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Asymptotics for diffusion first-passage laws

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By using Berg's Abelian theorem, Csáki extracted a sharp asymptotic estimate for a diffusion first-passage law from its Laplace transform. We extend the method and give a simple formulation for Itô diffusions.

Keywords: asymptotic expansion; diffusion; hitting law; Tauberian theorem

1. Introduction

We describe a method that extracts sharp estimates for $\mathbb{P}[T_x < t]$ from $\hat{\mu}(z,x) := \mathbb{E}[\mathrm{e}^{-zT_x}]$, where T_x is the first time a real diffusion, starting from zero, hits $x > 0$. Many such Laplace transforms are known explicitly – via solutions of Sturm–Liouville equations [11]. However, tight bounds on the probability law are more elusive. For example, Remark 6.6 in [15] points out that a convergent series for a Bessel process hitting law density fails to provide the exact asymptotic behavior of $\mathbb{P}[T_x < t]$ as $\tau := t/x^2 \downarrow 0$, likewise, the estimate [8], (1.18), derived from de Bruijn's Tauberian theorem ([2], page 254).

Csáki [4] observed the utility of Berg's Abelian theorem ([1], pages 112–113). This describes an expansion of $\rho \to \int \mathrm{e}^{-u^2} G(\rho + \mathrm{i}u\sqrt{\rho})\,\mathrm{d}u$ as $\rho \uparrow \infty$, provided G is holomorphic on a right half-plane and satisfies certain growth restrictions there. Since Csáki's diffusion has Brownian scaling, his G depends on z only, but for wider application, we employ a rule of thumb. We write

$$G(z) = G(z,x) = \frac{\kappa z \mathrm{e}^{2z} \hat{\mu}(\kappa z^2 + \gamma_0, x)}{\kappa z^2 + \gamma_0}, \qquad (\kappa, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

with γ_0 the leading eigenvalue determined as in (3.2) below and κ chosen to ensure $\log G(\rho) = o(\rho)$ for $\rho \uparrow \infty$. Our extension then applies transparently to a wide class of Itô diffusions and leads naturally to the computation of asymptotic expansions. We therefore state results for the latter, while granting that few, if any, applications will involve more than the leading term.

The following statement indicates what we have in mind. For Berg's conditions, see the hypotheses of Lemma 2.1 below. Throughout, $0 < \rho_0 \le \rho_1 \le \rho_2$ will denote constants independent of x.

Theorem 1.1. *Assume, for* $\min(\rho, \kappa\rho^2) \ge \rho_0$, *that G satisfies the conditions of Berg's Abelian theorem uniformly in x. Then, with error constant independent of x,*

$$\mathbb{P}[T_x < t] = \frac{\mathrm{e}^{\gamma_0 t - 1/\kappa t}}{\sqrt{\kappa t \pi}} \left[\sum_{m=0}^{n-1} \frac{G^{(2m)}(1/\kappa t, x)}{2^{2m} m!} (-\kappa t)^{-m} + O\bigl(G(1/\kappa t, x)(\kappa t)^{n(\beta-1)}\bigr) \right] \tag{1.2}$$

as $\min(1/\kappa t, 1/\kappa t^2) \uparrow \infty$.

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We prove this Tauberian theorem in Section 2. A major drawback for applications remains the difficulty of verifying its conditions, even in simple cases. Of course analyticity of G follows by the spectral representation ([11], Section 4.11) of $z \to \hat{\mu}(z,x)$, but we lack a similar soft argument for checking (A_α) and $(B_{\beta,n})$ in Lemma 2.1.

To this end, we formulate a more explicit, if less general, variant; prompted by the well-known fact ([7], Chapter IV) that one can solve a large class of Sturm–Liouville equations using asymptotic power series. In Section 3, we explain why the hitting law of an Itô process with sufficiently smooth coefficients has an asymptotic expansion of the form

$$\hat{\mu}(z,x) = \mathrm{e}^{-\sqrt{2(z-\gamma_0)}x} \sum_{n\geq 0} q_n(x)[2(z-\gamma_0)]^{-n/2} \tag{1.3}$$

as $\Re z = \rho \uparrow \infty$ sufficiently fast – usually $\inf(\rho, \rho x^{-K}) \uparrow \infty$ for some $K \geq 1$. Taking $\kappa = 2/x^2$, formula (1.1) then becomes

$$G(z,x) = \frac{4z}{4z^2 + 2\gamma_0 x^2} \sum_{n\geq 0} (x/2)^n q_n(x) z^{-n} := \frac{1}{z} \sum_{n\geq 0} u_n(x) z^{-n}, \tag{1.4}$$

where, for the last part we also require $\rho x^{-1} \uparrow \infty$.

Now recall (e.g., [7], Section 1.6) that asymptotic power series can be multiplied and composed. They can also be divided whenever the denominator contains a non-zero constant and differentiated if the function is holomorphic on a suitable sector – as we have here. Applied to (1.4), this shows that (A_0) and $(B_{2,n})$ of Lemma 2.1 hold automatically. Calculating the coefficients in (1.2), we deduce

Corollary 1.1. *Assume* (1.3) *holds as* $\min(\rho, \rho x^{-1}) \uparrow \infty$ *and define* $\tau = t/x^2$. *Then,*

$$\mathbb{P}[T_x < t] = \mathrm{e}^{\gamma_0 t - (2\tau)^{-1}} \sqrt{\frac{2\tau}{\pi}} \left[\sum_{m=0}^{n-1} a_m \tau^m + G(2/\tau, x) O(\tau^{n-1}) \right] \tag{1.5}$$

as $\tau(1+x) \downarrow 0$, *where*

$$a_n = \sum_{m=0}^{n} (-1)^{n-m} \frac{2^{m-n}(2n-m)!}{m!(n-m)!} x^m \sum_{k=0}^{[m/2]} q_{m-2k}(x)(-2\gamma_0)^k.$$

Remark 1.1. (1) The error in (1.2) agrees with [1], Section 49.3, and by $(B_{\beta,n})$ of Lemma 2.1, it has the order of the next term. Note that [4], (2.7), contains a misprint (irrelevant for his argument).

(2) The requirement $\tau(1+x) \downarrow 0$ comes from our convergence condition for (1.4). Compare Example 4.2.

(3) A recursive formula, listed at (3.5), gives an efficient method for computing coefficients in the reciprocal power series of (1.3).

(4) The above approach works more generally. If (1.4) holds for $z^{-\nu-1/2}G(z)$ with $\nu > -\frac{1}{2}$, then we can show G satisfies (A_1) and $(B_{2,n})$ with $\eta = 1$. See [10], page 435, for an example.

(5) We assume throughout that, in the terminology of [11], zero is either a regular point for X or else an entrance boundary for X on $[0, \infty)$.

(6) From (1.2), we have the probabilistic interpretation $\lim_{t\downarrow 0} t \log \mathbb{P}[T_x < t] = -1/\kappa$. Equivalence with our definition $\lim_{z\to\infty} z^{-1} \log \mathbb{E}[\mathrm{e}^{-z^2 T_x}] = -2/\sqrt{\kappa}$ follows from de Bruijn's Tauberian theorem ([2], page 254).

This article is organized as follows. Section 2 reworks the proof of Berg's theorem to determine the error constant required for the proof of our main result. In Section 3, we follow [7], Chapter IV, and discuss methods for computing (1.3). The final section looks at (1.2) for selected examples related to Brownian motion with constant drift and the Ornstein–Uhlenbeck process.

2. Proof of Theorem 1.1

The idea is to apply Berg's Abelian theorem by controlling the x-dependence in (1.1). We therefore rework his proof, but keep track of certain error constants. The following lemma (cf. [1], Section 49.3) uses

$$\mathcal{I}_{2n} := \int_{-\infty}^{\infty} u^{2n} \mathrm{e}^{-u^2} \frac{\mathrm{d}u}{\sqrt{\pi}} = \frac{(2n)!}{4^n n!}.$$

Lemma 2.1. *Let G be holomorphic on $\Re z = \rho > \rho_0 > 0$ and positive on $[\rho_0, \infty[$, and suppose there exist $\alpha \ge 0$, $\beta > 1$ and $\eta > \frac{1}{2}$ such that for all $\rho > \rho_0$,*

$$(A_\alpha) \qquad |G(\rho + \mathrm{i}u)| \le C_{A,\alpha} G(\rho) \mathrm{e}^{\alpha|u|/\sqrt{\rho}} \qquad \text{uniformly in } |u| \ge \rho^\eta,$$

$$(B_{\beta,n}) \qquad \left|G^{(2m)}(\rho + \mathrm{i}u)\right| \le C_{B,\beta,m} \rho^{-m\beta} G(\rho) \qquad \text{uniformly in } |u| \le \rho^\eta,\ 1 \le m \le n.$$

Then, for $\rho > \rho_1 = \rho_1(A_\alpha, B_{\beta,n}) \ge \rho_0$,

$$\left| \frac{1}{\sqrt{\pi\rho}} \int_{-\infty}^{\infty} G(\rho + \mathrm{i}u) \mathrm{e}^{-u^2/\rho}\, \mathrm{d}u - \sum_{m=0}^{n-1} \frac{G^{(2m)}(\rho)}{2^{2m} m!} (-\rho)^m \right| \le \frac{2C_{B,\beta,n}}{4^n n!} G(\rho) \rho^{-n(\beta-1)}. \tag{2.1}$$

Proof. By (A_α), the error in restricting the integral to $(-\rho^\eta, \rho^\eta)$ is $o(\mathrm{e}^{-(1/2)\rho^{2\eta-1}})G(\rho)$. Using a Taylor expansion, there exists $|\chi_u| \le 1$ such that

$$\begin{aligned} \frac{1}{\sqrt{\pi\rho}} \int_{-\rho^\eta}^{\rho^\eta} G(\rho + \mathrm{i}u) \mathrm{e}^{-u^2/\rho}\, \mathrm{d}u &= \sum_{m=0}^{2n-1} \frac{G^{(m)}(\rho)}{m!} \frac{1}{\sqrt{\pi\rho}} \int_{-\rho^\eta}^{\rho^\eta} (\mathrm{i}u)^m \mathrm{e}^{-u^2/\rho}\, \mathrm{d}u \\ &\quad + \frac{1}{\sqrt{\pi\rho}} \int_{-\rho^\eta}^{\rho^\eta} \frac{G^{(2n)}(\rho + \mathrm{i}\chi_u u)}{(2n)!} (\mathrm{i}u)^{2n} \mathrm{e}^{-u^2/\rho}\, \mathrm{d}u. \end{aligned} \tag{2.2}$$

We estimate each term, starting with the series, whose (even) terms are bounded by

$$\frac{G^{(2m)}(\rho)}{(2m)!}(-1)^m \int_{-\rho^\eta}^{\rho^\eta} u^{2m} \mathrm{e}^{-u^2/\rho} \frac{\mathrm{d}u}{\sqrt{\pi\rho}}$$
$$= \frac{G^{(2m)}(\rho)}{4^m m!}(-\rho)^m - 2\frac{G^{(2m)}(\rho)}{(2m)!}(-1)^m \int_{\rho^\eta}^{\infty} u^{2m} \mathrm{e}^{-u^2/\rho} \frac{\mathrm{d}u}{\sqrt{\pi\rho}},$$

where $(B_{\beta,n})$ shows the final term is $o(\mathrm{e}^{-(1/2)\rho^{2\eta-1}})G(\rho)$. Similarly, we bound the remainder in (2.2) by $C_{B,\beta,n}$ times

$$2\frac{\rho^{-\beta n} G(\rho)}{(2n)!\sqrt{\pi\rho}} \int_0^\infty u^{2n} \mathrm{e}^{-u^2/\rho}\, \mathrm{d}u = \rho^{-n(\beta-1)} G(\rho) \frac{\mathcal{I}_{2n}}{(2n)!} = \rho^{-n(\beta-1)} \frac{G(\rho)}{4^n n!}.$$

It remains to note that for $\rho > \rho_1$ sufficiently large, this estimate dominates the sum of the other error terms. $\square$

Berg's result follows by change of variable in (2.1). Taking first $z = \rho + \mathrm{i}u$, and then $w = z^2$, we find

$$\frac{\mathrm{e}^\rho}{\mathrm{i}\sqrt{\pi\rho}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} G(z)\mathrm{e}^{-2z+z^2/\rho}\, \mathrm{d}z = \frac{\mathrm{e}^\rho}{2\mathrm{i}\sqrt{\pi\rho}} \int_{\Gamma_\rho} \frac{G(\sqrt{w})\mathrm{e}^{-2\sqrt{w}+w/\rho}}{\sqrt{w}}\, \mathrm{d}w,$$

where is Γ_ρ is the parabola $v^2 = 4\rho^2(\rho^2 - u)$ in the $w = z^2 = u + \mathrm{i}v$ plane (see Figure 1).

The integrand is holomorphic on the exterior of Γ_ρ and, by using a simple estimate, we can deform Γ_ρ into $\mathbb{L}_c = \{c + \mathrm{i}s : s \in \mathbb{R}\}$ without prejudice to the error constant in (2.1). The argument uses condition $(B_{\beta,1})$ and $\Im G = 0$ on $[\rho_0, \infty[$ to deduce $G(\rho) \le G(\rho_0)\mathrm{e}^{\varepsilon\rho}$ for $\rho \ge \rho_2$; see [13] for details. Our version of [1], Section 49.5, then reads as follows.

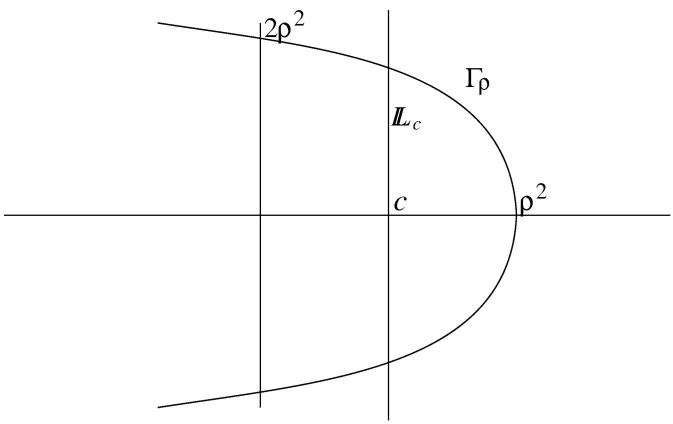

Figure 1. w plane.

Berg's Abelian theorem. *For G as in the lemma, $\exists \rho_2 \geq \rho_1$ such that for any $n \geq 1$,*

$$\begin{aligned}&\left| \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{G(\sqrt{w})\mathrm{e}^{-2\sqrt{w}+w/\rho}}{\sqrt{w}} \,\mathrm{d}w - \mathrm{e}^{-\rho}\sqrt{\frac{\rho}{\pi}} \sum_{m=0}^{n-1} \frac{G^{(2m)}(\rho)}{2^{2m} m!} (-\rho)^m \right| \\ &\quad \leq 2\frac{C_{B,\beta,n}}{4^n n!} \mathrm{e}^{-\rho} \rho^{-n(\beta-1)} \sqrt{\rho} G(\rho) \qquad \forall \rho > \rho_2,\end{aligned} \tag{2.3}$$

provided $c > \rho_2^2$.

Remark 2.1. If G is bounded on $[\rho_1, \infty)$, then we can take $\rho_2 = \rho_1$.

We prove Theorem 1.1 by applying (2.3) to the function (1.1). There are three items. First, by hypotheses, the expansion (2.1) holds when $\min(\rho, \rho^2\kappa) \geq \rho_1$. Second, as explained above, (2.3) follows since $(B_{\beta,1})$ applies uniformly in x. Third, choosing $c > 0$ such that $\kappa c + \gamma_0 > \rho_2^2$, we transform

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\kappa \hat{\mu}(\kappa w + \gamma_0, x)}{\kappa w + \gamma_0} \mathrm{e}^{w/\rho} \,\mathrm{d}w$$

by substituting $\rho = 1/\kappa t$ and $z = \kappa w + \gamma_0$. Laplace transform inversion then yields

$$\frac{\mathrm{e}^{-\gamma_0 t}}{2\pi\mathrm{i}} \int_{\kappa c+\gamma_0-\mathrm{i}\infty}^{\kappa c+\gamma_0+\mathrm{i}\infty} \frac{\hat{\mu}(z, x)}{z} \mathrm{e}^{zt} \,\mathrm{d}z = \mathrm{e}^{-\gamma_0 t} \mathbb{P}[T_x < t]$$

since we already know the integral converges, while the convergence condition appears as desired.

3. Expansion of the Laplace transform

We consider the computation of (1.3) for an Itô diffusion with generator

$$\mathcal{G} = \frac{1}{2} a^2(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x) \frac{\mathrm{d}}{\mathrm{d}x}, \tag{3.1}$$

having coefficients $b \in C^1$ and $0 < a \in C^2$. To determine $\hat{\mu}(z, x) = \mathbb{E}[\mathrm{e}^{-zT_x}]$, it suffices to solve for $f(z, x) := 1/\hat{\mu}(z, x)$ as the unique increasing solution of $\mathcal{G}f = zf$ subject to $f(z, 0) = 1$. We therefore seek conditions for f to possess an asymptotic expansion in powers of $z^{-1/2}$ as $\rho := \Re z \uparrow \infty$. Computing the reciprocal power series will then give us (1.3).

As in [7], Chapter IV, we first reduce $\mathcal{G}f = zf$ to a convenient canonical form. Noting that $g = f\mathrm{e}^{\int (b/a^2)}$ solves $g'' = [(b/a^2)' + (b/a^2)^2 + 2za^{-2}]g$, rescaling by $y = \int^x a^{-1}$ shows that $h(y) := a^{-1/2} g$ satisfies

$$h'' = (r + 2z)h, \qquad r = a^2[(b/a^2)' + (b/a^2)^2] - (a^{1/2})'' a^{3/2}.$$

Erdélyi [7] expands h in powers of $z^{-1/2}$. However, we need the increasing solution. So, introducing $2\gamma_0 = -\inf_{x>0} r(x)$, we rearrange as

$$h'' = (\chi^2 + \bar{r})h, \qquad \bar{r} = r + 2\gamma_0 \geq 0, \qquad \chi^2 = 2(z - \gamma_0) \tag{3.2}$$

and look to expand h in powers of $1/\chi$ for $\chi \uparrow \infty$. For this, [7] proposes the following methods.

3.1. Volterra's integral equation

For $\bar{r}$ continuous, any continuous solution of Volterra's integral equation

$$h(x) = h(\chi, x) = c_1 \mathrm{e}^{\chi x} + c_2 \mathrm{e}^{-\chi x} + \frac{1}{\chi}\int_0^x \sinh \chi(x - t)\bar{r}(t)h(t)\,\mathrm{d}t$$

solves (3.2). The obvious candidate $h = \sum_{n\geq 0} h_n$ is defined by

$$h_0 = c_1 \mathrm{e}^{\chi x} + c_2 \mathrm{e}^{-\chi x}; \qquad h_{n+1}(x) = \frac{1}{\chi}\int_0^x \sinh \chi(x - t)\bar{r}(t)h_n(t)\,\mathrm{d}t.$$

Introducing $w_n(x) = w_n(\chi, x) = \mathrm{e}^{-\chi x} h_n(\chi, x)$ with $w_0 \equiv 1$, we have

$$w_{n+1}(x) = \frac{1}{2\chi}\int_0^x \left[1 - \mathrm{e}^{-2\chi(x-t)}\right]\bar{r}(t)w_n(t)\,\mathrm{d}t \tag{3.3}$$

and $h(x) = \mathrm{e}^{\chi x}\sum_{n\geq 0} w_n(x)$ is the increasing solution of (3.2) normalized by $h(0) = 1$. By induction

$$|w_n(\chi, x)| \leq \frac{M^n(x)x^n}{\chi^n n!}, \qquad M(x) := \sup_{0<t<x} |\bar{r}(t)|, \tag{3.4}$$

which gives a simple test for convergence of

$$\frac{1}{\hat{\mu}(z, x)} = f(z, x) = \mathrm{e}^{-\int_0^x b(u)\mathrm{d}u + \sqrt{2(z-\gamma_0)}x} \sum_{n\geq 0} w_n\big([2(z - \gamma_0)]^{1/2}, x\big).$$

Beware, however, the practical difficulty of computing $\sum_{n\geq 0} w_n(\chi, x)$ and its subsequent rearrangement into $\sum_{n\geq 0} k_n(x)\chi^{-n}$. The next method does better on both counts.

3.2. A direct recursion

This method requires $\bar{r}$ to be sufficiently differentiable. The idea is to substitute the required expansion $h(x) = \sum_{n=0}^{\infty} k_n(x)\chi^{-n}\mathrm{e}^{\chi q(x)}$ into (3.2) and to then compare coefficients thus:

$$\begin{aligned}
\chi^2 &: (q')^2 k_0 = k_0;\\
\chi &: 2k_0' q' + k_0 q'' + k_1 (q')^2 = k_1;\\
\chi^{-n} &: k_n'' + 2k_{n+1}' q' + k_{n+1} q'' + k_{n+2}(q')^2 = \bar{r}k_n + k_{n+2}, \qquad n \geq 0.
\end{aligned}$$

Since these equations do not determine k_0 and k_1, we will use

$$k_0 \equiv 1, \qquad k_1(x) = \tfrac{1}{2}\int_0^x \bar{r}(t)\,\mathrm{d}t$$

from (3.3). Taking $q(x) = x$ to get the increasing solution, the above becomes $2k'_{n+1} = \bar{r}k_n - k'_n$, which we write as

$$2k_{n+1}(x) = \int_0^x \bar{r}(t)k_n(t)\,\mathrm{d}t - k'_n(x), \qquad n \geq 1. \tag{3.5}$$

Compared with (3.3), the recursion (3.5) offers a more efficient route to computing

$$\frac{1}{\hat{\mu}(z,x)} = f(z,x) = \mathrm{e}^{-\int_0^x b(u)\mathrm{d}u + \sqrt{2(z-\gamma_0)}x}\sum_{n=0}^{\infty} k_n(x)\chi^{-n/2}.$$

Assuming $\bar{r}$ to be sufficiently differentiable, and for x restricted to a bounded interval, Erdélyi [7] proved the validity of this expansion as $\chi \uparrow \infty$. However, our examples may involve more general conditions.

Remark 3.1. The above presupposes that $\bar{r}$ is integrable at zero. Otherwise, although $z \to \hat{\mu}(z,x)$ may still have an expansion, this will not always look like (1.3). For example, [10], page 435, considers the Bessel process of dimension $2\nu + 2$ where $r(x) = (\nu^2 - \frac{1}{4})x^{-2}$ and, using [3], page 387,

$$\hat{\mu}(z,x) = \mathbb{E}[\mathrm{e}^{-zT_x}] = \frac{2^{-\nu}(x\sqrt{2z})^{\nu}}{\Gamma(\nu+1)I_\nu(x\sqrt{2z})}.$$

Then [16], page 203, shows that $z^{(-1/4)(2\nu+1)}\hat{\mu}(z,x)$ has an expansion of the form (1.3), so we can apply Remark 1.1(4).

4. Examples

All processes start at zero, B is Brownian motion and $\rho = \Re z \uparrow \infty$. We use $X^\bullet$ (resp. X°) for the maximum (resp. minimum) of our diffusion X. In particular, $(X^\bullet_t > x) = (T_x < t)$. Recall that $\tau := t/x^2$ throughout.

We noted in the Introduction that a series expansion for a Bessel process hitting time fails to give the asymptotic estimate stated in Conjecture 6.7 of [15]. Gruet and Shi [10] established the latter using Csáki's observation [4], but did not dispel the notion, implicit in [15] and also in [5], that expansions like the well-known [9], Section 8.254,

$$\mathbb{P}[B^\bullet_t > x] = 2\mathbb{P}[B_t > x] = \mathrm{e}^{-1/2\tau}\sqrt{\frac{2}{\tau\pi}}\left[1 + \sum_{m=1}^{n}\frac{(2m)!}{2^m m!}(-\tau)^m + O(\tau^{n+1})\right], \tag{4.1}$$

as $\tau = t/x^2 \downarrow 0$, are somehow exceptional.

We claim the contrary: Berg's theorem gives hitting law asymptotics for many of the diffusions one meets in practice. We illustrate with three examples. In the first, which looks at the range of Brownian motion with drift, we start from Williams' formula [17] for $\hat{\mu}$ and determine the coefficients for (1.5) by hand. The next example considers the Ornstein–Uhlenbeck process where, despite knowing $\hat{\mu}$ in terms of parabolic cylinder functions, we found the methods of Section 3 provide a better way of obtaining (1.5).

Our final example examines the hitting law for a Bessel process with drift [14]. Here, the methods of Section 3 fail due to the singularity of r at zero. Nevertheless, G has an expansion, albeit one containing powers of $z^{-1}\log z$, which we can use to verify (A_1) and $(B_{2,n})$ of Lemma 2.1.

Example 4.1. Define $X = Y - Y^{\circ}$, where $Y_t = B_t + \lambda t$ for $\lambda > 0$. Then, (X, ℓ) is the unique positive solution of the reflecting SDE

$$X_t = B_t + \lambda t + \ell_t, \qquad \int_0^t X_s\,\mathrm{d}\ell_s = 0,$$

so ℓ is the local time of X at zero. Using the local martingale $f(X_t)\mathrm{e}^{-\lambda t}$ defined by $f'' + 2\lambda f' = 2zf$ and $f'(0) = 0$, Williams [17] showed

$$z\int_0^\infty \mathrm{e}^{-zt}\mathbb{P}[X_t^\bullet > x]\,\mathrm{d}t = \hat{\mu}(z,x) = \frac{\chi\mathrm{e}^{\lambda x}}{\chi\cosh\chi x + \lambda\sinh\chi x}, \qquad \chi^2 = 2z + \lambda^2.$$

We use his formula to compute the full asymptotic expansion of $\mathbb{P}[X_t^\bullet > x]$ as $\tau_x := \tau(1+x) \downarrow 0$, thereby improving a result in [5] (the authors found the leading term as $x \uparrow \infty$ with $t = 1$). We start by noting $\gamma_0 = -\frac{1}{2}\lambda^2$, so for $\kappa = 2/x^2$ we find

$$G(z,x) = \frac{\mathrm{e}^{2z}}{2\cosh 2z} \times \frac{2z}{2z + \lambda x\tanh 2z} \times \frac{8z\mathrm{e}^{\lambda x}}{4z^2 - x^2\lambda^2}.$$

To calculate (1.4), we first replace G with a simpler function having the same expansion. In fact, $1/(1-\mathrm{e}^{-4z}) = \sum_{n\geq 0}\mathrm{e}^{-4nz}$ can be ignored since its derivatives are all $O(\mathrm{e}^{-4\rho})$. Also, the second factor in G satisfies

$$\frac{2z}{2z+\lambda x\tanh 2z} - \frac{2z}{2z+\lambda x} = \frac{2z\lambda x}{(2z+\lambda x)} \times \frac{\mathrm{e}^{-4z}}{2z+\lambda x+(2z-\lambda x)\mathrm{e}^{-4z}},$$

where all derivatives of the last term decrease exponentially fast. Thus, G has the same asymptotic power series as

$$H(z) := \frac{8z\mathrm{e}^{\lambda x}}{4z^2 - x^2\lambda^2} \times \frac{2z}{2z+\lambda x} = \frac{2\mathrm{e}^{\lambda x}}{z}\left[1 - \frac{\lambda^2 x^2}{4z^2}\right]^{-1}\left[1 + \frac{\lambda x}{2z}\right]^{-1},$$

whose expansion converges as $\min(\rho, \rho x^{-1}) \uparrow \infty$. Applying Corollary 1.1,

$$\mathbb{P}[X_t^\bullet > x] = \mathrm{e}^{-(\lambda t - x)^2/2t}\sqrt{\frac{8\tau}{\pi}}\left[\sum_{m=0}^{n-1} a_m(-\tau)^m + O(\tau^n)\right], \qquad \tau_x := \tau(1+x) \downarrow 0,$$

where, for $v_n = \#\{k \geq 0 : 2k \leq n\}$,

$$a_n := \sum_{m=0}^{n} b_{n,m}(\lambda x)^m, \qquad b_{n,m} := v_m \frac{(2n-m)!}{2^{n-m} m!(n-m)!}.$$

Looking at the first few terms,

$$\begin{aligned}
&\mathbb{P}[X_t^\bullet > x] \\
&\quad = \mathrm{e}^{-(\lambda t - x)^2/2t} \sqrt{8\tau/\pi}[1 - (1+\lambda x)\tau + (2\lambda^2 x^2 + 3\lambda x + 3)\tau^2 \\
&\qquad\qquad - (2\lambda^3 x^3 + 12\lambda^2 x^2 + 15\lambda x + 15)\tau^3 \\
&\qquad\qquad + (3\lambda^4 x^4 + 20\lambda^3 x^3 + 90\lambda^2 x^2 + 105\lambda x + 105)\tau^4 \\
&\qquad\qquad - (3\lambda^5 x^5 + 45\lambda^4 x^4 + 210\lambda^3 x^3 + 840\lambda^2 x^2 + 945\lambda x + 945)\tau^5 \\
&\qquad\qquad + O(\tau_x^6)],
\end{aligned}$$

we guess, and can easily prove, that $b_{2n,2n} = v_{2n} = n + 1 = b_{2n+1,2n+1}$.

Remark 4.1. If $\lambda = 0$, then by Lévy's theorem, $X^\bullet := (B - B^\circ)^\bullet \overset{\text{law}}{=} |B|^\bullet := B^*$ and, comparing our expansion with (4.1), we find $\mathbb{P}[B_t^* > x]$ and $2\mathbb{P}[B_t^\bullet > x]$ have the same asymptotic power series for $\tau \downarrow 0$. A simpler way to see this uses the strong Markov property.

Example 4.2. For the Ornstein–Uhlenbeck process solving $X_t = B_t - \lambda \int_0^t X_s \, \mathrm{d}s$, we have $a \equiv 1$ and $b = -\lambda x$ in (3.1). Thus, $\gamma_0 = \frac{1}{2}\lambda > 0$ and $\bar{r}(x) = \lambda^2 x^2$. Estimate (3.4) now gives

$$|w_n(\chi, x)| \leq C \frac{x^{3n}}{n!|\chi|^n}, \qquad \chi^{-1} x^3 = O(1) \quad \Rightarrow \quad \sum_{n \geq 0} |w_n(\chi, x)| < \infty,$$

showing that $\sum_{n\geq 0} w_n(2z/x, x)$ converges provided $\inf(\rho, \rho x^{-4}) \uparrow \infty$. Since $\sum_{m=0}^{N-1} w_m(\chi, x)$ differs from $\sum_{m=0}^{N-1} k_n(x)\chi^{-m}$ by a finite number of higher order terms, this proves convergence of the latter and hence of its reciprocal series (1.3). Computing the coefficients in (1.5) from (3.5), we see

$$\begin{aligned}
&\mathbb{P}[X_t^\bullet > x] \\
&\quad = \mathrm{e}^{(1/2)\lambda t - (2\tau)^{-1}} \sqrt{\frac{2}{\pi}} \bigg[1 - \left(1 + \frac{\lambda^2 x^4}{6}\right)\tau + \left(3 - \frac{\lambda x^2}{2} + \frac{3}{4}\lambda^2 x^4 + \frac{\lambda^4 x^8}{6^2 2!}\right)\tau^2 \\
&\qquad\qquad - \left(15 - 3\lambda x^2 + \frac{17}{4}\lambda^2 x^4 - \frac{\lambda^3 x^6}{12} + \frac{\lambda^4 x^8}{10} + \frac{\lambda^6 x^{12}}{6^3 3!}\right)\tau^3 + \cdots \bigg],
\end{aligned}$$

valid for $\tau(1 + x^4) \downarrow 0$. To verify that $\tau^n \lambda^{2n} x^{4n}$ has coefficient $-(6^n n!)^{-1}$, one can argue by induction. We have no general expression for the other non-constant coefficients.

Example 4.3. The Bessel process of dimension $2\nu+2$ and drift λ solves

$$X_t = B_t + \frac{2\nu+1}{2}\int_0^t \frac{\mathrm{d}s}{X_s} + \lambda\,\mathrm{d}t$$

for $\nu > -\frac{1}{2}$ (Kendall's pole-seeking Brownian motion [12] corresponds to $\nu = 0$ and $\lambda < 0$). Here the methods of Section 3 fail because, in the notation of (3.1), $r = b' + b^2$ is not integrable at zero. Nevertheless, from [14], page 363, and using the relation [6], 6.9(1), we find

$$\hat{\mu}(z,x) = \mathbb{E}[\mathrm{e}^{-zT_x}] = \frac{\mathrm{e}^{x(\lambda+\chi)}}{{}_1F_1((\nu+1/2)(1-(\lambda/2\chi)), 2\nu+1; 2\chi x)}, \qquad \chi^2 = 2z + \lambda^2.$$

Choosing $\gamma_0 = -\frac{1}{2}\lambda^2$ and $\kappa = 2/x^2$ in (1.1) yields

$$G(z) = \frac{4z}{4z^2 - \lambda^2 x^2}\frac{\mathrm{e}^{4z+\lambda x}}{{}_1F_1(\alpha(4z), 2\nu+1; 4z)}, \qquad \alpha(z) := \bigl(\nu + \tfrac{1}{2}\bigr)\bigl(1 - (\lambda x/4z)\bigr),$$

so we need the expansion of $z \to {}_1F_1(\alpha(z), 2\nu+1; z)$ as $z \to \infty$. We follow the derivation of [6], 6.13(2), starting from the functional relation [6], 6.7(7),

$${}_1F_1(\alpha(z), \beta; z) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\mathrm{e}^{\mathrm{i}\pi\alpha}\Psi(\alpha,\beta;z) + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\mathrm{e}^{z+\mathrm{i}\pi(\alpha-\beta)}\Psi(\beta-\alpha,\beta;-z)$$

for Tricomi's function

$$\Psi(\alpha,\beta;z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{-zt}t^{\alpha-1}(1+t)^{\beta-\alpha-1}\,\mathrm{d}t.$$

By expanding $(1+t)^{\beta-\alpha-1}$ in a Taylor series and estimating the contribution of the remainder, Watson's lemma ([7], page 34) gives the behavior of Ψ as $z \to \infty$. The method also works for $z \to -\infty$ since we can analytically continue in z and t to ensure the integral remains absolutely convergent. Thus (cf. [6], 6.13.1(2)), ${}_1F_1(\alpha(z), \beta; z)$ has expansion

$$z^{-1-2\nu+\alpha(-z)}\mathrm{e}^z\frac{\Gamma(2\nu+1)}{\Gamma(\alpha(z))}\left[\sum_{k=0}^{N-1}\frac{(\alpha(-z)-1)_k(2\nu-\alpha(-z))_k}{(-z)^k k!} + O(|z|^{-N})\right]$$

with $(a)_n = a(a+1)\dots(a+n-1)$. Writing $\alpha(z) = \nu + \frac{1}{2} + (\lambda' x/z)$, we find

$$\begin{aligned} &G(z)(4z)^{\lambda' x/4z}(4z)^{-\nu-(1/2)} \\ &\quad = \frac{1}{z}\frac{4z^2\mathrm{e}^{\lambda x}}{4z^2-\lambda^2x^2}\frac{\Gamma(\nu+(1/2)+(\lambda' x/4z))}{\Gamma(2\nu+1)} \\ &\qquad \times \left[\sum_{k=0}^{N-1}\frac{(\nu-(1/2)+(\lambda' x/4z))_k(\nu+(1/2)-(\lambda' x/4z))_k}{k!(-4z)^k} + O(|z|^{-N})\right]^{-1}, \end{aligned}$$

so the left-hand side has an expansion of the form (1.4). As pointed out in Remark 1.1(4), it follows that $G(z)(4z)^{\lambda' x/4z}$ satisfies (A_1) and $(B_{2,n})$ for $\eta = 1$. Expanding $(4z)^{\lambda' x/4z}$ in powers of $z^{-1}\log z$, we deduce the same for G. Applying Theorem 1.1 and doing some elementary simplification, we end up with

$$\mathbb{P}[X_t^\bullet > x] = \frac{2\mathrm{e}^{-(\lambda t - x)^2/2\tau}}{(2\tau)^\nu \Gamma(\nu+1)}\bigl(1 + O(\tau)\bigr), \qquad \tau(1+x) \downarrow 0;$$

see [10], Theorem 1.1 for the case $\lambda = 0$.

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