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ON THE COMPLEX ŁOJASIEWICZ INEQUALITY WITH PARAMETER

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To Piotr Tworzewski on the occasion of his 60th birthday.

Abstract

We prove a continuity property in the sense of currents of a continuous family of holomorphic functions which allows us to obtain a Łojasiewicz inequality with an effective exponent independent of the parameter.

1. Introduction

The *Łojasiewicz inequality* introduced in [12] is one of the most important tools in singularity theory, both complex and real. The first result concerning a parametrized family—but, of course, with *an exponent that is independent of the parameter*—is due to Łojasiewicz and Wachta [13]. Fairly recently, we have obtained in [8] an effective Łojasiewicz inequality with parameter in complex analytic geometry, using only complex analytic methods. This article is somehow a continuation of that work, inspired to some extent by the observations made in [7] and the intersection theory results introduced in [18].

Our best results are presented in the following theorem. Throughout the paper we assume that the topological space T is *1st countable*.

Theorem 1.1. *Assume that* $f : T \times \Omega \to \mathbf{C}$ *is a continuous function where* T *is a locally compact, connected topological space,* $\Omega \subset \mathbf{C}^m$ *is a domain, and for all* $t \in T$, $f_t \in \mathcal{O}(\Omega)$ *does not vanish identically. Assume moreover that* $0 \in \Omega$ *and* $f_t(0) = 0$ *for any* t. *Then*

(1) $Z_{f_t} \to Z_{f_{t_0}}$ *in the sense of currents, where* Z_{f_t} *denotes the cycle of zeroes of* f_t;

(2) *there is a neighbourhood* $U \subset \Omega$ *of zero in which, for all* t *close enough to* t_0,

$$|f_t(x)| \geq c(t)\ \mathrm{dist}(x, f_t^{-1}(0))^{\alpha},$$

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where $c(t) > 0$ is a constant depending on the parameter, but the exponent $\alpha = \mathrm{ord}_0 f_0$ is uniform.

For the convenience of the reader let us recall two basic notions of convergence of sets, especially useful in analytic geometry (see e.g. [4] and [19]). We consider the following situation: T is a topological space and $E \subset T \times \mathbf{R}^n$ is a set with closed sections $E_t = \{x \in \mathbf{R}^n \mid (t, x) \in E\}$ and we put $F := \pi(E)$ for $\pi(t, x) = t$. Assume that t_0 is an accumulation point of F.

Definition 1.2 (see e.g. [4]). We say that E_t *converges in the sense of Kuratowski* to a set A, when $t \to t_0$, if

- for any $x \in A$, for any neighbourhood U of x, there is a neighbourhood V of t_0 such that $U \cap E_t \neq \emptyset$ for all $t \in V \cap F \setminus \{t_0\}$, i.e. $A \subset \liminf_{t \to t_0} E_t$ (*the lower Kuratowski limit*);
- if x is such that for any neighbourhood $U \ni x$ and any neighbourhood $V \ni t_0$ there is a point $t \in V \setminus \{t_0\}$ such that $U \cap E_t \neq \emptyset$, then $x \in A$, i.e. $A \supset \limsup_{t \to t_0} E_t$ (*the upper Kuratowski limit*).

We write then $E_t \xrightarrow{K} A$.

If for each t_0, $E_t \xrightarrow{K} E_{t_0}$, then we say that E *has continuously varying fibres*.

Remark 1.3. It is easy to see (cf. [19], [4]) that this convergence for the graphs of a sequence continuous functions is precisely the *local uniform convergence* of the functions themselves.

We have the following straightforward observation:

Lemma 1.4. *If any point in T has a countable basis of neighbourhoods, then $E_t \xrightarrow{K} A$ when $t \to t_0$ iff*

- *if $x \in A$, then for any sequence $t_\nu \to t_0$ we can find points $E_{t_\nu} \ni x_\nu \to x$;*
- *if x is such that there is a sequence $t_\nu \to t_0$ and points $E_{t_\nu} \ni x_\nu \to x$, then $x \in A$.*

In complex analytic geometry this kind of convergence is very useful for different purposes (Bishop's Theorem, algebraic approximation as in [1] or algebraicity criteria as in [10]). We may refine it taking into account multiplicities (cf. [18] and [2]). In order to do so, consider a sequence of *positive pure k-dimensional analytic cycles*[1] Z_ν, $\nu = 0, 1, 2, \ldots$ in some open set $\Omega \subset \mathbf{C}^m$ (of course, everything can be carried over to manifolds).

Definition 1.5 (Tworzewski [18]). We say that Z_ν *converges to Z_0 in the sense of Tworzewski*, which we denote by $Z_\nu \xrightarrow{T} Z_0$, if

[1] A positive pure k-dimensional cycle Z is a formal sum $\sum \alpha_\iota S_\iota$ where $\alpha_\iota > 0$ are integers and $\{S_\iota\}$ is a locally finite family of irreducible k-dimensional analytic sets; then the analytic set $|Z| := \bigcup S_\iota$ is called the *support* of Z; for details see [18].

- the supports $|Z_\nu| \xrightarrow{K} |Z_0|$;
- for any regular point $a \in \mathrm{Reg}|Z_0|$ and any relatively compact manifold M of complementary dimension, transversal to $|Z_0|$ at a and such that $\overline{M} \cap |Z_0| = \{a\}$, we have for the *total number of intersection*[2] $\deg(Z_\nu \cdot M) = \deg(Z_0 \cdot M)$ from some index ν_0 onwards.

We will call M *a testing manifold for* Z_0 *at* a.

Remark 1.6. As noted by Alain Yger [22], this convergence is precisely *the weak convergence of the corresponding integration currents* $[Z_\nu]$. See also the general though not very precise discussion in [2] and the elegant construction in [15].

By [18] Lemma 3.2 it is sufficient to consider testing manifolds at a dense subset of the regular points of $|Z_0|$.

Of course, the definition may be extended to families $\{Z_t\}$ where t belongs to a topological space T.

It will be useful to state clearly the following observation being a mere corollary to the result of [19]:

Proposition 1.7. *If X_0, Y_0 are analytic subsets of an open set $\Omega \subset \mathbf{C}^m$ of pure dimensions p, q respectively, and if $X_0 \cap Y_0$ has pure dimension $p+q-m$, then for any sequences $X_\nu \xrightarrow{K} X_0$ and $Y_\nu \xrightarrow{K} Y_0$ of analytic subsets of Ω of pure dimension p and q respectively, locally the intersections $X_\nu \cap Y_\nu$ are proper* (*i.e. of pure dimension $p+q-m$*) *for all indices large enough.*

Proof. By [19] we know that $X_\nu \cap Y_\nu \xrightarrow{K} X_0 \cap Y_0$. Besides, at any $a \in X_\nu \cap Y_\nu$ we obviously have $\dim_a X_\nu \cap Y_\nu \geq p+q-m$.

Now fix a point $a \in X_0 \cap Y_0$ and choose coordinates in such a way that in a bounded neighbourhood $W = U \times V \subset \mathbf{C}^{p+q-m} \times \mathbf{C}^{2m-p-q}$ of a the natural projection onto U restricted to the set $Z_0 = X_0 \cap Y_0$ is a branched covering. We may ask that $(\overline{U} \times \partial V) \cap Z_0 = \emptyset$. Write $Z_\nu := X_\nu \cap Y_\nu \cap W$. Then, by the convergence, for all indices large enough, $(\overline{U} \times \partial V) \cap Z_\nu = \emptyset$, whereas $Z_\nu \neq \emptyset$.

This means that any such Z_ν projects properly on U. Therefore, if we pick a point $z \in Z_\nu$ and an arbitrarily small polydisc around it, then by the Remmert Proper Map Theorem, $\dim_z Z_\nu \leq p+q-m$. This implies that all the Z_ν's have pure dimension $p+q-m$.

Since any subsequence of $X_\nu \cap Y_\nu$ converges to $X_0 \cap Y_0$ the proof is accomplished. □

Finally, we briefly recall the notion of *c-holomorphic functions* (cf. [16] and [21]) i.e. complex continuous functions that are defined on an analytic set A

[2] By [19], almost all interesections $|Z_\nu| \cap M$ are discrete and so finite. Then the total number of intersection is the formal sum of the intersection points with their respective Draper intersection indices [11] taken into account.

and holomorphic at its regular points Reg A. We denote by $\mathcal{O}_c(A)$ their ring for a fixed A. Their study from the geometric point of view was carried to some extent in [5]–[8]. They share many a property of holomorphic functions, though they form a larger class without really useful differential properties. Their main feature is the fact that they are characterized among all the continuous functions $A \to \mathbf{C}$ by the analycity of their graphs (see [21]). That allows the use of geometric methods. In particular there is an identity principle on irreducible sets (cf. [6]) and we can consider the *order of vanishing* (see [5] where it is introduced and studied) at a point $f(a) = 0$ (when $f \not\equiv 0$) as

$$\mathrm{ord}_a f := \max\{\eta > 0 \,|\, |f(x)| \le \mathrm{const.}\|x\|^\eta, \text{ in a neighbourhood of } a \in A\}.$$

For a holomorphic function defined in an open set this coincides with the degree of the first non-zero form in the expansion into homogeneous forms at a.

2. Continuity principle

LEMMA 2.1. *Let $E \subset \mathbf{R}^k_t \times \mathbf{R}^n_x$ be a closed, nonempty set with continuously varying sections E_t over $F := \pi(E)$ where $\pi(t,x) = t$. Then the function*

$$\delta(t,x) := \mathrm{dist}(x, E_t), \quad (t,x) \in F \times \mathbf{R}^n$$

is continuous.

Proof. The function $\delta(t,\cdot)$ is 1-Lipschitz which means that $\lim_{x\to x_0} \delta(t,x) = \delta(t,x_0)$ is uniform with respect to t. Therefore, in view of the Iterated Limits Theorem, we need only to check that $t \mapsto \delta(t,x)$ is continuous for all x. Indeed, then

$$\lim_{(t,x)\to(t_0,x_0)} \delta(t,x) = \lim_{x\to x_0} \delta(t_0,x) = \delta(t_0,x_0).$$

Fix (t_0,x_0). We know that $E_t \to E_{t_0}$ in the sense of Kuratowski. Then let $d := d(x_0, E_{t_0})$. In particular, for any $\varepsilon > 0$,

$$(K) \qquad \mathbf{B}(x_0, d+\varepsilon) \cap E_{t_0} \neq \emptyset \quad \text{and} \quad \overline{\mathbf{B}}(x_0, d-\varepsilon) \cap E_{t_0} = \emptyset.$$

Then, the convergence implies (cf. [4] Lemma 2.1) that for all t sufficiently close to t_0, condition (K) holds for E_t instead of E_{t_0}. That in turn implies that for all such t,

$$d - \varepsilon < \mathrm{dist}(x_0, E_t) < d + \varepsilon$$

and the proof is complete. □

Remark 2.2. Of course, the lemma is true for a product of metric spaces. In particular we can replace the parameter space $\mathbf{R}^k$ by a 1st countable topological space T, since for such a T the following general Iterated Limits

Theorem holds[3]: if $f : T \times X \to Y$ where X, Y are metric spaces with Y complete, is such that

- $\exists \lim_{t\to t_0} f(t,x) = \varphi(x)$ for any $x \in X$;
- $\exists \lim_{x\to x_0} f(t,x) = \psi(t)$ uniformly in t,

then there exists $\lim_{(t,x)\to(t_0,x_0)} f(t,x) = \lim_{x\to x_0} f(t_0,x) = \psi(t_0)$.

PROPOSITION 2.3. *Consider a pure $(k+n)$-dimensional analytic set $A \subset U \times V \times \mathbf{C}^p$ with proper projection $\pi(t,z,w) = (t,z)$ onto the product domain $U \times V \subset \mathbf{C}^k \times \mathbf{C}^n$. Then*

(1) *The sections A_t vary continuously*;
(2) *The function $\delta : U \times (V \times \mathbf{C}^p) \ni (t,x) \mapsto \mathrm{dist}(x, A_t) \in \mathbf{R}$ is continuous.*

Proof. Since A is closed, the sections A_t are upper semi-continuous, by [4] Proposition 2.7, i.e. for any t_0,

$$\limsup_{t\to t_0} A_t \subset A_{t_0}.$$

We need to check that $A_{t_0} \subset \liminf_{t\to t_0} A_t$. This amounts to proving that for any $x \in A_{t_0}$ and any $t_\nu \to t_0$ we can find points $x_\nu \in A_{t_\nu}$ converging to x. Since π is a branched covering on A, we see that the fibres $\pi^{-1}(\pi(t_\nu,x)) \cap A$ converge to the fibre $\pi^{-1}(\pi(t_0,x)) \cap A$ containing (t_0,x) which gives exactly what we need and the proof of (1) is complete.

Now (2) follows from the previous lemma. □

Remark 2.4. We stress once again that (2) is a simple consequence of (1).

LEMMA 2.5. *Let T be a locally compact topological space and $X \subset \mathbf{C}^m$ a nonempty, locally closed set. If $f : T \times X \to \mathbf{C}$ is continuous and we write $f_t(x) = f(t,x)$, then $t \to t_0$ in T implies the convergence of graphs*:

$$\Gamma_{f_t} \xrightarrow{K} \Gamma_{f_{t_0}}.$$

Proof. In view of Remark 1.3 we need only to check that for any $t_\nu \to t_0$, $f_{t_\nu} \to f_{t_0}$ locally uniformly on X. Take a compact set $K \subset X$. Then $K' = \{t_0\} \times K$ is compact and for a fixed $\varepsilon > 0$ and any $x \in K$ we find neighbourhoods $U_x \times \mathbf{B}(x, r_x)$ of (t_0,x) at points (t,y) for which

$$|f(t,y) - f(t_0,x)| < \varepsilon.$$

By compacity we choose a finite covering $K' \subset \bigcup_{i=1}^p U_i \times \mathbf{B}(x_i, r_i)$ and put $U := \bigcap_{i=1}^p U_i$, then for any $(t,x) \in U \times K$ we have $(t,x) \in U_i \times \mathbf{B}(x_i, r_i)$ for some i and so

$$|f(t,x) - f(t_0,x)| < \varepsilon.$$

This ends the proof. □

[3] We do not have a reference for this fact, but the proof is obvious.

Proposition 2.6. *Let T be a locally compact, connected topological space, A a pure k-dimensional analytic subset of some open set $\Omega \subset \mathbf{C}^m$ and $f : T \times A \to \mathbf{C}$ a continuous function such that for each $t \in T$, $f_t(x) := f(x,t)$ is c-holomorphic on A. Then $t \to t_0$ in T implies*

$$\Gamma_{f_t} \xrightarrow{T} \Gamma_{f_{t_0}}.$$

Proof. By Lemma 2.5 we have

$$\Gamma_{f_t} \xrightarrow{K} \Gamma_{f_{t_0}}.$$

This means that on $\operatorname{Reg} A$, for any $t_\nu \to t_0$, we have a sequence of holomorphic functions converging locally uniformly.

Now, observe that for any $g \in \mathcal{O}_c(A)$, $\Gamma_{g|_{\operatorname{Reg} A}} \subset \operatorname{Reg} \Gamma_g$ is dense. For a testing M at $a \in \Gamma_{f_{t_0}|_{\operatorname{Reg} A}}$ we have the equality $T_a M \cap T_a \Gamma_{f_{t_0}} = \{0\}$ where $T_a \Gamma_{f_{t_0}}$ denotes the tangent space at a, and so $\deg(M \cdot \Gamma_{f_{t_0}}) = 1$. But since in the holomorphic case, the local uniform convergence is a convergence with the tangents, we easily conclude that for sufficiently large indices ν, M is transversal to the manifold (near a) Γ_{f_t} and so $\deg(M \cdot \Gamma_{f_t}) = 1$, too (there are no multiplicities attached to the graphs). To be somewhat more precise, if $a = (a', f_{t_0}(a'))$, then

$$T_{(a', f_{t_\nu}(a'))} \Gamma_{f_{t_\nu}} \xrightarrow{K} T_{(a', f_{t_0}(a'))} \Gamma_{f_{t_0}}$$

and we apply [19] to conclude that M intersects Γ_{f_t} transversally. □

Recall (cf. [5]–[7]) that if $f \in \mathcal{O}_c(A)$ does not vanish identically on any irreducible component of A, where A is a pure k-dimensional analytic subset of a domain $D \subset \mathbf{C}^m$, then we define the *cycle of zeroes* as the Draper proper intersection cycle ([11])

$$Z_f := \Gamma_f \cdot (D \times \{0\}).$$

In the same way we may define the *fibre cycle*, namely

$$[f^{-1}(f(a))] := \Gamma_f \cdot (D \times \{f(a)\})$$

and consider this as a cycle in D.

Now we can state the following Hurwitz-type theorem:

Theorem 2.7. *Let T be a connected topological space, A a pure k-dimensional analytic subset of some domain $D \subset \mathbf{C}^m$, $f : T \times A \to \mathbf{C}$ a continuous function such that for each $t \in T$, $f_t(x) := f(x,t)$ is c-holomorphic on A. Then if $f_{t_0} \not\equiv 0$ on any irreducible component of A and $f_{t_0}^{-1}(0) \neq \emptyset$, we have*

$$Z_{f_t} \xrightarrow{T} Z_{f_{t_0}}, \quad t \to t_0.$$

Proof. By the previous Proposition we have

$$\Gamma_{f_t} \xrightarrow{T} \Gamma_{f_{t_0}}.$$

Of course, $f_{t_0}^{-1}(0)$ is a hypersurface in A (cf. the identity principle from [6]) which means that the intersection $\Gamma_{f_{t_0}} \cap (D \times \{0\})$ is proper (i.e of the minimal dimension possible: $k-1$). By [18] Lemma 3.5 (cf. Proposition 1.7) we conclude that for any sequence $t_\nu \to t_0$,

$$\Gamma_{f_{t_\nu}} \cdot (D \times \{0\}) \xrightarrow{T} \Gamma_{f_{t_0}} \cdot (D \times \{0\}).$$

This ends the proof. □

Corollary 2.8. *Let* $g \in \mathcal{O}_c(A)$, $g \neq \text{const.}$ *on any irreducible component of* $A \subset D$, *where* A *is pure* k*-dimensional. Then for any* $t_0 \in A$,

$$[g^{-1}(t)] \xrightarrow{T} [g^{-1}(t_0)], \quad t \to t_0.$$

Proof. Let $f : A \times \mathbf{C} \ni (x,t) \mapsto g(x) - t \in \mathbf{C}$. By [6], we conclude that all the nonempty fibres of g have pure dimension $k-1$. Then f satisfies the assumptions of the preceding Theorem and

$$\begin{aligned} Z_{f_t} &= \Gamma_{f_t} \cdot (D \times \{0\}) \\ &= \Gamma_g \cdot (D \times \{t\}) \\ &= [g^{-1}(t)], \end{aligned}$$

since $\Phi(x,s) = (x, s+t)$ is an automorphism of $D \times \mathbf{C}$ sending Γ_{f_t} to Γ_g and $D \times \{0\}$ to $D \times \{t\}$. This ends the proof. □

Before the next corollary recall that for any positive cycle $Z = \sum \alpha_\iota S_\iota$ we define its *local degree* at $a \in |Z|$ as $\deg_a Z := \sum \alpha_\iota \deg_a S_\iota$, where $\deg_a S_\iota$ is the usual local degree (Lelong number) with the convention that $\deg_a S_\iota = 0$ if $a \notin S_\iota$.

Corollary 2.9. *Under the assumptions of the preceding Theorem suppose in addition that* $f_t(a) = 0$ *for all* $t \in T$ *and some fixed* $a \in A$. *Then for all* t *close enough to* t_0,

$$\deg_a Z_{f_t} \leq \deg_a Z_{f_{t_0}},$$

for the local degrees at a.

Proof. Take any affine subspace L through a, of dimension $m-k+1$ and such that

$$L \cdot Z_{f_{t_0}} = \deg_a Z_{f_{t_0}} \cdot \{a\}.$$

Then by Theorem 2.7 together with [18] Lemma 3.5,

$$L \cdot Z_{f_t} \xrightarrow{T} L \cdot Z_{f_{t_0}}$$

which ends the proof, since

$$L \cdot Z_{f_t} = \sum_{b \in L \cap f_t^{-1}(0)} i(L \cdot Z_{f_t}, b)\{b\}$$

and for each Draper intersection index (multiplicity) $i(L \cdot Z_{f_t}, b)$ we have

$$i(L \cdot Z_{f_t}, b) \geq \deg_b Z_{f_t},$$

for $\deg_b L = 1$. Therefore, we obtain by the convergence, for all t sufficiently close to t_0,

$$\begin{aligned}\deg_a Z_{f_{t_0}} &= \deg(L \cdot Z_{f_{t_0}})\\ &= \deg(L \cdot Z_{f_t})\\ &= \sum_{b \in L \cap f_t^{-1}(0)} i(L \cdot Z_{f_t}, b)\\ &\geq i(L \cdot Z_{f_t}, a) \geq \deg_a Z_{f_t},\end{aligned}$$

as $a \in L \cap f_t^{-1}(0)$ (for all t). □

3. On the Łojasiewicz inequality and the total degree

We recall one result from [17] (see also [7] Theorem 2.3) which is the basis which we shall work upon.

THEOREM 3.1 ([17] Theorem 1). *Let $f : \Omega \to \mathbf{C}$ be holomorphic in a (connected) neighbourhood Ω of $0 \in \mathbf{C}^m$. If f is non-constant and $f(0) = 0$ then there is a neighbourhood U of zero such that the following Łojasiewicz inequality holds*:

$$|f(x)| \geq \text{const.dist}(x, f^{-1}(0))^{\operatorname{ord}_0 f}, \quad x \in U$$

where $\operatorname{ord}_0 f$ denotes the order of vanishing of f at zero. Moreover, this is the best exponent possible.

As before we consider the intersection cycle of zeroes $Z_f = \Gamma_f \cdot (\Omega \times \{0\})$.

PROPOSITION 3.2 ([7] Proposition 2.1). *In the setting introduced above, $\deg_0 Z_f = \operatorname{ord}_0 f$.*

We easily generalize these results to c-holomorphic functions, although only in a weak sense (compare the following theorem with the results of [8]). Consider a pure k-dimensional ($k \geq 2$) analytic subset A of a neighbourhood Ω of $0 \in \mathbf{C}^m$ with $0 \in A$. Assume that $f \in \mathcal{O}_c(A)$ satisfies $f(0) = 0$ and does not vanish identically on any irreducible component of A containing zero.

THEOREM 3.3. *In the c-holomorphic setting introduced above, there is a neighbourhood W of zero such that*

$$|f(z)| \geq \text{const.dist}(z, f^{-1}(0))^{\deg_0 Z_f \cdot \deg_0 f^{-1}(0)}, \quad z \in W \cap A.$$

Proof. Write $\mathbf{C}^m = \mathbf{C}^{k-1} \times \mathbf{C}^{m-k+1}$ with coordinates (x, y).

We may assume that the coordinates are chosen in such a way that the projection $\pi(x, y) = x$ onto the first $k-1$ coordinates is proper on $Z := f^{-1}(0) \cap (U \times V)$ with covering number equal to the local degree $\deg_0 f^{-1}(0) =: d$. Here $U \times V$ is a neighbourhood of the origin satisfying $(\{0\} \times \overline{V}) \cap f^{-1}(0) = \{0\}$.

Applying Proposition 2.2 from [3] we find a holomorphic mapping $F : U \times \mathbf{C}^{m-k+1} \to \mathbf{C}^p$ such that $F^{-1}(0) = f^{-1}(0) \cap (U \times V)$ and

$$(*) \qquad \|F(x, y)\| \geq \operatorname{dist}((x, y), Z)^d, \quad (x, y) \in U \times \mathbf{C}^{m-k+1}.$$

If we write $F = (F_1, \ldots, F_p)$ we observe that $F_j^{-1}(0) \cap A \supset f^{-1}(0) \cap (U \times V)$ for all j. The intersection of the graph Γ_f with $\Omega \times \{0\}$ being proper, we can now apply the c-holomorphic Nullstellensatz from [6]. In other words, we find a neighbourhood $W \subset U \times V$ of zero and p c-holomorphic functions h_j on $W \cap A$ for which

$$(**) \qquad F_j^\delta = h_j f \quad \text{on } A \cap W, \quad j = 1, \ldots, p$$

with $\delta = \deg_0 Z_f$.

Combining $(*)$ and $(**)$ we eventually obtain the inequality looked for. □

Proposition 3.4. *Under the assumptions of the previous theorem,*

$$\deg_0 Z_f \cdot \deg_0 f^{-1}(0) \geq \operatorname{ord}_0 f.$$

Proof. This follows from Lemma 4.8 in [5]. □

Using Corollary 2.9 and Proposition 3.2 or simply looking at the expansion into a (Hartogs) power series, we easily obtain

Lemma 3.5. *If $f = f(t, x) \in \mathcal{O}_{k+m}$ is such that $f_t(0) := f(t, 0) = 0$ for all t small enough and $f_0 = f(0, \cdot)$ is non-constant, then*

$$\operatorname{ord}_0 f_t \leq \operatorname{ord}_0 f_0$$

for all t sufficiently close to zero.

Example 3.6. The inequality may be strict as we easily see by taking $f(t, x) = tx + x^2$; then for $t \neq 0$, $\operatorname{ord}_0 f_t = 1 < \operatorname{ord}_0 f_0 = 2 = \operatorname{ord}_0 f$. But of course there is no direct relation with $\operatorname{ord}_0 f$, it suffices to take $f(t, x) = tx + x^3$ in order to have $\operatorname{ord}_0 f_t = 1 < \operatorname{ord}_0 f = 2 < \operatorname{ord}_0 f_0$.

The proof of Theorem 3.1 suggests the following result.

Proposition 3.7. *Let $V \times W \Subset \mathbf{C}^{m-1} \times \mathbf{C}$ be a bounded, connected neighbourhood of zero (a polydisc) and let $P \in \mathcal{O}(V)[t]$ be unitary and such that*

$P^{-1}(0) \subset (V \times W)$ projects properly onto V. Then in $V \times W$ there is

$$|P(x,t)| \geq \mathrm{dist}((x,t), P^{-1}(0))^{\delta}$$

with $\delta = \deg((\{0\}^{m-1} \times W) \cdot Z_P)$.

Proof. Recall from [7] that $Z_P = \sum \alpha_j S_j$ where S_j are the irreducible components of $P^{-1}(0)$ and $\alpha_j = \min\{\mathrm{ord}_z\, P \,|\, z \in \mathrm{Reg}\, S_j\}$ is the generic order of vanishing of P along S_j. Note that each S_j projects onto the whole of V.

Now, since the intersections $(\{x\} \times W) \cap P^{-1}(0)$ are proper, by [18] (see also [2]) we conclude that for any $x_\nu \to 0$ we have

$$(\{x_\nu\} \times W) \cdot Z_P \xrightarrow{T} (\{0\}^{m-1} \times W) \cdot Z_P$$

and so $\deg((\{x_\nu\} \times W) \cdot Z_P) = \delta$ for sufficiently large ν.

Observe that for the generic $x \in V$ we have the following situation: $\{x\} \times W$ intersects $P^{-1}(0)$ transversally at d regular points $b^{(i)} = (x, t^{(i)})$, where d is the multiplicity of the branched covering $P^{-1}(0) \to V$, each of these points belongs to exactly one S_j, all the S_j's appear in this assignment, and $\mathrm{ord}_{b^{(i)}}\, P = \alpha_j$ for the unique j such that $b^{(i)} \in S_j$. Therefore, we may write

$$\delta = \sum_{b \in (\{x\} \times W) \cap P^{-1}(0)} \mathrm{ord}_b\, P.$$

On the other hand, for any such point x we have

$$P(x,t) = \prod_{i=1}^{d} (t - t^{(i)})^{n_i}$$

with n_i independent of the point chosen. We observe that $n_i = \mathrm{ord}_{b^{(i)}}\, P$. Indeed, if we write $\{x\} \times W$ as the zero-set of an affine mapping $\ell = (\ell_1, \ldots, \ell_{m-1})$ restricted to $V \times W$, then the transversality of the intersection $(\{x\} \times W) \cap P^{-1}(0)$ implies by the Tsikh-Yuzhakov result (see [2]) that the multiplicity $m_{b^{(i)}}(P, \ell)$ at each point $b^{(i)}$ of the proper mapping germ (P, ℓ) is equal to the product of the orders of P and the ℓ_j's, i.e. to $\mathrm{ord}_{b^{(i)}}\, P$. On the other hand, by [2] pp. 107–108 we easily see that

$$m_{b^{(i)}}(P, \ell) = \mathrm{ord}_{t^{(i)}}\, P|_{\{x\} \times W} = n_i.$$

Therefore, $\delta = \sum_{i=1}^{d} n_i$. This allows us to write, for the generic $x \in V$, the following inequalities:

$$\begin{aligned} |P(x,t)| &= \prod_{i=1}^{d} |t - t^{(i)}|^{n_i} \\ &= \prod_{i=1}^{d} \|(x,t) - (x, t^{(i)})\|^{n_i} \\ &\geq \mathrm{dist}((x,t), P^{-1}(0))^{\sum_{i=1}^{d} n_i}. \end{aligned}$$

Extending this by continuity to the whole of $V \times W$ ends the proof. □

Remark 3.8. The proof above is in fact an extrapolation of the proof of Theorem 3.1, where we use the Weierstrass Preparation in a neighbourhood of zero such that $(\{0\} \times W) \cap f^{-1}(0) = \{0\}$ and $\mathrm{ord}_0 f = \mathrm{ord}_0 P$.

Corollary 3.9. *If* $f : V \times W \to \mathbf{C}$ *is a holomorphic function such that* $f^{-1}(0)$ *projects properly onto* V, *then for some possibly smaller neighbourhood* $U \subset V \times W$ *of zero,* f *satisfies the Łojasiewicz inequality in* U *with exponent* $\deg((\{0\} \times W) \cdot Z_f)$.

Proof. In $V \times W$ we can apply the Weierstrass Preparation Theorem and write $f = hP$ with a holomorphic function h such that $h^{-1}(0) = \emptyset$. Shrinking the neighbourhood (actually, we need only to shrink V if any), we may assume that $\inf|h| > 0$. Then $Z_f = Z_P$, since $\mathrm{ord}_b f = \mathrm{ord}_b P$. The preceding Proposition gives the result. □

4. The Łojasiewicz inequality with parameter

Eventually, we are ready to prove the main result.

Theorem 4.1. *Assume that* $f : T \times \Omega \to \mathbf{C}$ *is a continuous function where* T *is a locally compact, connected topological space,* $\Omega \subset \mathbf{C}^m$ *is a domain, and for all* $t \in T$, $f_t \in \mathcal{O}(\Omega)$ *does not vanish identically. Assume moreover that* $0 \in \Omega$ *and* $f_t(0) = 0$ *for any* t. *Then there is a neighbourhood* $U \subset \Omega$ *of zero such that, for all* t *close enough to* t_0,

$$|f_t(x)| \geq c(t)\,\mathrm{dist}(x, f_t^{-1}(0))^{\alpha}, \quad x \in U$$

where $c(t) > 0$ *is a constant depending on the parameter, but the exponent*

$$\alpha = \mathrm{ord}_0 f_{t_0}$$

is uniform.

Proof. By Theorem 2.7 we know in particular that $f_t^{-1}(0) \xrightarrow{K} f_{t_0}^{-1}(0)$. Of course these sets are hypersurfaces. The type of convergence implies that we can choose coordinates in $\mathbf{C}^m$ in such a way that for some neighbourhood $V \times W \subset \mathbf{C}^{m-1} \times \mathbf{C}$ of zero, V connected and W a disc, we have

$$f_t^{-1}(0) \cap (V \times \partial W) = \emptyset$$

for all t close enough to t_0. This means that the zero-sets intersected with $V \times W$ project properly onto V. Moreover, we may assume that

$$(\{0\}^{m-1} \times W) \cdot Z_{f_{t_0}} = \mathrm{ord}_0 f_{t_0}\{0\}.$$

In the situation considered, the proof of Proposition 3.7 shows that the Łojasiewicz inequality for f_t is satisfied in $V \times W$ with the exponent $d_t =$

$\deg((\{0\}^{m-1} \times W) \cdot Z_{f_t})$:

$$(*) \qquad |f_t(x)| \geq c(t)\,\mathrm{dist}(x, f_t^{-1}(0))^{d_t}, \quad x \in V \times W$$

where $c(t) > 0$ is a constant.

But then, for t close enough to t_0, the numbers d_t fortunately coincide with $\deg(\{0\}^{m-1} \times W) \cdot Z_{f_{t_0}} = \mathrm{ord}_0 f_{t_0}$ by the convergence (Theorem 2.7).

This ends the proof. □

It seems hard to obtain a satisfactory c-holomorphic counter-part to this Theorem due to the use of the Nullstellensatz with parameter. The best we were able to obtain is the following Theorem.

Theorem 4.2. *Assume that* $f: T \times A \to \mathbf{C}$ *is a continuous function where* T *is a locally compact, connected topological space,* A *is a pure* k*-dimensional analytic subset of an open set* $\Omega \subset \mathbf{C}^m$, $0 \in A$, *and for all* $t \in T$, $f_t \in \mathcal{O}_c(A)$ *does not vanish identically on any irreducible component of* A *through zero. Assume moreover that* $f_t(0) = 0$ *for any* t. *Then there is a neighbourhood* $U \subset \Omega$ *of zero such that, for all* t *close enough to* t_0,

$$|f_t(x)| \geq c(t)\,\mathrm{dist}(x, f_t^{-1}(0))^{\alpha}, \quad x \in A \cap U$$

where $c(t) > 0$ *is a constant depending on the parameter, but the exponent*

$$\alpha = (\deg_0 Z_{f_{t_0}})^2$$

is uniform.

Proof. We give the proof in several steps.

Step 1. Choose coordinates in $\mathbf{C}^m$ in such a way that A projects properly onto the first k coordinates and, moreover,

$$i((\{0\}^{k-1} \times \mathbf{C}^{m-k+1}) \cdot Z_{f_{t_0}}; 0) = \deg_0 Z_{f_{t_0}}.$$

Let $\ell: \mathbf{C}^m \to \mathbf{C}^{k-1}$ be the linear epimorphism whose kernel is exactly $\{0\}^{k-1} \times \mathbf{C}^{m-k+1}$. Write

$$\varphi_t : A \ni x \mapsto (f_t(x), \ell(x)) \in \mathbf{C} \times \mathbf{C}^{k-1}$$

for $t \in T$. Fix a polydisc $V \times W \subset \mathbf{C}^{k-1} \times \mathbf{C}^{m-k+1}$ centred at zero such that

$$(\{0\}^{k-1} \times \overline{W}) \cap f_{t_0}^{-1}(0) = \{0\}.$$

In particular we may assume that $f_{t_0}^{-1}(0)$ projects properly onto V.

Step 2. The latter intersection corresponds to $(\overline{V \times W} \times \{0\}^k) \cap \Gamma_{\varphi_{t_0}}$ which means that there is a polydisc $P \subset \mathbf{C}^k$ such that the pure k-dimensional analytic set $(V \times W \times P) \cap \Gamma_{\varphi_{t_0}}$ projects properly *onto* P along $V \times W$. In other words, $\varphi_{t_0}|_{(V \times W) \cap A}$ is proper with image P.

As in Lemma 2.5, the continuity of

$$\Phi : T \times A \ni (t, x) \mapsto \varphi_t(x) \in \mathbf{C}^k$$

implies the Kuratowski convergence of the graphs $\Gamma_{\varphi_t} \xrightarrow{K} \Gamma_{\varphi_{t_0}}$ as $t \to t_0$. Therefore, by the same argument as in Proposition 1.7, we conclude that for all t close enough to t_0, the restrictions of the natural projection

$$\pi_t : (V \times W \times P) \cap \Gamma_{\varphi_t} \to P$$

are branched coverings. In particular, all these φ_t have the same image P. Let q_t denote the multiplicity of the branched covering $\varphi_t|_{A\cap(V\times W)}$.

STEP 3. By the choice of $V \times W$ and Theorem 2.7, we know (cf. the proof of the previous Theorem) that for all t close enough to t_0, the zero-sets $f_t^{-1}(0) \cap (V \times W)$ project properly onto V. Let d_t denote the multiplicity of such a branched covering.

Since by Theorem 2.7 we know that the cycles of zeroes of the restrictions $f_t|_{A\cap(V\times W)}$ converge with $t \to t_0$ in the sense of Tworzewski, we easily conclude from [18] Lemma 3.5 and [19] that

$$(\star) \qquad d_t \leq \deg((\{0\}^{k-1} \times W) \cdot Z_{f_t}) = \deg((\{0\}^{k-1} \times W) \cdot Z_{f_{t_0}}) = \deg_0 Z_{f_{t_0}}.$$

On the other hand, we observe that $q_t = \deg((\{0\}^{k-1} \times W) \cdot Z_{f_t})$ and so

$$(\star\star) \qquad q_t \leq \deg_0 Z_{f_{t_0}}.$$

Indeed, it is easy to see that q_t is in fact the multiplicity $\tilde{q}_t$ of the projection

$$\pi : \mathbf{C}^{k-1} \times \mathbf{C}^{m-k+1} \times \mathbf{C} \ni (u, v, w) \mapsto (u, w) \in \mathbf{C}^{k-1} \times \mathbf{C}$$

over P when restricted to $\Gamma_t := \Gamma_{f_t} \cap (V \times W \times \mathbf{C})$, because for a generic point $(x_0, w_0) \in P$, we have

$$\begin{aligned}
\tilde{q}_t &= \#\{(x, y, f_t(x, y)) \mid (x, y) \in V \times W, \pi(x, y, f_t(x, y)) = (x_0, w_0)\} \\
&= \#\{y \in W \mid w_0 = f_t(x_0, y)\} = \# f_t^{-1}(w_0) \cap (\{x_0\} \times W) \\
&= \#\{(x, y, f_t(x, y), \ell(x, y)) \mid (x, y) \in V \times W, w_0 = f_t(x, y), \ell(x, y) = x_0\} \\
&= \#\{(x, y, \varphi_t(x, y)) \mid (x, y) \in V \times W, \pi_t(x, y, \varphi_t(x, y)) = (x_0, w_0)\} = q_t.
\end{aligned}$$

The multiplicity $\tilde{q}_t$, in turn, by the classical Stoll Formula[4], coincides with the total degree of the intersection cycle $\pi^{-1}(0) \cdot \Gamma_t$. In other words, we obtain

$$q_t = \deg((\{0\}^{k-1} \times W \times \{0\}) \cdot \Gamma_t).$$

[4] If the natural projection $\pi : D \times \mathbf{C}^p \to D$ onto the domain $D \subset \mathbf{C}^k$ is proper on the pure k-dimensional analytic set $X \subset D \times \mathbf{C}^p$ with covering degree d, then Stoll's Formula states that for any $y \in D$, $d = \sum_{x\in\pi^{-1}(y)\cap X} m_x(\pi|_X)$ where $m_x(\pi|_X)$ denotes the local multiplicity of the projection at the point x of the fibre. As already observed in [11], $m_x(\pi|_X) = i(X \cdot \pi^{-1}(y); x)$, which means that $d = \deg(X \cdot \pi^{-1}(y))$.

However, in view of [20] Theorem 2.2, we can write

$$\begin{aligned}(\{0\}^{k-1} \times W \times \{0\}) \cdot \Gamma_t \\ &= (\{0\}^{k-1} \times W \times \{0\}) \cdot_{V\times W\times\{0\}} ((V \times W \times \{0\}) \cdot_{V\times W\times\mathbf{C}} \Gamma_t) \\ &= (\{0\}^{k-1} \times W \times \{0\}) \cdot_{V\times W\times\{0\}} Z_{f_t|_{A\cap(V\times W)}} \\ &= (\{0\}^{k-1} \times W) \cdot_{V\times W} Z_{f_t},\end{aligned}$$

whence $q_t = \deg((\{0\}^{k-1} \times W) \cdot Z_f)$ as required.

Step 4. As in the proof of Theorem 3.3, by [3] Proposition 2.2 we know that for each t close to t_0 there are $p_t = d_t(m-k)+1$ holomorphic functions $F_{t,j} : V \times \mathbf{C}^{m-k+1} \to \mathbf{C}$ whose common zeroes form coincide with the set $f_t^{-1}(0) \cap (V \times W)$ and for which

$$\|(F_{t,1}, \dots, F_{t,p_t})(x)\| \geq \mathrm{dist}(x, f_t^{-1}(0) \cap (V \times W))^{d_t}$$

for all $x \in V \times W$.

Now, we can apply Lemma 3.1 from [6] (compare [14]) in order to get *on the whole* of $A \cap (V \times W)$,

$$F_{t,j}^{q_t} = h_{t,j} f_t, \quad j = 1, \dots, p_t,$$

with some functions $h_{t,j} \in \mathcal{O}_c(A \cap (V \times W))$.

This leads to the inequalities

$$(\#) \qquad |f_t(x)| \geq c(t)\, \mathrm{dist}(x, f_t^{-1}(0))^{d_t q_t}, \quad x \in A \cap (V \times W)$$

for all t close to t_0 and some constants $c(t) > 0$.

Step 5. Thanks to the continuity of the zero-sets (cf. Theorem 2.7), Proposition 2.3 (cf. Remark 2.4) allows us to choose an arbitrarily small neighbourhood T_0 of t_0 and a neighbourhood $U \subset V \times W$ of zero such that for all $t \in T_0$ and all $x \in U$, we have

$$\mathrm{dist}(x, f_t^{-1}(0)) < 1.$$

Therefore, we may increase *ad libitum* the exponent in (#), provided $x \in A \cap U$. The estimates ($\star$) and ($\star\star$) end the proof. □

Remark 4.3. In both theorems in this section the assumption that for any $t \in T$, f_t does not vanish identically on the irreducible components of the domain is automatically satisfied, if we just assume that f_{t_0} does not vanish identically on the irreducible componens of the domain (cf. Proposition 1.7 and Theorem 2.7).

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