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Asymptotics for the small fragments of the fragmentation at nodes

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We consider the fragmentation at nodes of the Lévy continuous random tree introduced in a previous paper. In this framework we compute the asymptotic behaviour of the number of small fragments at time θ. This limit is increasing in θ and discontinuous. In the α-stable case the fragmentation is self-similar with index $1/\alpha$, with $\alpha \in (1, 2)$, and the results are close to those Bertoin obtained for general self-similar fragmentations but with an additional assumption which is not fulfilled here.

Keywords: continuous random tree; fragmentation; Lévy snake; local time; small fragments

1. Introduction

A fragmentation process is a Markov process which describes how an object with given total mass evolves as it breaks into several fragments randomly as time passes. Notice there may be loss of mass but no gain. Such processes have been widely studied in recent years; see Bertoin (2006) and references therein. To be more precise, the state space of a fragmentation process is the set of non-increasing sequences of masses with finite total mass:

$$\mathcal{S}^{\downarrow} = \left\{ s = (s_1, s_2, \dots);\ s_1 \geqslant s_2 \geqslant \dots \geqslant 0 \quad \text{and} \quad \Sigma(s) = \sum_{k=1}^{+\infty} s_k < +\infty \right\}.$$

If we denote by P_s the law of an $\mathcal{S}^{\downarrow}$-valued process $\Lambda = (\Lambda(\theta), \theta \geqslant 0)$ starting at $s = (s_1, s_2, \dots) \in \mathcal{S}^{\downarrow}$, we say that Λ is a fragmentation process if it is a Markov process such that $\theta \mapsto \Sigma(\Lambda(\theta))$ is non-increasing and if it satisfies the fragmentation property: the law of $(\Lambda(\theta), \theta \geqslant 0)$ under P_s is the non-increasing reordering of the fragments of independent processes of the respective laws $P_{(s_1,0,\dots)}$, $P_{(s_2,0,\dots)}$, ... In other words, each fragment after dislocation behaves independently of the others, and its evolution depends only on its initial mass. As a consequence, to describe the law of the fragmentation process with any initial condition, it suffices to study the laws $P_r := P_{(r,0,\dots)}$ for any $r \in (0, +\infty)$, that is, the law of the fragmentation process starting with a single mass r.

A fragmentation process is said to be self-similar of index α' if, for any $r > 0$, the process Λ under P_r is distributed as the process $(r\Lambda(r^{\alpha'}\theta), \theta \geqslant 0)$ under P_1. Bertoin

(2000b) proved that the law of a self-similar fragmentation is characterized by the index of self-similarity α', an erosion coefficient c which corresponds to a rate of mass loss, and a dislocation measure ν on $\mathcal{S}^{\downarrow}$ which describes sudden dislocations of a fragment of mass 1. The dislocation measure of a fragment of size r, ν_r is given by $\int F(s)\nu_r(\mathrm{d}s) = r^{\alpha'} \int F(rs)\nu(\mathrm{d}s)$.

When there is no loss of mass (which implies that $c = 0$ and $\alpha' > 0$), under some additional assumptions, the number of fragments at a fixed time is infinite. A natural question is therefore to study the asymptotic behaviour when ε goes down to 0 of $N^{\varepsilon}(\theta) = \text{Card}\,\{i, \Lambda_i(\theta) > \varepsilon\}$, where $\Lambda(\theta) = (\Lambda_1(\theta), \Lambda_2(\theta), \ldots)$ is the state of the fragmentation at time θ; see Bertoin (2004) and also Haas (2004) when α' is negative.

The goal of this paper is to study the same problem for the fragmentation at nodes of the Lévy continuous random tree constructed by Abraham and Delmas (2005).

Le Gall and Le Jan (1998a, 1998b) associated with a Lévy process with no negative jumps that does not drift to infinity, $X = (X_s, s \geqslant 0)$ with Laplace exponent ψ, a continuous state branching process (CSBP) and a Lévy continuous random tree (CRT) which keeps track of the genealogy of the CSBP. The Lévy CRT can be coded by the so-called height process $H = (H_s, s \geqslant 0)$. Informally, H_s gives the distance (which can be understood as the number of generations) between the individual labelled s and the root, 0, of the CRT. The precise definition of ψ we consider is given at the beginning of Section 2.1.

In order to construct a fragmentation process from this CRT, Abraham and Delmas (2005) mark the nodes, of the tree in a Poissonian manner. They then cut the CRT at these marked nodes, and the 'sizes' of the resulting subtrees give the state of the fragmentation at some time. As time θ increases, the parameter of the Poisson processes used to mark the nodes increases as well as the set of the marked nodes. This gives a fragmentation process with no loss of mass. When the initial Lévy process is stable – that is, when $\psi(\lambda) = \lambda^{\alpha}$, $\alpha \in (1, 2]$ – the fragmentation is self-similar with index $1/\alpha$ and with a zero erosion coefficient; see also Aldous and Pitman (1998) and Bertoin (2000a) for $\alpha = 2$, and Miermont (2005) for $\alpha \in (1, 2)$. For a general subcritical or critical CRT, there is no more scaling property, and the dislocation measure, which describes how a fragment of size $r > 0$ is cut into smaller pieces, cannot be expressed as a nice function of the dislocation measure of a fragment of size 1. Abraham and Delmas (2005) give the family of dislocation measures $(\nu_r, r > 0)$ for the fragmentation at nodes of a general subcritical or critical CRT. Intuitively ν_r describes the way a mass r breaks into smaller pieces.

We denote by $\mathbb{N}$ the excursion measure of the Lévy process X (the fragmentation process is then defined under this measure). We denote by σ the length of the excursion. We have (see Section 3.2.2 in Duquesne and Le Gall 2002) that

$$\mathbb{N}[1 - \mathrm{e}^{-\lambda\sigma}] = \psi^{-1}(\lambda), \tag{1}$$

and ψ^{-1} is the Laplace exponent of a subordinator (see Chapter VII in Bertoin 1996), whose Lévy measure we denote by π_*. The distribution of σ under $\mathbb{N}$ is given by π_*. As π_* is a Lévy measure, we have $\int_{(0,\infty)} (1 \wedge r)\, \pi_*(\mathrm{d}r) < \infty$. For $\varepsilon > 0$, we write

$$\bar{\pi}_*(\varepsilon) = \pi_*((\varepsilon, \infty)) = \mathbb{N}[\sigma > \varepsilon] \quad \text{and} \quad \varphi(\varepsilon) = \int_{(0,\varepsilon]} r\pi_*(\mathrm{d}r) = \mathbb{N}[\sigma \mathbf{1}_{\{\sigma \leqslant \varepsilon\}}].$$

If $\Lambda(\theta) = (\Lambda_1(\theta), \Lambda_2(\theta), \ldots)$ is the state of the fragmentation at time θ, we denote by $N^\varepsilon(\theta)$ the number of fragments of size greater than ε, given by

$$N^\varepsilon(\theta) = \sum_{k=1}^{+\infty} \mathbf{1}_{\{\Lambda_k(\theta) > \varepsilon\}} = \sup\{k \geqslant 1, \Lambda_k(\theta) > \varepsilon\},$$

with the convention $\sup \varnothing = 0$. And we denote by $M^\varepsilon(\theta)$ the mass of the fragments of size less than ε,

$$M^\varepsilon(\theta) = \sum_{k=1}^{+\infty} \Lambda_k(\theta) \mathbf{1}_{\{\Lambda_k(\theta) \leqslant \varepsilon\}} = \sum_{k=N^\varepsilon(\theta)+1}^{+\infty} \Lambda_k(\theta).$$

Let $\mathcal{J} = \{s \geqslant 0, X_s > X_{s-}\}$ and let $(\Delta_s, s \in \mathcal{J})$ be the set of jumps of X. Conditionally on $(\Delta_s, s \in \mathcal{J})$, let $(T_s, s \in \mathcal{J})$ be a family of independant random variables such that T_s has exponential distribution with mean $1/\Delta_s$. T_s is the time at which the node of the CRT associated with the jump Δ_s is marked in order to construct the fragmentation process. Under $\mathbb{N}$, we denote by $R(\theta)$ the mass of the marked nodes of the Lévy CRT:

$$R(\theta) = \sum_{s \in \mathcal{J} \cap [0,\sigma]} \Delta_s \mathbf{1}_{\{T_s \leqslant \theta\}}.$$

The main result of this paper is then the following theorem:

Theorem 1.1. *We have*

$$\lim_{\varepsilon \to 0} \frac{N^\varepsilon(\theta)}{\bar{\pi}_*(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{M^\varepsilon(\theta)}{\varphi(\varepsilon)} = R(\theta)$$

in $L^2(\mathbb{N}[\mathrm{e}^{-\beta\sigma} \cdot])$, *for any* $\beta > 0$.

We consider the stable case $\psi(\lambda) = \lambda^\alpha$, where $\alpha \in (1, 2)$. We have

$$\pi_*(\mathrm{d}r) = (\alpha\Gamma(1 - \alpha^{-1}))^{-1} r^{-1-1/\alpha}\, \mathrm{d}r,$$

which gives

$$\bar{\pi}_*(\varepsilon) = \Gamma(1 - \alpha^{-1})^{-1} \varepsilon^{-1/\alpha} \quad \text{and} \quad \varphi(\varepsilon) = \big((\alpha - 1)\Gamma(1 - \alpha^{-1})\big)^{-1} \varepsilon^{1-\alpha^{-1}}.$$

From the scaling property, there exists a version of $(\mathbb{N}_r, r > 0)$ such that, for all $r > 0$, we have $\mathbb{N}_r[F((X_s, s \in [0, r]))] = \mathbb{N}_1[F((r^{1/\alpha} X_{s/r}, s \in [0, r]))]$ for any non-negative measurable function F defined on the set of cadlag paths. The proof of the next proposition relies on a second-moment computation (see (20)). It is similar to the proof of Corollary 4.3 in Delmas (2006) and is not reproduced here.

Proposition 1.2. *Let* $\psi(\lambda) = \lambda^\alpha$, *for* $\alpha \in (1, 2)$. *For all* $\theta > 0$, *we have* $\mathbb{N}$-*almost everywhere or* $\mathbb{N}_1$-*almost surely,*

$$\lim_{\varepsilon\to 0}\Gamma\left(1-\frac{1}{\alpha}\right)\varepsilon^{1/\alpha}N^{\varepsilon}(\theta)=\lim_{\varepsilon\to 0}(\alpha-1)\Gamma\left(1-\frac{1}{\alpha}\right)\frac{M^{\varepsilon}(\theta)}{\varepsilon^{1-1/\alpha}}=R(\theta). \tag{2}$$

Remark 1.1. Notice the similarity with the results in Delmas (2006) on asymptotic behaviour of the small fragments for the fragmentation at height of the CRT: the local time of the height process is here replaced by the functional R.

Remark 1.2. Let us compare the result of Proposition 1.2 with the main theorem of Bertoin (2004), which we now recall. Let Λ be a self-similar fragmentation with index $\alpha > 0$, erosion coefficient $c = 0$ and dislocation measure ν. We set

$$\varphi_b(\varepsilon)=\int_{\mathcal{S}^{\downarrow}}\left(\sum_{i=1}^{\infty}\mathbf{1}_{\{x_i>\varepsilon\}}-1\right)\nu_1(\mathrm{d}x),$$

$$f_b(\varepsilon)=\int_{\mathcal{S}^{\downarrow}}\sum_{i=1}^{\infty}x_i\mathbf{1}_{\{x_i<\varepsilon\}}\nu_1(\mathrm{d}x),$$

$$g_b(\varepsilon)=\int_{\mathcal{S}^{\downarrow}}\left(\sum_{i=1}^{\infty}x_i\mathbf{1}_{\{x_i<\varepsilon\}}\right)^2\nu_1(\mathrm{d}x).$$

If there exists $\beta\in(0, 1)$ such that φ_b being regularly varying at 0 with index $-\beta$ (which is equivalent to f_b being regularly varying at 0 with index $1-\beta$), and if there exist two positive constants c, η such that

$$g_b(\varepsilon)\leqslant cf_b^2(\varepsilon)(\log 1/\varepsilon)^{-(1+\eta)}, \tag{3}$$

then almost surely

$$\lim_{\varepsilon\to 0}\frac{N^{\varepsilon}(\theta)}{\varphi_b(\varepsilon)}=\lim_{\varepsilon\to 0}\frac{M^{\varepsilon}(\theta)}{f_b(\varepsilon)}=\int_0^{\theta}\sum_{i=1}^{\infty}\Lambda_i(u)^{\alpha+\beta}\,\mathrm{d}u.$$

In our case, we have φ and $\bar{\pi}_*$ equivalent to φ_b and f_b (up to multiplicative constants; see Lemmas 5.1 and 5.2). The normalizations are consequently the same. However, we have here $g_b(\varepsilon)=O(f_b^2(\varepsilon))$ (see Lemma 5.3) and Bertoin's assumption (3) is not fulfilled. When this last assumption holds, remark that the limit process is an increasing continuous process (as θ varies). In our case this assumption does not hold and the limit process $(R(\theta), \theta\geqslant 0)$ is still increasing but discontinuous as $R(\theta)$ is a pure jump process (this is an increasing sum of marked masses).

The paper is organized as follows. In Section 2, we recall the definition and properties of the height and exploration processes that code the Lévy CRT and we recall the construction of the fragmentation process associated with the CRT. The proof of Theorem 1.1 is given in Section 3. Notice that the computations given in the proof of Lemma 3.1 based on Propositions 2.1 and 2.2 are enough to characterize the transition kernel of the fragmentation Λ. We characterize the law of the scaling limit $R(\theta)$ in Section 4. The computations needed for Remark 1.2 are given in Section 5.

2. Notation

2.1. The exploration process

Let ψ denote the Laplace exponent of $X : \mathbb{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}$, $\lambda > 0$. We shall assume there is no Brownian part, so that

$$\psi(\lambda) = \alpha_0\lambda + \int_{(0,+\infty)} \pi(d\ell)\left[e^{-\lambda\ell} - 1 + \lambda\ell\right],$$

where $\alpha_0 \geqslant 0$ and the Lévy measure π is a positive σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi(d\ell) < \infty$. Following Duquesne and Le Gall (2002), we shall also assume that X is of infinite variation almost surely, which implies that $\int_{(0,1)} \ell\pi(d\ell) = \infty$. Notice that these hypotheses are fulfilled in the stable case: $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2)$. For $\lambda \geqslant 1/\varepsilon > 0$, we have $e^{-\lambda\ell} - 1 + \lambda\ell \geqslant \frac{1}{2}\lambda\ell\mathbf{1}_{\{\ell\geqslant 2\varepsilon\}}$, which implies that $\lambda^{-1}\psi(\lambda) \geqslant \alpha_0 + \int_{(2\varepsilon,\infty)} \ell\,\pi(d\ell)$. We deduce that

$$\lim_{\lambda\to\infty} \frac{\lambda}{\psi(\lambda)} = 0. \tag{4}$$

The so-called exploration process $\rho = (\rho_t, t \geqslant 0)$ is a Markov process taking values in $\mathcal{M}_f$, the set of positive measures on $\mathbb{R}_+$. The height process at time t is defined as the supremum of the closed support of ρ_t (with the convention that $H_t = 0$ if $\rho_t = 0$). Informally, H_t gives the distance (which can be understood as the number of generations) between the individual labelled t and the root, 0, of the CRT. In some sense $\rho_t(dv)$ records the 'number' of brothers, with labels larger than t, of the ancestor of t at generation v.

We recall the definition and properties of the exploration process which are given in Le Gall and Le Jan (1998a, 1998b) and Duquesne and Le Gall (2002). The results of this section are mainly extracted from Duquesne and Le Gall.

Let $I = (I_t, t \geqslant 0)$ be the infimum process of X, $I_t = \inf_{0\leqslant s\leqslant t} X_s$. We will also consider, for every $0 \leqslant s \leqslant t$, the infimum of X over $[s, t]$:

$$I_t^s = \inf_{s\leqslant r\leqslant t} X_r.$$

There exists a sequence $(\varepsilon_n, n \in \mathbb{N}^*)$ of positive real numbers decreasing to 0 such that

$$\tilde{H}_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t \mathbf{1}_{\{X_s < I_t^s + \varepsilon_k\}}\, ds$$

exists and is finite almost surely for all $t \geqslant 0$.

The point 0 is regular for the Markov process $X - I$, $-I$ is the local time of $X - I$ at 0, and the right-continuous inverse of $-I$ is a subordinator with Laplace exponent ψ^{-1} (see Chapter VII in Bertoin 1996). Notice that this subordinator has no drift thanks to (4). Let π_* denote the corresponding Lévy measure. Let $\mathbb{N}$ be the associated excursion measure of the process $X - I$ out of 0, and $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under $\mathbb{N}$. Under $\mathbb{N}$, $X_0 = I_0 = 0$.

For $\mu \in \mathcal{M}_f$, we define $H^\mu = \sup\{x \in \text{supp}\ \mu\}$, where supp μ is the closed support of the measure μ. From Section 1.2 in Duquesne and Le Gall (2002), there exists an $\mathcal{M}_f$-valued process $\rho^0 = (\rho^0_t, t \geqslant 0)$, called the exploration process, such that:

- almost surely, for every $t \geqslant 0$, we have $\langle \rho^0_t, 1\rangle = X_t - I_t$, and the process ρ^0 is cadlag;
- the process $(H^0_s = H^{\rho^0_s}, s \geqslant 0)$ taking values in $[0, \infty]$ is lower semi-continuous;
- for each $t \geqslant 0$, almost surely $H^0_t = \tilde{H}_t$;
- for every measurable non-negative function f defined on $\mathbb{R}_+$,

$$\langle \rho^0_t, f\rangle = \int_{[0,t]} f(H^0_s)\, \mathrm{d}_s I^s_t,$$

or equivalently, with δ_x being the Dirac mass at x,

$$\rho^0_t(\mathrm{d}r) = \sum_{\substack{0<s\leqslant t\\ X_{s-}<I^s_t}} (I^s_t - X_{s-})\delta_{H^0_s}(\mathrm{d}r).$$

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ almost surely. To obtain the Markov property of ρ, we must define the process ρ starting at any initial measure $\mu \in \mathcal{M}_f$. For $a \in [0, \langle \mu, 1\rangle]$, we define the erased measure $k_a\mu$ by

$$k_a\mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1\rangle - a), \qquad \text{for } r \geqslant 0.$$

If $a > \langle \mu, 1\rangle$, we set $k_a\mu = 0$. In other words, the measure $k_a\mu$ is the measure μ erased by a mass a backward from H^μ.

For $\nu, \mu \in \mathcal{M}_f$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f$ of the two measures by

$$\langle [\mu, \nu], f\rangle = \langle \mu, f\rangle \;+\; \langle \nu, f(H^\mu + \cdot)\rangle,$$

for f non-negative measurable. Eventually, we set for every $\mu \in \mathcal{M}_f$ and every $t > 0$,

$$\rho_t = [k_{-I_t}\mu, \rho^0_t].$$

We say that $\rho = (\rho_t, t \geqslant 0)$ is the process ρ starting at $\rho_0 = \mu$, and write $\mathbb{P}_\mu$ for its law. We set $H_t = H^{\rho_t}$. The process ρ is cadlag (with respect to the weak convergence topology on $\mathcal{M}_f$) and strong Markov.

2.2. The fragmentation at nodes

We recall the construction of the fragmentation under $\mathbb{N}$ given in Abraham and Delmas (2005) in an equivalent but easier way to understand. Recall that $(\Delta_s, s \in \mathcal{J})$ is the set of jumps of X and T_s is the time at which the jump Δ_s is marked. Conditionally on $(\Delta_s, s \in \mathcal{J})$, $(T_s, s \in \mathcal{J})$ is a family of independent random variables such that T_s has exponential distribution with mean $1/\Delta_s$. We consider the family of measures (increasing in θ) defined for $\theta \geqslant 0$ and $t \geqslant 0$ by

$$\tilde{m}_t^\theta(\mathrm{d}r) = \sum_{\substack{0<s\leqslant t\\ X_{s-}<I_t^s}} \mathbf{1}_{\{T_s\leqslant\theta\}}\ \delta_{H_s}(\mathrm{d}r).$$

Intuitively, $\tilde{m}_t^\theta$ describes the marked masses of the measure ρ_t, that is, the marked nodes of the associated CRT.

Then we cut the CRT according to these marks to obtain the state of the fragmentation process at time θ. To construct the fragmentation, let us consider the following equivalence relation $\mathcal{R}^\theta$ on $[0, \sigma]$, defined under $\mathbb{N}$ or $\mathbb{N}_\sigma$ by

$$s\mathcal{R}^\theta t \ \Leftrightarrow\ \tilde{m}_s^\theta([H_{s,t}, H_s]) = \tilde{m}_t^\theta([H_{s,t}, H_t]) = 0, \tag{5}$$

where $H_{s,t} = \inf_{u\in[s,t]} H_u$. Intuitively, two points s and t belongs to the same equivalence class (i.e. the same fragment) at time θ if there is no cut on their lineage down to their most recent common ancestor, that is, $\tilde{m}_s^\theta$ puts no mass on $[H_{s,t}, H_s]$ nor $\tilde{m}_t^\theta$ on $[H_{s,t}, H_t]$. Notice that cutting occurs on branching points, that is, at nodes of the CRT. Each node of the CRT corresponds to a jump of the underlying Lévy process X. The fragmentation process at time θ is then the Lebesgue measures (ranked in non-increasing order) of the equivalence classes of $\mathcal{R}^\theta$.

Remark 2.1. In definition (14) of Abraham and Delmas (2005: 12), we use another family of measures $m_t^{(\theta)}$. From their construction, notice that $\tilde{m}_t^\theta$ is absolutely continuous with respect to $m_t^{(\theta)}$ and $m_t^{(\theta)}$ is absolutely continuous with respect to $\tilde{m}_t^\theta$, if we take $T_s = \inf\{V_{s,u}, u>0\}$, where $\sum_{u>0}\delta_{V_{s,u}}$ is a Poisson point measure on $\mathbb{R}_+$ with intensity $\Delta_s\mathbf{1}_{\{u>0\}}$; see Section 3.1 in Abraham and Delmas (2005). In particular, $\tilde{m}_t^\theta$ and $m_t^{(\theta)}$ define the same equivalence relation and therefore the same fragmentation.

In order to index the fragments, we define the 'generation' of a fragment. For any $s\leqslant\sigma$, let us define $H_s^0 = 0$ and, recursively for $k\in\mathbb{N}$,

$$H_s^{k+1} = \inf\{u\geqslant 0,\ \tilde{m}_s^\theta((\mathrm{H}_s^k, u]) > 0\},$$

with the usual convention that $\inf\varnothing = +\infty$. We set the 'generation' of s as

$$K_s = \sup\{j\in\mathbb{N},\ H_s^j < +\infty\}.$$

Remark 8.1 in Abraham and Delmas (2005) ensures that K_s is finite $\mathbb{N}$-almost everywhere and that the 'generation' is well defined. Notice that if $s\mathcal{R}^\theta t$, then $K_s = K_t$. In particular, all elements of a fragment have the same 'generation'. We also call this 'generation' the 'generation' of the fragment. Let $(\sigma^{i,k}(\theta), i\in I_k)$ be the family of lengths of fragments in 'generation' k. Notice that I_0 is reduced to one point, say 0, and we write

$$\tilde{\sigma}(\theta) = \sigma^{0,0}(\theta)$$

for the fragment which contains the root. The joint law of $(\tilde{\sigma}(\theta), \sigma)$ is given in Proposition 7.3 in Abraham and Delmas (2005).

Let $(r^{j,k+1}(\theta), j\in J_{k+1})$ be the family of sizes of the marked nodes attached to the snake of 'generation' k. More precisely,

$$\{r^{j,k+1}(\theta),\ j\in J_{k+1}\}=\{\Delta_s,\ T_s\leqslant\theta \text{ and } K_s=k+1\}.$$

We set, for $k\in\mathbb{N}$,

$$L_k(\theta)=\sum_{i\in I_k}\sigma^{i,k}(\theta),\qquad N_k^\varepsilon(\theta)=\sum_{i\in I_k}\mathbf{1}_{\{\sigma^{i,k}(\theta)>\varepsilon\}},\qquad M_k^\varepsilon(\theta)=\sum_{i\in I_k}\sigma^{i,k}(\theta)\mathbf{1}_{\{\sigma^{i,k}\leqslant\varepsilon\}},$$

and, for $k\in\mathbb{N}^*$,

$$R_k(\theta)=\sum_{j\in J_k} r^{i,k}(\theta).$$

We set $R_0=0$. Let us remark that we have $\sigma=\sum_{k\geqslant 0}L_k(\theta)$, $N^\varepsilon(\theta)=\sum_{k\geqslant 0}N_k^\varepsilon(\theta)$, $M^\varepsilon(\theta)=\sum_{k\geqslant 0}M_k^\varepsilon(\theta)$ and $R(\theta)=R_k(\theta)$.

Let $\mathcal{F}_k$ be the σ-field generated by $((\sigma^{i,l}(\theta),\ i\in I_l),\ R_l(\theta))_{0\leqslant l\leqslant k}$. As a consequence of the special Markov property (Theorem 5.2 of Abraham and Delmas 2005) and using the recursive construction of Lemma 8.6 of Abraham and Delmas (2005), we have the following propositions:

Proposition 2.1. *Under $\mathbb{N}$, conditionally on $\mathcal{F}_{k-1}$ and $R_k(\theta)$, $\sum_{i\in I_k}\delta_{\sigma^{i,k}(\theta)}$ is distributed as a Poisson point process with intensity $R_k(\theta)\mathbb{N}[\mathrm{d}\tilde{\sigma}(\theta)]$.*

Proposition 2.2. *Under $\mathbb{N}$, conditionally on $\mathcal{F}_{k-1}$, $\sum_{j\in J_k}\delta_{r^{j,k}(\theta)}$ is distributed as a Poisson point process with intensity $L_{k-1}(\theta)(1-\mathrm{e}^{-\theta r})\,\pi(\mathrm{d}r)$.*

Remark 2.2. These propositions allow the law of the fragmentation $\Lambda(\theta)$ to be computed for a given θ (see the Laplace transform computations in the proof of Lemma 3.1).

Let us recall that the key object in Abraham and Delmas (2005) is the tagged fragment which contains the root. Recall that its size is denoted by $\tilde{\sigma}(\theta)$. This fragment corresponds to the subtree of the initial CRT (after pruning) that contains the root. This subtree is a Lévy CRT and the Laplace exponent of the associated Lévy process is

$$\psi_\theta(\lambda):=\psi(\lambda+\theta)-\psi(\theta),\qquad \lambda\geqslant 0.$$

This implies $\psi_\theta^{-1}(v)=\psi^{-1}(v+\psi(\theta))-\theta$, and we deduce from (4) that

$$\lim_{\lambda\to\infty}\psi_\theta^{-1}(\lambda)/\lambda=0. \tag{6}$$

We also have

$$\mathbb{N}\big[1-\mathrm{e}^{-\beta\tilde{\sigma}(\theta)}\big]=\psi_\theta^{-1}(\beta)\quad\text{and}\quad \mathbb{N}\big[\tilde{\sigma}(\theta)\mathrm{e}^{-\beta\tilde{\sigma}(\theta)}\big]=\frac{1}{\psi_\theta'(\psi_\theta^{-1}(\beta))} \tag{7}$$

(see (1) for the first equality with ψ instead of ψ_θ).

3. Proofs

We fix $\theta > 0$. As θ is fixed, we will omit to mention the dependence with respect to θ of the different quantities in this section: for example, we write $\tilde{\sigma}$ and N^ε for $\tilde{\sigma}(\theta)$ and $N^\varepsilon(\theta)$. We set

$$\mathcal{N}^\varepsilon = N^\varepsilon - \mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} \quad \text{and} \quad \mathcal{M}^\varepsilon = M^\varepsilon - \tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}.$$

3.1. Proof of Theorem 1.1

The poof is in four steps. In the first step we compute the Laplace transform of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R, \sigma)$. From there we could prove the convergence of Theorem 1.1 in probability instead of in L^2. However, we need a convergence speed to get the almost sure convergence in the α-stable case of Proposition 1.2. In the second step, we check that the computed Laplace transform has the necessary regularity in order to derive, in the third step, the second moment of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R)$ under $\mathbb{N}[\mathrm{e}^{-\beta\sigma}\cdot]$. In the last step we check the convergence statement of the second moment.

Step 1. We give the joint law under $\mathbb{N}$ of $(\mathcal{N}^\varepsilon, \mathcal{M}^\varepsilon, R, \sigma)$ by computing for $x > 0$, $y > 0$, $\beta > 0$, $\gamma > 0$,

$$\mathbb{N}[\mathrm{e}^{-(x\mathcal{N}^\varepsilon+y\mathcal{M}^\varepsilon+\gamma R+\beta\sigma)}|\tilde{\sigma}].$$

By monotone convergence, we have

$$\mathbb{N}[\mathrm{e}^{-(x\mathcal{N}^\varepsilon+y\mathcal{M}^\varepsilon+\gamma R+\beta\sigma)}|\tilde{\sigma}] = \lim_{n\to\infty}\mathbb{N}[\mathrm{e}^{-\left(\beta\tilde{\sigma}+\sum_{l=1}^n (xN_l^\varepsilon+yM_l^\varepsilon+\gamma \mathrm{R}_l+\beta L_l)\right)}|\tilde{\sigma}]. \tag{8}$$

We define the function $H_{(x,y,\gamma)}$ by

$$H_{(x,y,\gamma)}(c) = G\big(\gamma + \mathbb{N}\big[1 - \mathrm{e}^{-(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}+c\tilde{\sigma})}\big]\big),$$

where, for $a \geqslant 0$,

$$G(a) = \int \pi(\mathrm{d}r)\big(1-\mathrm{e}^{-\theta r}\big)\big(1-\mathrm{e}^{-ar}\big) = \psi(\theta+a) - \psi(a) - \psi(\theta) = \psi_\theta(a) - \psi(a). \tag{9}$$

Recall that $\mathcal{F}_k$ is the σ-field generated by $((\sigma^{i,l}, i \in I_l), R_l)_{0\leqslant l\leqslant k}$. We then have the following lemma.

Lemma 3.1. *For $x, y, \gamma \in \mathbb{R}_+$, $\varepsilon > 0$, we have, for $k \in \mathbb{N}^*$,*

$$\mathbb{N}[\mathrm{e}^{-(xN_k^\varepsilon+yM_k^\varepsilon+cL_k+\gamma R_k)}|\mathcal{F}_{k-1}] = \mathrm{e}^{-H_{(x,y,\gamma)}(c)L_{k-1}}.$$

***Proof*.** As a consequence of Proposition 2.1, we have

$$\mathbb{N}[\mathrm{e}^{-(xN_k^\varepsilon+yM_k^\varepsilon+cL_k+\gamma R_k)}|\mathcal{F}_{k-1}, R_k] = \mathrm{e}^{-R_k(\gamma+\mathbb{N}[1-\exp(-(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}+c\tilde{\sigma}))])}.$$

As a consequence of Proposition 2.2, we have

$$\mathbb{N}[\mathrm{e}^{-R_k(\gamma+\mathbb{N}[1-\exp(-(x\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leqslant\varepsilon\}}+c\tilde\sigma))])}|\mathcal{F}_{k-1}] = \mathrm{e}^{-H_{(x,y,\gamma)}(c)L_{k-1}}.$$

□

We define the constants $c_{(k)}$ by induction:

$$c_{(0)} = 0, \qquad c_{(k+1)} = H_{(x,y,\gamma)}(c_{(k)} + \beta).$$

An immediate backward induction yields (recall $L_0 = \tilde\sigma$) that, for every integer $n \geqslant 1$, we have

$$\mathbb{N}[\mathrm{e}^{-\left(\sum_{l=1}^n (xN_l^\varepsilon + yM_l^\varepsilon + \gamma R_l + \beta L_l)\right)}|\tilde\sigma] = \mathrm{e}^{-c_{(n)}\tilde\sigma}.$$

Notice that the function G is of class $\mathcal{C}^\infty$ on $(0, \infty)$ and is concave increasing, and that the function

$$c \mapsto \mathbb{N}\left[1 - \mathrm{e}^{-(x\mathbf{1}_{\{\tilde\sigma>\varepsilon\}}+y\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leqslant\varepsilon\}}+(\beta+c)\tilde\sigma)}\right]$$

is of class $\mathcal{C}^\infty$ on $[0, \infty)$ and is concave increasing. This implies that $H_{(x,y,\gamma)}$ is concave increasing and of class $\mathcal{C}^\infty$. Notice that

$$x\mathbf{1}_{\{\tilde\sigma>\varepsilon\}} + y\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leqslant\varepsilon\}} + c\tilde\sigma \leqslant \left(\frac{x}{\varepsilon} + y + c\right)\tilde\sigma.$$

In particular, we have $H_{(x,y,\gamma)}(c) \leqslant G(\gamma + \psi_\theta^{-1}(x/\varepsilon + y + c))$. As $\lim_{a\to\infty} G'(a) = 0$, this implies that $\lim_{a\to\infty} G(a)/a = 0$. Since $\lim_{\lambda\to\infty}\psi_\theta^{-1}(\lambda) = \infty$, we deduce, thanks to (6), that

$$\lim_{c\to\infty} \frac{H_{(x,y,\gamma)}(c)}{c} = 0. \tag{10}$$

For $\gamma > 0$, notice $H_{(x,y,\gamma)}(0) > 0$. As the function $H_{(x,y,\gamma)}$ is increasing and continuous, we deduce that the sequence $(c_{(n)}, n \geqslant 0)$ is increasing and converges to the unique root, say c', of $c = H_{(x,y,\gamma)}(c+\beta)$. And we deduce from (8) that

$$\mathbb{N}[\mathrm{e}^{-(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R + \beta\sigma)}|\tilde\sigma] = \mathrm{e}^{-(\beta+c')\tilde\sigma}. \tag{11}$$

Step 2. We look at the dependency of the root of $c = H_{(x,y,\gamma)}(c+\beta)$ in (x, y, γ).

Let $\varepsilon, x, y, \beta, \gamma \in (0, \infty)$ be fixed. There exists $a > 0$ small enough such that for all $z \in (-a, a)$, we have $z\gamma + \mathbb{N}[1 - \mathrm{e}^{-\beta\tilde\sigma/2}] > 0$ and, for all $\tilde\sigma \geqslant 0$,

$$z(x\mathbf{1}_{\{\tilde\sigma>\varepsilon\}} + y\tilde\sigma\mathbf{1}_{\{\tilde\sigma\leqslant\varepsilon\}}) + \beta\tilde\sigma \geqslant \beta\tilde\sigma/2.$$

We consider the function J defined on $(-\beta/2, \infty) \times (-a, a)$ by

$$J(c, z) = H_{zx,zy,z\gamma}(c+\beta) - c.$$

From the regularity of G, we deduce that the function J is of class $\mathcal{C}^\infty$ on $(-\beta/2, \infty) \times (-a, a)$ and that the function $c \mapsto J(c, z)$ is concave. Notice that $J(0, z) > 0$ for all $z \in (-a, a)$. Together with (10), this implies that there exists a unique solution $c(z)$ to the equation $J(c, z) = 0$ and that $\partial J(c(z), z)/\partial c < 0$ for all $z \in (-a, a)$. The implicit function theorem implies that the function $z \mapsto c(z)$ is of class $\mathcal{C}^\infty$ on $(-a, a)$. In particular, we have $c(z) = c_0 + zc_1 + z^2c_2/2 + o(z^2)$. We deduce from (11) that, for all $z \in [0, a)$,

$$\mathbb{N}[\mathrm{e}^{-z(x\mathcal{N}^\varepsilon+y\mathcal{M}^\varepsilon+\gamma R)-\beta\sigma}|\tilde{\sigma}] = \mathrm{e}^{-(\beta+c(z))\tilde{\sigma}} = \mathrm{e}^{-(\beta+c_0+zc_1+z^2c_2/2+o(z^2))\tilde{\sigma}}.$$

Step 3. We investigate the second moment $\mathbb{N}[(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R)^2\mathrm{e}^{-\beta\sigma}]$. Standard results on Laplace transforms imply that the second moment is finite and

$$\mathbb{N}[(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R)^2\mathrm{e}^{-\beta\sigma}|\tilde{\sigma}] = \mathrm{e}^{-(\beta+c_0)\tilde{\sigma}}(c_1^2\tilde{\sigma} - c_2)\tilde{\sigma}. \tag{12}$$

Next, we compute c_0, c_1 and c_2. By the definition of $c(z)$, we have

$$c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2) = G\Big(z\gamma + \mathbb{N}\Big[1 - \mathrm{e}^{-z(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})-(\beta+c_0+zc_1+z^2c_2/2+o(z^2))\tilde{\sigma}}\Big]\Big).$$

We compute the expansion in z of the right-hand-side term of this equality. We set

$$\begin{aligned}
a_0 &= \mathbb{N}\big[1 - \mathrm{e}^{-(\beta+c_0)\tilde{\sigma})}\big] = \psi_\theta^{-1}(\beta + c_0),\\
a_1 &= \gamma + \mathbb{N}\big[\mathrm{e}^{-(\beta+c_0)\tilde{\sigma}}(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}} + c_1\tilde{\sigma})\big],\\
a_2 &= \mathbb{N}\big[\mathrm{e}^{-(\beta+c_0)\tilde{\sigma}}(c_2\tilde{\sigma} - (x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}} + c_1\tilde{\sigma})^2)\big],
\end{aligned}$$

so that standard results on Laplace transform yield

$$c_0 + zc_1 + \frac{z^2}{2}c_2 + o(z^2) = G\left(a_0 + za_1 + \frac{z^2}{2}a_2 + o(z^2)\right).$$

We deduce that

$$\begin{aligned}
c_0 &= G(a_0) = G\big(\mathbb{N}\big[1 - \mathrm{e}^{-(\beta+c_0)\tilde{\sigma}}\big]\big),\\
c_1 &= a_1G'(a_0), && (13)\\
c_2 &= a_2G'(a_0) + a_1^2G''(a_0) = a_2G'(a_0) + \frac{c_1^2G''(a_0)}{G'(a_0)^2}. && (14)
\end{aligned}$$

Using (9) and (7), we have $c_0 = G(\psi_\theta^{-1}(\beta + c_0)) = \beta + c_0 - \psi(\psi_\theta^{-1}(\beta + c_0))$, that is

$$h_\beta := \beta + c_0 = \psi_\theta(\psi^{-1}(\beta)) \quad \text{and} \quad a_0 = \psi_\theta^{-1}(\beta + c_0) = \psi^{-1}(\beta).$$

Notice that $h_\beta > 0$. And we have, thanks to the second equality of (7),

$$G'(\psi^{-1}(\beta))\mathbb{N}\big[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}\big] = \frac{\psi_\theta'(\psi^{-1}(\beta)) - \psi'(\psi^{-1}(\beta))}{\psi_\theta'(\psi^{-1}(\beta))} < 1.$$

(This last inequality is equivalent to saying that $\partial J(c(z), z)/\partial c < 0$ at $z = 0$.) From (13), we obtain

$$c_1 = G'(\psi^{-1}(\beta))\frac{\gamma + \mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})]}{1 - G'(\psi^{-1}(\beta))\mathbb{N}\big[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}\big]}, \tag{15}$$

and from (14),

$$c_2 = \frac{-G'(\psi^{-1}(\beta))\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}} + c_1\tilde{\sigma})^2] + c_1^2 G''(\psi^{-1}(\beta))/G'(\psi^{-1}(\beta))^2}{1 - G'(\psi^{-1}(\beta))\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}\right]}. \tag{16}$$

We obtain

$$\mathbb{N}\left[(x\mathcal{N}^\varepsilon + y\mathcal{M}^\varepsilon + \gamma R)^2\mathrm{e}^{-\beta\sigma}\right] = c_1^2\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}^2\right] - c_2\mathbb{N}\left[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}\right], \tag{17}$$

where c_1 and c_2 defined by (15) and (16) are polynomials of respective degree 1 and 2 in x, y and γ. In particular, (17) also holds for $x, y, \gamma \in \mathbb{R}$.

Step 4. We look at asymptotics as ε decreases to 0. Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $\gamma = -(\lambda_1 + \lambda_2)$. We set

$$x_\varepsilon = \lambda_1/\bar{\pi}_*(\varepsilon) \quad \text{and} \quad y_\varepsilon = \lambda_2/\varphi(\varepsilon).$$

We recall from Lemma 4.1 in Delmas (2006) that

$$\lim_{\varepsilon\to 0}\frac{1}{\bar{\pi}_*(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{\varepsilon}{\varphi(\varepsilon)} = 0. \tag{18}$$

Lemma 7.2 in Abraham and Delmas (2005) tells that for any non-negative measurable function F, we have $\mathbb{N}[F(\tilde{\sigma})] = \mathbb{N}[\mathrm{e}^{-\theta\sigma}F(\sigma)]$. We define

$$\begin{aligned}\Delta_\varepsilon &:= \gamma + \mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x_\varepsilon\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y_\varepsilon\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})]\\ &= -x_\varepsilon\mathbb{N}[(1-\mathrm{e}^{-(h_\beta+\theta)\sigma})\mathbf{1}_{\{\sigma>\varepsilon\}}] - \varepsilon y_\varepsilon\mathbb{N}[(1-\mathrm{e}^{-(h_\beta+\theta)\sigma})\frac{\sigma}{\varepsilon}\mathbf{1}_{\{\sigma\leqslant\varepsilon\}})].\end{aligned}$$

In particular, we have $\Delta_\varepsilon = O(1/\bar{\pi}_*(\varepsilon) + \varepsilon/\varphi(\varepsilon))$ and, from (18), $\lim_{\varepsilon\to 0}\Delta_\varepsilon = 0$. From (15), we obtain $c_1 = O(\Delta_\varepsilon) = O(1/\bar{\pi}_*(\varepsilon) + \varepsilon/\varphi(\varepsilon))$. From (16), we also have, for some finite constant C independent of ε,

$$|c_2| \leqslant 2\frac{G'(\psi^{-1}(\beta))\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x_\varepsilon\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y_\varepsilon\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})^2]}{1 - G'(\psi^{-1}(\beta))\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}]} + Cc_1^2.$$

Notice that

$$\begin{aligned}\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x_\varepsilon\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y_\varepsilon\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})^2] &= \mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}(x_\varepsilon^2\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}} + y_\varepsilon^2\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}})]\\ &\leqslant \frac{\lambda_1}{\bar{\pi}_*(\varepsilon)}\ x_\varepsilon\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}] + \frac{\lambda_2\varepsilon}{\varphi(\varepsilon)}\ y_\varepsilon\mathbb{N}[\mathrm{e}^{-h_\beta\tilde{\sigma}}\tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}]\\ &= O\left(\frac{1}{\bar{\pi}_*(\varepsilon)} + \frac{\varepsilon}{\varphi(\varepsilon)}\right).\end{aligned} \tag{19}$$

We deduce that $c_2 = O(1/\bar{\pi}_*(\varepsilon) + \varepsilon/\varphi(\varepsilon))$. Equation (17) implies that

$$\mathbb{N}\left[\left(\lambda_1\frac{\mathcal{N}^\varepsilon}{\bar{\pi}_*(\varepsilon)} + \lambda_2\frac{\mathcal{M}^\varepsilon}{\varphi(\varepsilon)} - (\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right] = O\left(\frac{1}{\bar{\pi}_*(\varepsilon)} + \frac{\varepsilon}{\varphi(\varepsilon)}\right).$$

As $\sigma \geqslant \tilde{\sigma}$, we have that

$$\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}\right)\mathrm{e}^{-\beta\sigma}\right]\leqslant\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}\right)\mathrm{e}^{-\beta\tilde{\sigma}}\right]$$
$$=O\left(\frac{1}{\bar{\pi}_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right),$$

where we used (19) for the last equation (with β instead of h_β). Recall that $N^\varepsilon(\theta) = \mathcal{N}^\varepsilon + \mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}$ and $M^\varepsilon(\theta) = \mathcal{M}^\varepsilon + \tilde{\sigma}\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}$ and thus

$$\mathbb{N}\left[\left(\lambda_1\frac{N^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{M^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]$$
$$\leqslant 2\mathbb{N}\left[\left(\lambda_1\frac{\mathcal{N}^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{\mathcal{M}^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]$$
$$+2(\lambda_1^2+\lambda_2^2)\mathbb{N}\left[\left(\frac{1}{\bar{\pi}_*(\varepsilon)^2}\mathbf{1}_{\{\tilde{\sigma}>\varepsilon\}}+\frac{1}{\varphi(\varepsilon)^2}\tilde{\sigma}^2\mathbf{1}_{\{\tilde{\sigma}\leqslant\varepsilon\}}\right)\mathrm{e}^{-\beta\sigma}\right].$$

We deduce that

$$\mathbb{N}\left[\left(\lambda_1\frac{N^\varepsilon}{\bar{\pi}_*(\varepsilon)}+\lambda_2\frac{M^\varepsilon}{\varphi(\varepsilon)}-(\lambda_1+\lambda_2)R\right)^2\mathrm{e}^{-\beta\sigma}\right]=O\left(\frac{1}{\bar{\pi}_*(\varepsilon)}+\frac{\varepsilon}{\varphi(\varepsilon)}\right). \tag{20}$$

which, thanks to (18), says precisely that $\lim_{\varepsilon\to 0}(N^\varepsilon(\theta)/\bar{\pi}_*(\varepsilon)) = \lim_{\varepsilon\to 0}(M^\varepsilon(\theta)/\varphi(\varepsilon)) = R$ in $L^2(\mathbb{N}[\mathrm{e}^{-\beta\sigma}\cdot])$. □

4. Law of $R(\theta)$

Lemma 4.1. *Let $\beta \geqslant 0$, $\gamma \leqslant 0$. We have*

$$\mathbb{N}\left[1-\mathrm{e}^{-\beta\sigma-\gamma R(\theta)}\right]=v,$$

where v is the unique non-negative root of

$$\beta+\psi(\gamma+\theta+v)=\psi(v+\theta)+\psi(v+\gamma). \tag{21}$$

Remark 4.1. For the limit case $\psi(\lambda)=\lambda^2$ (which is excluded here), we obtain that the unique non-negative root of (21) is $v=\sqrt{\lambda+2\gamma\theta}$. This would imply $R(\theta)=2\theta\sigma$ $\mathbb{N}$-almost everywhere and $R(\theta)=2\theta$ $\mathbb{N}_1$-almost surely. This agrees with the result in Bertoin (2004), where the limit which appears for (2) is almost surely equal to 2θ.

Proof. Take $x=y=0$ in (11), integrate with respect to $\mathbb{N}$ and use (7) to obtain

$$\mathbb{N}\left[1-\mathrm{e}^{-\gamma R(\theta)-\beta\sigma}\right]=\mathbb{N}\left[1-\mathrm{e}^{-(\beta+c)\tilde{\sigma}(\theta)}\right]=\psi_\theta^{-1}(\beta+c),$$

where c is the unique root of $c=H_{(0,0,\gamma)}(c)$, that is, of $c=G(\gamma+\psi_\theta^{-1}(\beta+c))$. If we set

$v = \psi_\theta^{-1}(\beta + c)$, we have that v is the unique non-negative root of the equation $G(\gamma + v) = \psi_\theta(v) - \beta$, that is, (21). □

5. Computations for Remark 1.2

Let $\alpha \in (1, 2)$. Recall from Abraham and Delmas (2005: Corollary 9.3) or Miermont (2005) that the fragmentation is self-similar with index $1/\alpha$ and dislocation measure given by

$$\int_{\mathcal{S}^\downarrow} F(x)\nu_1(\mathrm{d}x) = \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)}\mathbb{E}\left[S_1\ F\left(\left(\frac{\Delta S_t}{S_1},\ t \leqslant 1\right)\right)\right],$$

where F is any non-negative measurable function on $\mathcal{S}^\downarrow$, and $(\Delta S_t,\ t \geqslant 0)$ are the jumps of a stable subordinator $S = (S_t,\ t \geqslant 0)$ of Laplace exponent $\psi^{-1}(\lambda) = \lambda^{1/\alpha}$, ranked by decreasing size.

In this section we shall compute the functions f_b, φ_b and g_b defined in Bertoin (2004) and recalled in Remark 1.2 for the self-similar fragmentation at nodes.

Lemma 5.1. *We have*

$$f_b(\varepsilon) = \frac{1}{\Gamma(1+1/\alpha)}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha}.$$

Proof. The Lévy measure of S is given by

$$\pi_*(\mathrm{d}r) = \frac{1}{\alpha\Gamma(1-1/\alpha)}\frac{\mathrm{d}r}{r^{1+1/\alpha}}\,\mathrm{d}r.$$

For $\beta \in (0, 1)$, we have

$$\int_{(0,\infty)}\frac{\mathrm{d}y}{y^{1+\beta}}(1-\mathrm{e}^{-y\lambda}) = \lambda^\beta\frac{\Gamma(1-\beta)}{\beta}.$$

We deduce that

$$\mathbb{E}[S_1^\beta] = \frac{\beta}{\Gamma(1-\beta)}\mathbb{E}\left[\int_0^\infty\frac{\mathrm{d}y}{y^{1+\beta}}(1-\mathrm{e}^{-yS_1})\right] = \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty\frac{\mathrm{d}y}{y^{1+\beta}}(1-\mathrm{e}^{-y^{1/\alpha}}) = \frac{\Gamma(1-\alpha\beta)}{\Gamma(1-\beta)}.$$

Standard computations for Poisson measure yield

$$
\begin{aligned}
f_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} x_i \mathbf{1}_{\{x_i<\varepsilon\}} \nu_1(\mathrm{d}x) \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \sum_{t\leqslant 1} \frac{\Delta S_t}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\sum_{t\leqslant 1} \Delta S_t \mathbf{1}_{\{\Delta S_t<\varepsilon(S_1-\Delta S_t)/(1-\varepsilon)\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\int \pi_*(\mathrm{d}r) r \mathbf{1}_{\{r<\varepsilon S_1/(1-\varepsilon)\}} \right] \\
&= \frac{\alpha}{\Gamma(2-\alpha)} \mathbb{E}[S_1^{1-1/\alpha}] \left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha} \\
&= \frac{1}{\Gamma(1+1/\alpha)} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{1-1/\alpha}.
\end{aligned}
$$

□

Lemma 5.2. *We have*

$$
\lim_{\varepsilon\to 0} \varepsilon^{1/\alpha} \varphi_b(\varepsilon) = \frac{\alpha-1}{\Gamma(1+1/\alpha)}.
$$

Proof. We have

$$
\begin{aligned}
\varphi_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow} \left(\sum_{i=1}^{\infty} \mathbf{1}_{\{x_i>\varepsilon\}} - 1 \right) \nu_1(\mathrm{d}x) \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \sum_{t\leqslant 1} \mathbf{1}_{\{\Delta S_t>\varepsilon S_1\}} - S_1 \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[\sum_{t\leqslant 1} (S_1-\Delta S_t) \mathbf{1}_{\{\Delta S_t>\varepsilon(S_1-\Delta S_t)/(1-\varepsilon)\}} - \Delta S_t \mathbf{1}_{\{\Delta S_t\leqslant\varepsilon(S_1-\Delta S_t)/(1-\varepsilon)\}} \right] \\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 \int \pi_*(\mathrm{d}r) \mathbf{1}_{\{r>\varepsilon S_1/(1-\varepsilon)\}} \right] - f_b(\varepsilon) \\
&= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1^{1-1/\alpha} \right] \left(\frac{\varepsilon}{1-\varepsilon}\right)^{-1/\alpha} - f_b(\varepsilon) \\
&= \frac{\alpha-1}{\Gamma(1+1/\alpha)} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{-1/\alpha} - f_b(\varepsilon).
\end{aligned}
$$

□

Lemma 5.3. *The limit* $\lim_{\varepsilon\to 0}(g_b(\varepsilon)/f_b(\varepsilon)^2)$ *exists and belongs to* $(0,\infty)$.

Proof. We have

$$
\begin{aligned}
g_b(\varepsilon) &= \int_{\mathcal{S}^\downarrow}\left(\sum_{i=1}^{\infty} x_i \mathbf{1}_{\{x_i<\varepsilon\}}\right)^2 \nu_1(\mathrm{d}x)\\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)}\mathbb{E}\left[S_1\left(\sum_{t\leqslant 1}\frac{\Delta S_t}{S_1}\mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}}\right)^2\right]\\
&= \frac{\alpha(\alpha-1)\Gamma(1-1/\alpha)}{\Gamma(2-\alpha)}\left(\mathbb{E}\left[\sum_{t\leqslant 1}\frac{(\Delta S_t)^2}{S_1}\mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}}\right]\right.\\
&\quad\left.+\,\mathbb{E}\left[\sum_{t\leqslant 1, s\leqslant 1, s\neq t}\frac{\Delta S_t \Delta S_s}{S_1}\mathbf{1}_{\{\Delta S_t<\varepsilon S_1, \Delta S_s<\varepsilon S_1\}}\right]\right).
\end{aligned}
$$

For the first term, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sum_{t\leqslant 1}\frac{(\Delta S_t)^2}{S_1}\mathbf{1}_{\{\Delta S_t<\varepsilon S_1\}}\right] &\leqslant \mathbb{E}\left[\sum_{t\leqslant 1}\frac{(\Delta S_t)^2}{S_1-\Delta S_t}\mathbf{1}_{\{\Delta S_t<\varepsilon(S_1-\Delta S_t)/(1-\varepsilon)\}}\right]\\
&= \mathbb{E}\left[\frac{1}{S_1}\int \pi_*(\mathrm{d}r) r^2 \mathbf{1}_{\{r<\varepsilon S_1/(1-\varepsilon)\}}\right]\\
&= \frac{1}{(2\alpha-1)\Gamma(1-1/\alpha)}\mathbb{E}\left[S_1^{1-1/\alpha}\right]\left(\frac{\varepsilon}{1-\varepsilon}\right)^{2-1/\alpha}\\
&= o(\varepsilon^{2-2/\alpha}).
\end{aligned}
$$

For the second term we notice that, for $r, s, S \in \mathbb{R}_+$,

$$
\begin{aligned}
\{r \leqslant \varepsilon S/(1-\varepsilon), v \leqslant \varepsilon S/(1-\varepsilon)\} &\subset \{r \leqslant \varepsilon(S+r+v), v\leqslant \varepsilon(S+r+v)\}\\
&\subset \{r\leqslant \varepsilon S/(1-2\varepsilon), v \leqslant \varepsilon S/(1-2\varepsilon)\},
\end{aligned}
$$

and we obtain

$$\mathbb{E}\left[\sum_{t\leqslant 1, s\leqslant 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1, \Delta S_s<\varepsilon S_1\}}\right]$$
$$\leqslant \mathbb{E}\left[\sum_{t\leqslant 1, s\leqslant 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1-\Delta S_t-\Delta S_s} \mathbf{1}_{\{\Delta S_t<\varepsilon(S_1-\Delta S_t-\Delta S_s)/(1-2\varepsilon), \Delta S_s<\varepsilon(S_1-\Delta S_t-\Delta S_s)/(1-2\varepsilon)\}}\right]$$
$$= \mathbb{E}\left[\frac{1}{S_1}\left(\int \pi_*(\mathrm{d}r) r \mathbf{1}_{\{r<\varepsilon S_1/(1-2\varepsilon)\}}\right)^2\right]$$
$$= c_\alpha \left(\frac{\varepsilon}{1-2\varepsilon}\right)^{2-2/\alpha},$$

with

$$c_\alpha = \frac{\Gamma(3-\alpha)}{(\alpha-1)^2\Gamma(2/\alpha)\Gamma(1-1/\alpha)^2},$$

as well as

$$\mathbb{E}\left[\sum_{t\leqslant 1, s\leqslant 1, s\neq t} \frac{\Delta S_t \Delta S_s}{S_1} \mathbf{1}_{\{\Delta S_t<\varepsilon S_1, \Delta S_s<\varepsilon S_1\}}\right]$$
$$\geqslant \mathbb{E}\left[\sum_{t\leqslant 1, s\leqslant 1, s\neq t} \frac{\Delta S_t \Delta S_s}{(S_1-\Delta S_t-\Delta S_s)(1+2\varepsilon)/(1-\varepsilon)} \mathbf{1}_{\{\Delta S_t<\varepsilon(S_1-\Delta S_t-\Delta S_s)/(1-\varepsilon), \Delta S_s<\varepsilon(S_1-\Delta S_t-\Delta S_s)/(1-\varepsilon)\}}\right]$$
$$= \frac{1-\varepsilon}{1+2\varepsilon}\mathbb{E}\left[\frac{1}{S_1}\left(\int \pi_*(\mathrm{d}r) r 1_{r<\varepsilon S_1/(1-\varepsilon)\}}\right)^2\right]$$
$$= c_\alpha \frac{1-\varepsilon}{1+2\varepsilon}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{2-2/\alpha}.$$

In particular, we have that $g_b(\varepsilon) = c_\alpha \varepsilon^{2-2/\alpha}(1+o(1))$. We deduce that $\lim_{\varepsilon\to 0}(g_b(\varepsilon)/f_b(\varepsilon)^2) \in (0, \infty)$. □

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