Bernoulli **16**(4), 2010, 926–952
DOI: 10.3150/09-BEJ244

Hausdorff and packing dimensions of the images of random fields

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Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. For any finite Borel measure μ and analytic set $E \subset \mathbb{R}^N$, the Hausdorff and packing dimensions of the image measure μ_X and image set $X(E)$ are determined under certain mild conditions. These results are applicable to Gaussian random fields, self-similar stable random fields with stationary increments, real harmonizable fractional Lévy fields and the Rosenblatt process.

Keywords: Hausdorff dimension; images; packing dimension; packing dimension profiles; real harmonizable fractional Lévy motion; Rosenblatt process; self-similar stable random fields

1. Introduction

Fractal dimensions such as Hausdorff dimension, box-counting dimension and packing dimension are very useful in characterizing roughness or irregularity of stochastic processes and random fields which, in turn, serve as stochastic models in various scientific areas including image processing, hydrology, geostatistics and spatial statistics. Many authors have studied the Hausdorff dimension and exact Hausdorff measure of the image sets of Markov processes and Gaussian random fields. We refer to Taylor (1986) and Xiao (2004) for extensive surveys on results and techniques for Markov processes, and to Adler (1981) and Kahane (1985) for results on Gaussian random fields.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$, which will simply be called an (N, d)-*random field*. For any finite Borel measure μ on $\mathbb{R}^N$, the image measure of μ under X is defined by $\mu_X := \mu \circ X^{-1}$. Similarly, for every $E \subset \mathbb{R}^N$, the image set is denoted by $X(E) = \{X(t), t \in E\} \subset \mathbb{R}^d$. This paper is concerned with the Hausdorff and packing dimensions of the image measures and image sets of random fields which are, in a certain sense, comparable to a self-similar process. Recall that $X = \{X(t), t \in \mathbb{R}^N\}$ is said to be H-*self-similar* if, for every constant $c > 0$, we have

$$\{X(ct), t \in \mathbb{R}^N\} \stackrel{d}{=} \{c^H X(t), t \in \mathbb{R}^N\} \tag{1.1}$$

and X is said to have *stationary increments* if, for every $h \in \mathbb{R}^N$,

$$\{X(t+h) - X(h), t \in \mathbb{R}^N\} \stackrel{d}{=} \{X(t) - X(0), t \in \mathbb{R}^N\}, \tag{1.2}$$

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where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions. If X satisfies both (1.1) and (1.2), then it is called an *H-SSSI random field*. Samorodnitsky and Taqqu (1994) give a systematic account of self-similar stable processes. The main results of this paper show that the Hausdorff and packing dimensions of the images of an H-SSSI random field X are determined by the self-similarity index H and essentially do not depend on the distributions of X.

An important example of an H-SSSI (N, d)-random field is fractional Brownian motion $X = \{X(t), t \in \mathbb{R}^N\}$ of index H $(0 < H < 1)$, which is a centered Gaussian random field with the covariance function $\mathbb{E}[X_i(t)X_j(s)] = \frac{1}{2}\delta_{i,j}(\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H})$ for all $s, t \in \mathbb{R}^N$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. It is well known (see Kahane (1985), Chapter 18) that for every Borel set $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}} E\right\} \qquad \text{a.s.,} \tag{1.3}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. On the other hand, Talagrand and Xiao (1996) proved that, when $N > Hd$, the packing dimension analog of (1.3) fails in general. Xiao (1997) proved that

$$\dim_{\mathrm{P}} X(E) = \frac{1}{H}\mathrm{Dim}_{Hd}\, E \qquad \text{a.s.,} \tag{1.4}$$

where $\dim_{\mathrm{P}}$ denotes packing dimension and $\mathrm{Dim}_s\, E$ is the packing dimension profile of E defined by Falconer and Howroyd (1997) (see Section 2 for its definition). Results (1.3) and (1.4) show that there are significant differences between Hausdorff dimension and packing dimension, and both dimensions are needed for characterizing the fractal structures of $X(E)$.

There have been various efforts to extend (1.3) to other non-Markovian processes or random fields, but with only partial success; see Kôno (1986), Lin and Xiao (1994), Benassi, Cohen and Istas (2003) and Xiao (2007). In order to establish a Hausdorff dimension result similar to (1.3) for a random field X, it is standard to determine upper and lower bounds for $\dim_{\mathrm{H}} X(E)$ separately. While the capacity argument (based on Frostman's theorem) is useful for determining lower bounds, the methods based on the classical covering argument for establishing an upper bound for $\dim_{\mathrm{H}} X(E)$ are quite restrictive and usually require strong conditions to be imposed on X. As such, the aforementioned authors have only considered random fields which either satisfy a uniform Hölder condition of appropriate order on compact sets or have at least the first moment. In particular, the existing methods are not enough, even for determining $\dim_{\mathrm{H}} X([0, 1]^N)$, when $X = \{X(t), t \in \mathbb{R}^N\}$ is a general stable random field.

Given a random field $X = \{X(t), t \in \mathbb{R}^N\}$ and a Borel set $E \subset \mathbb{R}^N$, it is usually more difficult to determine the packing dimension of the image set $X(E)$. Recently, Khoshnevisan and Xiao (2008a) and Khoshnevisan, Schilling and Xiao (2009) have solved the above problem when $X = \{X(t), t \geq 0\}$ is a Lévy process in $\mathbb{R}^d$. However, their method depends crucially on the strong Markov property of Lévy processes and cannot be applied directly to random fields.

This paper is motivated by the need to develop methods for determining the Hausdorff and packing dimensions of the image measure μ_X and image set $X(E)$ under minimal conditions on the random field X. By applying measure-theoretic methods and the theory of packing dimension profiles, we are able to solve the problems for the Hausdorff and packing dimensions of the image measure μ_X under mild conditions (namely, (C1) and (C2) in Section 3). The main results are

Theorem 3.8 and Theorem 3.12. When X satisfies certain uniform Hölder conditions, Theorems 3.8 and 3.12 can be applied directly to compute the Hausdorff and packing dimensions of $X(E)$. More generally, we also provide a method for determining the Hausdorff dimension of $X(E)$ under conditions (C1) and (C2) (see Theorem 4.9). However, we have not been able to solve the problem of determining $\dim_{\mathrm{P}} X(E)$ in general.

The rest of this paper is organized as follows. In Section 2, we recall the definitions and some basic properties of Hausdorff dimension, packing dimension and packing dimension profiles of sets and Borel measures. In Section 3, we determine the Hausdorff and packing dimensions of the image measure μ_X under general conditions (C1) and (C2). In Section 4, we study the Hausdorff and packing dimensions of the image set $X(E)$, where $E \subset \mathbb{R}^N$ is an analytic set (i.e., E is a continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$ or, equivalently, E is a continuous image of a Borel set). Section 5 contains applications of the theorems in Sections 3 and 4 to SSSI stable random fields, real harmonizable fractional Lévy fields and the Rosenblatt process.

Throughout this paper, we will use $\langle x, y\rangle$ to denote the inner product and $\|\cdot\|$ to denote the Euclidean norm in $\mathbb{R}^n$, no matter what the value of n is. For any $s, t \in \mathbb{R}^n$ such that $s_j < t_j$ $(j = 1, \dots, n)$, $[s, t] = \prod_{j=1}^n [s_j, t_j]$ is called a *closed interval*. We will use K to denote an unspecified positive constant which may differ from line to line. Specific constants in Section i will be denoted by $K_{i,1}, K_{i,2}, \dots$.

2. Preliminaries

In this section, we recall briefly the definitions and some basic properties of Hausdorff dimension, packing dimension and packing dimension profiles. More information on Hausdorff and packing dimensions can be found in Falconer (1990) and Mattila (1995).

2.1. Hausdorff dimension of sets and measures

For any $\alpha > 0$, the α-dimensional Hausdorff measure of $E \subset \mathbb{R}^N$ is defined by

$$s^\alpha\text{-}m(E) = \lim_{\varepsilon \to 0} \inf\left\{\sum_{i=1}^{\infty} (2r_i)^\alpha \colon E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon\right\}, \tag{2.1}$$

where $B(x, r) = \{y \in \mathbb{R}^N \colon |y - x| < r\}$. The Hausdorff dimension of E is defined as $\dim_{\mathrm{H}} E = \inf\{\alpha > 0 \colon s^\alpha\text{-}m(E) = 0\}$. For a finite Borel measure μ on $\mathbb{R}^N$, its Hausdorff dimension is defined by $\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} E \colon \mu(E) > 0$ and $E \subset \mathbb{R}^N$ is a Borel set$\}$ and its upper Hausdorff dimension is defined by $\dim_{\mathrm{H}}^* \mu = \inf\{\dim_{\mathrm{H}} E \colon \mu(\mathbb{R}^N \setminus E) = 0$ and $E \subset \mathbb{R}^N$ is a Borel set$\}$. Hu and Taylor (1994) proved that

$$\dim_{\mathrm{H}} \mu = \sup\left\{\beta > 0 \colon \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^\beta} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}, \tag{2.2}$$

$$\dim_{\mathrm{H}}^* \mu = \inf\left\{\beta > 0 \colon \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^\beta} > 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}. \tag{2.3}$$

The Hausdorff dimensions of an analytic set $E \subset \mathbb{R}^N$ and finite Borel measures on E are related by the following identity (which can be verified by (2.2) and Frostman's lemma):

$$\dim_{\mathrm{H}} E = \sup\{\dim_{\mathrm{H}} \mu\colon\ \mu \in \mathcal{M}_c^+(E)\}, \tag{2.4}$$

where $\mathcal{M}_c^+(E)$ denotes the family of all finite Borel measures with compact support in E.

2.2. Packing dimension of sets and measures

Packing dimension was introduced by Tricot (1982) as a dual concept to Hausdorff dimension and has become a useful tool for analyzing fractal sets and sample paths of stochastic processes; see Taylor and Tricot (1985), Taylor (1986), Talagrand and Xiao (1996), Falconer and Howroyd (1997), Howroyd (2001), Xiao (1997, 2004, 2009), Khoshnevisan and Xiao (2008a, 2008b), Khoshnevisan, Schilling and Xiao (2009) and the references therein for more information.

For any $\alpha > 0$, the α-dimensional packing measure of $E \subset \mathbb{R}^N$ is defined as

$$s^\alpha\text{-}p(E) = \inf\left\{\sum_n \phi\text{-}P(E_n)\colon\ E \subset \bigcup_n E_n\right\},$$

where $s^\alpha\text{-}P$ is the set function on subsets of $\mathbb{R}^N$ defined by

$$s^\alpha\text{-}P(E) = \lim_{\varepsilon\to 0} \sup\left\{\sum_i (2r_i)^\alpha\colon\ \overline{B}(x_i, r_i) \text{ are disjoint, } x_i \in E, r_i < \varepsilon\right\}.$$

The packing dimension of E is defined by $\dim_{\mathrm{P}} E = \inf\{\alpha > 0\colon\ s^\alpha\text{-}p(E) = 0\}$. It is well known that $0 \le \dim_{\mathrm{H}} E \le \dim_{\mathrm{P}} E \le N$ for every set $E \subset \mathbb{R}^N$.

The packing dimension of a finite Borel measure μ on $\mathbb{R}^N$ is defined by $\dim_{\mathrm{P}} \mu = \inf\{\dim_{\mathrm{P}} E\colon\ \mu(E) > 0 \text{ and } E \subset \mathbb{R}^N \text{ is a Borel set}\}$ and the upper packing dimension of μ is defined by $\dim_{\mathrm{P}}^* \mu = \inf\{\dim_{\mathrm{P}} E\colon\ \mu(\mathbb{R}^N \setminus E) = 0 \text{ and } E \subset \mathbb{R}^N \text{ is a Borel set}\}$. In analogy to (2.4), Falconer and Howroyd (1997) proved, for every analytic set $E \subset \mathbb{R}^N$, that

$$\dim_{\mathrm{P}} E = \sup\{\dim_{\mathrm{P}} \mu\colon\ \mu \in \mathcal{M}_c^+(E)\}. \tag{2.5}$$

2.3. Packing dimension profiles

Next, we recall some aspects of the packing dimension profiles of Falconer and Howroyd (1997) and Howroyd (2001). For a finite Borel measure μ on $\mathbb{R}^N$ and for any $s > 0$, let

$$F_s^\mu(x, r) = \int_{\mathbb{R}^N} \psi_s\left(\frac{x-y}{r}\right) \mathrm{d}\mu(y)$$

be the potential with respect to the kernel $\psi_s(x) = \min\{1, \|x\|^{-s}\}, \forall x \in \mathbb{R}^N$.

Falconer and Howroyd (1997) defined the packing dimension profile and the upper packing dimension profile of μ as

$$\mathrm{Dim}_s\,\mu = \sup\left\{\beta \geq 0:\ \liminf_{r\to 0} \frac{F_s^{\mu}(x,r)}{r^{\beta}} = 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\} \tag{2.6}$$

and

$$\mathrm{Dim}_s^*\,\mu = \inf\left\{\beta > 0:\ \liminf_{r\to 0} \frac{F_s^{\mu}(x,r)}{r^{\beta}} > 0 \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}, \tag{2.7}$$

respectively. Further, they showed that $0 \leq \mathrm{Dim}_s\,\mu \leq \mathrm{Dim}_s^*\,\mu \leq s$ and, if $s \geq N$, then

$$\mathrm{Dim}_s\,\mu = \dim_{\mathrm{P}}\mu, \qquad \mathrm{Dim}_s^*\,\mu = \dim_{\mathrm{P}}^*\mu. \tag{2.8}$$

Motivated by (2.5), Falconer and Howroyd (1997) defined the s-dimensional packing dimension profile of $E \subset \mathbb{R}^N$ by

$$\mathrm{Dim}_s\,E = \sup\{\mathrm{Dim}_s\,\mu:\ \mu \in \mathcal{M}_c^+(E)\}. \tag{2.9}$$

It follows that

$$0 \leq \mathrm{Dim}_s\,E \leq s \quad \text{and} \quad \mathrm{Dim}_s\,E = \dim_{\mathrm{P}} E \qquad \text{if } s \geq N. \tag{2.10}$$

By the above definition, it can be verified (see Falconer and Howroyd (1997), page 286) that for every Borel set $E \subset \mathbb{R}^N$ with $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, we have

$$\mathrm{Dim}_s\,E = \min\{s, \dim_{\mathrm{P}} E\}. \tag{2.11}$$

The following lemma is a consequence of Proposition 18 in Falconer and Howroyd (1997).

Lemma 2.1. *Let μ be a finite Borel measure on $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ be bounded and non-empty. Let $\sigma : \mathbb{R}_+ \to [0, N]$ be any one of the functions* $\mathrm{Dim}_s\,\mu$, $\mathrm{Dim}_s^*\,\mu$ *or* $\mathrm{Dim}_s\,E$ *in s. Then $\sigma(s)$ is non-decreasing and continuous.*

3. Hausdorff and packing dimensions of the image measures

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-random field defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume throughout this paper that X is separable (i.e., there exists a countable and dense set $T^* \subset \mathbb{R}^N$ and a zero probability event Υ_0 such that for every open set $F \subset \mathbb{R}^N$ and closed set $G \subset \mathbb{R}^d$, the two events $\{\omega:\ X(t,\omega) \in G \text{ for all } t \in F \cap T^*\}$ and $\{\omega:\ X(t,\omega) \in G \text{ for all } t \in F\}$ differ from each other only by a subset of Υ_0; in this case, T^* is called a *separant* for X) and $(t,\omega) \mapsto X(t,\omega)$ is $\mathcal{B}(\mathbb{R}^N) \times \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}^N)$ is the Borel σ-algebra of $\mathbb{R}^N$.

For any Borel measure μ on $\mathbb{R}^N$, the image measure μ_X of μ under $t \mapsto X(t)$ is

$$\mu_X(B) := \mu\left\{t \in \mathbb{R}^N:\ X(t) \in B\right\} \qquad \text{for all Borel sets } B \subset \mathbb{R}^d.$$

In this section, we derive upper and lower bounds for the Hausdorff and packing dimensions of the image measures of X, which rely, respectively, on the following conditions (C1) and (C2). Analogous problems for the image set $X(E)$ will be considered in Section 4.

(C1) There exist positive and finite constants H_1 and β such that

$$\mathbb{P}\Big\{\sup_{\|s-t\|\le h}\|X(s)-X(t)\|\ge h^{H_1}u\Big\}\le K_{3,1}u^{-\beta} \tag{3.1}$$

for all $t\in\mathbb{R}^N$, $h\in(0,h_0)$ and $u\ge u_0$, where h_0, u_0 and $K_{3,1}$ are positive constants.

(C2) There exists a positive constant H_2 such that for all $s,t\in\mathbb{R}^N$ and $r>0$,

$$\mathbb{P}\big\{\|X(s)-X(t)\|\le\|s-t\|^{H_2}r\big\}\le K_{3,2}\min\{1,r^d\}, \tag{3.2}$$

where $K_{3,2}>0$ is a finite constant.

Remark 3.1. Since (C1) and (C2) play essential roles in this paper, we will now make some relevant remarks about them.

- Condition (C1) is a type of local maximal inequality and is easier to verify when the random field X has a certain approximate self-similarity. For example, if X is H_1-self-similar, then condition (C1) is satisfied whenever the tail probability of $\sup_{\|s-t\|\le 1}\|X(s)-X(t)\|$ decays no slower than a polynomial rate; see Proposition 3.2 below and Section 5. It can also be verified directly for Gaussian or more general infinitely random fields by using large deviations techniques without appealing to self-similarity.
- There may be different pairs of (H_1,β) for which (C1) is satisfied. We note that the formulae for Hausdorff and packing dimensions of the images do not depend on the constant $\beta>0$, it is $\sup\{H_1\colon$ (C1) holds for some $(H_1,\beta)\}$ that determines the best upper bounds for the Hausdorff and packing dimensions of the image measures.
- For every point $t\in\mathbb{R}^N$, the local Hölder exponent of X at t is defined as

$$\alpha_X(t)=\sup\Big\{\gamma>0\colon\ \lim_{\|s-t\|\to 0}\frac{\|X(s)-X(t)\|}{\|s-t\|^{\gamma}}=0\Big\}.$$

  Condition (C1) and the Borel–Cantelli lemma imply that $\alpha_X(t)\ge H_1$ almost surely (see (3.9) below). However, (C1) does not even imply sample path continuity of X.
- In Section 4, the following, slightly weaker, form of condition (C2) will be sufficient:

  (C2′) There exist positive constants H_2 and $K_{3,2}$ such that (3.2) holds for all $s,t\in\mathbb{R}^N$ satisfying $\|t-s\|\le 1$ and $r>0$.

- Condition (C2) (or (C2′)) is satisfied if, for all $s,t\in\mathbb{R}^N$ (or those satisfying $\|s-t\|\le 1$), the random vector $(X(s)-X(t))/\|s-t\|^{H_2}$ has a density function which is uniformly bounded in s and t. As shown by Proposition 3.3 below, (C2) is significantly weaker than the latter.

The following proposition gives a simple sufficient condition for an SSSI process $X=\{X(t),t\in\mathbb{R}\}$ to satisfy condition (C1). More precise information can be obtained if further distributional properties of X are known; see Section 5.

Proposition 3.2. *Let $X = \{X(t), t \in \mathbb{R}\}$ be a separable, H-SSSI process with values in $\mathbb{R}^d$. If there exist positive constants $\beta > 0$ and $K_{3,3}$ such that $H\beta > 1$ and*

$$\mathbb{P}\{\|X(1)\| \geq u\} \leq K_{3,3} u^{-\beta} \qquad \forall u \geq 1, \tag{3.3}$$

then there exists a positive constant $K_{3,4}$ such that for all $u \geq 1$,

$$\mathbb{P}\Big\{\sup_{t \in [0,1]} \|X(t)\| \geq u\Big\} \leq K_{3,4} u^{-\beta}. \tag{3.4}$$

In particular, condition (C1) *is satisfied with $H_1 = H$ and the same β as in* (3.3).

Proof. Without loss of generality, we can assume $d = 1$. Since the self-similarity index $H > 0$, we have $X(0) = 0$ a.s. Let $T^* = \{t_n, n \geq 0\}$ be a separant for $X = \{X(t), t \in [0, 1]\}$. We assume that $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots$. For any $n \geq 2$, consider the random variables Y_k $(1 \leq k \leq n)$ defined by $Y_k = X(t_k) - X(t_{k-1})$. For $1 \leq i < j \leq n$, let $S_{i,j} = \sum_{k=i}^{j} Y_k$. By the stationarity of increments and self-similarity of X and (3.3), we derive that for any $u \geq 1$,

$$\mathbb{P}\left\{\left|\sum_{k=i}^{j} Y_k\right| \geq u\right\} = \mathbb{P}\left\{|X(1)| \geq \frac{u}{(t_j - t_{i-1})^H}\right\} \leq K_{3,3} u^{-\beta} (t_j - t_{i-1})^{H\beta}. \tag{3.5}$$

Thus, condition (3.4) of Theorem 3.2 of Moricz, Serfling and Stout (1982) is satisfied with $g(i, j) = t_j - t_{i-1}$, $\alpha = H\beta$ and $\phi(t) = t^\beta$. It is easy to see that the non-negative function $g(i, j)$ satisfies their condition (1.2) (i.e., $g(i, j) \leq g(i, j+1)$ and $g(i, j) + g(j+1, k) \leq Qg(i, k)$ for $1 \leq i \leq j < k \leq n$) with $Q = 1$. It therefore follows from Theorem 3.2 of Moricz, Serfling and Stout (1982) that there exists a constant $K_{3,4}$ (independent of n) such that for all $u \geq 1$,

$$\mathbb{P}\Big\{\max_{1 \leq j \leq n} |X(t_j)| \geq u\Big\} = \mathbb{P}\left\{\max_{1 \leq j \leq n}\left|\sum_{k=1}^{j} Y_k\right| \geq u\right\} \leq K_{3,4} u^{-\beta}. \tag{3.6}$$

Letting $n \to \infty$ yields (3.4), which, in turn, implies that (C1) holds for $H_1 = H$. □

Next, we provide a necessary and sufficient condition for an (N, d)-random field $X = \{X(t), t \in \mathbb{R}^N\}$ to satisfy condition (C2) (or (C2′)). For any $r > 0$, let

$$\phi_r(x) = \prod_{j=1}^{d} \frac{1 - \cos(2rx_j)}{2\pi r x_j^2}, \qquad x \in \mathbb{R}^d.$$

Proposition 3.3. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. Condition* (C2) (*or* (C2′)) *then holds if and only if there exists a positive constant $K_{3,5}$ such that for all $r > 0$ and all $s, t \in \mathbb{R}^N$ (or for those satisfying $\|t - s\| \leq 1$),*

$$\int_{\mathbb{R}^d} \phi_r(x) \mathbb{E}\big(e^{i\langle x, X(t) - X(s)\rangle / \|t-s\|^{H_2}}\big)\, dx \leq K_{3,5} \min\{1, r^d\}. \tag{3.7}$$

Remark 3.4. Since $\phi_r(x) = \mathrm{O}(\|x\|^{-2})$ as $\|x\| \to \infty$, condition (3.7) is significantly weaker than assuming that $(X(t) - X(s))/\|t - s\|^{H_2}$ has a bounded density and can be applied conveniently to SSSI processes. We mention that (3.7) is also weaker than the integrability condition in Assumption 1 on page 269 of Benassi, Cohen and Istas (2003). It can be shown that Theorem 2.1 in Benassi, Cohen and Istas (2003) still holds under (3.7) and their Assumption 2.

Proof of Proposition 3.3. Note that for every $r > 0$, the function $\phi_r(x)$ is non-negative and is in $L^1(\mathbb{R}^d)$. The Fourier transform of ϕ_r is

$$\widehat{\phi_r}(z) = \prod_{j=1}^{d} \left(1 - \frac{|z_j|}{2r}\right)^+ \qquad \forall z \in \mathbb{R}^d.$$

In the above, $a^+ := \max(a, 0)$ for all $a \in \mathbb{R}$. Since $z \in B(0, r)$ implies that $1 - (2r)^{-1}|z_j| \ge \frac{1}{2}$, we have $\mathbb{1}_{B(0,r)}(z) \le 2^d \widehat{\phi_r}(z)$ for all $z \in \mathbb{R}^d$. Here, and in the sequel, $\mathbb{1}_A$ denotes the indicator function (or random variable) of the set (or event) A. By Fubini's theorem, we obtain

$$\begin{aligned}\mathbb{P}\{\|X(s) - X(t)\| \le \|s - t\|^{H_2} r\} &\le 2^d \mathbb{E}\left[\widehat{\phi_r}\left(\frac{X(t) - X(s)}{\|t - s\|^{H_2}}\right)\right] \\ &= 2^d \int_{\mathbb{R}^d} \phi_r(x) \mathbb{E}\left(\mathrm{e}^{\mathrm{i}\langle x, X(t) - X(s)\rangle / \|t-s\|^{H_2}}\right) \mathrm{d}x.\end{aligned}$$

Hence, (3.7) implies condition (C2). On the other hand, we have $\widehat{\phi_r}(z) \le \mathbb{1}_{B(0, 2\sqrt{d} r)}(z)$ for all $z \in \mathbb{R}^d$. Consequently,

$$\mathbb{E}\left[\widehat{\phi_r}\left(\frac{X(t) - X(s)}{\|t - s\|^{H_2}}\right)\right] \le \mathbb{P}\left\{\|X(s) - X(t)\| \le 2\sqrt{d}\|s - t\|^{H_2} r\right\}.$$

Therefore, condition (C2) implies (3.7). This completes the proof. □

3.1. Hausdorff dimensions of the image measures

First, we consider the upper bounds for the Hausdorff dimensions of the image measure μ_X.

Proposition 3.5. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. If condition* (C1) *is satisfied, then for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\begin{aligned}&\dim_{\mathrm{H}} \mu_X \le \min\left\{d, \frac{1}{H_1} \dim_{\mathrm{H}} \mu\right\} \quad \textit{and} \\ &\dim_{\mathrm{H}}^* \mu_X \le \min\left\{d, \frac{1}{H_1} \dim_{\mathrm{H}}^* \mu\right\} \qquad \text{a.s.}\end{aligned} \tag{3.8}$$

Proof. Let $\lambda > 1/\beta$ be a constant. For any fixed $s \in \mathbb{R}^N$ and the sequence $h_n = 2^{-n}$ ($n \ge 1$), it follows from condition (C1) that for all integers $n \ge \max\{\log(1/h_0), \frac{1}{\log 2} u_0^{1/\lambda}\}$,

$$\mathbb{P}\Big\{\sup_{\|t-s\|\le 2^{-n}} \|X(t) - X(s)\| \ge 2^{-H_1 n} (\log 2^n)^\lambda\Big\} \le K n^{-\beta\lambda}.$$

Since $\sum_{n=1}^\infty n^{-\beta\lambda} < \infty$, the Borel–Cantelli lemma implies that almost surely

$$\sup_{\|t-s\|\le 2^{-n}} \|X(t) - X(s)\| \le (\log 2)^\lambda 2^{-H_1 n} n^\lambda \qquad \forall n \ge n_0, \tag{3.9}$$

where $n_0 = n_0(\omega, s)$ depends on ω and s. By Fubini's theorem, we derive that, for any finite Borel measure μ on $\mathbb{R}^N$, almost surely (3.9) holds for μ-a.a. $s \in \mathbb{R}^N$.

We now fix an $\omega \in \Omega$ such that (3.9) is valid for μ-a.a. $s \in \mathbb{R}^N$ and prove that both inequalities in (3.8) hold. In the sequel, ω will be suppressed.

To prove the first inequality in (3.8), since $\dim_{\mathrm{H}} \mu_X \le d$ holds trivially, we only need to prove that $\dim_{\mathrm{H}} \mu_X \le \frac{1}{H_1} \dim_{\mathrm{H}} \mu$. Without loss of generality, we assume $\dim_{\mathrm{H}} \mu_X > 0$ and take any $\gamma \in (0, \dim_{\mathrm{H}} \mu_X)$. Then, by (2.2), we have

$$\limsup_{r\to 0} r^{-\gamma} \int_{\mathbb{R}^d} \mathbb{1}_{\{\|y-x\|\le r\}} \, \mathrm{d}\mu_X(y) = 0 \qquad \text{for } \mu_X\text{-a.a. } x \in \mathbb{R}^d. \tag{3.10}$$

Equivalently to (3.10), we have

$$\limsup_{r\to 0} r^{-\gamma} \int_{\mathbb{R}^N} \mathbb{1}_{\{\|X(t)-X(s)\|\le r\}} \, \mathrm{d}\mu(t) = 0 \qquad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N. \tag{3.11}$$

Let us fix $s \in \mathbb{R}^N$ such that both (3.9) and (3.11) hold. For any $\varepsilon > 0$, we choose $n_1 \ge n_0$ such that $n^\lambda \le 2^{\varepsilon n}$ for all $n \ge n_1$. By (3.9), we can write

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbb{1}_{\{\|X(t)-X(s)\|\le r\}} \, \mathrm{d}\mu(t) &\ge \sum_{n=n_1}^\infty \int_{2^{-n-1}\le\|t-s\|<2^{-n}} \mathbb{1}_{\{\|X(t)-X(s)\|\le r\}} \, \mathrm{d}\mu(t) \\ &\ge \int_{\|t-s\|<2^{-n_1}} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}} \, \mathrm{d}\mu(t).\end{aligned} \tag{3.12}$$

Hence, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}} \, \mathrm{d}\mu(t) \le& \int_{\mathbb{R}^N} \mathbb{1}_{\{\|X(t)-X(s)\|\le r\}} \, \mathrm{d}\mu(t) \\ &+ \int_{\|t-s\|\ge 2^{-n_1}} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}} \, \mathrm{d}\mu(t).\end{aligned} \tag{3.13}$$

For the last integral, we have

$$\lim_{r\to 0} r^{-\gamma} \int_{\|t-s\|\ge 2^{-n_1}} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}} \, \mathrm{d}\mu(t) = 0 \tag{3.14}$$

because the indicator function takes the value 0 when $r > 0$ is sufficiently small.

It follows from (3.11), (3.13) and (3.14) that with $r = \rho^{H_1-\varepsilon}$,

$$\begin{aligned}\limsup_{\rho\to 0} \rho^{-(H_1-\varepsilon)\gamma} \int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le\rho\}}\,\mathrm{d}\mu(t) &= \limsup_{r\to 0} r^{-\gamma} \int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}}\,\mathrm{d}\mu(t)\\ &\le \limsup_{r\to 0} r^{-\gamma} \int_{\mathbb{R}^N} \mathbb{1}_{\{\|X(t)-X(s)\|\le r\}}\,\mathrm{d}\mu(t) = 0.\end{aligned} \tag{3.15}$$

We have thus proven that (3.15) holds almost surely for μ-a.a. $s \in \mathbb{R}^N$. This implies that $\dim_{\mathrm{H}} \mu \ge (H_1 - \varepsilon)\gamma$ almost surely. Since $\varepsilon > 0$ and $\gamma < \dim_{\mathrm{H}} \mu_X$ are arbitrary, (3.8) follows.

To prove the second inequality in (3.8), it is sufficient to show that $\dim^*_{\mathrm{H}} \mu_X \le \frac{1}{H_1} \dim^*_{\mathrm{H}} \mu$ a.s. Let $\omega \in \Omega$ be fixed as above. We take an arbitrary $\beta > \dim^*_{\mathrm{H}} \mu$. By (2.3), we have

$$\limsup_{\rho\to 0} \rho^{-\beta} \int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le\rho\}}\,\mathrm{d}\mu(t) > 0 \qquad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N. \tag{3.16}$$

By using (3.12), we derive that for $x = X(s)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbb{1}_{\{\|y-x\|\le r\}}\,\mathrm{d}\mu_X(y) \ge{}& \int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}}\,\mathrm{d}\mu(t)\\ &- \int_{\|t-s\|\ge 2^{-n_1}} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}}\,\mathrm{d}\mu(t).\end{aligned} \tag{3.17}$$

It follows from (3.17), (3.14) and (3.16) that

$$\limsup_{r\to 0} \frac{\int_{\mathbb{R}^d} \mathbb{1}_{\{\|y-x\|\le r\}}\,\mathrm{d}\mu_X(y)}{r^{\beta/(H_1-\varepsilon)}} \ge \limsup_{r\to 0} \frac{\int_{\mathbb{R}^N} \mathbb{1}_{\{\|t-s\|\le r^{1/(H_1-\varepsilon)}\}}\,\mathrm{d}\mu(t)}{r^{\beta/(H_1-\varepsilon)}} > 0 \tag{3.18}$$

for all $s \in \mathbb{R}^N$ that satisfy (3.16). This implies that $\dim^*_{\mathrm{H}} \mu_X \le \beta/(H_1 - \varepsilon)$ a.s. Letting $\varepsilon \downarrow 0$ and $\beta \downarrow \dim^*_{\mathrm{H}} \mu$ yields the second inequality in (3.8). This completes the proof of Proposition 3.5. □

Remark 3.6. Note that in (3.8), the exceptional null probability events depend on μ. For several purposes, it is more useful to have a single exceptional null probability event Ω_0 such that, for all $\omega \notin \Omega_0$, both inequalities in (3.8) hold *simultaneously* for all finite Borel measures μ on $\mathbb{R}^N$. By slightly modifying the proof of Proposition 3.5 (see (3.12)), one can show that this is indeed true if, for every $\varepsilon > 0$ and every compact interval I, the sample function $X(t)$ satisfies almost surely a uniform Hölder condition of order $H_1 - \varepsilon$ on I.

Next, we show that condition (C2) determines lower bounds for the Hausdorff dimensions of the image measures of the random field X.

Proposition 3.7. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-random field satisfying condition* (C2). *Then, for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{H}} \mu_X \geq \min\left\{d, \frac{1}{H_2} \dim_{\mathrm{H}} \mu\right\} \quad \text{and} \quad \dim_{\mathrm{H}}^* \mu_X \geq \min\left\{d, \frac{1}{H_2} \dim_{\mathrm{H}}^* \mu\right\} \qquad \text{a.s.} \tag{3.19}$$

Proof. In order to prove the first inequality in (3.19), we fix any constants $0 < \gamma < \gamma' < \min\{d, \frac{1}{H_2} \dim_{\mathrm{H}} \mu\}$. Since $\dim_{\mathrm{H}} \mu > \gamma' H_2$, it follows from (2.2) that

$$\limsup_{r \to 0} \frac{\mu(B(s, r))}{r^{\gamma' H_2}} = 0 \qquad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N. \tag{3.20}$$

Let $s \in \mathbb{R}^N$ be a fixed point such that (3.20) holds. By (C2), we derive

$$\begin{aligned} \mathbb{E}\mu_X(B(X(s), r)) &= \int_{\mathbb{R}^N} \mathbb{P}\big(\|X(t) - X(s)\| \leq r\big) \mu(\mathrm{d}t) \\ &\leq K_{3,2} \mu(B(s, r^{1/H_2})) + K_{3,2} \int_{\|t-s\| > r^{1/H_2}} \left(\frac{r}{\|t - s\|^{H_2}}\right)^d \mu(\mathrm{d}t). \end{aligned} \tag{3.21}$$

Let κ be the image measure of μ under the mapping $t \mapsto \|t - s\|$ from $\mathbb{R}^N$ to $\mathbb{R}_+$. Then, by using an integration-by-parts formula and (3.20), we have

$$\begin{aligned} \int_{\|t-s\| > r^{1/H_2}} \left(\frac{r}{\|t - s\|^{H_2}}\right)^d \mu(\mathrm{d}t) &= \int_{r^{1/H_2}}^{\infty} \frac{r^d}{\rho^{H_2 d}} \kappa(\mathrm{d}\rho) \\ &\leq H_2 d \int_{r^{1/H_2}}^{\infty} \frac{r^d}{\rho^{H_2 d + 1}} \mu(B(s, \rho))\, \mathrm{d}\rho \\ &\leq K r^{\gamma'} \end{aligned} \tag{3.22}$$

for all $r > 0$ small enough, where the last inequality follows from (3.20) and the fact that $\gamma' < d$. Combining (3.21) and (3.22), we see that $\mathbb{E}\mu_X(B(X(s), r)) \leq K r^{\gamma'}$ for $r > 0$ small. This, and the Markov inequality, imply that for all n large enough,

$$\mathbb{P}\big(\mu_X(B(X(s), 2^{-n})) \geq 2^{-n\gamma}\big) \leq K 2^{-n(\gamma' - \gamma)}.$$

It follows from the Borel–Cantelli lemma that a.s. $\mu_X(B(X(s), 2^{-n})) < 2^{-n\gamma}$ for all n large enough. It should be clear the above implies that for all $0 < \gamma < \min\{d, \frac{1}{H_2} \dim_{\mathrm{H}} \mu\}$,

$$\limsup_{r \to 0} \frac{\mu_X(B(x, r))}{r^{\gamma}} = 0 \qquad \text{for } \mu_X\text{-a.a. } x \in \mathbb{R}^d$$

almost surely. Thus, $\dim_{\mathrm{H}} \mu_X \geq \gamma$ a.s., and (3.19) follows from the arbitrariness of γ.

To prove the second inequality in (3.19), let $0 < \gamma < \gamma' < \min\{d, \frac{1}{H_2} \dim_{\mathrm{H}}^* \mu\}$. By (2.3), there exists a Borel set $A \subset \mathbb{R}^N$ such that $\mu(A) > 0$ and $\limsup_{r \to 0} r^{-\gamma' H_2} \mu(B(s, r)) = 0$ for all

$s \in A$. The proof above shows that a.s. $\limsup_{r\to 0} r^{-\gamma}\mu_X(B(x,r)) = 0$ for all $x \in X(A)$. Since $\mu_X(X(A)) > 0$ a.s., we derive $\dim_{\mathrm{H}}^* \mu_X \ge \gamma$ a.s. and the proof is completed. □

Combining Propositions 3.5 and 3.7, we have the following theorem, whose proof is omitted.

Theorem 3.8. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-random field and let H be a positive constant. If, for every $\varepsilon > 0$, X satisfies condition* (C1) *with $H_1 = H - \varepsilon$, some $\beta = \beta(\varepsilon) > 0$ and* (C2) *with $H_2 = H + \varepsilon$, then for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{H}} \mu_X = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}} \mu\right\} \quad \textit{and} \quad \dim_{\mathrm{H}}^* \mu_X = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}}^* \mu\right\} \qquad \text{a.s.} \tag{3.23}$$

3.2. Packing dimensions of the image measures

We now study the problem of determining the packing dimensions $\dim_{\mathrm{P}} \mu_X$ and $\dim_{\mathrm{P}}^* \mu_X$. The following upper bounds for the image measures are proved by Schilling and Xiao (2009).

Proposition 3.9. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. If condition* (C1) *is satisfied, then for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{P}} \mu_X \le \frac{1}{H_1}\mathrm{Dim}_{H_1 d}\, \mu \quad \textit{and} \quad \dim_{\mathrm{P}}^* \mu_X \le \frac{1}{H_1}\mathrm{Dim}_{H_1 d}^*\, \mu \qquad \text{a.s.} \tag{3.24}$$

Similarly to Remark 3.6, we have the following.

Remark 3.10. If, for every $\varepsilon > 0$ and every compact interval $I \subset \mathbb{R}^N$, $X(t)$ satisfies almost surely a uniform Hölder condition of order $H_1 - \varepsilon$ on I, then almost surely both inequalities in (3.24) hold for all finite Borel measures μ on $\mathbb{R}^N$.

For the lower bounds of packing dimensions, we have the following proposition.

Proposition 3.11. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-random field satisfying condition* (C2). *Then, for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{P}} \mu_X \ge \frac{1}{H_2}\mathrm{Dim}_{H_2 d}\, \mu \quad \textit{and} \quad \dim_{\mathrm{P}}^* \mu_X \ge \frac{1}{H_2}\mathrm{Dim}_{H_2 d}^*\, \mu \qquad \text{a.s.} \tag{3.25}$$

Proof. We only prove the first inequality in (3.25); the proof of the second one is similar. We may, and will, assume that $\mathrm{Dim}_{H_2 d}\, \mu > 0$. For fixed $s \in \mathbb{R}^N$, Fubini's theorem implies that

$$\mathbb{E}F_d^{\mu_X}(X(s), r) = \int_{\mathbb{R}^N} \mathbb{E}\min\{1, r^d \|X(t) - X(s)\|^{-d}\}\, \mathrm{d}\mu(t). \tag{3.26}$$

The integrand in (3.26) can be written as

$$\begin{aligned}&\mathbb{E}\min\{1, r^d \|X(t)-X(s)\|^{-d}\}\\&\quad = \mathbb{P}\{\|X(t)-X(s)\| \le r\} + \mathbb{E}\big\{r^d\|X(t)-X(s)\|^{-d}\cdot \mathbb{1}_{\{\|X(t)-X(s)\|\ge r\}}\big\}.\end{aligned} \tag{3.27}$$

By condition (C2), we obtain that for all $s, t \in \mathbb{R}^N$ and $r > 0$,

$$\mathbb{P}\{\|X(t)-X(s)\| \le r\} \le K_{3,2}\min\left\{1, \frac{r^d}{\|t-s\|^{H_2 d}}\right\}. \tag{3.28}$$

Denote the distribution of $X(t)-X(s)$ by $\Gamma_{s,t}(\cdot)$. Let ν be the image measure of $\Gamma_{s,t}(\cdot)$ under the mapping $T : z \mapsto \|z\|$ from $\mathbb{R}^d$ to $\mathbb{R}_+$. The last term in (3.27) can then be written as

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{r^d}{\|z\|^d}\mathbb{1}_{\{\|z\|\ge r\}}\Gamma_{s,t}(\mathrm{d}z) &= \int_r^\infty \frac{r^d}{\rho^d}\nu(\mathrm{d}\rho)\\ &\le d\int_r^\infty \frac{r^d}{\rho^{d+1}}\mathbb{P}\{\|X(t)-X(s)\|\le \rho\}\,\mathrm{d}\rho,\end{aligned} \tag{3.29}$$

where the last inequality follows from an integration-by-parts formula.

By (3.28) and (3.29), we derive that the last term in (3.27) can be bounded by a constant multiple of

$$\begin{aligned}&\int_r^\infty \frac{r^d}{\rho^{d+1}}\min\left\{1, \frac{\rho^d}{\|t-s\|^{H_2 d}}\right\}\mathrm{d}\rho\\ &\quad\le \begin{cases} K, & \text{if } r \ge \|t-s\|^{H_2},\\ \dfrac{K r^d}{\|t-s\|^{H_2 d}}\log\left(\dfrac{\|t-s\|^{H_2}}{r}\right), & \text{if } r < \|t-s\|^{H_2}.\end{cases}\end{aligned} \tag{3.30}$$

It follows from (3.27), (3.28), (3.29) and (3.30) that for any $0 < \varepsilon < 1$ and $s, t \in \mathbb{R}^N$,

$$\mathbb{E}\min\{1, r^d\|X(t)-X(s)\|^{-d}\} \le K_{3,6}\min\left\{1, \frac{r^{d-\varepsilon}}{\|t-s\|^{H_2(d-\varepsilon)}}\right\}. \tag{3.31}$$

For any $\gamma \in (0, \mathrm{Dim}_{H_2 d}\,\mu)$, by Lemma 2.1, there exists $\varepsilon > 0$ such that $\gamma < \mathrm{Dim}_{H_2(d-\varepsilon)}\,\mu$. It follows from (2.6) that

$$\liminf_{r\to 0} r^{-\gamma/H_2}\int_{\mathbb{R}_+}\min\left\{1, \frac{r^{d-\varepsilon}}{\|t-s\|^{H_2(d-\varepsilon)}}\right\}\mathrm{d}\mu(t) = 0 \qquad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N. \tag{3.32}$$

By (3.26), (3.31), (3.32) and Fatou's lemma, we have that for μ-a.a. $s \in \mathbb{R}^N$,

$$\begin{aligned}&\mathbb{E}\Big(\liminf_{r\to 0} r^{-\gamma/H_2} F_d^{\mu_X}(X(s), r)\Big)\\ &\quad\le K_{3,6}\liminf_{r\to 0} r^{-\gamma/H_2}\int_{\mathbb{R}^N}\min\left\{1, \frac{r^{d-\varepsilon}}{\|t-s\|^{H_2(d-\varepsilon)}}\right\}\mathrm{d}\mu(t) = 0.\end{aligned} \tag{3.33}$$

By using Fubini's theorem again, we see that almost surely

$$\liminf_{r\to 0} r^{-\gamma/H_2} F_d^{\mu_X}(X(s), r) = 0 \qquad \text{for } \mu\text{-a.a. } s \in \mathbb{R}^N.$$

Hence, $\dim_{\mathrm{P}} \mu_X \geq \frac{\gamma}{H_2}$ a.s. Since γ can be arbitrarily close to $\mathrm{Dim}_{H_2 d}\,\mu$, we obtain (3.25). □

The following is a direct consequence of Propositions 3.9 and 3.11.

Theorem 3.12. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-random field and let H be a positive constant. If, for every $\varepsilon > 0$, X satisfies condition* (C1) *with $H_1 = H - \varepsilon$, some $\beta = \beta(\varepsilon) > 0$ and* (C2) *with $H_2 = H + \varepsilon$, then for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{P}} \mu_X = \frac{1}{H} \mathrm{Dim}_{Hd}\,\mu \quad \textit{and} \quad \dim_{\mathrm{P}}^* \mu_X = \frac{1}{H} \mathrm{Dim}_{Hd}^*\,\mu \qquad \text{a.s.} \tag{3.34}$$

4. Hausdorff and packing dimensions of the image sets

We now consider the Hausdorff and packing dimensions of the image set $X(E)$. We will see that general lower bounds for $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$ can be derived from the results in Section 3 by using a measure theoretic method. For random fields which satisfy uniform Hölder conditions on compact intervals, the upper bounds for $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$ can also be easily obtained. However, it is difficult to obtain upper bounds for $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$ under condition (C1) alone. We have only been able to provide a partial result on determining the upper bound for $\dim_{\mathrm{H}} X(E)$. The analogous problem for $\dim_{\mathrm{P}} X(E)$ remains open.

We will need the following lemmas. Lemma 4.1 is from Lubin (1974), which is more general than Theorem 1.20 in Mattila (1995).

Lemma 4.1. *Let $E \subset \mathbb{R}^N$ be an analytic set and let $f : \mathbb{R}^N \to \mathbb{R}^d$ be a Borel function. If ν is a finite Borel measure on $\mathbb{R}^d$ with support in $f(E)$, then $\nu = \mu_f$ for some $\mu \in \mathcal{M}_c^+(E)$.*

Lemma 4.2. *Let $E \subset \mathbb{R}^N$ be an analytic set. Then, for all Borel measurable functions $f : \mathbb{R}^N \to \mathbb{R}^d$, we have*

$$\dim_{\mathrm{H}} f(E) = \sup\{\dim_{\mathrm{H}} \mu_f : \mu \in \mathcal{M}_c^+(E)\}, \tag{4.1}$$

$$\dim_{\mathrm{P}} f(E) = \sup\{\dim_{\mathrm{P}} \mu_f : \mu \in \mathcal{M}_c^+(E)\}. \tag{4.2}$$

Proof. Denote the right-hand side of (4.1) by γ_E. By (2.4), we get $\dim_{\mathrm{H}} f(E) \geq \gamma_E$. Next, for any $\nu \in \mathcal{M}_c^+(f(E))$, Lemma 4.1 implies that $\nu = \mu_f$ for some $\mu \in \mathcal{M}_c^+(E)$. This and (2.4) together imply $\dim_{\mathrm{H}} f(E) \leq \gamma_E$. Hence, (4.1) is proved. The proof of (4.2) is similar and is therefore omitted. □

We first consider the lower bounds for the Hausdorff and packing dimensions of $X(E)$.

Proposition 4.3. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be an* (N, d)*-random field that satisfies condition* (C2′). *Then, for every analytic set* $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} X(E) \geq \min\left\{d, \frac{1}{H_2} \dim_{\mathrm{H}} E\right\} \quad \textit{and} \quad \dim_{\mathrm{P}} X(E) \geq \frac{1}{H_2} \mathrm{Dim}_{H_2 d} E \qquad \text{a.s.} \tag{4.3}$$

Proof. Since both $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ are σ-stable (see Falconer (1990)), we may, and will, assume that the diameter of E is at most 1. Hence, condition (C2′) will be enough to prove (4.3).

Let us prove the first inequality in (4.3). It follows from (2.4) that for any $0 < \gamma < \dim_{\mathrm{H}} E$, there exists a $\mu \in \mathcal{M}_c^+(E)$ such that $\dim_{\mathrm{H}} \mu \geq \gamma$. By Proposition 3.7 (which holds for any finite Borel measure whose support has diameter ≤ 1), we have $\dim_{\mathrm{H}} \mu_X \geq \min\{d, \frac{1}{H_2} \dim_{\mathrm{H}} \mu\}$ a.s. This and (4.1) together imply that $\dim_{\mathrm{H}} X(E) \geq \min\{d, \frac{1}{H_2}\gamma\}$ a.s. Since $\gamma < \dim_{\mathrm{H}} E$ is arbitrary, the desired inequality follows.

Next, we prove the second inequality in (4.3). Note that for any $0 < \gamma < \frac{1}{H_2} \mathrm{Dim}_{H_2 d} E$, by (2.9), there exists a Borel measure $\mu \in \mathcal{M}_c^+(E)$ such that $H_2 \gamma < \mathrm{Dim}_{H_2 d} \mu$. It follows from (3.25) that $\dim_{\mathrm{P}} \mu_X > \gamma$ a.s. Hence, by Lemma 4.2, we have $\dim_{\mathrm{P}} X(E) > \gamma$ a.s., which, in turn, implies that $\dim_{\mathrm{P}} X(E) \geq \frac{1}{H_2} \mathrm{Dim}_{H_2 d} E$ a.s. The proof is therefore completed. □

The following proposition gives upper bounds for $\dim_{\mathrm{H}} X(E)$ and $\dim_{\mathrm{P}} X(E)$.

Proposition 4.4. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be an* (N, d)*-random field. If for every* $\varepsilon > 0$, X *satisfies a uniform Hölder condition of order* $H_1 - \varepsilon$ *on all compact intervals of* $\mathbb{R}^N$ *almost surely, then, for all analytic sets* $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} X(E) \leq \min\left\{d, \frac{1}{H_1} \dim E\right\} \quad \textit{and} \quad \dim_{\mathrm{P}} X(E) \leq \frac{1}{H_1} \mathrm{Dim}_{H_1 d} E \qquad \text{a.s.} \tag{4.4}$$

Proof. Both inequalities in (4.4) follow from Remarks 3.6, 3.10 and Lemma 4.2. □

Combining Propositions 4.3 and 4.4 yields the following theorem.

Theorem 4.5. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *be an* (N, d)*-random field and let* $H \in (0, 1]$ *be a constant. If, for every* $\varepsilon > 0$, X *satisfies a uniform Hölder condition of order* $H - \varepsilon$ *on all compact intervals of* $\mathbb{R}^N$ *and condition* (C2′) *with* $H_2 = H + \varepsilon$, *then, for all analytic sets* $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{1}{H} \dim_{\mathrm{H}} E\right\} \quad \textit{and} \quad \dim_{\mathrm{P}} X(E) = \frac{1}{H} \mathrm{Dim}_{Hd} E \qquad \text{a.s.} \tag{4.5}$$

It is often desirable to compute $\dim_{\mathrm{P}} X(E)$ in terms of $\dim_{\mathrm{P}} E$. The following is the packing dimension analog of (1.3). Note that if $N > Hd$, then the conclusion of Corollary 4.6 does not hold in general; see Talagrand and Xiao (1996). In this sense, it is the best possible result of this kind.

Corollary 4.6. *Let* $X = \{X(t), t \in \mathbb{R}^N\}$ *and* $E \subset \mathbb{R}^N$ *be as in Theorem 4.5. If either* $N \leq Hd$ *or* E *satisfies* $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, *then* $\dim_{\mathrm{P}} X(E) = \min\{d, \frac{1}{H} \dim_{\mathrm{P}} E\}$ *a.s.*

Proof. If $N \leq Hd$, then (2.10) implies that for every analytic set $E \subset \mathbb{R}^N$, $\mathrm{Dim}_{Hd} E = \dim_{\mathrm{P}} E$. Hence, Theorem 4.5 yields $\dim_{\mathrm{P}} X(E) = \frac{1}{H} \dim_{\mathrm{P}} E$ a.s., as desired. On the other hand, if an analytic set $E \subset \mathbb{R}^N$ satisfies $\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E$, then (2.11) implies that $\mathrm{Dim}_{Hd} E = \min\{Hd, \dim_{\mathrm{P}} E\}$. Hence, again, the conclusion follows from Theorem 4.5. □

Since many random fields do not have continuous sample functions and, even if they do, it is known that $\dim_{\mathrm{H}} X(E)$ is not determined by the exponent of uniform modulus of continuity (a typical example being linear fractional stable motion – see Example 5.4 below), there have been various efforts to remove the uniform Hölder condition. However, except for Markov processes or random fields with certain Markov structure, no satisfactory method has been developed. The main difficulty lies in deriving a sharp upper bound for $\dim_{\mathrm{H}} X(E)$.

In the following, we derive an upper bound for $\dim_{\mathrm{H}} X(E)$ under condition (C1). This method is partially motivated by an argument in Schilling (1998) for Feller processes generated by pseudo-differential operators and, as far as we know, is more general than the existing methods in the literature.

Lemma 4.7. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. If condition* (C1) *holds for $H_1 > 0$ and $\beta > 0$, then, for all $t \in \mathbb{R}^N$, $h > 0$ and $\gamma > 0$,*

$$\mathbb{E}\big(D(t,h)^{\gamma} \mathrm{e}^{-D(t,h)}\big) \leq K_{4,1} h^{H_1(\gamma \wedge \beta)}, \tag{4.6}$$

where $D(t,h) = \sup_{\|s-t\| \leq h} \|X(s) - X(t)\|$ and $K_{4,1}$ is a constant independent of t and h.

Proof. We write

$$\begin{aligned} \mathbb{E}\big(D(t,h)^{\gamma} \mathrm{e}^{-D(t,h)}\big) &= \int_0^{\infty} u^{\gamma-1} \mathrm{e}^{-u} (\gamma - u) \mathbb{P}\{D(t,h) > u\}\, \mathrm{d}u \\ &\leq K \int_0^{\gamma} u^{\gamma-1} \mathrm{e}^{-u} (\gamma - u) \min\{1, (h^{-H_1} u)^{-\beta}\}\, \mathrm{d}u, \end{aligned} \tag{4.7}$$

where the inequality follows from (C1). It is elementary to verify that, up to a constant, the last integral is bounded by

$$\int_0^{h^{H_1}} u^{\gamma-1}\, \mathrm{d}u + h^{H_1 \beta} \int_{h^{H_1}}^{\gamma} u^{\gamma-\beta-1} (\gamma - u)\, \mathrm{d}u \leq K_{4,1} h^{H_1(\gamma \wedge \beta)}. \tag{4.8}$$

This proves (4.6). □

Proposition 4.8. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. Suppose that the sample function of X is a.s. bounded on all compact subsets of $\mathbb{R}^N$. If condition* (C1) *holds for $H_1 > 0$ and $\beta > 0$, then, for every analytic set $E \subset \mathbb{R}^N$ that satisfies* $\dim_{\mathrm{H}} E < \beta H_1$,

$$\dim_{\mathrm{H}} X(E) \leq \min\left\{d, \frac{1}{H_1} \dim_{\mathrm{H}} E\right\} \qquad \text{a.s.} \tag{4.9}$$

Proof. Without loss of generality, we assume that $E \subset [0,1]^N$. For any constant $\gamma \in (\dim_{\mathrm{H}} E, \beta H_1)$, there exists a sequence of balls $\{B(t_k, h_k), k \ge 1\}$ such that

$$E \subset \limsup_{k\to\infty} B(t_k, h_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (2h_k)^\gamma < \infty. \tag{4.10}$$

For a constant $M > 0$, let $\Omega_M = \{\omega\colon \sup_{t\in[0,1]^N} \|X(t)\| \le M\}$. Since the sample function of $X(t)$ is almost surely bounded on $[0,1]^N$, we have $\lim_{M\to\infty} \mathbb{P}(\Omega_M) = 1$. Note that $X(E) \subset \limsup_{k\to\infty} B(X(t_k), D(t_k, h_k))$ and, by Lemma 4.7, (4.10) and the fact that $\gamma < \beta H_1$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E}\big(D(t_k, h_k)^{\gamma/H_1} \mathbb{1}_{\Omega_M}\big) &\le \mathrm{e}^{2M} \sum_{k=1}^{\infty} \mathbb{E}\big(D(t_k, h_k)^{\gamma/H_1} \mathrm{e}^{-D(t_k, h_k)}\big) \\ &\le \mathrm{e}^{2M} K_{4,1} \sum_{k=1}^{\infty} h_k^\gamma < \infty. \end{aligned} \tag{4.11}$$

It follows from (4.11) that $\sum_{k=1}^{\infty} D(t_k, h_k)^{\gamma/H_1} < \infty$ almost surely on Ω_M. This implies that $\dim_{\mathrm{H}} X(E) \le \gamma/H_1$ almost surely on Ω_M. Letting $M \to \infty$ first and then $\gamma \downarrow \dim_{\mathrm{H}} E$ along the rational numbers proves (4.9). □

Putting Proposition 4.3 and Proposition 4.8 together, we derive the following theorem.

Theorem 4.9. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$ whose sample function is a.s. bounded on all compact subsets of $\mathbb{R}^N$. If there is a constant $H > 0$ such that for every $\varepsilon > 0$, X satisfies conditions* (C1) *with $H_1 = H - \varepsilon$ and* (C2′) *with $H_2 = H + \varepsilon$, then for every analytic set $E \subset \mathbb{R}^N$ that satisfies $\dim_{\mathrm{H}} E < \beta H$,*

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{1}{H} \dim_{\mathrm{H}} E\right\} \qquad \text{a.s.} \tag{4.12}$$

5. Applications

The general results in Sections 3 and 4 can be applied to wide classes of Gaussian or non-Gaussian random fields. Since the applications to Gaussian random fields can be carried out by extending Xiao (2007, 2009), we will focus on non-Gaussian random fields in this section.

5.1. Self-similar stable random fields

If $X = \{X(t), t \in \mathbb{R}_+\}$ is a stable Lévy process in $\mathbb{R}^d$, the Hausdorff dimensions of its image sets have been well studied; see Taylor (1986) and Xiao (2004) for historical accounts. The packing dimension results similar to those in Sections 3 and 4 have also been obtained by Khoshnevisan, Schilling and Xiao (2009) for Lévy processes. In this subsection, we will only consider non-Markov stable processes and stable random fields.

Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be an α-stable random field in $\mathbb{R}$ with the representation

$$X_0(t) = \int_F f(t,x) M(\mathrm{d}x), \tag{5.1}$$

where M is a symmetric α-stable (SαS) random measure on a measurable space $(F, \mathcal{F})$ with control measure m and $f(t,\cdot): F \to \mathbb{R}$ ($t \in \mathbb{R}^N$) is a family of functions on F satisfying

$$\int_F |f(t,x)|^\alpha m(\mathrm{d}x) < \infty \qquad \forall t \in \mathbb{R}^N.$$

For any integer $n \ge 1$ and $t_1, \ldots, t_n \in \mathbb{R}^N$, the characteristic function of the joint distribution of $X_0(t_1), \ldots, X_0(t_n)$ is given by

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{j=1}^n \xi_j X_0(t_j)\right) = \exp\left(-\left\|\sum_{j=1}^n \xi_j f(t_j,\cdot)\right\|_{\alpha,m}^\alpha\right),$$

where $\xi_j \in \mathbb{R}$ $(1 \le j \le n)$ and $\|\cdot\|_{\alpha,m}$ is the $L^\alpha(F,\mathcal{F},m)$-norm (or quasi-norm if $\alpha < 1$).

The class of α-stable random fields with representation (5.1) is broad. In particular, if a random field $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ is α-stable with $\alpha \ne 1$ or symmetric α-stable, and is *separable in probability* (i.e., there is a countable subset $T_0 \subset \mathbb{R}^N$ such that for every $t \in \mathbb{R}^N$, there exists a sequence $\{t_k\} \subset T_0$ such that $X_0(t_k) \to X_0(t)$ in probability), then X_0 has a representation (5.1); see Theorems 13.2.1 and 13.2.2 in Samorodnitsky and Taqqu (1994).

For a separable α-stable random field in $\mathbb{R}$ given by (5.1), Rosinski and Samorodnitsky (1993) investigated the asymptotic behavior of $\mathbb{P}\{\sup_{t\in[0,1]^N} |X_0(t)| \ge u\}$ as $u \to \infty$ (see also Samorodnitsky and Taqqu (1994)). The following lemma is a consequence of their result.

Lemma 5.1. *Let $X_0 = \{X_0(t), t \in \mathbb{R}^N\}$ be a separable α-stable random field in $\mathbb{R}$ given in the form* (5.1). *Assume that X_0 has a.s. bounded sample paths on* $[0,1]^N$. *There then exists a positive and finite constant $K_{5,1}$, depending on α, f and m only, such that for all* $u > 0$,

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\Big\{\sup_{t\in[0,1]^N} |X_0(t)| \ge u\Big\} = K_{5,1}. \tag{5.2}$$

Remark 5.2. In the above lemma, it is crucial to assume that X_0 has bounded sample paths on $[0,1]^N$ almost surely. Otherwise, (5.2) may not hold, as shown by the linear fractional stable motion X_0 with $0 < \alpha < 1$ (see Example 5.4 below).

We define an α-stable random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ by

$$X(t) = (X_1(t), \ldots, X_d(t)), \tag{5.3}$$

where $X_1, \ldots, X_d$ are independent copies of X_0.

The following result gives the Hausdorff and packing dimensions of the image measures of self-similar stable random fields.

Theorem 5.3. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a separable α-stable field with values in $\mathbb{R}^d$ defined by (5.3), where X_0 is given in the form (5.1). Suppose that X_0 is H-SSSI and its sample path is a.s. bounded on all compact subsets of $\mathbb{R}^N$. Then, for every finite Borel measure μ on $\mathbb{R}^N$,*

$$\dim_{\mathrm{H}} \mu_X = \min\left\{d, \frac{1}{H} \dim_{\mathrm{H}} \mu\right\} \quad \textit{and} \quad \dim_{\mathrm{P}} \mu_X = \frac{1}{H} \mathrm{Dim}_{Hd}\, \mu \qquad \text{a.s.} \tag{5.4}$$

Moreover, for every analytic set $E \subset \mathbb{R}^N$ that satisfies $\dim_{\mathrm{H}} E < \alpha H$, we have

$$\dim_{\mathrm{H}} X(E) = \min\left\{d, \frac{1}{H} \dim_{\mathrm{H}} E\right\} \qquad \text{a.s.}$$

Proof. It follows from the self-similarity and Lemma 5.1 that X satisfies condition (C1) with $H_1 = H$ and $\beta = \alpha$. On the other hand, condition (C2) with $H_2 = H$ is satisfied because X is H-self-similar and has stationary increments, and the α-stable variable $X(1)$ has a bounded continuous density function. Therefore, both equalities in (5.4) follow from Theorems 3.8 and 3.12. Finally, the last conclusion follows from Theorem 4.9. □

Next, we consider two important types of SSSI stable processes.

Example 5.4 (Linear fractional stable motion). Let $0 < \alpha < 2$ and $H \in (0, 1)$ be given constants. We define an α-stable process $X_0 = \{X_0(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}$ by

$$X_0(t) = \int_{\mathbb{R}} h_H(t, s) M_\alpha(\mathrm{d}s), \tag{5.5}$$

where M_α is a symmetric α-stable random measure on $\mathbb{R}$ with Lebesgue measure as its control measure and where

$$h_H(t, s) = a\{(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}\} + b\{(t-s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha}\}.$$

In the above, $a, b \in \mathbb{R}$ are constants with $|a| + |b| \neq 0$, $t_+ = \max\{t, 0\}$ and $t_- = \max\{-t, 0\}$. The α-stable process X_0 is then H-self-similar with stationary increments, which is called an (α, H)-*linear fractional stable motion*. If $H = \frac{1}{\alpha}$, then the integral in (5.5) is understood as $aM([0, t])$ if $t \geq 0$ and as $bM([t, 0])$ if $t < 0$. Hence, X_0 is an α-stable Lévy process.

Maejima (1983) proved that if $\alpha H < 1$, then X_0 is a.s. unbounded on any interval of positive length. On the other hand, if $\alpha H > 1$ (i.e., $1 < \alpha < 2$ and $1/\alpha < H < 1$), then Kolmogorov's continuity theorem implies that X_0 is a.s. continuous. In the latter case, Takashima (1989) further studied the local and uniform Hölder continuity of X_0. His Theorems 3.1 and 3.4 showed that the local Hölder exponent of X_0 equals H. However, the exponent of the uniform Hölder continuity cannot be bigger than $H - \frac{1}{\alpha}$.

Now, let $X = \{X(t), t \in \mathbb{R}_+\}$ be the (α, H)-linear fractional stable motion with values in $\mathbb{R}^d$ defined by (5.3). It follows from Theorem 5.3 that if $\alpha H > 1$, then for every finite Borel measure μ on $\mathbb{R}_+$,

$$\dim_{\mathrm{H}} \mu_X = \min\left\{d, \frac{1}{H} \dim_{\mathrm{H}} \mu\right\} \quad \text{and} \quad \dim_{\mathrm{P}} \mu_X = \frac{1}{H} \mathrm{Dim}_{Hd}\, \mu \qquad \text{a.s.,}$$

and for every analytic set $E \subset \mathbb{R}_+$, $\dim_{\mathrm{H}} X(E) = \min\{d, \frac{1}{H} \dim_{\mathrm{H}} E\}$ a.s. Note that the above dimension results do not depend on the uniform Hölder exponent of X.

There are several ways to define linear fractional α-stable random fields; see Kokoszka and Taqqu (1994). For example, for $H \in (0,1)$ and $\alpha \in (0,2)$, define

$$Z^H(t) = \int_{\mathbb{R}^N} \big(\|t-s\|^{H-N/\alpha} - \|s\|^{H-N/\alpha}\big) M_\alpha(\mathrm{d}s) \qquad \forall t \in \mathbb{R}^N, \tag{5.6}$$

where M_α is an SαS random measure on $\mathbb{R}^N$ with the N-dimensional Lebesgue measure as its control measure. This is the stable analog of the N-parameter fractional Brownian motion. However, it follows from Theorem 10.2.3 in Samorodnitsky and Taqqu (1994) that, whenever $N \geq 2$, the sample paths of Z^H are a.s. unbounded on any interval in $\mathbb{R}^N$. Thus, the results of this paper do not apply to Z^H when $N \geq 2$. In general, little is known about the sample path properties of Z^H.

Example 5.5 (Harmonizable fractional stable motion). Given $0 < \alpha < 2$ and $H \in (0,1)$, the harmonizable fractional stable field $\widetilde{Z}^H = \{\widetilde{Z}^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ is defined by

$$\widetilde{Z}^H(t) = \operatorname{Re} \int_{\mathbb{R}^N} \frac{\mathrm{e}^{\mathrm{i}\langle t,\lambda\rangle} - 1}{\|\lambda\|^{H+N/\alpha}} \widetilde{M}_\alpha(\mathrm{d}\lambda), \tag{5.7}$$

where $\widetilde{M}_\alpha$ is a complex-valued, rotationally invariant α-stable random measure on $\mathbb{R}^N$ with the N-dimensional Lebesgue measure as its control measure. It is easy to verify that the α-stable random field $\widetilde{Z}^H$ is H-self-similar with stationary increments.

It follows from Theorem 10.4.2 in Samorodnitsky and Taqqu (1994) (which covers the case $0 < \alpha < 1$) and Theorem 3 of Nolan (1989) (which covers $1 \leq \alpha < 2$) that $\widetilde{Z}^H$ has continuous sample paths almost surely. Moreover, it can be proven that $\widetilde{Z}^H$ satisfies the following uniform Hölder continuity: for any compact interval $I = [a,b] \subset \mathbb{R}^N$ and any $\varepsilon > 0$,

$$\lim_{h\to 0} \sup_{\substack{s,t\in I \\ \|s-t\|\leq h}} \frac{|\widetilde{Z}^H(t) - \widetilde{Z}^H(s)|}{\|t-s\|^H |\log\|t-s\||^{1/2+1/\alpha+\varepsilon}} = 0 \qquad \text{a.s.} \tag{5.8}$$

When $N = 1$, (5.8) is due to Kôno and Maejima (1991). In general, (5.8) follows from the results in Biermé and Lacaux (2009) or Xiao (2010). Note that the Hölder continuity of $\widetilde{Z}^H$ is different from that of the linear fractional stable motions.

Applying Theorem 4.5 to the harmonizable fractional stable motion in $\mathbb{R}^d$ defined as in (5.3), still denoted by $\widetilde{Z}^H$, we derive that for every analytic set $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} \widetilde{Z}^H(E) = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}} E\right\} \quad \text{and} \quad \dim_{\mathrm{P}} \widetilde{Z}^H(E) = \frac{1}{H}\operatorname{Dim}_{Hd} E \qquad \text{a.s.} \tag{5.9}$$

Remark 5.6. The results in this section are applicable to other self-similar stable random fields, including the Telecom process (Lévy and Taqqu (2000), Pipiras and Taqqu (2000)), self-similar fields of Lévy–Chentsov type (Samorodnitsky and Taqqu (1994), Shieh (1996)) and the stable sheet (Ehm (1981)). We leave the details to interested readers.

5.2. Real harmonizable fractional Lévy motion

We show that the results in Sections 3 and 4 can be applied to the real harmonizable fractional Lévy motion (RHFLM) introduced by Benassi, Cohen and Istas (2002). To recall their definition, let ν be a Borel measure on $\mathbb{C}$ which satisfies $\int_{\mathbb{C}} |z|^p \nu(\mathrm{d}z) < \infty$ for all $p \geq 2$. We assume that ν is rotationally invariant. Hence, if P is the map $z = \rho \mathrm{e}^{\mathrm{i}\theta} \mapsto (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_+$, then the image measure of ν under P can be written as $\nu_P(\mathrm{d}\theta, \mathrm{d}\rho) = \mathrm{d}\theta \, \nu_\rho(\mathrm{d}\rho)$, where $\mathrm{d}\theta$ is the uniform measure on $[0, 2\pi)$ and ν_ρ is a Borel measure on $\mathbb{R}_+$.

Let $N(\mathrm{d}\xi, \mathrm{d}z)$ be a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with mean measure $n(\mathrm{d}\xi, \mathrm{d}z) = \mathbb{E}(N(\mathrm{d}\xi, \mathrm{d}z)) = \mathrm{d}\xi \, \nu(\mathrm{d}z)$ and let $\widetilde{N}(\mathrm{d}\xi, \mathrm{d}z) = N(\mathrm{d}\xi, \mathrm{d}z) - n(\mathrm{d}\xi, \mathrm{d}z)$ be the compensated Poisson measure. Then, according to Definition 2.3 in Benassi, Cohen and Istas (2002), a real harmonizable fractional Lévy motion (without the Gaussian part) $X_0^H = \{X_0^H(t), t \in \mathbb{R}^N\}$ with index $H \in (0, 1)$ is defined by

$$X_0^H(t) = \int_{\mathbb{R}^N \times \mathbb{C}} 2 \operatorname{Re}\left(\frac{\mathrm{e}^{-\mathrm{i}\langle t, \xi\rangle} - 1}{\|\xi\|^{H+N/2}} z\right) \widetilde{N}(\mathrm{d}\xi, \mathrm{d}z) \qquad \text{for all } t \in \mathbb{R}^N. \tag{5.10}$$

As shown by Benassi, Cohen and Istas (2002), X_0^H has stationary increments, as well as moments of all orders; it behaves locally like fractional Brownian motion, but at the large scale, it behaves like harmonizable fractional stable motion $\widetilde{Z}^H$ in (5.7). Because of these multiscale properties, RHFLM's form a class of flexible stochastic models.

The following equation on characteristic functions of X_0^H was given by Benassi, Cohen and Istas (2002): for all integers $n \geq 2$, all $t^1, \ldots, t^n \in \mathbb{R}^N$ and all $u^1, \ldots, u^n \in \mathbb{R}$,

$$\mathbb{E} \exp\left(\mathrm{i} \sum_{j=1}^n u^j X_0^H(t^j)\right) = \exp\left(\int_{\mathbb{R}^N \times \mathbb{C}} \left[\mathrm{e}^{f_n(\xi, z)} - 1 - f_n(\xi, z)\right] \mathrm{d}\xi \, \nu(\mathrm{d}z)\right), \tag{5.11}$$

where

$$f_n(\xi, z) = \mathrm{i} 2 \operatorname{Re}\left(z \sum_{j=1}^n u^j \frac{\mathrm{e}^{-\mathrm{i}\langle t^j, \xi\rangle} - 1}{\|\xi\|^{H+N/2}}\right).$$

In particular, for any $s, t \in \mathbb{R}^N$ and $u \in \mathbb{R}$, (5.11) gives that

$$\mathbb{E} \exp\left(\mathrm{i} u \frac{X_0^H(t) - X_0^H(s)}{\|t - s\|^H}\right) = \exp\left(-2\pi \int_{\mathbb{R}^N} \psi\left(\frac{2u(1 - \cos\langle t - s, \xi\rangle)}{\|t - s\|^H \|\xi\|^{H+N/2}}\right) \mathrm{d}\xi\right), \tag{5.12}$$

where, for every $x \in \mathbb{R}$, $\psi(x)$ is defined by $\psi(x) = \int_0^\infty (1 - \cos(x\rho)) \nu_\rho(\mathrm{d}\rho)$. Note that the function ψ is non-negative and continuous. Moreover, up to a constant, it is the characteristic exponent of the infinitely divisible law in $\mathbb{C}$ with Lévy measure ν. For the proof of Theorem 5.7, we will make use of the following fact: there exists a positive constant K such that

$$\psi(x) \geq K^{-1} x^2 \int_0^{x^{-1}} \rho^2 \nu_\rho(\mathrm{d}\rho) \qquad \text{for all } x \in [0, 1]. \tag{5.13}$$

This is verified by using the inequality $1 - \cos x \geq K^{-1}x^2$ for all $x \in [0, 1]$.

Theorem 5.7. *Let $X^H = \{X^H(t), t \in \mathbb{R}^N\}$ be a separable real harmonizable fractional Lévy field in $\mathbb{R}^d$ defined by* (5.3), *where X_0^H is defined as in* (5.10). *Assume that ψ satisfies the following condition: there exists a constant $\delta \in (0, 1]$ such that*

$$\frac{\psi(ax)}{\psi(x)} \geq a^\delta \qquad \text{for all } a \geq 1 \text{ and } x \in \mathbb{R}. \tag{5.14}$$

Then, for every analytic set $E \subset \mathbb{R}^N$,

$$\dim_{\mathrm{H}} X^H(E) = \min\left\{d, \frac{1}{H}\dim_{\mathrm{H}} E\right\} \quad \text{and} \quad \dim_{\mathrm{P}} X^H(E) = \frac{1}{H}\operatorname{Dim}_{Hd} E \qquad \text{a.s.} \tag{5.15}$$

Proof. It follows from Proposition 3.3 in Benassi, Cohen and Istas (2002) that for every $\varepsilon > 0$, X^H satisfies almost surely a uniform Hölder condition of order $H - \varepsilon$ on all compact sets of $\mathbb{R}^N$. Hence, the upper bounds in (5.15) follow from Proposition 4.4.

In order to prove the desired lower bounds in (5.15), by Proposition 4.3, it suffices to show that X^H satisfies condition (C2′) with $H_2 = H$. This is done by showing that there exists a positive function $g \in L^1(\mathbb{R}^d)$ such that for all $s, t \in \mathbb{R}^N$ satisfying $\|s - t\| \leq 1$, we have

$$\left|\mathbb{E}\left(\mathrm{e}^{\mathrm{i}\langle u, X(t)-X(s)\rangle/\|t-s\|^H}\right)\right| \leq g(u) \quad \text{for all } u \in \mathbb{R}^d. \tag{5.16}$$

This and the Fourier inversion formula together imply that the density functions of $X(t) - X(s)/\|t - s\|^H$ are uniformly bounded for all $s, t \in \mathbb{R}^N$ satisfying $\|s - t\| \leq 1$.

Since the coordinate processes $X_1^H, \ldots, X_d^H$ are independent copies of X_0^H, it is sufficient to prove (5.16) for $d = 1$. Note that, by (5.16), we can take $g(u) = 1$ for all $|u| \leq 1$. For any u such that $|u| > 1$, condition (5.14) implies that

$$\begin{aligned}\int_{\mathbb{R}^N} \psi\left(\frac{2u(1 - \cos\langle t - s, \xi\rangle)}{\|t - s\|^H \|\xi\|^{H+N/2}}\right) \mathrm{d}\xi &\geq K|u|^\delta \int_{\mathbb{R}^N} \psi\left(\frac{1 - \cos\langle t - s, \xi\rangle}{\|t - s\|^H \|\xi\|^{H+N/2}}\right) \mathrm{d}\xi \\ &\geq K|u|^\delta \int_{\|\xi\| \geq \gamma\|t-s\|^{-1}} \psi\left(\frac{1 - \cos\langle t - s, \xi\rangle}{\|t - s\|^H \|\xi\|^{H+N/2}}\right) \mathrm{d}\xi,\end{aligned} \tag{5.17}$$

where $\gamma > 1$ is a constant whose value will be chosen later.

By a change of variable $\xi \mapsto \eta\|t - s\|^{-1}$, we see that the last integral becomes

$$\begin{aligned}&\int_{\|\eta\| \geq \gamma} \psi\left(\frac{\|t - s\|^{N/2}(1 - \cos\langle (t - s)/\|t - s\|, \eta\rangle)}{\|\eta\|^{H+N/2}}\right) \frac{\mathrm{d}\eta}{\|t - s\|^N} \\ &\quad \geq K \int_{\|\eta\| \geq \gamma} \frac{(1 - \cos\langle (t - s)/\|t - s\|, \eta\rangle)^2}{\|\eta\|^{2H+N}} \mathrm{d}\eta,\end{aligned} \tag{5.18}$$

where the inequality follows from (5.13), and we have used the fact that $\|t - s\| \leq 1$ and taken γ large. The last integral is a constant because the Lebesgue measure is rotationally invariant.

Thus, we have proven that for $|u| > 1$,

$$\mathbb{E}\exp\left(\mathrm{i}u\frac{X_0^H(t) - X_0^H(s)}{\|t-s\|^H}\right) \le \exp(-K_{5,2}|u|^\delta). \tag{5.19}$$

Therefore, when $d = 1$, (5.16) holds for the function g defined as $g(u) = 1$ if $|u| \le 1$ and $g(u) = \mathrm{e}^{-K_{5,2}|u|^\delta}$ if $|u| > 1$. This completes the proof of Theorem 5.7. □

We mention that Benassi, Cohen and Istas (2004) have introduced another interesting class of fractional Lévy fields, namely, the moving average fractional Lévy fields (MAFLF). Similarly to the contrast between linear fractional stable motion and harmonizable fractional stable motion, many properties of MAFLF's are different from those of RHFLM's. For example, the exponent of the uniform modulus of continuity of an MAFLF is strictly smaller than its local Hölder exponent. Nevertheless, we believe that the arguments in this paper are applicable to MAFLF's. This and some related problems will be dealt with elsewhere.

5.3. The Rosenblatt process

Given an integer $m \ge 2$ and a constant $\kappa \in (1/2 - 1/(2m), 1/2)$, the Hermite process $Y^{m,\kappa} = \{Y^{m,\kappa}(t), t \in \mathbb{R}_+\}$ of order m is defined by

$$Y^{m,\kappa}(t) = K_{5,3}\int'_{\mathbb{R}^m}\left\{\int_0^t \prod_{j=1}^m (s-u_j)_+^{\kappa-1}\,\mathrm{d}s\right\}\mathrm{d}B(u_1)\cdots\mathrm{d}B(u_m), \tag{5.20}$$

where $K_{5,3} > 0$ is a normalizing constant depending on m and κ only and the integral $\int'_{\mathbb{R}^m}$ is the m-tuple Wiener–Itô integral with respect to the standard Brownian motion excluding the diagonals $\{u_i = u_j\}$, $i \ne j$. The integral (5.20) is also well defined if $m = 1$; the process is a fractional Brownian motion for which the problem considered in this paper has been solved.

The Hermite process $Y^{m,\kappa}$ is H-SSSI and $H = 1 + m\kappa - \frac{m}{2} \in (0, 1)$. It is a non-Gaussian process and often appears in non-central limit theorems for processes defined as integrals or partial sums of nonlinear functionals of stationary Gaussian sequences with long-range dependence; see Taqqu (1975, 1979), Dobrushin and Major (1979) and Major (1981).

It follows from Theorem 6.3 of Taqqu (1979) that the Hermite process $Y^{m,\kappa}$ has the following equivalent representation:

$$Y^{m,\kappa}(t) = K_{5,4}\int'_{\mathbb{R}^m}\frac{\mathrm{e}^{\mathrm{i}t(u_1+\cdots+u_m)} - 1}{\mathrm{i}(u_1+\cdots+u_m)}\prod_{j=1}^m |u_j|^{\kappa-1} Z_G(\mathrm{d}u_1)\cdots Z_G(\mathrm{d}u_m), \tag{5.21}$$

where $K_{5,4} > 0$ is a normalizing constant and Z_G is a centered complex Gaussian random measure on $\mathbb{R}$ with Lebesgue measure as its control measure.

Mori and Oodaira (1986) studied the functional laws of the iterated logarithms for the Hermite process $Y^{m,\kappa}$. Lemma 5.8 follows from Lemma 6.3 in Mori and Oodaira (1986).

Lemma 5.8. *There exist positive constants $K_{5,5}$ and $K_{5,6}$, depending on m only, such that $\mathbb{P}\{\max_{t\in[0,1]} |Y^{m,\kappa}(t)| \geq u\} \leq \exp(-K_{5,6}u^{2/m})$ for all $u \geq K_{5,5}$.*

Using Lemma 5.8, one can derive easily a uniform modulus of continuity for $Y^{m,\kappa}$.

Lemma 5.9. *There exists a finite constant $K_{5,7}$ such that for all constants $0 \leq a < b < \infty$,*

$$\limsup_{h\downarrow 0} \sup_{a\leq t\leq b-h} \sup_{0\leq s\leq h} \frac{|Y^{m,\kappa}(t+s) - Y^{m,\kappa}(t)|}{h^H (\log 1/h)^{m/2}} \leq K_{5,7} \qquad \text{a.s.}, \tag{5.22}$$

where $H = 1 + m\kappa - \frac{m}{2}$.

Proof. For every $t \geq 0$ and $h > 0$, the self-similarity of $Y^{m,\kappa}$ and Lemma 5.8 together imply that

$$\mathbb{P}\{|Y^{m,\kappa}(t+h) - Y^{m,\kappa}(h)| > h^H u\} \leq \exp(-K_{5,6}u^{2/m}). \tag{5.23}$$

Hence, $Y^{m,\kappa} = \{Y^{m,\kappa}(t), t \geq 0\}$ satisfies the conditions of Lemmas 2.1 and 2.2 in Csáki and Csörgő (1992) with $\sigma(h) = h^H$ and $\beta = 2/m$. Consequently, (5.22) follows directly from Theorem 3.1 in Csáki and Csörgő (1992). □

The case $m = 2$ has recently received considerable attention. The process $Y^{2,\kappa}$ is called the *Rosenblatt process* by Taqqu (1975) (or *fractional Rosenblatt motion* by Pipiras (2004)). Its self-similarity index is given by $H = 2\kappa$. This non-Gaussian process in many ways resembles fractional Brownian motion. For example, since $H > 1/2$, fractional noise of $Y^{2,\kappa}$ exhibits long-range dependence. Besides its connections to non-central limit theorems, the Rosenblatt process also appears in limit theorems for some quadratic forms of random variables with long-range dependence. Albin (1998a, 1998b) has discussed distributional properties and the extreme value theory of $Y^{2,\kappa}$. In particular, Albin (1998b), Section 16, obtained sharp asymptotics on the tail probability of $\max_{t\in[0,1]} Y^{2,\kappa}(t)$. Pipiras (2004) established a wavelet-type expansion for the Rosenblatt process. Tudor (2008) has recently developed a stochastic calculus for $Y^{2,\kappa}$ based on both pathwise type calculus and Malliavin calculus.

We now consider the Rosenblatt process $X^{2,\kappa}$ with values in $\mathbb{R}^d$ by letting its component processes be independent copies of $Y^{2,\kappa}$. The following result determines the Hausdorff and packing dimensions of the image sets of $X^{2,\kappa}$.

Corollary 5.10. *Let $X^{2,\kappa} = \{X^{2,\kappa}(t), t \in \mathbb{R}_+\}$ be a Rosenblatt process in $\mathbb{R}^d$ as defined above. Then, for every analytic set $E \subset \mathbb{R}_+$, we have*

$$\dim_{\mathrm{H}} X^{2,\kappa}(E) = \min\left\{d, \frac{1}{2\kappa}\dim_{\mathrm{H}} E\right\} \quad \textit{and} \quad \dim_{\mathrm{P}} X^{2,\kappa}(E) = \frac{1}{2\kappa}\mathrm{Dim}_{2\kappa d} E \qquad \text{a.s.} \tag{5.24}$$

Proof. By Lemma 5.9, for any $\varepsilon > 0$, $X^{2,\kappa}$ satisfies a uniform Hölder condition of order $H - \varepsilon$ (where $H = 2\kappa$) on all compact intervals in $\mathbb{R}_+$. On the other hand, it is known that the random variable $Y^{2,\kappa}(1)$ has a bounded and continuous density (see Davydov (1990) or Albin (1998a)). Thus, $X^{2,\kappa}$ also satisfies condition (C2) with $H_2 = 2\kappa$. Therefore, the two equalities in (5.24) follow from Theorem 4.5. □

Acknowledgements

The authors thank Professor Davar Khoshnevisan for stimulating discussions on packing dimension profiles, Professors Patrik Albin, David Nualart and Frederi Viens for very helpful suggestions on the Rosenblatt process and the anonymous referee for comments and suggestions which have led to significant improvement of the manuscript. Research of N.-R. Shieh was partially supported by Taiwan NSC Grant 962115M002005MY3. Research of Y. Xiao was partially supported by the NSF Grant DMS-0706728 and the National Natural Science Foundation of China (No. 70871050).

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Received April 2009 and revised September 2009