$, $|x|^{-D}$, $e^{\mp \operatorname{sign}(x) i\pi D/2}$ appearing in the result below are all primary matrix functions. The same interpretation is also adopted throughout the rest of the paper, for example, with functions $\sin(\pi D/2)$, $\cos(\pi D/2)$ appearing in Theorem 3.2. (It should also be noted, in particular, that the definition of the matrix exponential based on a series is equivalent to that based on primary matrix functions.)

Proposition 3.1. *Under* (3.1) *and condition* (3.18) *in Theorem* 3.2,

$$\int_{\mathbb{R}} e^{iux}\bigl((t-u)_{\pm}^{D} - (-u)_{\pm}^{D}\bigr)\,du = \frac{e^{itx}-1}{ix}|x|^{-D}\Gamma(D+I)e^{\mp \operatorname{sign}(x) i\pi D/2}. \tag{3.17}$$

Next, we construct time domain representations for OFBMs, which provides the main result in this section. Further comments about the result can be found after the proof.

Theorem 3.2. *Let* $\{B_H(t)\}_{t\in\mathbb{R}}$ *be an OFBM with o.s.s. exponent* H *having the spectral representation* (3.2) *with* $A = A_1 + iA_2$, *where* $A_1, A_2 \in M(n,\mathbb{R})$.

(i) *Suppose that* $H \in M(n,\mathbb{R})$ *has characteristic roots satisfying* (3.1) *and*

$$\operatorname{Re}(h_k) \neq \frac{1}{2}, \qquad k = 1,\dots,n. \tag{3.18}$$

There are then $M_+, M_- \in M(n,\mathbb{R})$ *such that*

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} \bigl(\bigl((t-u)_{+}^{D} - (-u)_{+}^{D}\bigr)M_+ + \bigl((t-u)_{-}^{D} - (-u)_{-}^{D}\bigr)M_-\bigr)B(du)\right\}_{t\in\mathbb{R}}, \tag{3.19}$$

where $\{B(u)\}_{u\in\mathbb{R}}$ *is a vector-valued process consisting of independent Brownian motions and such that* $EB(du)B(du)^* = du$. *Moreover, the matrices* M_+, M_- *can be taken as*

$$M_{\pm} = \sqrt{\frac{\pi}{2}}\left(\sin\left(\frac{\pi D}{2}\right)^{-1}\Gamma(D+I)^{-1}A_1 \pm \cos\left(\frac{\pi D}{2}\right)^{-1}\Gamma(D+I)^{-1}A_2\right). \tag{3.20}$$

(ii) *Suppose that* $H = (1/2)I$. *There then exist* $M, N \in M(n,\mathbb{R})$ *such that*

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} \left(\bigl(\operatorname{sign}(t-u) - \operatorname{sign}(-u)\bigr)M + \log\left(\frac{|t-u|}{|u|}\right)N\right)B(du)\right\}_{t\in\mathbb{R}}, \tag{3.21}$$

where $\{B(u)\}_{u\in\mathbb{R}}$ *is as in* (3.19). *Moreover, the matrices* M, N *can be taken as*

$$M=\sqrt{\frac{\pi}{2}}A_1,\qquad N=-\sqrt{\frac{2}{\pi}}A_2. \tag{3.22}$$

Proof. (i) Denote the process on the right-hand side of (3.19) by X_H. By using the Jordan decomposition of D, it is easy to show that X_H is well defined. It suffices to show that there are $M_\pm$ such that the covariance structure of X_H matches that of the OFBM B_H given by its spectral representation (3.2) with $A=A_1+\mathrm{i}A_2$. By using the Plancherel identity, note first that

$$\begin{aligned}&EX_H(s)X_H(t)^*\\&\quad=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{(\mathrm{e}^{\mathrm{i}sx}-1)(\mathrm{e}^{-\mathrm{i}tx}-1)}{|x|^2}\big(|x|^{-D}\Gamma(D+I)\big(\mathrm{e}^{-\operatorname{sign}(x)\mathrm{i}\pi D/2}M_++\mathrm{e}^{\operatorname{sign}(x)\mathrm{i}\pi D/2}M_-\big)\big)\\&\qquad\qquad\times\big(\big(M_+^*\mathrm{e}^{\operatorname{sign}(x)\mathrm{i}\pi D^*/2}+M_-^*\mathrm{e}^{-\operatorname{sign}(x)\mathrm{i}\pi D^*/2}\big)\Gamma(D+I)^*|x|^{-D^*}\big)\,\mathrm{d}x.\end{aligned}$$

Meanwhile, for B_H, we have

$$EB_H(s)B_H(t)^*=\int_{\mathbb{R}}\frac{(\mathrm{e}^{\mathrm{i}sx}-1)(\mathrm{e}^{-\mathrm{i}tx}-1)}{|x|^2}\big(x_+^{-D}AA^*x_+^{-D^*}+x_-^{-D}\overline{AA^*}x_-^{-D^*}\big)\,\mathrm{d}x. \tag{3.23}$$

Thus, by using the relation $\mathrm{e}^{\mathrm{i}\Theta}=\cos(\Theta)+\mathrm{i}\sin(\Theta)$, $\Theta\in M(n)$, it is sufficient to find $M_\pm\in M(n,\mathbb{R})$ such that

$$\begin{aligned}AA^*&=\frac{1}{2\pi}\Gamma(D+I)(\mathrm{e}^{-\mathrm{i}\pi D/2}M_++\mathrm{e}^{\mathrm{i}\pi D/2}M_-)\\&\quad\times(M_+^*\mathrm{e}^{\mathrm{i}\pi D^*/2}+M_-^*\mathrm{e}^{-\mathrm{i}\pi D^*/2})\Gamma(D+I)^*\\&=\frac{1}{2\pi}\Gamma(D+I)\bigg(\sin\Big(\frac{\pi D}{2}\Big)(M_+-M_-)(M_+^*-M_-^*)\sin\Big(\frac{\pi D^*}{2}\Big)\\&\qquad+\cos\Big(\frac{\pi D}{2}\Big)(M_++M_-)(M_+^*+M_-^*)\cos\Big(\frac{\pi D^*}{2}\Big)\\&\qquad+\mathrm{i}\bigg(\cos\Big(\frac{\pi D}{2}\Big)(M_++M_-)(M_+^*-M_-^*)\sin\Big(\frac{\pi D^*}{2}\Big)\\&\qquad-\sin\Big(\frac{\pi D}{2}\Big)(M_+-M_-)(M_+^*+M_-^*)\cos\Big(\frac{\pi D^*}{2}\Big)\bigg)\bigg)\Gamma(D+I)^*.\end{aligned} \tag{3.24}$$

On the other hand,

$$AA^*=(A_1A_1^*+A_2A_2^*)+\mathrm{i}(A_2A_1^*-A_1A_2^*). \tag{3.25}$$

By comparing (3.25) and (3.24), a natural way to proceed is to consider M_+ and M_- as solutions of the system

$$A_1 = \frac{1}{\sqrt{2\pi}}\Gamma(D+I)\sin\left(\frac{\pi D}{2}\right)(M_+ - M_-),$$
$$A_2 = \frac{1}{\sqrt{2\pi}}\Gamma(D+I)\cos\left(\frac{\pi D}{2}\right)(M_+ + M_-). \tag{3.26}$$

By assumption (3.18), $\sin(\frac{\pi D}{2})$, $\cos(\frac{\pi D}{2})$ and $\Gamma(D+I)$ are invertible, and we obtain the solution given by (3.20).

(ii) In this case, one can readily compute the inverse Fourier transform of the integrand in (3.2), that is (up to $(2\pi)^{-1}$),

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux}\left(\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\right)\bigl(1_{\{x>0\}}A + 1_{\{x<0\}}\overline{A}\bigr)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}}\left(\frac{\cos((t-u)x) - \cos((t-u)x) + \mathrm{i}(\sin((t-u)x) + \sin(ux))}{\mathrm{i}x}\right)\bigl(1_{\{x>0\}}A + 1_{\{x<0\}}\overline{A}\bigr)\,\mathrm{d}x.$$

As shown in Appendix B, this becomes

$$-2\log\left(\frac{|t-u|}{|u|}\right)A_2 + \bigl(\operatorname{sign}(t-u) - \operatorname{sign}(-u)\bigr)\pi A_1.$$

Then, by considering second moments and using Plancherel's identity, representation (3.21) holds with $M = (2\pi)^{-1/2}\pi A_1$ and $N = (2\pi)^{-1/2}(-2)A_2$. It is well defined because the integrand comes from the inverse Fourier transform of a square-integrable function and hence is also square-integrable. □

Remark 3.3. Note that the invertibility of M or N in (3.19) is not a requirement for the process to be proper. A simple example would be that of a bivariate OFBM B_H whose time domain representation (3.19) has matrix parameters $H = hI$, $h \in (0,1)\backslash\{1/2\}$,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The two components of B_H are two independent (univariate) FBMs with exponent h. Thus, B_H is proper.

Example 3.1. When (3.18) does not hold and $H \neq (1/2)I$, the general form of time domain representations can be quite intricate. For example, with

$$D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \left(H = \begin{pmatrix} 1/2 & 0 \\ 1 & 1/2 \end{pmatrix}\right),$$

the calculation of the inverse Fourier transform (up to $(2\pi)^{-1}$)

$$\int_{\mathbb{R}} e^{-iux}\left(\frac{e^{itx}-1}{ix}\right)(x_+^{-D}A + x_-^{-D}\overline{A})\,dx \tag{3.27}$$

in Appendix B shows that B_H has the time domain representation

$$\{B_H(t)\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} \big(f_1(t,u)M + f_2(t,u)N\big)B(du)\right\}_{t\in\mathbb{R}}, \tag{3.28}$$

where $M = \sqrt{\frac{\pi}{2}}A_1$, $N = -\sqrt{\frac{2}{\pi}}A_2$,

$$f_1(t,u) = \begin{pmatrix} \operatorname{sign}(t-u) - \operatorname{sign}(-u) & 0 \\ (C + \log|t-u|)\operatorname{sign}(t-u) - (C + \log|u|)\operatorname{sign}(-u) & \operatorname{sign}(t-u) - \operatorname{sign}(-u) \end{pmatrix},$$

$$f_2(t,u) = \begin{pmatrix} \log\left(\frac{|t-u|}{|u|}\right) & 0 \\ \log\left(\frac{|t-u|}{|u|}\right)\left(C + \frac{1}{2}\log(|t-u||u|)\right) & \log\left(\frac{|t-u|}{|u|}\right) \end{pmatrix},$$

where C is Euler's constant. Note that, without taking the Fourier transform of (3.28), it is by no means obvious why its right-hand side has stationary increments and is o.s.s.

4. Conditions for properness

We now provide sufficient conditions for a process with spectral and time domain representations (3.2) and (3.19), respectively, to be proper and, thus, to be an OFBM.

Proposition 4.1. *Let $\{X(t)\}_{t\in\mathbb{R}}$ be a process with spectral domain representation (3.2), where the characteristic roots of H satisfy (3.1). If $\operatorname{Re}(AA^*)$ is a full rank matrix, then $\{X(t)\}_{t\in\mathbb{R}}$ is proper (i.e., it is an OFBM).*

Proof. We must show that

$$EX(t)X(t)^* = \int_{\mathbb{R}} \left|\frac{e^{itx}-1}{ix}\right|^2 (x_+^{-D}AA^*x_+^{-D^*} + x_-^{-D}\overline{AA^*}x_-^{-D^*})\,dx, \qquad t \neq 0,$$

is a full rank matrix. For simplicity, let $d\mu(x) = |\frac{e^{itx}-1}{ix}|^2\,dx$. Then

$$\begin{aligned} EX(t)X(t)^* &= \int_{\mathbb{R}} x_+^{-D}AA^*x_+^{-D^*}\,d\mu(x) + \int_{\mathbb{R}} x_+^{-D}\overline{AA^*}x_+^{-D^*}\,d\mu(x) \\ &= 2\int_{\mathbb{R}} x_+^{-D}\operatorname{Re}(AA^*)x_+^{-D^*}\,d\mu(x). \end{aligned}$$

The matrix $\int_{\mathbb{R}} x_+^{-D} \operatorname{Re}(AA^*) x_+^{-D^*} \, \mathrm{d}\mu(x)$ is Hermitian positive semidefinite. Moreover, for any $v \in \mathbb{C}^n \backslash \{0\}$,

$$v^* \biggl(\int_{\mathbb{R}} x_+^{-D} \operatorname{Re}(AA^*) x_+^{-D^*} \, \mathrm{d}\mu(x) \biggr) v > 0,$$

where the strict inequality follows from the fact that $(v^* x_+^{-D}) \operatorname{Re}(AA^*)(x_+^{-D^*} v) > 0$ for all $x > 0$, the latter being a consequence of the invertibility of x_+^{-D} and the assumption that $\operatorname{Re}(AA^*)$ has full rank. □

Based on Proposition 4.1, we can easily obtain conditions for properness based on time domain parameters. Consider a process $\{X(t)\}_{t\in\mathbb{R}}$ with time domain representation (3.19), where the characteristic roots of H satisfy (3.1) and (3.18). If

$$M_+ + M_-, \qquad M_+ - M_-$$

are full rank matrices, then $\{X(t)\}_{t\in\mathbb{R}}$ is proper (i.e., it is an OFBM).

Remark 4.1. $\operatorname{Re}(AA^*)$ having full rank does not imply that AA^* has full rank since $\mathrm{i}(A_2 A_1^* - A_1 A_2^*)$ may have negative eigenvalues. Also, note that $\operatorname{Re}(AA^*)$ being a full rank matrix is not a necessary condition for properness. For example, consider the process $\{X(t)\}_{t\in\mathbb{R}}$ with spectral representation (3.2), where

$$AA^* = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad h_1, h_2 \in (0,1).$$

Then

$$\begin{aligned} EX(t)X(t)^* &= \int_{\mathbb{R}} \biggl| \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} \biggr|^2 \begin{pmatrix} |x|^{-2(h_1 - 1/2)} & 2|x|^{-((h_1-1/2)+(h_2-1/2))} \\ 2|x|^{-((h_1-1/2)+(h_2-1/2))} & 4|x|^{-2(h_2-1/2)} \end{pmatrix} \mathrm{d}x \\ &= \begin{pmatrix} |t|^{2h_1} C_2(h_1)^2 & 2|t|^{h_1+h_2} C_2\Bigl(\dfrac{h_1+h_2}{2}\Bigr)^2 \\ 2|t|^{h_1+h_2} C_2\Bigl(\dfrac{h_1+h_2}{2}\Bigr)^2 & 4|t|^{2h_2} C_2(h_2)^2 \end{pmatrix}, \end{aligned}$$

where

$$C_2(h)^2 = \frac{\pi}{h\Gamma(2h)\sin(h\pi)} \tag{4.1}$$

(see, e.g., [31], page 328). Therefore, $\det(EX(t)X(t)^*) = 0$ if and only if

$$C_2(h_1)^2 C_2(h_2)^2 = \biggl(C_2\Bigl(\frac{h_1+h_2}{2}\Bigr)^2 \biggr)^2,$$

which generally does not hold.

5. Time-reversibility of OFBMs

We shall provide here conditions for an OFBM to be time-reversible. Recall that a process X is said to be time-reversible if

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{X(-t)\}_{t\in\mathbb{R}}. \tag{5.1}$$

When X is a zero-mean multivariate Gaussian stationary process, (5.1) is equivalent to

$$EX(s)X(t)^* = EX(-s)X(-t)^*, \qquad s,t\in\mathbb{R},$$

which, in turn, is equivalent to

$$EX(s)X(t)^* = EX(t)X(s)^*, \qquad s,t\in\mathbb{R}.$$

The next proposition provides necessary and sufficient conditions for time-reversibility in the case of Gaussian processes with stationary increments. It is stated without proof since the latter is elementary.

Proposition 5.1. *Let X be a Gaussian process with stationary increments and spectral representation*

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\widetilde{Y}(\mathrm{d}x)\right\}_{t\in\mathbb{R}},$$

where $\widetilde{Y}(\mathrm{d}x)$ is an orthogonal-increment random measure in $\mathbb{C}^n$. The following statements are equivalent:

(i) *X is time-reversible*;
(ii) $E\widetilde{Y}(\mathrm{d}x)\widetilde{Y}(\mathrm{d}x)^* = E\widetilde{Y}(-\mathrm{d}x)\widetilde{Y}(-\mathrm{d}x)^*$;
(iii) $EX(s)X(t)^* = EX(t)X(s)^*$, $s,t\in\mathbb{R}$.

The following result on time-reversibility of OFBMs is a direct consequence of Proposition 5.1.

Theorem 5.1. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM with exponent H and spectral representation* (3.2). *Let $A = A_1 + \mathrm{i}A_2$, where $A_1, A_2 \in M(n,\mathbb{R})$. Then B_H is time-reversible if and only if*

$$AA^* = \overline{AA^*} \quad \text{or} \quad A_2A_1^* = A_1A_2^*. \tag{5.2}$$

Proof. From Proposition 5.1(ii), time-reversibility is equivalent to

$$\begin{aligned} &E\big((x_+^{-D}A + x_-^{-D}\overline{A})\widetilde{B}(\mathrm{d}x)\widetilde{B}(\mathrm{d}x)^*(A^*x_+^{-D^*} + \overline{A^*}x_-^{-D^*})\big) \\ &\quad = E\big((x_-^{-D}A + x_+^{-D}\overline{A})\widetilde{B}(-\mathrm{d}x)\widetilde{B}(-\mathrm{d}x)^*(A^*x_-^{-D^*} + \overline{A^*}x_+^{-D^*})\big) \end{aligned}$$

or

$$x_+^{-D}AA^*x_+^{-D} + x_-^{-D}\overline{AA^*}x_-^{-D^*} = x_-^{-D}AA^*x_-^{-D} + x_+^{-D}\overline{AA^*}x_+^{-D^*} \qquad \mathrm{d}x\text{-a.e.}$$

Since $|x|^D$ is invertible for $x > 0$, this is equivalent to (5.2). □

Corollary 5.1. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM with time domain representation given by* (3.19) *and exponent H satisfying* (3.1) *and* (3.18). *Then B_H is time-reversible if and only if*

$$\begin{aligned}&\cos\left(\frac{\pi D}{2}\right)(M_+ + M_-)(M_+^* - M_-^*)\sin\left(\frac{\pi D^*}{2}\right)\\&\quad = \sin\left(\frac{\pi D}{2}\right)(M_+ - M_-)(M_+^* + M_-^*)\cos\left(\frac{\pi D^*}{2}\right).\end{aligned} \tag{5.3}$$

Proof. As in the proof of Theorem 3.2, under (3.18), the matrices $\sin(\pi D/2)$, $\cos(\pi D/2)$ and $\Gamma(D+I)$ are invertible and, thus, by using (3.26), one can equivalently re-express condition (5.2) as (5.3). □

A consequence of Theorem 5.1 is that time-irreversible OFBMs can only emerge in the multivariate context since condition (5.2) is always satisfied in the univariate context. Another elementary consequence of Proposition 5.1 is the following result, which partially justifies the interest in time-reversibility in the case of OFBMs.

Proposition 5.2. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM with H satisfying* (3.1). *If $\{B_H(t)\}_{t\in\mathbb{R}}$ is time-reversible, then its covariance structure is given by the function*

$$\begin{aligned}EB_H(s)B_H(t)^* = \frac{1}{2}\big(&|t|^H\Gamma(1,1)|t|^{H^*} + |s|^H\Gamma(1,1)|s|^{H^*}\\&- |t-s|^H\Gamma(1,1)|t-s|^{H^*}\big),\end{aligned} \tag{5.4}$$

where $\Gamma(1,1) = EB_H(1)B_H(1)^*$. Conversely, an OFBM with covariance function (5.4) *is time-reversible.*

Proof. This follows from Proposition 5.1(iii). □

Remark 5.1. One should note that for a fixed exponent H, not every positive definite matrix $\Gamma(1,1)$ leads to a valid covariance function (5.4) for time-reversible OFBMs.

In fact, fix $\Gamma(1,1) = I$, $n = 2$. We will show that, for an exponent of the form

$$H = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix}, \qquad h \in (0,1),$$

there does not exist a time-reversible OFBM B_H such that $EB_H(1)B_H(1)^* = \Gamma(1,1)$.

From Theorem 5.1,

$$EB_H(1)B_H(1)^* = \int_{\mathbb{R}}\left|\frac{e^{ix}-1}{ix}\right|^2 |x|^{-D}AA^*|x|^{-D^*}\,dx, \tag{5.5}$$

where $(s_{ij})_{i,j=1,2} := AA^* \in M(n,\mathbb{R})$. We have

$$|x|^{-D}AA^*|x|^{-D^*} = |x|^{-2d}\begin{pmatrix} 1 & \\ \log(x) & 1 \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}\begin{pmatrix} 1 & \log(x) \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_{11} & s_{11}\log(x)+s_{12} \\ s_{11}\log(x)+s_{12} & s_{11}(\log(x))^2+2s_{12}\log(x)+s_{22} \end{pmatrix}.$$

For notational simplicity, let

$$r_k(d) = \int_{\mathbb{R}} \left|\frac{e^{ix}-1}{ix}\right|^2 (\log(x))^k |x|^{-2d}\,dx, \qquad k=0,1,2.$$

We obtain

$$EB_H(1)B_H(1)^* = \begin{pmatrix} s_{11}r_0(d) & s_{11}r_1(d)+s_{12}r_0(d) \\ s_{11}r_1(d)+s_{12}r_0(d) & s_{11}r_2(d)+2s_{12}r_1(d)+s_{22}r_0(d) \end{pmatrix}.$$

On the other hand, for any real symmetric matrix, the condition for it to have equal eigenvalues is that the discriminant of the characteristic polynomial is zero. In terms of $EB_H(1)B_H(1)^*$, this means that

$$s_{11}r_0(d) = s_{11}r_2(d)+2s_{12}r_1(d)+s_{22}r_0(d), \qquad s_{11}r_1(d)+s_{12}r_0(d)=0.$$

Therefore, $s_{11}=s_{12}=s_{22}=0$, which contradicts the assumption that $\Gamma(1,1)=I$.

This issue is a problem, for instance, in the context of simulation methods that require knowledge of the covariance function. For time-reversible OFBMs with diagonalizable H, one natural way to parameterize $\Gamma(1,1)$ is by means of the formula (5.5) since, in this case, the former can be explicitly computed (see [16]).

Finally, we provide a result (Proposition 5.3) characterizing time-reversibility of some OFBMs in terms of their symmetry group G_1 (see Section 2.2). This result will be used several times in the next section.

Proposition 5.3. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM such that $hI \in \mathcal{E}(B_H)$ for some $h\in(0,1)$. Then $\{B_H(t)\}_{t\in\mathbb{R}}$ is time-reversible if and only if $G_1(B_H)$ is conjugate to $O(n)$.*

Proof. Regarding necessity, note that if such B_H is time-reversible, then, by Theorem 5.1, its covariance function can be written as

$$\Gamma(t,s) = \int_{\mathbb{R}} \left(\frac{e^{itx}-1}{ix}\right)\left(\frac{e^{-isx}-1}{-ix}\right)|x|^{-2dI} S\,dx$$

for some positive definite $S \in M(n)$ (note that if S is only positive semidefinite, then the process is not proper).

For sufficiency, consider the covariance function of the OFBM with exponent $H = hI$, $h \in (0, 1)$,

$$\Gamma(t, s) = \int_{\mathbb{R}} \left(\frac{e^{itx} - 1}{ix}\right)\left(\frac{e^{-isx} - 1}{-ix}\right)(x_+^{-2dI} AA^* + x_-^{-2dI} \overline{AA^*})\, dx.$$

Define $\widetilde{B}_H = W^{-1} B_H$, where $WO(n)W^{-1} = G_1(B_H)$ for a positive definite W. Then, for any $O \in O(n)$,

$$\{O\widetilde{B}_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{\widetilde{B}_H(t)\}_{t\in\mathbb{R}}.$$

By the uniqueness of the spectral distribution function, this implies that $O(W^{-1}AA^*W^{-1})O^* = W^{-1}AA^*W^{-1}$, that is, $O(W^{-1}AA^*W^{-1}) = (W^{-1}AA^*W^{-1})O$. Since O is any matrix in $O(n)$, it follows that $W^{-1}AA^*W^{-1} = cI$, $c \in \mathbb{C}\backslash\{0\}$ (for a proof of this technical result, see [8]). Thus, $AA^* = cW^2$ and $c > 0$. Hence, $AA^* = \overline{AA^*}$. □

6. The dichotomy principle

We now take a closer look at the increments of an OFBM, which form a stationary process.

Definition 6.1. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM. The increment process*

$$\{Y_H(t)\}_{t\in T} \stackrel{\mathcal{L}}{=} \{B_H(t+1) - B_H(t)\}_{t\in T}, \qquad \text{where } T = \mathbb{Z} \text{ or } \mathbb{R},$$

is called operator fractional Gaussian noise (OFGN).

From Theorem 3.1, the spectral representation of OFGN in continuous time is

$$\{Y_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} e^{itx} \frac{e^{ix} - 1}{ix} (x_+^{-D} A + x_-^{-D} \overline{A}) \widetilde{B}(dx)\right\}_{t\in\mathbb{R}}. \tag{6.1}$$

The spectral density of $\{Y_H(t)\}_{t\in\mathbb{R}}$ is then

$$f_{Y_H}(x) = \frac{|e^{ix} - 1|^2}{|x|^2} (x_+^{-D} AA^* x_+^{-D^*} + x_-^{-D} \overline{AA^*} x_-^{-D^*}), \qquad x \in \mathbb{R}, \tag{6.2}$$

since the cross terms are zero.

In discrete time, analogously to the univariate expression,

$$\begin{aligned} &EY_H(0)Y_H(n)^* \\ &\quad = \int_{-\pi}^{\pi} e^{inx} \sum_{k=-\infty}^{\infty} f_{Y_H}(x + 2\pi k)\, dx, \qquad n \in \mathbb{Z}. \end{aligned} \tag{6.3}$$

The spectral density of $\{Y_H(n)\}_{n\in\mathbb{Z}}$ is then

$$g_{Y_H}(x) = 2\big(1-\cos(x)\big) \times \sum_{k=-\infty}^{\infty} \frac{1}{|x+2\pi k|^2}\big((x+2\pi k)_+^{-D} AA^*(x+2\pi k)_+^{-D^*} + (x+2\pi k)_-^{-D}\overline{AA^*}(x+2\pi k)_-^{-D^*}\big), \qquad x\in[-\pi,\pi]. \tag{6.4}$$

The form (6.4) of the spectral density leads to the following result.

Theorem 6.1. *Let H be an exponent with (possibly repeated) characteristic roots h_l, $l = 1,\dots,n$, such that*

$$1/2 < \mathrm{Re}(h_l) < 1, \qquad l = 1,\dots,n. \tag{6.5}$$

Let $g_{Y_H}(x) = \{g_{Y_H}(x)_{ij}\}$ be the spectral density (6.4) of OFGN in discrete time. Then, for fixed i, j, either:

(i) $|g_{Y_H}(x)_{ij}| \to \infty$, *as* $x\to 0$; *or*
(ii) $g_{Y_H}(x)_{ij} \equiv 0$ *on* $[-\pi,\pi]$.

Proof. Let d_l and N be as in (2.2) and (2.3), and take $x>0$. By assumption (6.5), $0<\mathrm{Re}(d_l)< 1/2$. For a given $z>0$, if we take $-D$ in Jordan canonical form PJP^{-1}, we obtain that

$$z^{-D} = P\,\mathrm{diag}(z^{J_{-d_1}},\dots,z^{J_{-d_N}})P^{-1},$$

where J_{-d_l} is a Jordan block in J, $l=1,\dots,N\le n$. Without loss of generality, for $k\ge 0$, each term of the summation (6.4) involves the matrix expression

$$P\,\mathrm{diag}\big((x+2\pi k)_+^{J_{-d_1}},\dots,(x+2\pi k)_+^{J_{-d_N}}\big)P^{-1}A \times A^*(P^*)^{-1}\,\mathrm{diag}\big((x+2\pi k)_+^{J^*_{-d_1}},\dots,(x+2\pi k)_+^{J^*_{-d_N}}\big)P^*. \tag{6.6}$$

Denote the entries of the matrix-valued function (6.6) by $h(x+2\pi k)_{ij}$, $i,j=1,\dots,n$. As shown in Appendix D, $h(x+2\pi k)_{ij}$ is a linear combination (with complex coefficients) of terms of the form

$$p(x+2\pi k)(x+2\pi k)^{-d_l}q(x+2\pi k)(x+2\pi k)^{-\overline{d}_m}, \qquad l,m=1,\dots,N,$$

where $p(x), q(x)$ are polynomials (with complex coefficients) in $\log(x)$. Thus,

$$\sup_{x\in[-\pi,\pi]}\left|\sum_{k=1}^{\infty}\frac{1}{|x+2\pi k|^2}h(x+2\pi k)_{ij}\right| < \infty, \qquad i,j=1,\dots,n, \tag{6.7}$$

and, therefore,

$$\lim_{x\to 0} 2\big(1-\cos(x)\big)\left(\frac{1}{|x|^2}h(x)_{ij} + \sum_{k=1}^{\infty}\frac{1}{|x+2\pi k|^2}h(x+2\pi k)_{ij}\right)$$
$$= \lim_{x\to 0} 2\big(1-\cos(x)\big)\left(\frac{1}{|x|^2}h(x)_{ij}\right).$$

On the other hand, since $\mathrm{Re}(d_l) > 0$ for $l = 1,\dots,n$, $h(x)_{ij}$ diverges as the power function (times some $p(x)q(x)$) as $x \to 0$ unless it is identically zero for all x (in particular, for $x + 2\pi k$). Thus, by (6.7) and the fact that $\frac{2(1-\cos(x))}{x^2} \to 1$ as $x \to 0$, the claim follows. □

In the univariate context, the range $(1/2, 1)$ for H is commonly known as that of long-range dependence (LRD). In the multivariate context, characteristic roots of H with real parts between $1/2$ and 1 have the potential to generate divergence of the spectrum at zero. Theorem 6.1 thus states that if OFGN is long-range dependent in the sense of (6.5), then the cross correlation between any two components is characterized by the following *dichotomy*:

- it either has a divergent spectrum at zero, a characteristic usually associated with LRD; or
- it is identically equal to zero.

Obtaining a similar dichotomy principle for a larger range of characteristic roots than that in (6.5) is much more delicate. The following two examples illustrate some of the potential difficulties. Example 6.1 shows that if one of the characteristic roots of $D = H - (1/2)I$ is 0, then the dichotomy may not hold. Example 6.2 shows that certain cancellations may occur in the cross spectrum if the characteristic roots of D have opposite signs.

Example 6.1. If, for instance, $D = P\,\mathrm{diag}(d,0)P^{-1}$, where $0 < d < 1/2$,

$$P = \begin{pmatrix} 1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{pmatrix}$$

and $A := P$, then, as $x \to 0$,

$$g_{Y_H}(x) \sim \begin{pmatrix} x^{-2d} + 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

where $\sim$ indicates entrywise asymptotic equivalence. As a consequence, if one of the components of OFBM behaves like Brownian motion, then this may create cross short-range dependence among the components. This example is a direct consequence of a more general operator parameter D whose eigenspaces are not the canonical axes. If we take, instead, $D = \mathrm{diag}(d,0)$, whose eigenspaces are the canonical axes, each term of the summation (6.4) has the form

$$\mathrm{diag}\big((x+2\pi k)_{\pm}^{-d},0\big)AA^*\,\mathrm{diag}\big((x+2\pi k)_{\pm}^{-d},0\big) = \begin{pmatrix} s_{11}(x+2\pi k)_{\pm}^{-2d} & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$(s_{ij})_{i,j=1,2} := AA^* \tag{6.8}$$

and thus the dichotomy holds.

Example 6.2. Consider $A \in GL(n, \mathbb{R})$ and $D = \mathrm{diag}(d, -d)$, where $d \in (0, 1/2)$. Using the notation (6.8), we have

$$g_{Y_H}(x) \sim P\,\mathrm{diag}(x^{-d}, x^{d})P^* AA^* P\,\mathrm{diag}(x^{-d}, x^{d})P^* = \begin{pmatrix} s_{11}x^{-2d} & s_{12} \\ s_{12} & s_{22}x^{2d} \end{pmatrix}$$

as $x \to 0$. Here, the multivariate differencing effects of the operator D cancel out in the cross-entries.

From a practical perspective, Theorem 6.1 raises the question of whether the class of OFGNs is flexible enough to capture multivariate LRD structures. This, and related issues regarding multivariate discrete time series, will be explored in future work.

7. Operator Brownian motions

In this section, we shall adopt the following definition of multivariate Brownian motion and establish some of its properties.

Definition 7.1. *The proper process $\{B_H(t)\}_{t\in\mathbb{R}}$ is an operator Brownian motion (OBM) if it is a Gaussian o.s.s. process which has stationary and independent increments and satisfies $B_H(0) = 0$ a.s.*

In place of the condition $B_H(0) = 0$ a.s., we can assume that the characteristic roots of the o.s.s. exponent H have positive real parts, which implies the former condition. Another important way to motivate Definition 7.1 is as follows: since $\{B_H(t)\}_{t\in\mathbb{R}}$ is L^2-continuous (see the beginning of the proof of Theorem 3.1), by Hudson and Mason [18], Theorem 4, and Hudson and Mason [19], Theorem 7, our Definition 7.1 implies that $(1/2)I$ can always be taken as an exponent of OBM.

The next proposition and example show that an OFBM B_H with $(1/2)I \in \mathcal{E}(B_H)$ is not necessarily an OBM. This stands in contrast with the univariate case.

Proposition 7.1. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OFBM with exponent $H = (1/2)I$. Consider its time domain representation (3.21) with parameters M and N, or its spectral domain representation (3.2) with $A = A_1 + \mathrm{i}A_2$. Then $\{B_H(t)\}_{t\in\mathbb{R}}$ is an OBM if and only if the following two equivalent conditions hold*:

(i) $MN^* = NM^*$;
(ii) $A_2A_1^* = A_1A_2^*$.

Proof. Since $(1/2)I \in \mathcal{E}(B_H)$, it follows that $B_H(0) = 0$ a.s. Therefore, we only have to establish that the increments are independent if and only if (i) holds. Demonstrating the equivalence between (i) and (ii) is straightforward by using the relation (3.22).

Write the time domain representation (3.21) of B_H as

$$\int_{\mathbb{R}} \bigl(2\bigl(1_{\{t-u>0\}} - 1_{\{-u>0\}}\bigr)M + (\log|t-u| - \log|-u|)N\bigr)B(\mathrm{d}u). \tag{7.1}$$

Take $s_1 < t_1 < s_2 < t_2$. For the increments of the process B_H, we have

$$\begin{aligned} &E\bigl(B_H(t_1) - B_H(s_1)\bigr)\bigl(B_H(t_2) - B_H(s_2)\bigr)^* \\ &= \int_{\mathbb{R}} \biggl(4 \cdot 1_{\{s_1<u<t_1\}} 1_{\{s_2<u<t_2\}} MM^* + \log\frac{|t_1-u|}{|s_1-u|} \log\frac{|t_2-u|}{|s_2-u|} NN^* \\ &\quad + 2 \cdot 1_{\{s_1<u<t_1\}} \log\frac{|t_2-u|}{|s_2-u|} MN^* + 2 \cdot 1_{\{s_2<u<t_2\}} \log\frac{|t_1-u|}{|s_1-u|} NM^*\biggr)\,\mathrm{d}u. \end{aligned} \tag{7.2}$$

From the univariate time domain representation of Brownian motion, the first two of the four terms in (7.2) have zero integral. Define $\varphi(u) = u(\log(u) - 1)$. The right-hand side of the expression (7.2) then becomes

$$\begin{aligned} &\bigl(\varphi(t_2 - s_1) - \varphi(t_2 - t_1) - \varphi(s_2 - s_1) + \varphi(s_2 - t_1)\bigr)MN^* \\ &\quad + \bigl(\varphi(t_2 - t_1) - \varphi(s_2 - t_1) - \varphi(t_2 - s_1) + \varphi(s_2 - s_1)\bigr)NM^*, \end{aligned}$$

which is identically zero if and only if $MN^* = NM^*$. □

Example 7.1. Consider a bivariate process X defined by the expression (7.1). Set $M = I$ and let $N = L \in so(2)\backslash\{0\}$. Then $MN^* = -NM^* \neq 0$ (from which the cross terms in expression (7.2) cancel out when $s_1 = s_2 = 0$ and $t_1 = t_2 = t$) and

$$EX(t)X(t)^* = 4|t|I + \pi^2|t|L(-L) = |t|(4I - \pi^2 L^2),$$

which is a full rank matrix for $t \neq 0$ (to obtain the constant π^2, one can use, e.g., Proposition 9.2 in [33] and Proposition 5.1 in [32]). Hence, X is proper. This gives an example of an OFBM for which $(1/2)I \in \mathcal{E}(X)$ but which is *not* an OBM. Moreover, it is an example of an OFBM with an exponent of the form hI, $h \in (0, 1)$, but which is *not* time-reversible and for which $G_1 \cong O(2)$ does *not* hold by Proposition 5.3 (in contrast, by Hudson and Mason [19], Theorem 6, $G_1(X) \cong O(n)$ implies that $hI \in \mathcal{E}(X)$ for some h).

The following result is a direct consequence of Theorem 5.1 and Proposition 7.1. It shows that in the class of OFBMs with exponent $H = (1/2)I$, time-reversibility is equivalent to independence of increments.

Corollary 7.1. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be a time-reversible OFBM with exponent $H = (1/2)I$. Then $\{B_H(t)\}_{t\in\mathbb{R}}$ is an OBM. Conversely, let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OBM. It is then time-reversible (and has exponent $H = (1/2)I$).*

Remark 7.1. Note that, as a consequence of Proposition 5.3, time-reversibility may be replaced in Corollary 7.1 by the condition that G_1 is conjugate to $O(n)$. In other words, an OBM is elliptically symmetric.

We conclude by providing a spectral representation for OBM.

Proposition 7.2. *Let $\{B_H(t)\}_{t\in\mathbb{R}}$ be an OBM. Then*

$$\{B_H(t)\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \left\{\int_{\mathbb{R}} \frac{e^{itx}-1}{ix} W \widetilde{B}(dx)\right\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \{WB(t)\}_{t\in\mathbb{R}} \tag{7.3}$$

for some positive definite operator W, where $\{B(t)\}_{t\in\mathbb{R}}$ is a vector of independent standard BMs.

Proof. Consider the spectral domain representation of B_H with parameter $A = A_1 + iA_2$. Set $W := (A_1A_1^* + A_2A_2^*)^{1/2}$. The result follows from Proposition 7.1(ii), and relations (3.23) and (3.25). □

Appendix A: Fourier transforms of OFBM kernels

In this appendix, the goal is to prove Proposition 3.1. First, we state a condensed version of Horn and Johnson [17], Theorems 6.2.9 and 6.2.10, pages 412–416, which will be useful in the subsequent derivations. We shall use the notation introduced before Definition 3.1.

Theorem A.1. *Let $f, g : U \to \mathbb{C}$ be two stem functions and let $\mathcal{M}_{fg} = \mathcal{M}_f \cap \mathcal{M}_g$. Then:*

(i) *the primary matrix function $f : \mathcal{M}_f \to M(n,\mathbb{C})$ is well defined in the sense that the value of $f(\Lambda)$, $\Lambda \in \mathcal{M}_f$, is independent of the particular Jordan canonical form (i.e., block permutation) used to represent it;*
(ii) *$f(\Lambda) = g(\Lambda)$ if and only if $f^{(j)}(\lambda_k) = g^{(j)}(\lambda_k)$ for $j = 0, 1, \dots, r_k - 1$, $k = 1, \dots, N$ and $\Lambda \in \mathcal{M}_{fg}$;*
(iii) *for $q(z) := f(z)g(z)$, we have $q(\Lambda) = f(\Lambda)g(\Lambda) = g(\Lambda)f(\Lambda)$ for $\Lambda \in \mathcal{M}_{fg}$;*
(iv) *for $s(z) := f(z) + g(z)$, we have $s(\Lambda) = f(\Lambda) + g(\Lambda)$ for $\Lambda \in \mathcal{M}_{fg}$.*

Throughout this section, we assume (3.1) and (3.18). Denote by $\mathcal{F}$ the Fourier transform operator. For $d \in \mathbb{C}$ such that

$$\operatorname{Re}(d) \in (-1/2, 1/2)\backslash\{0\}, \tag{A.1}$$

define

$$f_\pm(t,u,d) = (t-u)_\pm^d - (-u)_\pm^d$$

and

$$h_\pm(t,x,d) = \frac{e^{itx}-1}{ix}|x|^{-d}\Gamma(d+1)e^{\mp \operatorname{sign}(x) i\pi d/2}.$$

It is well known that

$$\mathcal{F}(f_\pm(t,\cdot,d))(x) = h_\pm(t,x,d) \tag{A.2}$$

when $d \in (-1/2, 1/2)\backslash\{0\}$ (see, e.g., [29], page 175). One can show that (A.2) also holds under (A.1) (see Remark A.1).

For the purpose of calculating Fourier transforms of primary matrix functions associated with the stem functions $f_{\pm}$ and $h_{\pm}$, we will need to consider derivatives of the latter with respect to d. Note that, for fixed x, the functions $\Gamma(d+1)$, $e^{\mp \operatorname{sign}(x) i\pi d/2}$ and $|x|^{-d}$ are holomorphic on the domain $-\frac{1}{2} < \operatorname{Re}(d) < \frac{1}{2}$. Thus, so are the functions $h_{\pm}(t, x, d)$. Note that, for fixed t and u, since $(t-u)_{\pm}^{d}$ and $(-u)_{\pm}^{d}$ are holomorphic on the domain $-1/2 < \operatorname{Re}(d) < 1/2$, then so are $f_{\pm}(t, u, d)$.

As a consequence, by Theorem A.1(i)–(iv),

$$h_{\pm}(t, x, D) = \frac{e^{itx} - 1}{ix} |x|^{-D} \Gamma(D + I) e^{\mp \operatorname{sign}(x) i\pi D/2}$$

and

$$f_{\pm}(t, u, D) = (t-u)_{\pm}^{D} - (-u)_{\pm}^{D}.$$

We now need to show that

$$\mathcal{F}(f_{\pm}(t, u, D))(x) = h_{\pm}(t, x, D), \tag{A.3}$$

where $\mathcal{F}$ is the *entrywise* Fourier transform operator.

Proof of Proposition 3.1. We will break up the proof into three cases:

Case 1: $-1/2 < \operatorname{Re}(d_k) < 0$, $k = 1, \ldots, N$. We will develop the calculations for h_+, which can be easily adapted to h_-. By Theorem A.1(ii), in the case of h_+, (A.3) is equivalent to

$$\frac{\partial^j}{\partial d^j} h_+(t, x, d) = \frac{\partial^j}{\partial d^j} \int_{\mathbb{R}} e^{iux} f_+(t, u, d)\, du = \int_{\mathbb{R}} e^{iux} \frac{\partial^j}{\partial d^j} f_+(t, u, d)\, du \tag{A.4}$$

at $d = d_k$, for $k = 1, \ldots, N$, $j = 0, 1, \ldots, r_k - 1$. Consider the domain $\Delta(\underline{d}, \overline{d}) := \{d \in \mathbb{C} : \underline{d} < \operatorname{Re}(d) < \overline{d})\}$, where $-1/2 < \underline{d} < \overline{d} < 0$, which is open and convex. Consider $j = 1$, that is, the first derivative. Fix $d^* \in \Delta(\underline{d}, \overline{d})$ and take a sequence $\{d_m\}_{m \in \mathbb{N}} \subseteq \Delta(\underline{d}, \overline{d})$ such that $d_m \to d^*$. For each m, by the mean value theorem for holomorphic functions ([13], Theorem 2.2), there exist constants $\delta_i(m)$, where

$$\delta_i(m) = \alpha_{m,i} d_m + (1 - \alpha_{m,i}) d^*, \qquad \alpha_{m,i} \in (0, 1), i = 1, 2,$$

such that

$$\begin{aligned} \frac{f_+(t, u, d_m) - f_+(t, u, d^*)}{d_m - d^*} &= \operatorname{Re}\left(\frac{f_+(t, u, d_m) - f_+(t, u, d^*)}{d_m - d^*}\right) \\ &\quad + i \operatorname{Im}\left(\frac{f_+(t, u, d_m) - f_+(t, u, d^*)}{d_m - d^*}\right) \\ &= \operatorname{Re}\left(\frac{\partial}{\partial d} f_+(t, u, \delta_1(m))\right) + i \operatorname{Im}\left(\frac{\partial}{\partial d} f_+(t, u, \delta_2(m))\right). \end{aligned}$$

We will now obtain an integrable function that majorizes $|\frac{\partial}{\partial d} f_+(t,\cdot,d)|$ for all $d \in \Delta(\underline{d},\overline{d})$. Assume, without loss of generality, that $t > 0$, and take $d \in \Delta(\underline{d},\overline{d})$ and $\delta > 0$ such that $-1/2 < \mathrm{Re}(\underline{d}) - \delta$ and $\mathrm{Re}(\overline{d}) + \delta < 0$. From the continuity of $\frac{\partial}{\partial d} f_+(t,u,d)$ for $0 \le u < t$, there exist constants K_1 and η_1 such that

$$\begin{aligned} \left|\frac{\partial}{\partial d} f_+(t,u,d)\right| &\le |\log(t-u)_+||(t-u)_+^d| \\ &\le K_1 1_{[0,t-\eta_1]}(u) + \left|(t-u)_+^{\mathrm{Re}(\underline{d})-\delta}\right| 1_{(t-\eta_1,t)}(u), \qquad u \ge 0. \end{aligned} \tag{A.5}$$

Also, there exists a constant K_2 such that

$$|\log(t-u)_+(t-u)_+^d - \log(-u)_+(-u)_+^d| \le K_2 + \left|(-u)_+^{\mathrm{Re}(\underline{d})-\delta}\right|, \qquad -1 \le u < 0. \tag{A.6}$$

One can show that there exist constants K_3 and $\eta_2 < -1$ such that

$$|\log(t-u)_+(t-u)_+^d - \log(-u)_+(-u)_+^d| \le K_3(-u)_+^{\mathrm{Re}(\overline{d})+\delta-1}, \qquad u < \eta_2. \tag{A.7}$$

From (A.5), (A.6) and (A.7), and from the fact that $\frac{\partial}{\partial d} f_+(t,u,d)$ is bounded on $\eta_2 \le u \le -1$ uniformly in $d \in \Delta(\underline{d},\overline{d})$, we conclude that the ratio $\frac{f_+(t,\cdot,d_m)-f_+(t,\cdot,d^*)}{d_m-d^*}$ is bounded by a function in $L^1(\mathbb{R})$. Thus, by the dominated convergence theorem (for $\mathbb{C}$-valued functions), we have

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} \frac{f_+(t,u,d_m) - f_+(t,u,d^*)}{d_m - d^*}\,\mathrm{d}u \to \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} \frac{\partial}{\partial d} f_+(t,u,d^*)\,\mathrm{d}u, \qquad m \to \infty.$$

We can always assume that $t \ne 0$ and the case of $t < 0$ can be dealt with in a similar fashion. The extension of the above argument for derivatives of higher order j poses no additional technical difficulties. This establishes (A.4).

Case 2: $0 < \mathrm{Re}(d_k) < 1/2$, $k = 1,\dots,N$. In this range, the upper bound in (A.7) is not in $L^1(\mathbb{R})$ so we need a slightly different procedure. Since $h_\pm(t,\cdot,d) \in L^2(\mathbb{R})$, we can apply $\mathcal{F}^{-1}$ on both sides of equation (A.2) and obtain

$$f_\pm(t,u,d) = \mathcal{F}^{-1}(h_\pm(t,\cdot,d))(u).$$

Therefore, it suffices to show that

$$f_\pm(t,u,D) = \mathcal{F}^{-1}(h_\pm(t,x,D)), \tag{A.8}$$

where $\mathcal{F}^{-1}$ is the *entrywise* inverse Fourier transform.

Note that expression (A.8) is equivalent to

$$\frac{\partial^j}{\partial d^j} f_\pm(t,x,d) = \frac{\partial^j}{\partial d^j} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} h_\pm(t,x,d)\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} \frac{\partial^j}{\partial d^j} h_\pm(t,x,d)\,\mathrm{d}x$$

at $d = d_k$, for $k = 1, \dots, N$, $j = 0, 1, \dots, r_k - 1$. To show this, one may proceed as in the case of $-1/2 < \mathrm{Re}(d) < 0$. The existence of an upper bound in $L^1(\mathbb{R})$ is ensured by the fact that

$$\begin{aligned}\left|\frac{\partial^j}{\partial d^j}\left(\left(\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\right)|x|^{-d}\right)\right| &\leq \left|\left(\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\right)|\log|x||^j|x|^{-\mathrm{Re}(d)}\right| \\ &\leq \left|\left(\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\right)|\log|x||^j|x|^{-\mathrm{Re}(\overline{d})}1_{\{0<|x|\leq 1\}}\right| \\ &\quad + \left|\left(\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\right)|\log|x||^j|x|^{-\mathrm{Re}(\underline{d})}1_{\{1<|x|<\infty\}}\right|,\end{aligned} \tag{A.9}$$

which is integrable for all $d \in \Delta(\underline{d}, \overline{d})$, $0 < \underline{d} < \overline{d} < 1/2$.

General case: As a consequence of (A.3),

$$\mathcal{F}(f_\pm(t, \cdot, J))(x) = h_\pm(t, x, J) \tag{A.10}$$

holds, where J is a matrix in Jordan canonical form with characteristic roots satisfying (3.1) and (3.18). Now pre- and post-multiply equation (A.10) by P and P^{-1}, respectively. Since

$$P\Gamma(-J)P^{-1} = \Gamma(-D), \qquad P\mathrm{e}^{(\mathrm{i}\pi/2)(J+I)}P^{-1} = \mathrm{e}^{(\mathrm{i}\pi/2)(D+I)},$$

it follows that

$$Pf_\pm(t, u, J)P^{-1} = f_\pm(t, u, D), \qquad Ph_\pm(t, u, J)P^{-1} = h_\pm(t, u, D),$$

from which we obtain equation (A.3). □

Remark A.1. A common way to prove that (A.2) also holds for $d \in \mathbb{C}$ satisfying (A.1) is by analytic continuation. In particular, this requires the ability to differentiate under the integral sign in the Fourier transform. The latter could be achieved by following the argument in the proof of Proposition 3.1.

Appendix B: Some useful integrals

In this appendix, we calculate the inverse Fourier transforms used in the proof of Theorem 3.2(ii) and Example 3.1. We shall use several formulas from [14]:

$$\int_{\mathbb{R}} 1_{\{x<0\}}\frac{\sin(ax)}{x}\,\mathrm{d}x = \int_{\mathbb{R}} 1_{\{x>0\}}\frac{\sin(ax)}{x}\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{sign}(a) \qquad \text{(page 423)}, \tag{B.1}$$

$$\int_{\mathbb{R}} 1_{\{x>0\}}\frac{\cos(ax)-\cos(bx)}{x}\,\mathrm{d}x = \log\frac{|b|}{|a|} \qquad \text{(page 447)} \tag{B.2}$$

$$\left(\text{therefore, } \int_{\mathbb{R}} 1_{\{x<0\}}\frac{\cos(ax)-\cos(bx)}{x}\,\mathrm{d}x = -\log\frac{|b|}{|a|}\right), \tag{B.3}$$

$$\int_0^\infty \log(x)\sin(ax)\frac{\mathrm{d}x}{x} = -\frac{\pi}{2}\big(C+\log(a)\big), \qquad a>0 \qquad \text{(page 594)} \tag{B.4}$$

$$\bigg(\text{therefore, with } a\in\mathbb{R},\ \int_0^\infty \log(x)\sin(ax)\frac{\mathrm{d}x}{x} = \int_{-\infty}^0 \log(x_-)\sin(ax)\frac{\mathrm{d}x}{x} \\ = -\frac{\pi}{2}(C+\log|a|)\operatorname{sign}(a)\bigg), \tag{B.5}$$

$$\int_0^\infty \log(x)\big(\cos(ax)-\cos(bx)\big)\frac{\mathrm{d}x}{x} = \log\Big(\frac{a}{b}\Big)\Big(C+\frac{1}{2}\log(ab)\Big), \qquad a,b>0 \\ \text{(page 594)}, \tag{B.6}$$

where C is Euler's constant

$$\bigg(\text{therefore, } \int_{-\infty}^0 \log(x_-)\big(\cos(ax)-\cos(bx)\big)\frac{\mathrm{d}x}{x} = -\log\Big(\frac{|a|}{|b|}\Big)\Big(C+\frac{1}{2}\log(|ab|)\Big)\bigg). \tag{B.7}$$

Using these formulas, we obtain that, for $x>0$,

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux}\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}1_{\{x>0\}}\,\mathrm{d}x \\ = \int_{\mathbb{R}} \frac{1}{\mathrm{i}x}\Big(\cos\big((t-u)x\big)-\cos(ux)+\mathrm{i}\big(\sin\big((t-u)x\big)+\sin(ux)\big)\Big)1_{\{x>0\}}\,\mathrm{d}x \\ = \frac{1}{\mathrm{i}}\log\Big(\frac{|u|}{|t-u|}\Big)+\frac{\pi}{2}\big(\operatorname{sign}(t-u)-\operatorname{sign}(-u)\big).$$

Similarly, for $x<0$,

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux}\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}1_{\{x<0\}}\,\mathrm{d}x \\ = \int_{\mathbb{R}} \frac{1}{\mathrm{i}x}\Big(\cos\big((t-u)x\big)-\cos(ux)+\mathrm{i}\big(\sin\big((t-u)x\big)+\sin(ux)\big)\Big)1_{\{x<0\}}\,\mathrm{d}x \\ = -\frac{1}{\mathrm{i}}\log\Big(\frac{|u|}{|t-u|}\Big)+\frac{\pi}{2}\big(\operatorname{sign}(t-u)-\operatorname{sign}(-u)\big).$$

Therefore,

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux}\frac{\mathrm{e}^{\mathrm{i}tx}-1}{\mathrm{i}x}\big(1_{\{x>0\}}A+1_{\{x<0\}}\overline{A}\big)\,\mathrm{d}x \\ = \big(\operatorname{sign}(t-u)-\operatorname{sign}(-u)\big)\frac{1}{2}\operatorname{Re}(A)+\log\Big(\frac{|u|}{|t-u|}\Big)\frac{1}{\pi}\operatorname{Im}(A), \tag{B.8}$$

which is the formula used in the proof of Theorem 3.2(ii).

We now turn to the calculations of the inverse Fourier transform (3.27) in Example 3.1. Note that

$$|x|^{-D} = \begin{pmatrix} 1 & 0 \\ -\log|x| & 1 \end{pmatrix}, \qquad x > 0.$$

We only need to calculate the inverse Fourier transform of the log term on the lower off-diagonal.

For $x > 0$, using the formulas above,

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} \log(x_+) 1_{\{x>0\}} \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{1}{\mathrm{i}x} \big(\cos\big((t-u)x\big) - \cos(ux) + \mathrm{i}\big(\sin\big((t-u)x\big) + \sin(ux)\big)\big) \log(x_+) 1_{\{x>0\}} \,\mathrm{d}x \\
&= \frac{1}{\mathrm{i}} \log\bigg(\frac{|t-u|}{|u|}\bigg)\bigg(C + \frac{1}{2}\log(|t-u||u|)\bigg) \\
&\quad - \frac{\pi}{2}\big((C + \log|t-u|)\operatorname{sign}(t-u) - (C + \log|u|)\operatorname{sign}(-u)\big).
\end{aligned}$$

Similarly, for $x < 0$,

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} \log(x_-) 1_{\{x<0\}} \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{1}{\mathrm{i}x} \big(\cos\big((t-u)x\big) - \cos(ux) + \mathrm{i}\big(\sin\big((t-u)x\big) + \sin(ux)\big)\big) \log(x_-) 1_{\{x<0\}} \,\mathrm{d}x \\
&= -\frac{1}{\mathrm{i}} \log\bigg(\frac{|t-u|}{|u|}\bigg)\bigg(C + \frac{1}{2}\log(|t-u||u|)\bigg) \\
&\quad - \frac{\pi}{2}\big((C + \log|t-u|)\operatorname{sign}(t-u) - (C + \log|u|)\operatorname{sign}(-u)\big).
\end{aligned}$$

Therefore, for $a \in \mathbb{C}$,

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{i}x} \big(-\log(x_+) 1_{\{x>0\}} a - \log(x_-) 1_{\{x<0\}} \overline{a}\big) \,\mathrm{d}x \\
&= \big((C + \log|t-u|)\operatorname{sign}(t-u) - (C + \log|u|)\operatorname{sign}(-u)\big)\frac{1}{2}\operatorname{Re}(a) \\
&\quad + \log\bigg(\frac{|t-u|}{|u|}\bigg)\bigg(C + \frac{1}{2}\log(|t-u||u|)\bigg)\bigg(-\frac{1}{\pi}\bigg)\operatorname{Im}(a).
\end{aligned} \tag{B.9}$$

By combining (B.9) and (B.8), one obtains the time domain kernels on the right-hand side of (3.28).

Appendix C: Nonexistence of OFBM for certain exponents

Proposition C.1 is mentioned in Remark 3.2.

Proposition C.1. *There does not exist an OFBM with exponent*

$$H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proof. Assume that such an OFBM exists. For notational simplicity, denote the process by X, and its entrywise processes by X_1 and X_2. We will show that X is not a proper process.

Note that, for $c > 0$, from the matrix expression for c^H and o.s.s.,

$$\left\{\begin{pmatrix} X_1(ct) \\ X_2(ct) \end{pmatrix}\right\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \left\{\begin{pmatrix} cX_1(t) \\ c\log(c)X_1(t) + cX_2(t) \end{pmatrix}\right\}_{t\in\mathbb{R}}. \tag{C.1}$$

In particular, this implies that X_1 is FBM with Hurst exponent 1. Thus, $X_1(t) = tZ$ a.s., where Z is a Gaussian random variable (e.g., [33]). By plugging this into (C.1), we obtain

$$\left\{\begin{pmatrix} ctZ \\ X_2(ct) \end{pmatrix}\right\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \left\{\begin{pmatrix} ctZ \\ c\log(c)tZ + cX_2(t) \end{pmatrix}\right\}_{t\in\mathbb{R}}. \tag{C.2}$$

In particular, by taking $c = t$ and $t = 1$, (C.2) implies that

$$EX_2(t)Z = E\big(t\log(t)Z + tX_2(1)\big)Z = t\log(t)EZ^2 + tEX_2(1)Z.$$

Thus,

$$\begin{aligned} &E\big(X_2(t+h) - X_2(h)\big)\big(X_1(1+h) - X_1(h)\big) \\ &= EX_2(t+h)Z - EX_2(h)Z \\ &= \big((t+h)\log(t+h) - h\log(h)\big)EZ^2 + tEX_2(1)Z. \end{aligned} \tag{C.3}$$

By stationarity of the increments, the expression (C.3) does not depend on h, which is possible only when

$$EZ^2 = 0.$$

As a consequence, $\{X_2(ct)\}_{t\in\mathbb{R}} \overset{\mathcal{L}}{=} \{cX_2(t)\}_{t\in\mathbb{R}}$. Thus,

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ tY \end{pmatrix} \qquad \text{a.s.,}$$

where Y is a Gaussian random variable. In particular, X is not proper. □

Appendix D: The exponential of a matrix in Jordan canonical form

Initially, let $J_\lambda \in M(n,\mathbb{C})$ be a Jordan block of size n_λ, whose expression is

$$J_\lambda = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda \end{pmatrix}. \tag{D.1}$$

We have

$$z^{J_\lambda} = \begin{pmatrix} z^\lambda & 0 & 0 & \dots & 0 \\ (\log z) z^\lambda & z^\lambda & 0 & \dots & 0 \\ \frac{(\log z)^2}{2!} z^\lambda & (\log z) z^\lambda & z^\lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{(\log z)^{n_\lambda - 1}}{(n_\lambda - 1)!} z^\lambda & \frac{(\log z)^{n_\lambda - 2}}{(n_\lambda - 2)!} z^\lambda & \dots & (\log z) z^\lambda & z^\lambda \end{pmatrix}. \tag{D.2}$$

The expression for z^J, where J is, more generally, a matrix in Jordan canonical form (i.e., whose diagonal is made up of Jordan blocks), follows immediately. In particular, the series-based notion of the matrix exponential is consistent with the primary matrix function-based notion of the matrix exponential.

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