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CURVATURE PROPERTIES OF HOMOGENEOUS REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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Abstract

In this paper, we study curvature properties of all homogeneous real hypersurfaces in nonflat complex space forms, and determine their minimalities and the signs of their sectional curvatures completely. These properties reflect the sign of the constant holomorphic sectional curvature c of the ambient space. Among others, for the case of $c < 0$ there exist homogeneous real hypersurfaces with positive sectional curvature and also ones with negative sectional curvature, whereas for the case of $c > 0$ there do not exist any homogeneous real hypersurfaces with nonpositive sectional curvature.

1. Introduction

We denote by $\tilde{M}_n(c)$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c $(\neq 0)$, which is called an n-dimensional *nonflat complex space form*. It is well-known that $\tilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbf{C}P^n(c)$ or an n-dimensional complex hyperbolic space $\mathbf{C}H^n(c)$ according as c is positive or negative. In Riemannian submanifold theory, homogeneous real hypersurfaces M^{2n-1} of a nonflat complex space form are one of fundamental examples, and have been studied actively. Here, those hypersurfaces M^{2n-1} are orbits of some subgroups of the full isometry group of the ambient space. Although there exists a duality between $\mathbf{C}P^n(c)$ and $\mathbf{C}H^n(c)$, real hypersurfaces in these spaces present different aspects according to the sign of the holomorphic sectional curvatures c of the ambient spaces. For instance, in $\mathbf{C}H^n(c)$ we have many homogeneous real hypersurfaces which are *not* Hopf

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hypersurfaces, whereas all homogeneous real hypersurfaces in $\mathbf{C}P^n(c)$ must be Hopf (for details, see Section 2).

In this paper, we study curvature properties of homogeneous real hypersurfaces of these ambient spaces. Note that such hypersurfaces have completely been classified, and the principal curvatures have also been calculated. Therefore, one can determine the minimality by direct calculations. The main contribution of this paper is to determine the signs of the sectional curvatures for all homogeneous real hypersurfaces in $\mathbf{C}P^n(c)$ and in $\mathbf{C}H^n(c)$.

We here summarize our results. For the case of $c > 0$, Takagi ([14]) classified homogeneous real hypersurfaces in $\mathbf{C}P^n(c)$. According to his result, such hypersurfaces can be classified into six cases, namely of types (A_1), (A_2), (B), (C), (D) and (E). For example, homogeneous real hypersurfaces of type (A_1) are geodesic spheres $G(r)$ of radius r, and those of type (A_2) are tubes of radius r around a totally geodesic $\mathbf{C}P^{\ell}(c)$. Unifying these two types, we call them of type (A) (see Section 2 for details and other hypersurfaces). By direct computations we have the following fact (*cf.* [10, 11]).

Theorem 1. *In the class of all homogeneous real hypersurfaces M of $\mathbf{C}P^n(c)$ with $n \geqq 2$, the following hold*:

(1) *For each family of types* (A_1), (A_2), (B), (C), (D) *and* (E), *we have just one example which is minimal. Hence, minimal homogeneous real hypersurfaces are classified into six types*;
(2) *M has nonnegative sectional curvature at its each point if and only if M is of type* (A). *In particular, M has positive sectional curvature at its each point if and only if M is of type* (A_1);
(3) *There exists no example M all of whose sectional curvatures are nonpositive at its each point.*

On the other hand, for the case of $c < 0$, Berndt and the second author ([4]) classified all homogeneous real hypersurfaces in $\mathbf{C}H^n(c)$ into eight cases of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$), (A_2), (B), (S), (W_1) and (W_2). For instance, the type ($\mathrm{A}_{1,0}$) is a class of geodesic spheres $G(r)$ of radius r. The type (S) is a class consisting of the minimal homogeneous ruled real hypersurface S determined by a horocycle in a totally geodesic $\mathbf{R}H^2(c/4)$, and equidistant hypersurfaces from S at distance r (see Section 2 for details). We shall establish a theorem corresponding to Theorem 1 in the case of $c < 0$.

Theorem 2. *In the class of all homogeneous real hypersurfaces M of $\mathbf{C}H^n(c)$ with $n \geqq 2$, the following hold*:

(1) *There exists just one example which is minimal in $\mathbf{C}H^n(c)$. It is the homogeneous ruled real hypersurface S, which is one of examples of type* (S);
(2) *M has nonnegative* (*resp. positive*) *sectional curvature at its each point if and only if M is a geodesic sphere $G(r)$ of sufficiently small radius r. In these cases, M is of type* ($\mathrm{A}_{1,0}$);

(3) *M has nonpositive* (*resp. negative*) *sectional curvature at its each point if and only if M is either the homogeneous ruled real hypersurface S or an equidistant hypersurface at sufficiently small distance r from S. Then, M belongs to the class of type* (S).

Note that the estimation on the distance r in Theorem 2 (2) and (3) are given explicitly in Propositions 4.3 and 7.4, respectively.

Our theorems completely determine the minimality, and the signs of the sectional curvatures, for all homogeneous real hypersurfaces in $\mathbf{C}P^n(c)$ and also in $\mathbf{C}H^n(c)$. This would lay a foundation for the study of real hypersurfaces in nonflat complex space forms. Furthermore, from our results one could find out some clear differences between the class of real hypersurfaces in $\mathbf{C}P^n(c)$ and that in $\mathbf{C}H^n(c)$, which can be summarized as follows.

Remark 1.1. For homogeneous real hypersurfaces in $\tilde{M}_n(c)$, one has the following:

(1) $\mathbf{C}H^n(c)$ admits just one minimal example, whereas $\mathbf{C}P^n(c)$ admits several minimal examples (the number of minimal examples depends on the dimension n of ambient space $\mathbf{C}P^n(c)$).
(2) $\mathbf{C}H^n(c)$ admits homogeneous real hypersurfaces with positive sectional curvature, and also those with negative curvature. On the other hand, $\mathbf{C}P^n(c)$ admits homogeneous real hypersurfaces with positive sectional curvature, but does not admit those with nonpositive curvatures.

We describe the contents of this paper. In Section 2 we recall some fundamental notions and the classification of all homogeneous real hypersurfaces in $\tilde{M}_n(c)$. In Section 3 we will sketch out the proof of Theorem 1. The proof of Theorem 2 will be broken up into some separate sections. Section 4 deals with the case of homogeneous Hopf hypersurfaces of $\mathbf{C}H^n(c)$. Sections 5, 6 and 7 take up the cases of types (W_1), (W_2) and (S), respectively.

2. Preliminaries

Let $\tilde{M}_n(c)$ be an n ($\geqq 2$)-dimensional nonflat complex space form and M be a real hypersurface of $\tilde{M}_n(c)$ through an isometric immersion. In this section, we recall some fundamental notions and prepare some known formulas, in order to compute the sectional curvatures of M in terms of the shape operators. We also recall the classification of all homogeneous real hypersurfaces in $\tilde{M}_n(c)$.

First of all we set up some notations. For $\tilde{M}_n(c)$, denote by g the standard Riemannian metric and by J the canonical Kähler structure. For a real hypersurface M, denote by $\mathcal{N}$ a unit normal local vector field, and also by the same notation g the induced Riemannian metric for simplicity. Then it is well-known that an almost contact metric structure (ϕ, ξ, η, g) on M, associated with $\mathcal{N}$, can be canonically defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N}, \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N})$$

for arbitrary vector fields X and Y on M. The structure satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$\eta(\xi) = 1, \quad \phi\xi = 0 \quad \text{and} \quad \eta(\phi X) = 0.$$

We call ξ the *characteristic vector field*. Denote by A the shape operator of M in $\tilde{M}_n(c)$. Then one knows the equation of Gauss, which represents the curvature tensor of M in terms of g, ϕ and A. Therefore, one has a similar expression of the sectional curvature as follows. For more details, we refer to [12].

Lemma 2.1. *Let M be a real hypersurface of $\tilde{M}_n(c)$, and use the above notations. Then, the sectional curvature $K(X, Y)$ of the real plane spanned by a pair $\{X, Y\}$ of orthonormal vectors is given by*

$$K(X, Y) = (c/4)(1 + 3g(\phi X, Y)^2) + g(AX, X)g(AY, Y) - g(AX, Y)^2.$$

An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\tilde{M}_n(c)$, and an eigenvalue of A is called a *principal curvature* of M in $\tilde{M}_n(c)$. Let V_λ denote the eigenspace associated with the principal curvature λ. That is, we set $V_\lambda = \{X \in TM \mid AX = \lambda X\}$. We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. We need the following lemma, which is one of the fundamental properties of principal curvatures of a Hopf hypersurface M in $\tilde{M}_n(c)$.

Lemma 2.2 ([7, 9]). *Let M be a Hopf hypersurface of a nonflat complex space form $\tilde{M}_n(c)$ with $n \geqq 2$. If a nonzero vector $X \in TM$ orthogonal to ξ satisfies $AX = \lambda X$, then $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$ holds, where δ is the principal curvature associated with ξ.*

Now, we survey the classification of homogeneous real hypersurfaces in a nonflat complex space form $\tilde{M}_n(c)$. In the case of $c > 0$, by virtue the works of Takagi and Kimura ([8, 14, 15]), we can see that a homogeneous real hypersurface in $\mathbf{C}P^n(c)$ with $n \geqq 2$ is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;

(A_2) A tube of radius r around a totally geodesic $\mathbf{C}P^\ell(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius r around a complex hyperquadric $\mathbf{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;

(C) A tube of radius r around the Segre embedding of $\mathbf{C}P^1(c) \times \mathbf{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and n ($\geqq 5$) is odd;

(D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbf{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;

(E) A tube of radius r around a Hermitian symmetric space $\mathrm{SO}(10)/\mathrm{U}(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

For the notational convention as stated in the introduction, unifying types (A_1) and (A_2) we call them of type (A).

In the case of $c < 0$, let M be a homogeneous real hypersurface in $\mathbf{C}H^n(c)$ with $n \geqq 2$. Then, due to [4], we know that M is locally congruent to one of the following:

(A_0) A horosphere in $\mathbf{C}H^n(c)$;

($\mathrm{A}_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;

($\mathrm{A}_{1,1}$) A tube of radius r around a totally geodesic $\mathbf{C}H^{n-1}(c)$, where $0 < r < \infty$;

(A_2) A tube of radius r around a totally geodesic $\mathbf{C}H^{\ell}(c)$ with $1 \leqq \ell \leqq n-2$, where $0 < r < \infty$;

(B) A tube of radius r around a totally real totally geodesic $\mathbf{R}H^n(c/4)$, where $0 < r < \infty$;

(S) The homogeneous ruled real hypersurface S determined by a horocycle in a totally geodesic $\mathbf{R}H^2(c/4)$ in $\mathbf{C}H^n(c)$, or an equidistant hypersurface from S at distance r, where $0 < r < \infty$;

(W_1) A tube of radius r around the minimal ruled submanifold W^{2n-k} with $k \in \{2, \dots, n-1\}$, where $0 < r < \infty$;

(W_2) A tube of radius r around the minimal ruled submanifold W_{φ}^{2n-k} for some $\varphi \in (0, \pi/2)$ and $k \in \{2, \dots, n-1\}$, where k is even and where $0 < r < \infty$.

Unifying real hypersurfaces of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$) and (A_2), we call them hypersurfaces of type (A). Note that, in the above list, all examples of types (A) and (B) are Hopf hypersurfaces and others are non-Hopf.

In what follows, we put $\tilde{r} := \sqrt{|c|}r$. We use this convention throughout the paper for the purpose of simplicity.

3. Sketch of the proof of Theorem 1

In this section, we shall outline the proof of Theorem 1. First, we recall the principal curvatures of homogeneous real hypersurfaces in $\mathbf{C}P^n(c)$.

Lemma 3.1 (*cf.* [15]). *The principal curvatures of homogeneous real hypersurfaces in* $\mathbf{C}P^n(c)$ *are given as follows*:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}\right)$	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}\right)$	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}-\dfrac{\pi}{4}\right)$	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}-\dfrac{\pi}{4}\right)$
λ_2	—	$-\dfrac{\sqrt{c}}{2}\tan\left(\dfrac{\tilde{r}}{2}\right)$	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}+\dfrac{\pi}{4}\right)$	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}+\dfrac{\pi}{4}\right)$
λ_3	—	—	—	$\dfrac{\sqrt{c}}{2}\cot\left(\dfrac{\tilde{r}}{2}\right)$
λ_4	—	—	—	$-\dfrac{\sqrt{c}}{2}\tan\left(\dfrac{\tilde{r}}{2}\right)$
δ	$\sqrt{c}\cot\tilde{r}$	$\sqrt{c}\cot\tilde{r}$	$\sqrt{c}\cot\tilde{r}$	$\sqrt{c}\cot\tilde{r}$

The multiplicities of these principal curvatures are given as follows:

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\delta)$	1	1	1	1	1	1

Note that the principal curvature δ in the above table is associated with the characteristic vector ξ, that is, $A\xi = \delta\xi$. Next, the following proves the first assertion (1) of Theorem 1.

Proposition 3.2. *A homogeneous real hypersurface M in $\mathbf{C}P^n(c)$ with $n \geqq 2$ is minimal if and only if it is congruent to either of type* (A_1), (A_2), (B), (C), (D) *or* (E), *and the radius r satisfies the following cases, respectively*:

(A_1) $\cot(\tilde{r}/2) = 1/\sqrt{2n-1}$;
(A_2) $\cot(\tilde{r}/2) = \sqrt{(2\ell+1)/(2n-2\ell-1)}$;
(B) $\cot(\tilde{r}/2) = \sqrt{n} + \sqrt{n-1}$;
(C) $\cot(\tilde{r}/2) = (\sqrt{n} + \sqrt{2})/\sqrt{n-2}$;
(D) $\cot(\tilde{r}/2) = \sqrt{5}$;
(E) $\cot(\tilde{r}/2) = (\sqrt{15} + \sqrt{6})/3$.

Proof. One can directly calculate the mean curvatures Trace A in terms of the tables of the principal curvatures in Lemma 3.1. Then, by using the equality

$$\sqrt{c}\cot\tilde{r} = (\sqrt{c}/2)\cot(\tilde{r}/2) - (\sqrt{c}/2)\tan(\tilde{r}/2)$$

and solving the equation Trace $A = 0$, one can complete the proof of the lemma. $\square$

In order to see Theorem 1 (2), (3), we prepare the following two propositions which give information on sectional curvatures of hypersurfaces of type (A) in $\mathbf{C}P^n(c)$ (for the proof, see [10, 11]).

Proposition 3.3. *Let M be a real hypersurface of type* (A_1) *in* $\mathbf{C}P^n(c)$ *with* $n \geqq 2$. *Then M has positive sectional curvature. More precisely, the sectional curvature K of M satisfies the following, where the both equalities are attained*:

$$0 < (c/4)\cot^2(\tilde{r}/2) \leqq K \leqq c + (c/4)\cot^2(\tilde{r}/2).$$

Proposition 3.4. *Let M be a real hypersurface of type* (A_2) *in* $\mathbf{C}P^n(c)$ *with $n \geqq 3$. Then M has nonnegative sectional curvature. More precisely, the sectional curvature K of M satisfies the following, where the both equalities are attained*:

$$0 \leqq K \leqq c + (c/4)\max\{\cot^2(\tilde{r}/2), \tan^2(\tilde{r}/2)\}.$$

For the case that M is of either type (B), (C), (D) or (E), we have

Proposition 3.5. *Let M be a real hypersurface of either type* (B), (C), (D) *or* (E) *in* $\mathbf{C}P^n(c)$. *Then the sectional curvature K of M can take both a positive sign and a negative sign.*

Proof. Every real hypersurface of either type (B), (C), (D) or (E) has three common principal curvatures λ_1, λ_2 and δ (see Lemma 3.1). Then, for unit vectors $X \in V_{\lambda_1}$ and $Y \in V_{\lambda_2}$, Lemma 2.1 yields that

$$K(X,\xi) = \frac{c}{4} + \lambda_1\delta = \frac{c}{4} - \frac{c}{4}\frac{(1+\tan(\tilde{r}/2))^2}{\tan(\tilde{r}/2)} < 0,$$

$$K(Y,\xi) = \frac{c}{4} + \lambda_2\delta = \frac{c}{4} + \frac{c}{4}\frac{(1-\tan(\tilde{r}/2))^2}{\tan(\tilde{r}/2)} > 0.$$

Therefore the sectional curvature K can take both signs. $\square$

Thus we obtain the statements (2), (3) of Theorem 1.

4. The case of homogeneous Hopf hypersurfaces in $\mathbf{C}H^n(c)$

In what follows, we shall prove Theorem 2. As a first step to do that, in this section we examine homogeneous Hopf hypersurfaces, namely, real hypersurfaces which are either of type (A) or type (B) in $\mathbf{C}H^n(c)$. The principal curvatures of those hypersurfaces are given by the following lemma.

Lemma 4.1 ([1]). *The principal curvatures of real hypersurfaces of types* (A) *and* (B) *are given as follows*:

	(A_0)	$(\mathrm{A}_{1,0})$	$(\mathrm{A}_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\tilde{r}}{2}\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\tilde{r}}{2}\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\tilde{r}}{2}\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\tilde{r}}{2}\right)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\tilde{r}}{2}\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\tilde{r}}{2}\right)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth\tilde{r}$	$\sqrt{\lvert c\rvert}\coth\tilde{r}$	$\sqrt{\lvert c\rvert}\coth\tilde{r}$	$\sqrt{\lvert c\rvert}\tanh\tilde{r}$

Here, *the principal curvature* δ *is associated with* ξ. *The multiplicities of these principal curvatures are given as follows*:

	(A_0)	$(\mathrm{A}_{1,0})$	$(\mathrm{A}_{1,1})$	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\delta)$	1	1	1	1	1

It follows directly that a real hypersurface of type (A) or (B) has two distinct principal curvatures if and only if it is of type (A_0), $(\mathrm{A}_{1,0})$, $(\mathrm{A}_{1,1})$, or of type (B) with radius $r = (1/\sqrt{|c|})\log(2+\sqrt{3})$ (in the last case $\lambda_1 = \delta = \sqrt{3|c|}/2$ holds). For other cases it has three distinct principal curvatures.

Note that all of the principal curvatures of every homogeneous Hopf hypersurface M are positive constants at each point of M. This implies that $\operatorname{Trace} A > 0$ on such a real hypersurface M. Thus we have

Proposition 4.2. *All homogeneous real hypersurfaces of types* (A) *and* (B) *in* $\mathbf{C}H^n(c)$ *with* $n \geqq 2$ *are not minimal.*

Next, we study the sectional curvatures of homogeneous Hopf hypersurfaces M one by one. First of all we study the case of type $(\mathrm{A}_{1,0})$.

Proposition 4.3 (*cf.* [6]). *Let* M *be a real hypersurface of type* $(\mathrm{A}_{1,0})$, *namely*, *a geodesic sphere* $G(r)$ *in* $\mathbf{C}H^n(c)$ *with* $n \geqq 2$. *Then*, *the sectional curvature* K *of* M *satisfies the following*, *where the both equalities are attained*:

$$c - (c/4)\coth^2(\tilde{r}/2) \leqq K \leqq (-c/4)\coth^2(\tilde{r}/2). \tag{4.1}$$

Hence, *we have the following*:

(1) K *is nonnegative if and only if* $0 < r \leqq (1/\sqrt{|c|})\log 3$;
(2) K *is positive if and only if* $0 < r < (1/\sqrt{|c|})\log 3$;
(3) K *can take both signs if and only if* $r > (1/\sqrt{|c|})\log 3$.

Proof. According to the table in Lemma 4.1, one knows that M has two distinct constant principal curvatures λ_1 with multiplicity $2n-2$ ($\geqq 2$), and δ with multiplicity 1, where $A\xi = \delta\xi$. We take a pair $\{X, Y\}$ of orthonormal vectors that are orthogonal to ξ in order to estimate the sectional curvature K of M. Since $X, Y \in V_{\lambda_1}$, we have

$$K(\sin\theta \cdot X + \cos\theta \cdot \xi, Y) = (c/4)\{\sin^2\theta(1 + 3g(\phi X, Y)^2) - \coth^2(\tilde{r}/2)\}$$

by use of Lemma 2.1. This gives the desired inequalities (4.1). Furthermore, for each unit vector X orthogonal to ξ, we have

$$K(X, \phi X) = c - (c/4)\coth^2(\tilde{r}/2), \quad K(X, \xi) = (-c/4)\coth^2(\tilde{r}/2).$$

Hence the both equalities of (4.1) are attained. The remaining assertions of the proposition immediately follow from the first assertion. □

In the case that M is of type (A_0), we remark that a horosphere can be obtained as a limit of a geodesic sphere $G(r)$ by taking $r \to \infty$. Taking $r \to \infty$ in (4.1), we have the following:

Proposition 4.4. *Let M be a real hypersurface of type* (A_0) *in* $\mathbf{C}H^n(c)$ *with* $n \geqq 2$. *Then the sectional curvature of the horosphere M can take both signs. More precisely, the sectional curvature K of M satisfies the following, where the both equalities are attained*:

$$3c/4 \leqq K \leqq -c/4.$$

For the other types, we have

Proposition 4.5. *Let M be a real hypersurface of either type* ($\mathrm{A}_{1,1}$), (A_2) *or* (B) *in* $\mathbf{C}H^n(c)$ *with* $n \geqq 2$. *Then the sectional curvature of M can take both signs.*

Proof. (i) The case that M is of type ($\mathrm{A}_{1,1}$). Then, according to the table in Lemma 4.1, we see that M has two distinct constant principal curvatures λ_1 and δ, where $A\xi = \delta\xi$. For a unit principal curvature vector $X \in V_{\lambda_1}$, we have $\phi X \in V_{\lambda_1}$. Hence, one finds from Lemma 2.1 that

$$K(X, \phi X) = c + \lambda_1^2 = c + (|c|/4)\tanh^2(\tilde{r}/2) < 0.$$

Next, we shall compute $K(X, \xi)$, where X is the above unit vector. Lemma 2.1 yields that

$$\begin{aligned} K(X, \xi) &= (c/4) + \lambda_1\delta = (c/4) + (|c|/2)\tanh(\tilde{r}/2)\coth\tilde{r} \\ &= (c/4) + (|c|/4)(1 + \tanh^2(\tilde{r}/2)) = (|c|/4)\tanh^2(\tilde{r}/2) > 0. \end{aligned}$$

(ii) The case that M is of type (A_2). In this case M has three distinct constant principal curvatures λ_1, λ_2 and δ with $A\xi = \delta\xi$. Applying Lemma 2.2, we find that $X \in V_{\lambda_2}$ implies $\phi X \in V_{\lambda_2}$. So, the same computations as above tell us that

$$K(X, \phi X) < 0, \quad K(X, \xi) > 0$$

for a unit vector $X \in V_{\lambda_2}$.

(iii) The case that M is of type (B). In this case, by Lemma 2.2, for a unit vector $X \in V_{\lambda_1}$ we have $\phi X \in V_{\lambda_2}$, so that

$$K(X, \phi X) = c + \lambda_1 \lambda_2 = c + (|c|/4) < 0.$$

The same vector X satisfies

$$\begin{aligned} K(X, \xi) &= (c/4) + \lambda_1 \delta = (c/4) + (|c|/2) \coth(\tilde{r}/2) \tanh \tilde{r} \\ &= \frac{c}{4} + \frac{|c|}{1 + \tanh^2(\tilde{r}/2)} > \frac{c}{4} + \frac{|c|}{2} > 0. \end{aligned}$$

Thus the sectional curvature K can take both signs. □

5. The case of type (W_1)

Recall that a real hypersurface M of type (W_1) is a tube of radius r around the minimal ruled submanifold W^{2n-k} with $k \in \{2, \dots, n-1\}$, where $0 < r < \infty$. In this section, we investigate this real hypersurface.

The principal curvatures of M have completely been calculated by Berndt and Díaz-Ramos ([3], Subsection 4.2), by giving an explicit matrix representation of the shape operator. Although they calculated it under the normalization $c = -1$, one can easily see the following.

Lemma 5.1 ([3]). *Let M be a real hypersurface of type* (W_1), *that is, a tube of radius r around W^{2n-k}. Let $p \in M$ and put $\tilde{r} := \sqrt{|c|}r$. Denote by A the shape operator of M. Then, there exists an orthogonal decomposition*

$$T_pM = \mathrm{Span}\{Z, J\mathcal{N}\} \oplus V_3 \oplus V_4$$

of T_pM into A-invariant subspaces of T_pM with a unit vector $Z \in T_pM$ perpendicular to $J\mathcal{N}$, such that the matrix representation of A with respect to this decomposition satisfies

$$\begin{aligned} A|_{\mathrm{Span}\{Z, J\mathcal{N}\}} &= \frac{\sqrt{|c|}}{2} \begin{pmatrix} \tanh^3(\tilde{r}/2) & -\mathrm{sech}^3(\tilde{r}/2) \\ -\mathrm{sech}^3(\tilde{r}/2) & (2 + \mathrm{sech}^2(\tilde{r}/2)) \tanh(\tilde{r}/2) \end{pmatrix}, \\ A|_{V_3} &= (\sqrt{|c|}/2) \tanh(\tilde{r}/2) I_{2n-2-k}, \\ A|_{V_4} &= (\sqrt{|c|}/2) \coth(\tilde{r}/2) I_{k-1}. \end{aligned}$$

By calculating the eigenvalues of the above matrices, the principal curvatures of real hypersurfaces of type (W_1) have been determined completely in [3]. The above matrix representations of A also tell us that Trace $A > 0$. Hence, we have

Proposition 5.2. *All real hypersurfaces of type* (W_1) *are not minimal.*

Moreover, the above matrices enable us to determine the signs of the sectional curvatures.

Proposition 5.3. *Let M be a real hypersurface of type* (W_1). *Then the sectional curvature of M can take both signs.*

Proof. Let M be a tube of radius r around W^{2n-k}, where $k \in \{2, \dots, n-1\}$. Then the shape operator A is described in Lemma 5.1. We use Z and $\xi = -J\mathcal{N}$. One also has a unit vector $X \in V_4$ by $\dim V_4 = k - 1 \geqq 1$. Then, since $\{Z, \xi\}$ is orthonormal, we have

$$g(AZ, Z) = (\sqrt{|c|}/2)\tanh^3(\tilde{r}/2),$$

$$g(A\xi, \xi) = (\sqrt{|c|}/2)(2 + \operatorname{sech}^2(\tilde{r}/2))\tanh(\tilde{r}/2).$$

Recall $c < 0$. Then it yields that

$$g(AZ, Z)g(AX, X) = -(c/4)\tanh^2(\tilde{r}/2),$$

$$g(A\xi, \xi)g(AX, X) = -(c/4)(2 + \operatorname{sech}^2(\tilde{r}/2)).$$

Now it follows form the formula in Lemma 2.1 that

$$K(Z, X) = (c/4)(3g(\phi Z, X)^2 + \operatorname{sech}^2(\tilde{r}/2)) < 0,$$

$$K(\xi, X) = -(c/4)(1 + \operatorname{sech}^2(\tilde{r}/2)) > 0.$$

Therefore, the sectional curvature K can take both signs. □

6. The case of type (W_2)

In this section, we study a real hypersurface M of type (W_2), that is, a tube of radius r around the minimal ruled submanifold W_{φ}^{2n-k}, where $\varphi \in (0, \pi/2)$, $k \in \{2, \dots, n-1\}$ with k even, and $0 < r < \infty$.

Berndt and Díaz-Ramos calculated the principal curvatures of M completely ([3], Subsection 4.3). However, in their paper, they omitted some entries of the matrix representation of the shape operator. Therefore, first of all, we describe the shape operator completely, and also investigate the characteristic vector.

Lemma 6.1 (*cf.* [3]). *Let M be a real hypersurface of type* (W_2), *i.e., a tube of radius r around W_{φ}^{2n-k} and $p \in M$. Set $\tilde{r} := \sqrt{|c|}r$ and denote by A the shape operator A of M. Then, there exists an orthogonal decomposition*

$$T_pM = \operatorname{Span}\{Z^*, P^*, F^*\} \oplus V_4 \oplus V_5$$

of T_pM into A-invariant subspaces of T_pM with a triplet $\{Z^*, P^*, F^*\}$ of orthonormal vectors in T_pM, such that the following properties are satisfied:

(1) *The matrix representation* (b_{ij}) *of* $A|_{\mathrm{Span}\{Z^*,P^*,F^*\}}$ *with respect to the basis* $\{Z^*,P^*,F^*\}$ *satisfies*

$$b_{11} = (\sqrt{|c|}/2)\{-\sin^2\varphi + \cosh^2(\tilde{r}/2)\}\,\mathrm{sech}^2(\tilde{r}/2)\tanh(\tilde{r}/2),$$

$$b_{12} = b_{21} = -(\sqrt{|c|}/2)\{\sin^2\varphi + \cos^2\varphi\cosh(\tilde{r}/2)\}\sin\varphi\,\mathrm{sech}^3(\tilde{r}/2),$$

$$b_{13} = b_{31} = \sqrt{|c|}\cos\varphi\sin^2\varphi\,\mathrm{sech}^3(\tilde{r}/2)\sinh^2(\tilde{r}/4),$$

$$\begin{aligned} b_{22} = (\sqrt{|c|}/2)\{&\sin^4\varphi + (1+\cos^2\varphi)\sin^2\varphi\cosh(\tilde{r}/2) + (1+\sin^2\varphi)\\ &\times(1+\cosh(\tilde{r}/2))\cosh^2(\tilde{r}/2)\}\,\mathrm{sech}^3(\tilde{r}/2)\tanh(\tilde{r}/4),\end{aligned}$$

$$\begin{aligned} b_{23} = b_{32} = (\sqrt{|c|}/2)\{&\sin^2\varphi + \cos^2\varphi\cosh(\tilde{r}/2) + \cosh^2(\tilde{r}/2) + \cosh^3(\tilde{r}/2)\}\\ &\times\sin\varphi\cos\varphi\,\mathrm{sech}^3(\tilde{r}/2)\tanh(\tilde{r}/4),\end{aligned}$$

$$\begin{aligned} b_{33} = (\sqrt{|c|}/2)\{&-\sin^2\varphi\cos^2\varphi + 2\sin^2\varphi\cos^2\varphi\cosh(\tilde{r}/2)\\ &-(1+\sin^2\varphi)\cos^2\varphi\cosh^2(\tilde{r}/2)\\ &+(1+\cos^2\varphi)\cosh^4(\tilde{r}/2)\}\,\mathrm{csch}(\tilde{r}/2)\,\mathrm{sech}^3(\tilde{r}/2);\end{aligned}$$

(2) *The matrix representations of* $A|_{V_4}$ *and* $A|_{V_5}$ *satisfy*

$$A|_{V_4} = (\sqrt{|c|}/2)\tanh(\tilde{r}/2)I_{2n-2-k},$$

$$A|_{V_5} = (\sqrt{|c|}/2)\coth(\tilde{r}/2)I_{k-2};$$

(3) *The characteristic vector is given by* $\xi = \sin\varphi\cdot P^* + \cos\varphi\cdot F^*$;

(4) $JZ^* \in V_4$.

Proof. We assume that the ambient complex hyperbolic space is normalized as $c = -1$. First of all, we recall some facts on the minimal ruled submanifold W_φ^{2n-k} in $\mathbf{C}H^n(-1)$. We denote by νW_φ^{2n-k} the normal bundle of W_φ^{2n-k}. Take $o \in W_\varphi^{2n-k}$ and let $v \in \nu_o W_\varphi^{2n-k}$ be a unit normal vector. We decompose Jv into

$$Jv = Pv + Fv,$$

where $Pv \in T_o W_\varphi^{2n-k}$ and $Fv \in \nu_o W_\varphi^{2n-k}$. Since $\nu_o W_\varphi^{2n-k}$ has constant Kähler angle $\varphi \in (0,\pi/2)$, one has $\|Pv\| = \sin\varphi \neq 0$ and $\|Fv\| = \cos\varphi \neq 0$. We then put

$$\bar{P}v := Pv/\sin\varphi, \quad \bar{F}v := Fv/\cos\varphi.$$

It has been known that there exists a unit vector $Z \in T_o W_\varphi^{2n-k}$ such that the second fundamental form II of W_φ^{2n-k} is given by the trivial symmetric bilinear extension of $II(Z,Pw) = (\sin^2(\varphi)/2)w$ for all $w \in \nu_o W_\varphi^{2n-k}$. Then one can see that the eigenvalues of the shape operator S_v of W_φ^{2n-k} with respect to v are $\sin(\varphi)/2$, $-\sin(\varphi)/2$ and 0, and the corresponding eigenspaces are

$$\mathbf{R}(Z+\bar{P}v), \quad \mathbf{R}(-Z+\bar{P}v), \quad T_o W_\varphi^{2n-k} \ominus (\mathbf{R}Z + \mathbf{R}\bar{P}v),$$

respectively, where $\ominus$ denotes the orthogonal complement. It has also been known that the vector Z satisfies $JZ \in T_o W_\varphi^{2n-k} \ominus (\mathbf{R}Z + \mathbf{R}\bar{P}v)$.

We here consider the tube M around W_φ^{2n-k} at distance $r > 0$. Let $\gamma_v = \gamma_v(t)$ be the geodesic in $\mathbf{C}H^n(-1)$ given by the initial conditions $\gamma_v(0) = o$ and $\dot{\gamma}_v(0) = v$. For any $X \in T_o\mathbf{C}H^n(-1)$, we denote by $B_X(t)$ the unique parallel field along the geodesic γ_v with $B_X(0) = X$. Then we can put

$$p := \gamma_v(r), \quad \mathcal{N} := -\gamma_v(r).$$

Moreover, we define

$$Z^* := B_Z(r), \quad P^* := B_{\bar{P}v}(r), \quad F^* := B_{\bar{F}v}(r),$$
$$V_4 := B_{T_o W_\varphi^{2n-k} \ominus (\mathbf{R}Z + \mathbf{R}\bar{P}v)}(r), \quad V_5 := B_{\nu_o W_\varphi^{2n-k} \ominus (\mathbf{R}v + \mathbf{R}\bar{F}v)}(r),$$

where B_V denotes the parallel translation of any vector subspace $V \subset T_o\mathbf{C}H^n(-1)$ along γ_v.

The assertions (1) and (2) can be proved by calculating the shape operator A in terms of the Jacobi field theory. The calculation is long but exactly same as the one in [5, Subsection 4.2], which will be omitted.

Note that the characteristic vector ξ at p is given by

$$\xi = -J\mathcal{N} = J(\dot{\gamma}_v(r)) = J(B_v(r)) = B_{Jv}(r) = B_{Pv+Fv}(r).$$

One knows $Pv + Fv = \sin\varphi \cdot \bar{P}v + \cos\varphi \cdot \bar{F}v$. Therefore, the assertion (3) follows from the linearity of B. The assertion (4) follows from

$$JZ^* = J(B_Z(r)) = B_{JZ}(r) \in V_4.$$

This completes the proof of the lemma. □

The matrix representations in Lemma 6.1 provide the following result.

PROPOSITION 6.2. *All real hypersurfaces of type* (W$_2$) *are not minimal.*

Proof. Let M be a real hypersurface of type (W$_2$) in $\mathbf{C}H^n(c)$, that is, the tube of radius r around W_φ^{2n-k}. Recall that $\tilde{r} := \sqrt{|c|}r > 0$. Then, straightforward calculations show that

$$b_{11} + b_{22} + b_{33} = (\sqrt{|c|}/2)(-1 + 2\cosh\tilde{r})\operatorname{csch}\tilde{r} > 0,$$

and all of other diagonal entries of A are positive. We thus have Trace $A > 0$, and hence M is not minimal. □

Moreover, we can also determine the signs of the sectional curvatures of real hypersurfaces of type (W$_2$).

PROPOSITION 6.3. *Let M be a real hypersurface of type* (W$_2$). *Then the sectional curvature of M can take both signs.*

Proof. We first remark that dim $V_4 = 2n - k - 2 \geqq 2$. Then one can take a unit vector X in V_4 orthogonal to JZ^* $(\in V_4)$ and can see that

$$g(\phi X, JZ^*) = g(X, Z^*) = 0.$$

We thus have

$$\begin{aligned} K(X, JZ^*) &= (c/4)(1 + 3g(\phi X, JZ^*)^2) + g(AX, X)g(AJZ^*, JZ^*) - g(AX, JZ^*)^2 \\ &= (c/4) - (c/4)\tanh^2(\tilde{r}/2) = (c/4)\,\mathrm{sech}^2(\tilde{r}/2) < 0. \end{aligned}$$

We next calculate $K(P^*, F^*)$. It follows from Lemma 6.1 (3) that

$$0 = \phi\xi = \sin\varphi \cdot \phi P^* + \cos\varphi \cdot \phi F^*,$$

which yields $g(\phi P^*, F^*) = 0$. It then follows from Lemma 2.1 and long calculations that

$$\begin{aligned} K(P^*, F^*) &= (c/4) + b_{22}b_{33} - b_{23}^2 \\ &= (|c|/4)(-\cos 2\varphi + \cosh^2(\tilde{r}/2))\,\mathrm{sech}^2(\tilde{r}/2) \\ &> (|c|/4)(-1 + \cosh^2(\tilde{r}/2))\,\mathrm{sech}^2(\tilde{r}/2) = (|c|/4)\tanh^2(\tilde{r}/2) > 0. \end{aligned}$$

This completes the proof. □

7. The case of type (S)

This section will be devoted to the study of real hypersurfaces of type (S). Such hypersurfaces are either the homogeneous ruled real hypersurface S determined by a horocycle in a totally geodesic $\mathbf{R}H^2(c/4)$ in $\mathbf{C}H^n(c)$, or an equidistant hypersurface from S at distance r, where $0 < r < \infty$.

Geometry of these hypersurfaces have been studied in detail by Berndt ([2]), and also by Hamada, Hoshikawa and the second author ([5]). By virtue of their works we can obtain the following lemmas.

LEMMA 7.1 ([2, 5]). *Let M be an equidistant hypersurface from S at distance r with $0 < r < \infty$ and $p \in M$. Set $\tilde{r} := \sqrt{|c|}r$ and let A denote the shape operator of M. Then, there exists an orthogonal decomposition*

$$T_pM = \mathrm{Span}\{Z_0, Y_1\} \oplus V_3$$

of T_pM into A-invariant subspaces of T_pM with a pair $\{Z_0, Y_1\}$ of orthonormal vectors in T_pM, such that the matrix representation of A with respect to this decomposition satisfies

$$A|_{\mathrm{Span}\{Z_0, Y_1\}} = \frac{\sqrt{|c|}}{2}\begin{pmatrix} 2\tanh(\tilde{r}/2) & -\mathrm{sech}(\tilde{r}/2) \\ -\mathrm{sech}(\tilde{r}/2) & \tanh(\tilde{r}/2) \end{pmatrix},$$

$$A|_{V_3} = (\sqrt{|c|}/2)\tanh(\tilde{r}/2)I_{2n-3}.$$

LEMMA 7.2 ([5]). *Let K be the sectional curvature of an equidistant hypersurface M from S at distance r $(0 < r < \infty)$ in $\mathbf{C}H^n(c)$ with $n \geqq 2$. Set $\tilde{r} := \sqrt{|c|}r$ and $t := \tanh(\tilde{r}/2)$. Then, the maximum value of K is given as follows*:

$$\max K = \begin{cases} (c/8)\{2 - 3t^2 - t\sqrt{4 - 3t^2}\} & (n \geqq 3), \\ (c/8)\{5 - 3t^2 - \sqrt{-15t^4 + 22t^2 + 9}\} & (n = 2). \end{cases} \tag{7.1}$$

We first note that the homogeneous ruled real hypersurface S is exactly coincident with the hypersurface obtained as a limit of an equidistant hypersurface by taking $r \to 0$. It follows immediately from Lemma 7.1 that

$$\mathrm{Trace}\, A > 0, \quad \lim_{r \to 0} \mathrm{Trace}\, A = 0.$$

Thus we have

PROPOSITION 7.3 ([2]). *A homogeneous real hypersurface of type* (S) *is minimal if and only if it is the homogeneous ruled real hypersurface S.*

We shall prove the following result as an application of Lemmas 7.1 and 7.2.

PROPOSITION 7.4. *Let M be an equidistant hypersurface at distance r $(0 < r < \infty)$ from S in $\mathbf{C}H^n(c)$ with $n \geqq 2$. Let us define $r_0 = r_0(n)$ by*

$$r_0 := \begin{cases} (1/\sqrt{|c|}) \log(2 + \sqrt{3}) & (n \geqq 3), \\ (1/\sqrt{|c|}) \log\{(2\sqrt{3} + \sqrt{13 - \sqrt{73}})(2\sqrt{3} - \sqrt{13 - \sqrt{73}})\} & (n = 2). \end{cases}$$

Then we have the following:

(1) *The sectional curvature of M is nonpositive if and only if $0 < r \leqq r_0$*;
(2) *The sectional curvature of M is negative if and only if $0 < r < r_0$*;
(3) *The sectional curvature of M can take both signs if and only if $r > r_0$.*

Proof. It follows from (7.1) that the maximum value of K is a monotone increasing function of the distance r in each case. By elementary computations we observe that the equation $\max K = 0$ implies $r = r_0$, where r_0 is that given in the proposition. This proves (1) and (2). In order to prove (3), it remains to show that the sectional curvature K can always take negative sign. Let Y_1 be the unit vector given in Lemma 7.1. Then, for a unit vector $X \in V_3$, we have

$$\begin{aligned} g(AY_1, Y_1) &= g(AX, X) = (\sqrt{|c|}/2) \tanh(\tilde{r}/2), \\ g(AY_1, X) &= 0. \end{aligned}$$

Then one can see that

$$\begin{aligned} K(Y_1, X) &= (c/4)(1 + 3g(\phi Y_1, X)^2) + (|c|/4) \tanh^2(\tilde{r}/2) \\ &\leqq -(|c|/4)(1 - \tanh^2(\tilde{r}/2)) < 0. \end{aligned}$$

Therefore, the proposition is proved. $\square$

Lastly, we determine the sign of sectional curvature of the homogeneous ruled real hypersurface S. By taking $r \to 0$ in (7.1), we have

Proposition 7.5. *The sectional curvature K of the homogeneous ruled real hypersurface S in $\mathbf{C}H^n(c)$ with $n \geqq 2$ has a negative sign.*

Altogether, we have now proved Theorem 2.

Remark 7.1. We make mention of examples in the case that real hypersurfaces are not necessarily homogeneous. The first author and the third author [13, 16] studied the properties of sectional curvatures of ruled real hypersurfaces in nonflat complex space forms. Needless to say that $\mathbf{C}P^n(c)$ does not admit any homogeneous ruled real hypersurfaces and the only example of such hypersurface in $\mathbf{C}H^n(c)$ is the former example S of type (S). From their results we can see that the sectional curvature K of every ruled real hypersurface in $\mathbf{C}P^n(c)$ satisfies $-\infty < K \leqq c$, whereas ruled real hypersurfaces of $\mathbf{C}H^n(c)$ are classified into two cases with regard to the range of K: $-\infty < K \leqq c/4$ and $c \leqq K \leqq c/4$. The homogeneous ruled real hypersurface S belongs to the latter class (for detail, see [13]).

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