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Limiting distributions of the non-central *t*-statistic and their applications to the power of *t*-tests under non-normality

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Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. Let X be an independent copy of X_1. Define $\mathbb{T}_n = \sqrt{n}\overline{X}/\mathrm{S}$, where $\overline{X}$ and S^2 are the sample mean and the sample variance, respectively. We refer to $\mathbb{T}_n$ as the central or non-central (Student's) t-statistic, depending on whether $\mathrm{E}X = 0$ or $\mathrm{E}X \neq 0$, respectively. The non-central t-statistic arises naturally in the calculation of powers for t-tests. The central t-statistic has been well studied, while there is a very limited literature on the non-central t-statistic. In this paper, we attempt to narrow this gap by studying the limiting behaviour of the non-central t-statistic, which turns out to be quite complicated. For instance, it is well known that, under finite second-moment conditions, the limiting distributions for the central t-statistic are normal while those for the non-central t-statistic can be non-normal and can critically depend on whether or not $\mathrm{E}X^4 = \infty$. As an application, we study the effect of non-normality on the performance of the t-test.

Keywords: domain of attraction; limit theorems; non-central t-statistic; power of t-test

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) non-degenerate random variables with a common mean, $\mu = \mathrm{E}(X_1)$, and variance, $\sigma^2 = \mathrm{E}(X_1 - \mu)^2$. Let X be an independent copy of X_1. Student's t-statistic is defined as

$$\mathbb{T}_n = \frac{\sqrt{n}\overline{X}}{S_n},$$

where $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $S_n^2 := (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. We call the statistic $\mathbb{T}_n$ *non-central* if $\mu \neq 0$, and *central* if $\mu = 0$.

There is a very extensive literature on the central t-statistic, and its limiting behaviour is now very well understood. See, for example, Logan *et al.* (1973), Chistyakov and Götze (2004), Giné *et al.* (1997) and Griffin (2002) for weak convergence, Bentkus and Götze (1996) and Bentkus *et al.* (1996) for Berry–Esséen bounds, Shao (1997, 1999) for large deviation, Wang and Jing (1999) and Jing *et al.* (2003) for non-uniform exponential Berry–Esséen bounds, and Jing *et al.* (2004) for saddlepoint approximation.

In contrast to the central t-statistic, there have been few studies on the limiting behaviour of the non-central t-statistic. This is somewhat surprising, considering the wide applications of the non-central t-statistic in statistical inference. See, for example, Walsh (1948), Scheuer and Spurgeon (1963), Owen (1965), Bagui (1993), Akahira (1995) and Thabane and Drekic (2003). One of the purposes of this paper is to investigate the limiting behaviour of the non-central t-statistic and to give a systematic description of its limiting distribution. It turns out that there are interesting and unexpected phenomena associated with the non-central t-statistic. For instance, by only assuming that $\mathrm{E}X^2 < \infty$, the limiting distribution of the non-central t-statistic can be non-normal while those of the central t-statistic are known to be asymptotically normal. In fact, the limiting behaviour of the non-central t-statistic critically depends on whether or not $\mathrm{E}X^4 = \infty$: if $\mathrm{E}X^4 < \infty$, the limit can be normal or a square of normal; if $\mathrm{E}X^4 = \infty$, the limit is related to stable distributions.

The non-central t-statistic arises naturally in calculating powers of Student's t-test for a location shift in the population mean. These studies are usually based on the assumption that the sample comes from a normal distribution. See, for example, Neyman (1935), Neyman and Tokarska (1936) and Johnson and Welch (1940). However, the normality assumption may not always be reasonable in practice. It is therefore of great interest to study the effect of non-normality on the power function of Student's t-test.

The rest of this paper is organized as follows. In Section 2, we provide limiting distributions for the non-central t-statistic. These results are then used in Section 3 to study the powers of t-tests under non-normality. All proofs are presented in Section 4.

2. Limiting distributions of the non-central t-statistic

In this section, we describe the domains of attraction for the non-central Student's t-statistic by providing a list of possible limiting distributions and sufficient (and perhaps necessary) conditions for X to belong to a domain of attraction. The limiting distributions for the non-central t-statistic will be presented in Sections 2.1 and 2.2 for the two cases of $\mathrm{E}X^2 < \infty$ and $\mathrm{E}X^2 = \infty$, respectively. In Section 2.3, we study the limiting distributions when $\mu = \mu_n$ depends on n.

Throughout this paper, we assume that $\mathrm{E}|X| < \infty$ and $\mu \neq 0$. We define $\nu = \mu/\sigma$. Let Z_τ follow a stable law with an index $\tau \in (0, 2]$ whose characteristic function is given by

$$\mathrm{E}\exp(\mathrm{i}tZ_\tau) = \begin{cases} \exp\{-t^2/2, & \text{if } \tau = 2, \\ \exp\{-|t|^\tau(1 - \mathrm{i}(\operatorname{sign} t)\tan(\pi\tau/2))\}, & \text{if } \tau \in (1, 2) \cup (0, 1), \\ \exp\{-|t|(1 + 2(\mathrm{i}/\pi)(\operatorname{sign} t)\ln|t|)\}, & \text{if } \tau = 1. \end{cases}$$

When $\tau = 2$, we write $Z = Z_2$, which is a standard normal random variable. We write $X \in \mathrm{DA}(\tau)$ if X belongs to the domain of attraction of the stable law of Z_τ.

2.1. Limiting distributions when $\mathrm{E}X^2 < \infty$

When $\mathrm{E}X^2 < \infty$, the asymptotic behaviour of the non-central t-statistic turns out to be quite different for the following two cases: depending on whether (a) $\mathrm{E}X^4 < \infty$ or (b) $\mathrm{E}X^4 = \infty$. For instance, $\mathbb{T}_n$ is attracted to the normal distribution (or its square) in case (a) and to other stable distributions in case (b). All of them have very different convergence rates, as given below.

Case (a): $\mathrm{E}X^4 < \infty$. We write $Y \in \mathcal{B}(p, 0, 1)$ $(0 < p < 1)$ if Y is a standardized Bernoulli random variable and write $X \in \mathcal{B}(p, \mu, \sigma^2)$ if $X = \sigma Y + \mu$, where $Y \in \mathcal{B}(p, 0, 1)$. It is easy to see that $Y \in \mathcal{B}(p, 0, 1)$ if and only if

$$P\left\{Y = -\frac{p}{\sqrt{pq}}\right\} = q, \qquad P\left\{Y = \frac{q}{\sqrt{pq}}\right\} = p,$$

where $q = 1 - p$.

Our first theorem shows that if $\mathrm{E}X^4 < \infty$, the limiting distributions of $\mathbb{T}_n$ are related to the normal distribution.

Theorem 2.1. *Assume that* $\mathrm{E}X^4 < \infty$.

(i) *For* $X \sim \mathcal{B}(p, \mu, \sigma^2)$ *such that* $\mu/\sigma = 2\sqrt{pq}/(q-p)$ and $p \neq \frac{1}{2}$, *we have*

$$a_n(\mathbb{T}_n - \nu_0\sqrt{n-1}) \xrightarrow{\mathcal{D}} N^2(0, 1),$$

where $\nu_0 = 2\sqrt{pq}/(q-p)$ *and* $a_n = 2\nu_0\sqrt{n}/(1+\nu_0^2)\}$.

(ii) *For any random variable* X *other than the one given in* (i), *we have*

$$\sigma_0^{-1}(\mathbb{T}_n - \nu\sqrt{n}) \xrightarrow{\mathcal{D}} N(0, 1),$$

where $\sigma_0^2 = 1 - \nu\alpha_3 + \nu^2(\alpha_4 - 1)/4$, *and* $\alpha_k = \mathrm{E}(X-\mu)^k/\sigma^k$, $k = 3, 4$.

Remark 2.1. The results of Theorem 2.1 can be rewritten as

$$\mathbb{T}_n \overset{\mathcal{D}}{\approx} \begin{cases} \nu_0\sqrt{n-1} + \dfrac{1+\nu_0^2}{2\nu_0\sqrt{n}} Z^2, & \text{in case (i)}, \\ \nu\sqrt{n} + \sigma_0 Z, & \text{in case (ii)}. \end{cases}$$

In case (ii), we note that $\sigma_0^2 = 1$ if $X = \mu + \sigma Y$, where $P\{Y = \pm 1\} = \frac{1}{2}$. If $X \sim N(\mu, \sigma^2)$, we have $\sigma_0^2 = 1 + \nu^2/2 > 1$. In general, we have $0 \leqslant \sigma_0^2 < \infty$, where $\sigma_0^2 = 0$ if and only if $X \sim \mathcal{B}(p, \mu, \sigma^2)$ for $p \neq \frac{1}{2}$ and $\mu/\sigma = 2\sqrt{pq}/(q-p)$.

Remark 2.2. Note that, for given $\nu = \mu/\sigma$, the norming sequence a_n in Theorem 2.1(i) has the same sign (not necessarily positive) as that of ν_0 for all n. In order to maintain our convention, $a_n > 0$, one can replace the result by $|a_n|(T - \nu_0\sqrt{n}) \xrightarrow{\mathcal{D}} \mathrm{sign}(\nu_0)N^2(0, 1)$.

Case (b): $\mathrm{E}X^4 = \infty$. First, we standardize $\{X, X_k, k \geqslant 1\}$ by $Y = (X - \mu)/\sigma$ and $Y_k = (X_k - \mu)/\sigma$ for $k \geqslant 1$. Assume that $X^2 \in \mathrm{DA}(\tau)$ with $\tau \in [1, 2]$. If $\tau \in (1, 2]$, then $Y^2 \in DA(\tau)$ and

$$c_n n^{-1/\tau}(Y_1^2 + \ldots + Y_n^2 - n) \xrightarrow{\mathcal{D}} Z_\tau, \qquad \text{as } n \to \infty, \tag{2.1}$$

where c_n is a slowly varying sequence (see Feller 1971; or Ibragimov and Linnik 1971). In the case $\{\tau = 1\}$, we have

$$c_n n^{-1}(Y_1^2 + \ldots + Y_n^2) - d_n \xrightarrow{\mathcal{D}} Z_1, \qquad \text{as } n \to \infty, \tag{2.2}$$

with slowly varying $c_n > 0$. The centring constants d_n satisfy $d_n \to \infty$.

Theorem 2.2. *Assume that* $\mu \neq 0$, $\mathrm{E}X^2 < \infty$, *and* $\mathrm{E}X^4 = \infty$. *Further assume that* $X^2 \in DA(\tau)$ *for* $1 \leqslant \tau \leqslant 2$ *and that* $(d_n - c_n)^2 = o(c_n)$ *when* $\tau = 1$. *Define*

$$a_n = 2c_n n^{1/2 - 1/\tau}/\nu,$$
$$b_n = \begin{cases} \nu\sqrt{n}, & \text{if } 1 < \tau \leqslant 2, \\ \nu\sqrt{n}(3/2 - d_n/(2c_n)), & \text{if } \tau = 1. \end{cases} \tag{2.3}$$

(i) *If* $\tau = 2$, *then*

$$a_n(\mathbb{T}_n - b_n) \xrightarrow{\mathcal{D}} Z.$$

(ii) *If* $\tau \in [1, 2)$, *then*

$$a_n(\mathbb{T}_n - b_n) \xrightarrow{\mathcal{D}} -Z_\tau.$$

Again, the non-central t-statistics can be attracted to non-normal (i.e. stable) as well as normal distributions (cf. Theorem 2.1).

Remark 2.3. When $Y^2 \in DA(1)$, then there is a slowly varying function $\ell(x)$ such that

$$P(Y^2 > x) = \frac{1}{x\ell(x)}, \qquad \text{for } x > 1;$$

see Feller (1971: 574–580), for instance. One can verify that condition $(d_n - c_n)^2 = o(c_n)$ is satisfied if $\int_x^\infty 1/(t\ell(t))\mathrm{d}t = o(1/\ell^{1/2}(x))$ as $x \to \infty$; see the end of Section 4 for a proof. This remark also applies to (2.6) in Theorem 2.5 below.

Remark 2.4. We believe that the results in this paper can be generalized to certain dependent cases, such as martingale and mixing sequences. However, the sample standard deviation S_n needs to be replaced by an estimator of $(1/n)\mathrm{var}(S_n)$. We refer to Davis and Mikosch (1998) and Mikosch and Straumann (2006) for stable limits of dependent sequences.

2.2. Limiting distributions when $\mathrm{E}X^2 = \infty$

If $\mathrm{E}X^2 = \infty$, we redefine the random variables Y and Y_k as $Y = X - \mu$ and $Y_k = X_k - \mu$ for $k \geqslant 1$. If $Y^2 \in DA(\tau)$ with $\tau \in (0, 1]$, then

$$c_n n^{-1/\tau}(Y_1^2 + \ldots + Y_n^2) - d_n \xrightarrow{\mathcal{D}} Z_\tau, \qquad \text{as } n \to \infty, \tag{2.4}$$

where c_n is a slowly varying sequence, and $d_n = 0$ if $\tau \in (0, 1)$ and $d_n \to \infty$ if $\tau = 1$.

Theorem 2.3. *Assume that* $\mathrm{E}|X| < \infty$ *and* $\mathrm{E}X^2 = \infty$. *Assume that* $Y^2 \in DA(\tau)$ *with* $\tau \in [\frac{1}{2}, 1]$. *If* $\tau = \frac{1}{2}$, *we further assume that the Feller's condition holds, that is,*

$$\limsup_{x\to\infty} \frac{x|\mathrm{E}YI\{|Y| \leqslant x\}|}{x^2 P(|Y| > x) + \mathrm{E}Y^2 I\{|Y| \leqslant x\}} < \infty. \tag{2.5}$$

Then,

$$a_n(\mathbb{T}_n - b_n) \xrightarrow{\mathcal{D}} \begin{cases} -Z_\tau, & \text{if } \tau = 1, \\ 1/\sqrt{Z_\tau}, & \text{if } \tau \in [\frac{1}{2}, 1), \end{cases}$$

where

$$a_n = \frac{2d_n^{3/2}}{\mu\sqrt{nc_n}}, \qquad b_n = \frac{\mu\sqrt{nc_n}}{\sqrt{d_n}}, \qquad \text{if } \tau = 1,$$

$$a_n = \frac{n^{-1+1/(2\tau)}}{\mu\sqrt{c_n}}, \quad b_n = 0, \qquad \text{if } \tau \in [\tfrac{1}{2}, 1).$$

Remark 2.5. *We observe that, to have a limiting distribution for the central* $\mathbb{T}_n$, *we need* $X \in DA(\tau)$ *with* $\tau \in (\frac{1}{2}, 1]$, *whereas for the non-central* $\mathbb{T}_n$ *a weaker assumption that* X^2 *is in a domain of attraction is sufficient.*

2.3. Limiting distributions when $\mu = \mu_n \to 0$

Here, we study the limiting distributions of the non-central t-statistic when $\mu = \mu_n$ depends on n. Theorem 2.4 below provides a sufficient condition for the convergence $\mathbb{T}_n - \nu_n\sqrt{n} \xrightarrow{\mathcal{D}} N(0, 1)$, where $\nu_n = \mu_n/\sigma$.

Let $\{Y, Y_i, i \geqslant 1\}$ be i.i.d. random variables with $\mathrm{E}Y = 0$ and $\mathrm{E}Y^2 = 1$, and let $X_i := X_{n,i} = \sigma Y_i + \mu_n$.

Theorem 2.4. *Assume that* $\mathrm{E}(Y^2)^\tau < \infty$ *for some* $1 \leqslant \tau \leqslant 2$. *Then*

$$\mathbb{T}_n - \nu_n\sqrt{n} \xrightarrow{\mathcal{D}} N(0, 1),$$

provided that $\mu_n \to 0$ *if* $\tau = 2$ *and* $\sup_n \mu_n n^{1/\tau - 1/2} < \infty$ *if* $1 \leqslant \tau < 2$.

The next theorem specifies the limiting distribution of $\mathbb{T}_n$ when $\mathrm{E}Y^4 = \infty$ and Y^2 is in the domain of attraction of a stable law with an index $\tau \in [1, 2]$. The limit may be a standard normal distribution, a stable distribution, or a mixture of normal and stable distributions.

Theorem 2.5. *Assume that* $\mathrm{E}(Y^4) = \infty$ *and* $Y^2 \in DA(\tau)$ *with index* $1 \leqslant \tau \leqslant 2$. *Let* c_n *and* d_n *be the sequences defined in* (2.1) *and* (2.2). *Put* $w_n = \nu_n n^{1/\tau - 1/2}/(2c_n)$. *Assume that* $\lim_{n\to\infty} w_n = w \in [0, \infty]$ *and that*

$$\begin{aligned} w_n(d_n - c_n)^2 &= o(c_n), && \text{if } w = 0 \text{ and } \tau = 1, \\ (d_n - c_n)^2 &= o(c_n), && \text{if } 0 < w \leqslant \infty \text{ and } \tau = 1. \end{aligned} \tag{2.6}$$

Write

$$a_n = \begin{cases} 1, & \text{if } w < \infty, \\ w_n^{-1}, & \text{if } w = \infty; \end{cases} \tag{2.7}$$

and

$$b_n = \begin{cases} \nu_n \sqrt{n}, & \text{if } 1 < \tau \leqslant 2, \\ \nu_n \sqrt{n}(1.5 - d_n/(2c_n)), & \text{if } \tau = 1. \end{cases} \tag{2.8}$$

Then

$$a_n(\mathbb{T}_n - b_n) \xrightarrow{\mathcal{D}} \begin{cases} Z, & \text{if } w = 0, \\ Z - wZ_\tau, & \text{if } 0 < w < \infty, \\ -Z_\tau, & \text{if } w = \infty, \end{cases} \tag{2.9}$$

where $Z \sim N(0, 1)$ *and is independent of* Z_τ.

3. Asymptotic powers and robustness of t-tests under non-normality

In this section, we shall apply the results obtained earlier to study the asymptotic powers of Student's t-test. Some performance criteria for the t-test are introduced in Section 3.1. In Section 3.2 we present asymptotic powers of the t-test under different situations.

3.1. Evaluation criteria for the t-test

Given a random sample, $\{X_1, \ldots, X_n\}$, from a population, $\mathrm{F}(\cdot)$, with $\mu = \mathrm{E}X_1$ and $\sigma^2 = \mathrm{var}(X_1) \in (0, \infty)$, we wish to test

$$H_0 : \mu = 0 \quad vs. \quad H_1 : \mu = \mu_n (\mu_n > 0).$$

(This is equivalent to testing $H_0 : \mu = 0$ versus $H_1 : \mu > 0$.) The most commonly used test is based on Student's t-statistic

$$\mathbb{T}_n = \frac{\sqrt{n}\,\overline{X}}{S_n},$$

where $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. The test rejects H_0 in favour of H_1 if $\mathbb{T}_n$ is large, $\mathbb{T}_n > C_n$, where C_n is the critical value. The *size* and the *power* of the test are given, respectively, by

$$\alpha_n := \alpha_n(C_n) = P_{H_0}(\mathbb{T}_n > C_n),$$

$$\mathcal{P}_n := \mathcal{P}_n(C_n, \mu_n) := 1 - \beta_n = P_{H_1}(\mathbb{T}_n > C_n).$$

Finally, we denote the asymptotic power of the test (if the limit exists) to be

$$\mathcal{P} := \lim_{n\to\infty} \mathcal{P}_n(C_n, \mu_n).$$

Assume first that the sample $\{X_1, \ldots, X_n\}$ comes from a normal distribution. Under H_0, $\mathbb{T}_n$ follows a t-distribution with $n-1$ degrees of freedom; under H_1, $\mathbb{T}_n$ has a non-central t-distribution with $n-1$ degrees of freedom and a non-centrality parameter, $\mu_n\sqrt{n}/\sigma$. The availability of tables of non-central t-distributions (see Bagui 1993) makes Student's t-test particularly easy to use in practice.

However, the normality assumption on the sample may not always be reasonable in practice. It is therefore of great interest to study the effect of non-normality on the power function of Student's t-test. Unfortunately, there are few papers on this issue. Srivastava (1958) investigated the effect of non-normality on the power of the t-test by assuming that the density function of the population can be represented by the first four terms in an Edgeworth series and concluded that 'the power of the t-test is not seriously invalidated even if the samples are from considerably non-normal populations'. However, given the limited scope and the restricted assumptions of that paper, the above conclusion may be questionable and certainly deserves a fresh look.

In order to investigate the performance of $\mathbb{T}_n$ under non-normality, we need to establish some performance criteria to evaluate it under different conditions. To do this, let us look at the various performance criteria that have been used in the literature to compare asymptotic relative efficiencies of two estimators (see Table 1; or Serfling 1981: 315). All these criteria require specifications regarding: (a) $\alpha = \lim_n \alpha_n$; (b) $\beta = \lim_n \beta_n$; (c) an alternative value, μ_n, depending on n or not.

The key observation is that *exponential moment conditions* have been assumed in all these approaches, except for Pitman's approach, where only a finite second-moment condition is assumed. Exponential moment conditions are clearly too strong in practice, which limits the usefulness of those criteria as shown in Table 1. Therefore, it is necessary to modify the performance criteria in Table 1 to evaluate the performance of $\mathbb{T}_n$ under *different* moment conditions. We are particularly interested in the effect of the tail probability on the performance of $\mathbb{T}_n$. For convenience, we shall focus our attention on those cases where the type II error probability approaches some positive limit, $\beta_n \to \beta > 0$, resulting in the behaviours described in Table 2.

It turns out that, in the first two cases in Table 2, where $\alpha_n \to \alpha > 0$ (somewhat related to Pitman's approach), the results are easy and very clear-cut. However, in the last two

Table 1. Different performance criteria

Contributor	Behaviour of α_n	β_n	μ_n
Pitman	$\alpha_n \to \alpha > 0$	$\beta_n \to \beta > 0$	$\mu_n \to 0$
Chernoff	$\alpha_n \to 0$	$\beta_n \to 0$	$\mu_n = \mu$ fixed
Bahadur	$\alpha_n \to 0$	$\beta_n \to \beta > 0$	$\mu_n = \mu$ fixed
Hodges and Lehmann	$\alpha_n \to \alpha > 0$	$\beta_n \to 0$	$\mu_n = \mu$ fixed
Hoeffding	$\alpha_n \to 0$	$\beta_n \to 0$	$\mu_n = \mu$ fixed
Rubin and Sethuraman	$\alpha_n \to 0$	$\beta_n \to 0$	$\mu_n \to 0$

Table 2. Performance criteria when $\beta_n \to \beta > 0$

Cases	Behaviour of α_n	μ_n
(i)	$\alpha_n \to \alpha > 0$	$\mu_n \to 0$
(ii)	$\alpha_n \to \alpha > 0$	$\mu_n = \mu > 0$ fixed
(iii)	$\alpha_n \to 0$	$\mu_n \to 0$
(iv)	$\alpha_n \to 0$	$\mu_n = \mu > 0$ fixed

cases where $\alpha_n \to 0$ (somewhat related to Bahadur's approach), the situations are rather complicated and the limiting behaviour can be quite different from what one might expect.

3.2. Asymptotic powers of the t-test

Here, we shall give some explicit expressions for the asymptotic powers, $\lim_{n\to\infty}\mathcal{P}_n(\mu_n, \alpha_n)$, for the four cases considered in Table 2. We shall see that these expressions are drastically different under different moment conditions. These results can be easily derived from the last section and hence are omitted here.

Cases (i) and (ii): $\alpha_n \to \alpha$ and $\mu_n \to 0$ or μ.

For these two cases, the asymptotic size of the test is non-zero. We have the following result.

Theorem 3.1. *Assume that* $\mathrm{E}X^2 < \infty$. *If* $\alpha_n \to \alpha$, $\mu_n \to 0$ *or* μ, *and* $\sqrt{n}\mu_n/\sigma \to d \in [0, \infty]$, *we have*

$$\mathcal{P} = 1 - \Phi[\Phi^{-1}(1-\alpha) - d],$$

where Φ *is the standard normal distribution function.*

It follows from Theorem 3.1 that

$$\mathcal{P} = \begin{cases} \alpha, & \text{if } d = 0, \\ 1 - \Phi[\Phi^{-1}(1-\alpha) - d], & \text{if } d \in (0, \infty), \\ 1, & \text{if } d = \infty. \end{cases}$$

In particular, the t-test is asymptotically unbiased as $\mathcal{P} \geqslant \alpha$.

Case (iii): $\alpha_n \to 0$ and $\mu_n \to \mu$. When $\alpha_n \to 0$, the situation is quite different from cases (i) and (ii), where $\alpha_n \to \alpha > 0$. The results in this case may be somewhat unexpected and they critically depend on whether $\mathrm{E}X^4 < \infty$ or $\mathrm{E}X^4 = \infty$.

If $\mathrm{E}X^4 < \infty$, the asymptotic power functions are related to normal distributions.

Theorem 3.2. *Assume that* $\mathrm{E}X^4 < \infty$.

(i) *Suppose that* $X \sim \mathcal{B}(p, \mu, \sigma^2)$ *with* $p \neq \frac{1}{2}$ *and* $\mu/\sigma = \nu_0 := 2\sqrt{p - p^2}/(1-2p)$. *If we choose the critical value to be* $C_n = \nu_0\sqrt{n-1} + x/a_n$ *with* $a_n = 2\nu_0\sqrt{n}/(1+\nu_0^2)$, *then*

$$\mathcal{P} = 2\Phi(-\sqrt{x}). \tag{3.1}$$

(ii) *Suppose that* $X \sim \mathcal{B}(p, \mu, \sigma^2)$ *with* $p \neq \frac{1}{2}$. *If we choose the critical value to be* $C_n = \nu\sqrt{n} + x\sigma_0$, *where* $\sigma_0^2 = 1 - \nu\alpha_3 + \nu^2(\alpha_4 - 1)/4$, $\alpha_k = \mathrm{E}(X-\mu)^k/\sigma^k$, $k = 3, 4$, *then*

$$\mathcal{P} = \Phi(-x). \tag{3.2}$$

If $\mathrm{E}X^4 = \infty$, the asymptotic power functions are related to stable distributions.

Theorem 3.3. *Under the assumptions of Theorem 2.2, for the critical value* $C_n = b_n + x/a_n$, *we have*

$$\mathcal{P} = P(Z_\tau \leqslant -x).$$

Case (iv): $\alpha_n \to 0$ *and* $\mu_n \to 0$. As in case (iii), the asymptotic power function also depends critically on whether $\mathrm{E}(X^4) < \infty$ or $\mathrm{E}(X^4) = \infty$. If $\mathrm{E}(X^4) < \infty$, we have the following theorem:

Theorem 3.4. *Assume that* $\mathrm{E}(X^4) < \infty$ *and* $\mu_n \to 0$. *If we choose the critical value to be* $C_n = \mu_n\sqrt{n}/\sigma + x$, *then*

$$\mathcal{P} = \Phi(-x).$$

However, if $\mathrm{E}(X^4) = \infty$, the limiting distributions are quite different.

Theorem 3.5. *Assume that the assumptions of Theorem 2.5 are satisfied. Then, for* $C_n = b_n + x/a_n$, *we have*

$$\mathcal{P} = \begin{cases} P(Z \geqslant x) & \textit{if } w = 0, \\ P(Z - wZ_\tau \geqslant x), & \textit{if } 0 < w < \infty, \\ P(-Z_\tau \geqslant x), & \textit{if } w = \infty, \end{cases}$$

where $Z \sim N(0, 1)$ *is independent of* Z_τ.

4. Proofs

For $\sigma^2 < \infty$ and $n = 1, 2, \ldots,$ write

$$Q_n = \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}, \qquad U_n = \sum_{i=1}^{n} \frac{(X_i - \mu)^2 - \sigma^2}{\sigma^2}.$$

Recall that $\nu = \mu/\sigma$. Elementary transformations lead to

$$\mathbb{T}_n = \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n} + \frac{Q_n}{\sqrt{n}}\right) \Big/ \sqrt{1 + \frac{U_n}{n} - \frac{Q_n^2}{n^2}}. \tag{4.1}$$

The following Taylor expansion will be used several times in this section: for $|u| \leqslant \frac{1}{2}$,

$$\frac{1}{\sqrt{1+u}} = 1 - \frac{u}{2} + \frac{3u^2}{8} + \vartheta' u^3, \qquad \text{for some } |\vartheta'| \leqslant 1, \tag{4.2}$$

$$= 1 - \frac{u}{2} + \vartheta u^2, \qquad \text{for some } |\vartheta| \leqslant 1. \tag{4.3}$$

Lemma 4.1. *Assume that* $\mathrm{E}X^2 < \infty$. *Then, we have*

$$\mathbb{T}_n = \nu\sqrt{n} + \frac{Q_n}{\sqrt{n}} - \frac{\nu U_n}{2\sqrt{n}} + 2\vartheta\nu\frac{U_n^2}{n^{3/2}} + o_p(1), \tag{4.4}$$

where ϑ *is a random variable with* $|\vartheta| \leqslant 1$, *and* $o_p(1)$ *indicates convergence to zero in probability. In particular, if* $\mathrm{E}|X|^r < \infty$ *for some* $1 \leqslant r \leqslant 8/3$, *then*

$$\mathbb{T}_n = \nu\sqrt{n} + \frac{Q_n}{\sqrt{n}} - \frac{\nu U_n}{2\sqrt{n}} + o_p(n^{(8-3r)/(2r)}). \tag{4.5}$$

Proof. Since $\mathrm{E}X^2 < \infty$, the law of large numbers and the central limit theorem imply that

$$Q_n = O_p(\sqrt{n}) \quad \text{and} \quad U_n = o_p(\mathrm{n}).$$

In view of this and the Taylor expansion (4.3), we have

$$\mathbb{T}_n = \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}\right)\left(1-\frac{U_n}{2n}+\frac{Q_n^2}{2n^2}+2\vartheta\left(\frac{U_n^2}{n^2}+\frac{{Q_n}^4}{n^4}\right)\right) \tag{4.6}$$

$$= \left(\nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}\right)\left(1-\frac{U_n}{2n}+2\vartheta\,\frac{U_n^2}{n^2}\right)+o_p(1)$$

$$= \nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}-\frac{\nu U_n}{2\sqrt{n}}+2\vartheta\nu\frac{U_n^2}{n^{3/2}}+\frac{Q_n}{\sqrt{n}}\left(-\frac{U_n}{2n}+2\vartheta\frac{U_n^2}{n^2}\right)+o_p(1)$$

$$= \nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}-\frac{\nu U_n}{2\sqrt{n}}+2\vartheta\nu\frac{U_n^2}{n^{3/2}}+o_p(1).$$

This proves (4.4).

When $\mathrm{E}|X|^r<\infty$, by the law of large numbers, $U_n=o_p(n^{2/r})$ and hence $U_n^2/n^{3/2}=o(n^{(8-3r)/(2r)})$. Now (4.5) follows from (4.4). □

Assuming that $\nu\neq 0$, consider the statistic

$$\mathbb{T}_n^* = \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}\right)\Big/\sqrt{1+\frac{2Q_n}{n\nu}-\frac{Q_n^2}{n^2}}. \tag{4.7}$$

Lemma 4.2. *Assume that $\mu\neq 0$ and $\mathrm{E}X^2<\infty$. Then, $\mathbb{T}_n^*$ has the stochastic expansion*

$$\mathbb{T}_n^* = \nu\sqrt{n-1}+\frac{Q_n^2(1+\nu^2)}{2n^{3/2}\nu}+O_p(n^{-1}).$$

Proof. Since $\mathrm{E}X^2<\infty$, we have $Q_n=O_p(n^{1/2})$. Applying the Taylor expansion (4.2) with $u=2Q_n/(n\nu)-Q_n^2/n^2$, we derive

$$\mathbb{T}_n^* = \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}\right)\left(1-\frac{Q_n}{n\nu}+\frac{Q_n^2}{2n^2}+\frac{3}{8}\left(\frac{2Q_n}{n\nu}-\frac{Q_n^2}{n^2}\right)^2+O_p(n^{-3/2})\right)$$

$$= \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n}+\frac{Q_n}{\sqrt{n}}\right)\left(1-\frac{Q_n}{n\nu}+\frac{Q_n^2}{2n^2}+\frac{3Q_n^2}{2n^2\nu^2}+O_p(n^{-3/2}\right)$$

$$= \sqrt{\frac{n-1}{n}}\left(\nu\sqrt{n}+\frac{Q_n^2}{2n^{3/2}\nu}+\frac{\nu Q_n^2}{2n^{3/2}}+O_p(n^{-1})\right)$$

$$= \nu\sqrt{n-1}+\frac{Q_n^2(1+\nu^2)}{2n^{3/2}\nu}+O_p(n^{-1}).$$

□

Lemma 4.3. *Let $\nu\in\mathbb{R}$ and let Y be a random variable such that $\mathrm{E}Y=0$ and $\mathrm{E}Y^2=1$. Let σ_0^2 be the variance of the random variable*

$$\xi = Y - \frac{\nu}{2}(Y^2 - 1),$$

that is, $\sigma_0^2 = \mathrm{E}\xi^2$. *Then* $\sigma_0 = 0$ *if and only if* $\nu = 2\alpha_3/(\alpha_4 - 1)$ *with* $\alpha_j = \mathrm{E}Y^j$, $j = 3, 4$, *and* $Y \sim \mathcal{B}(p, 0, 1)$ *with* $p \neq \frac{1}{2}$. *Furthermore,* $\sigma_0 = 0$ *implies* $\xi = 0$ *with probability one.*

Proof. We can assume that $\alpha_4 > 1$ in our proof. Indeed, $\alpha_4 \geqslant 1$, and the equality $\alpha_4 = 1$ is possible only if Y is a symmetric Bernoulli random variable such that $|Y| = 1$. But, in this case, $\sigma_0 = 1$.

Let us show that $\sigma_0 = 0$ implies that $Y \sim \mathcal{B}(p, 0, 1)$ with $p \neq \frac{1}{2}$, and that $\nu = 2\alpha_3/(\alpha_4 - 1)$. Indeed, if $\sigma_0 = 0$, then $\xi = 0$ with probability one, that is, $Y - \nu(Y^2 - 1)/2 = 0$ almost surely. Since the quadratic equation can have at most two roots, Y can take at most two different values. Hence, $\sigma_0 = 0$ yields $Y \sim \mathcal{B}(p, 0, 1)$ with $p \neq \frac{1}{2}$ (note that we cannot have $p = \frac{1}{2}$ since this implies that $\alpha_4 = 1$). It remains for us to check that $\nu = 2\alpha_3/(\alpha_4 - 1)$. It is easy to obtain that

$$\sigma_0^2 = 1 - \nu\alpha_3 + \frac{\nu^2}{4}(\alpha_4 - 1) \geqslant 0.$$

By minimizing σ_0^2 with respect to $\nu \in \mathbb{R}$, we see that σ_0 can be equal to 0 only if $\nu = 2\alpha_3/(\alpha_4 - 1)$.

Elementary calculations show that the variance $\sigma_0 = 0$ if $Y \sim \mathcal{B}(p, 0, 1)$ with $p \neq \frac{1}{2}$ and $\nu = 2\alpha_3/(\alpha_4 - 1)$. For such a Bernoulli random variable, we also have $\xi = 0$ with probability one. $\square$

Lemma 4.4. *Assume that* $\xi, \xi_1, \xi_2, \ldots$ *are i.i.d. non-negative random variables.*

(i) *If* $\xi \in DA(1)$, *that is,*

$$c_n n^{-1} \sum_{i=1}^{n} \xi_i - d_n \xrightarrow{\mathcal{D}} Z_1, \qquad \text{as } n \to \infty, \tag{4.8}$$

with some $c_n > 0$ *and* $d_n \in \mathbb{R}$, *then* c_n *is a slowly varying sequence,* $d_n \to \infty$ *and* $d_n = o(n^\varepsilon)$ *for all* $\varepsilon > 0$. *If, in addition,* $\mathrm{E}\xi < \infty$, *then* $c_n/d_n \to 1/\mathrm{E}\xi$.

(ii) *If* $\mathrm{E}\xi^\tau < \infty$ *for some* $0 < \tau < 1$, *and* $c_n n^{-1/\tau}(\xi_1 + \ldots + \xi_n) \xrightarrow{\mathcal{D}} Z_\tau$, *where* Z_τ *is a stable non-degenerate random variable, then* $c_n \to \infty$.

Proof. (i) It is well known that $\xi \in DA(1)$ implies that c_n must be a slowly varying sequence (see Ibragimov and Linnik 1971; Feller 1971).

Let us prove that $d_n \to \infty$. Assume that this is not the case. Then, there exists a subsequence, say $n_k \to \infty$, such that $\lim d_{n_k} = d$ with some $-\infty \leqslant d < \infty$. Let $S_n = \sum_{i=1}^n \xi_i$. We have

$$1 = P\{S_n \geqslant 0\} = P\{c_n n^{-1} S_n - d_n \geqslant -d_n\}.$$

Passing to the limit along the subsequence, we derive $1 = P\{Z_1 \geqslant -d\}$, which is impossible since, as is well known, $P\{Z_1 \geqslant x\} < 1$, for all $x > -\infty$ (see Ibragimov and Linnik 1971;

Samorodnitsky and Taqqu 1994). Noting that (4.8) implies $\mathrm{E}|\xi|^{1-\varepsilon} < \infty$ for all $0 < \varepsilon < 1$, one can apply the Marcinkiewicz law of large numbers and obtain that $d_n = o(n^{\varepsilon})$.

Next, we prove that $c_n/d_n \to 1/\mathrm{E}\xi$ if $\mathrm{E}\xi < \infty$. Since $d_n \to \infty$, (4.8) implies

$$\frac{c_n}{d_n}\frac{1}{n}\sum_{i=1}^{n}\xi_i \to 1 \qquad \text{in probability.}$$

On the other hand, by the law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \to \mathrm{E}\xi \qquad \text{in probability.}$$

Putting together the above two statements yields $c_n/d_n \to 1/\mathrm{E}\xi$.

(ii) To prove that $\mathrm{E}\xi^{\tau} < \infty$ and that $c_n n^{-1/\tau}(\xi_1 + \ldots + \xi_n) \xrightarrow{\mathcal{D}} Z_\tau$ implies $c_n \to \infty$, it suffices to use the Marcinkiewicz law of large numbers, namely, that $n^{-1/\tau}(\xi_1 + \ldots + \xi_n) \to 0$ with probability one. □

We are now ready to prove Theorems 2.1, 2.2, 2.4 and 2.5.

Proof of Theorem 2.1. We prove (i) first. Write $X = \sigma Y + \mu$, where $Y \sim \mathcal{B}(p, 0, 1)$. It is easy to check that

$$Y - \frac{\nu_0}{2}(Y^2 - 1) = 0, \qquad \text{where } \nu_0 = \frac{2\sqrt{pq}}{1-2p}.$$

Hence, $Y^2 - 1 = 2Y/\nu_0$ and $U_n = 2Q_n/\nu_0$. Replacing U_n and ν with $2Q_n/\nu_0$ and ν_0 respectively in (4.1), we obtain

$$\mathbb{T}_n = \sqrt{\frac{n-1}{n}}\left(\nu_0\sqrt{n} + \frac{Q_n}{\sqrt{n}}\right)\Big/\sqrt{1 + \frac{2Q_n}{n\nu_0} - \frac{Q_n^2}{n^2}}. \tag{4.9}$$

Applying Lemma 4.2 to $\mathbb{T}_n$ in (4.9), we have

$$\mathbb{T}_n = \nu_0\sqrt{n-1} + \frac{Q_n^2(1+\nu_0^2)}{2n^{3/2}\nu_0} + O_p(n^{-1}).$$

That is, $2\nu_0\sqrt{n}(1+\nu_0^2)^{-1}(\mathbb{T}_n - \nu_0\sqrt{n-1}) = Q_n^2/n + O_p(n^{-1/2})$. The proof of (i) then follows from the fact that $Q_n/\sqrt{n} \xrightarrow{\mathcal{D}} Z$ and Slutsky's theorem.

As to (ii), by Lemma 4.1, we have

$$\mathbb{T}_n - \nu\sqrt{n} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(Y_k - \frac{\nu}{2}(Y_k^2 - 1)) + o_p(1).$$

Since $\mathrm{E}Y^4 < \infty$, we can apply the central limit theorem to obtain $\mathbb{T}_n - \nu\sqrt{n} \xrightarrow{\mathcal{D}} \sigma_0 Z$, where

$$\sigma_0^2 = \mathrm{E}\Big(Y - \frac{\nu}{2}(Y^2 - 1)\Big)^2 \equiv 1 - \nu\alpha_3 + \nu^2(\alpha_4 - 1)/4.$$

It remains for us to check that $\sigma_0 = 0$ implies the condition of (i). By Lemma 4.3, $\sigma_0 = 0$

implies that Y is a Bernoulli random variable and $\nu = 2\alpha_3/(\alpha_4 - 1)$ with $\alpha_3 = \mathrm{E}Y^3$ and $\alpha_4 = \mathrm{E}Y^4$. Elementary calculations show that $\alpha_3/(\alpha_4 - 1) = \sqrt{p - p^2}/(1 - 2p)$. Hence, $\sigma_0 = 0$ if and only if the condition of (i) is satisfied. $\square$

Proof of Theorem 2.2. Consider three cases.
Case 1: $\tau = 2$. Since $X^2 \in DA(2)$, we have $\mathrm{E}|X|^{4-\varepsilon} < \infty$ for any $0 < \varepsilon < 1$ and hence, by (4.5),

$$\mathbb{T}_n - \nu\sqrt{n} = \frac{Q_n}{\sqrt{n}} - \frac{\nu U_n}{2\sqrt{n}} + o_p(1). \tag{4.10}$$

Since $\mathrm{E}X^4 = \infty$ and Y^2 is in the domain of attraction of a standard normal random variable, there exists a slowly varying sequence, $c_n \to 0$, such that $c_n U_n/\sqrt{n} \xrightarrow{\mathcal{D}} Z$ (see Feller 1971). Multiplying (4.10) by $a_n = 2c_n/\nu$ and noting that $Q_n/\sqrt{n} = O_p(1)$, we derive

$$a_n(\mathbb{T}_n - b_n) = -c_n U_n/\sqrt{n} + o_p(1).$$

It follows that $a_n(\mathbb{T}_n - b_n) \xrightarrow{\mathcal{D}} \mathrm{Z}$, which proves (i).
Case 2: $1 < \tau < 2$. Note that $X^2 \in DA(\tau)$ implies that $\mathrm{E}|X|^r < \infty$ for any $0 < r < 2\tau$. Also note that $1/2 - 1/\tau + (8 - 6\tau)/(4\tau) = 1/\tau - 1 < 0$. We can choose $r < 2\tau$ so that $\mathrm{E}|X|^r < \infty$ and $1/2 - 1/\tau + (8 - 3r)/(2r) < 0$. Thus, with $a_n = 2c_n n^{1/2-1/\tau}/\nu$ and $b_n = \nu\sqrt{n}$, by (4.5) and (2.1),

$$a_n(\mathbb{T}_n - b_n) = -c_n n^{-1/\tau} U_n + o_p(1) \xrightarrow{\mathcal{D}} -Z_\tau.$$

Case 3: $\tau = 1$. By (2.2),

$$c_n n^{-1} U_n + c_n - d_n \xrightarrow{\mathcal{D}} Z_1. \tag{4.11}$$

It follows from Lemma 4.4 that c_n is slowly varying, $c_n \to \infty$ and $d_n \sim c_n$. Hence, $U_n = O_p(n(1 + |d_n - c_n|)/c_n)$ and $U_n^2/n^{3/2} = O_p(n^{1/2}(1 + (d_n - c_n)^2)/c_n^2) = o_p(n^{1/2}/c_n)$ by the assumption $(d_n - c_n)^2 = o(c_n)$.

From (4.4), we obtain

$$\mathbb{T}_n = \nu\sqrt{n} - \frac{\nu U_n}{2\sqrt{n}} + o_p(n^{1/2}/c_n)$$

or

$$\frac{2c_n}{\nu\sqrt{n}}\left(\mathbb{T}_n - \nu\sqrt{n}\left(\frac{3}{2} - \frac{d_n}{2c_n}\right)\right) = -\left(\frac{c_n U_n}{n} + c_n - d_n\right) + o_p(1),$$

which, combining (4.11) and case 2, proves (ii). $\square$

Proof of Theorem 2.4. Note that the representation (4.4) remains valid with $\nu = \nu_n$. We thus have

$$\mathbb{T}_n = \nu_n\sqrt{n} + \frac{Q_n}{\sqrt{n}} - \frac{\nu_n U_n}{2\sqrt{n}} + \frac{2\vartheta\nu_n U_n^2}{n^{3/2}} + o_p(1). \tag{4.12}$$

Observe that $U_n/\sqrt{n} = O_p(1)$ if $\mathrm{E}Y^4 < \infty$ by the central limit theorem, and $U_n = o_p(n^{1/\tau})$ if $\mathrm{E}|Y|^{2\tau} < \infty$ for some $1 \leqslant \tau < 2$ by the law of large numbers. Hence,

$$\frac{\nu_n U_n}{\sqrt{n}} = o_p(1) \tag{4.13}$$

under the assumption of the theorem. The theorem now follows from the central limit theorem, $Q_n/\sqrt{n} \xrightarrow{\mathcal{D}} N(0, 1)$, (4.12) and (4.13). □

Proof of Theorem 2.5. Let $\xi_n = c_n n^{-1/\tau} U_n$ if $1 < \tau \leqslant 2$ and $\xi_n = c_n n^{-1}(U_n + n) - d_n$ if $\tau = 1$. Then, by (2.1) and (2.2),

$$\xi_n \xrightarrow{\mathcal{D}} Z_\tau. \tag{4.14}$$

We can also rewrite (4.12) as

$$\mathbb{T}_n - b_n = \frac{Q_n}{\sqrt{n}} - w_n \xi_n + \frac{2\vartheta \nu_n U_n^2}{n^{3/2}} + o_p(1). \tag{4.15}$$

We formulate the proof into six cases.

Case 1: $w = 0$ *and* $1 < \tau \leqslant 2$. We have

$$w_n \xi_n = o_p(1) \quad \text{and} \quad \frac{\nu_n U_n^2}{2n^{3/2}} = \frac{w_n \xi_n U_n}{n} = w_n \xi_n o_p(1). \tag{4.16}$$

Hence, (2.9) holds by (4.15).

Case 2: $w = 0$ *and* $\tau = 1$. In this case, we have $U_n = (n/c_n)(\xi_n + d_n - c_n)$ and

$$\begin{aligned}
\frac{|\nu_n| U_n^2}{n^{3/2}} &= \frac{|\nu_n|\sqrt{n}}{c_n^2}(\xi_n + d_n - c_n)^2 \\
&\leqslant \frac{2|\nu_n|\sqrt{n}}{c_n^2}(\xi_n^2 + (d_n - c_n)^2) \\
&= 4|w_n|\left(\frac{\xi_n^2}{c_n} + \frac{(d_n - c_n)^2}{c_n}\right) = o_p(1)
\end{aligned} \tag{4.17}$$

by (2.6), (4.14) and Lemma 4.4. This proves (2.9) by (4.15) and (4.17).

Case 3: $w = \infty$ *and* $1 < \tau \leqslant 2$. By (4.15) and (4.16), we see that $a_n(\mathbb{T}_n - b_n) = -\xi_n(1 + o_p(1)) + o_p(1) \xrightarrow{\mathcal{D}} -Z_\tau$.

Case 4: $w = \infty$ *and* $\tau = 1$. From the proof of (4.17) and by (2.6), we have

$$\frac{\nu_n U_n^2}{n^{3/2}} = o_p(w_n) \tag{4.18}$$

and hence $a_n(\mathbb{T}_n - b_n) = -\xi_n + o_p(1) \xrightarrow{\mathcal{D}} -Z_1$.

Case 5: $0 < w < \infty$ *and* $1 < \tau \leqslant 2$. By (4.14) and (4.16), we have

$$\mathbb{T}_n - b_n = \frac{Q_n}{\sqrt{n}} - w_n\xi_n(1 + o_p(1)) + o_p(1) = \frac{Q_n}{\sqrt{n}} - w_n\xi_n + o_p(1). \tag{4.19}$$

It is known that $(Q_n/\sqrt{n}, \xi_n) \xrightarrow{\mathcal{D}} (Z, Z_\tau)$, where $Z \sim N(0, 1)$ and Z and Z_τ are independent (see Resnick and Greenwood 1979). Therefore, by (4.19), we have $\mathbb{T}_n - b_n \xrightarrow{\mathcal{D}} Z - wZ_\tau$.

Case 6: $0 < w < \infty$ *and* $\tau = 1$. We have $\nu_n U_n^2/n^{3/2} = o_p(1)$ by (4.18). Thus, (4.14) reduces to

$$\mathbb{T}_n - b_n = \frac{Q_n}{\sqrt{n}} - w_n\xi_n + o_p(1).$$

Now (2.9) follows from the proof in case 5.

This completes the proof of Theorem 2.5. □

To prove Theorem 2.3, we need one more lemma. Let $Y = X - \mu$ and $Y_i = X_i - \mu$. Write $Q_n = \sum_{i=1}^n Y_i$ and $W_n = \sum_{i=1}^n Y_i^2$ and rewrite the representation (4.1) as

$$\mathbb{T}_n = \sqrt{\frac{n-1}{n}} \frac{n\mu + Q_n}{\sqrt{W_n(1 - Q_n^2/(nW_n))}}. \tag{4.20}$$

Lemma 4.5. *Assume that* $\mathrm{E}Y = 0$ *and* $Y^2 \in DA(\tau)$ *for* $\frac{1}{2} \leqslant \tau \leqslant 1$*. If* $\tau = \frac{1}{2}$*, we further assume that* (2.5) *is satisfied. Then,* $Q_n/\sqrt{W_n} = O_p(1)$*.*

Proof. By Griffin's (2002) necessary and sufficient condition for tightness of the Student's *t*-statistic, it suffices to show that

$$\limsup_{x\to\infty} \frac{x|\mathrm{E}YI\{|Y| \leqslant x\}|}{x^2P(|Y| > x) + \mathrm{E}Y^2I\{|Y| \leqslant x\}} < \infty. \tag{4.21}$$

This is exactly the assumption (2.5) when $\tau = \frac{1}{2}$. We thus only need to consider $\frac{1}{2} < \tau \leqslant 1$. Since $Y^2 \in DA(\tau)$, by Corollary 2, Section XVII.5 in Feller (1971), there exists a slowly varying function, $h(x)$, such that $P(Y^2 > x) = x^{-\tau}h(x)$. Since $\mathrm{E}Y = 0$, integration by parts yields

$$\begin{aligned} x|\mathrm{E}YI\{|Y| \leqslant x\}| = x|\mathrm{E}YI\{|Y| > x\}| &\leqslant x\left|\int_x^\infty t\,\mathrm{d}\,P(|Y| > t)\right| \\ &\leqslant x^2P(|Y| > x) + x\int_x^\infty P(|Y| > t)\mathrm{d}t. \end{aligned}$$

and

$$
\begin{aligned}
x\int_x^\infty P(|Y|>t)\mathrm{d}t &= x\int_x^\infty t^{-2\tau}h(t^2)\mathrm{d}t \\
&\leqslant x\int_x^\infty t^{-\tau-1/2}h(t)\mathrm{d}t \\
&= O(x^{2-2\tau}h(x^2)) = O(x^2P(|Y|>x)).
\end{aligned}
$$

This proves (4.21). □

Proof of Theorem 2.3. By Lemma 4.5, we have $Q_n/\sqrt{W_n} = O_p(1)$. Thus, by (4.20),

$$
\begin{aligned}
\mathbb{T}_n &= \sqrt{\frac{n-1}{n}}\frac{n\mu}{\sqrt{W_n}\sqrt{1-Q_n^2/(nW_n)}} + \sqrt{\frac{n-1}{n}}\frac{Q_n}{\sqrt{W_n}\sqrt{1-Q_n^2/(W_n)}} \\
&= \frac{n\mu}{\sqrt{W_n}}(1+O_p(1/n)) + O_p(1). \qquad (4.22)
\end{aligned}
$$

First, consider the case of $\frac{1}{2}\leqslant\tau<1$. Since c_n is slowly varying and $c_n\to\infty$ when $\tau=\frac{1}{2}$ by Lemma 4.4, $a_n = n^{-1+1/(2\tau)}/(\mu\sqrt{c_n})\to 0$ as $n\to\infty$. Therefore, by (4.22) and (2.4), we have

$$
a_n\mathbb{T}_n = \frac{1}{\sqrt{c_n n^{-1/\tau}W_n}}(1+O_p(n^{-1})) + o_p(1)\xrightarrow{\mathcal{D}}\frac{1}{\sqrt{Z_\tau}}. \qquad (4.23)
$$

For $\tau=1$, write $\xi_n = c_n n^{-1}W_n - d_n$. Then $W_n = c_n^{-1}n(\xi_n+d_n)$. Noting that $d_n\to\infty$ and $\xi_n\xrightarrow{\mathcal{D}}Z_1$, we have $\xi_n = o_p(d_n)$ and hence, by (4.22) and Taylor's expansion (4.3), we have

$$
\begin{aligned}
\mathbb{T}_n &= \frac{\mu\sqrt{nc_n}}{\sqrt{d_n}\sqrt{(1+\xi_n/d_n)}}\left(1+O_p\left(\frac{1}{n}\right)\right) + O_p(1) \\
&= \frac{\mu\sqrt{nc_n}}{\sqrt{d_n}}\left(1-\frac{\xi_n}{2d_n}+O_p\left(\frac{1}{d_n^2}\right)\right)\left(1+O_p\left(\frac{1}{n}\right)\right) + O_p(1) \\
&= \frac{\mu\sqrt{nc_n}}{\sqrt{d_n}}\left(1-\frac{\xi_n}{2d_n}+O_p\left(\frac{1}{d_n^2}\right)\right) + O_p(1).
\end{aligned}
$$

Therefore,

$$
a_n(\mathbb{T}_n - b_n) = -\xi_n + O_p\left(\frac{1}{d_n}\right) + O_p(a_n)\xrightarrow{\mathcal{D}} -Z_1,
$$

where $a_n\to 0$ comes from Lemma 4.4. This completes the proof of the theorem. □

Proof of Remark 2.3. Let $a_n = n/c_n$. By Feller (1971: 574–580), we have

$$
d_n = n\mathrm{E}(\sin(Y^2/a_n))
$$

and

$$a_n\ell(a_n) \approx n.$$

Hence $c_n \approx \ell(a_n)$. Noting that $E(Y^2) = 1$ and $\ell(x) \to \infty$ as $x \to \infty$, we have

$$\begin{aligned}|c_n - d_n| &= n|\mathrm{E}(Y^2/a_n - \sin(Y^2/a_n))| \\ &\leqslant 2(n/a_n)\mathrm{E}(Y^2 I(Y^2 \geqslant a_n)) + (n/a_n^3)\mathrm{E}(Y^6 I(Y^2 \leqslant a_n)) \\ &= o(n/(a_n\ell^{1/2}(a_n))) + O(n/(a_n\ell(a_n))) \\ &= o(c_n/\ell^{1/2}(a_n)) + O(c_n/\ell(n)) = o(c_n^{1/2}),\end{aligned}$$

as desired. □

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