M.-F. CHEN, Z.-S. GAO AND J.-L. ZHANG
KODAI MATH. J.
41 (2018), 264–283

ULTRA-DISCRETE EQUATIONS AND TROPICAL COUNTERPARTS OF SOME COMPLEX ANALYSIS RESULTS

Min-Feng Chen†, Zong-Sheng Gao and Ji-Long Zhang

Abstract

A tropical version of Nevanlinna theory describes value distribution of continuous piecewise linear functions of a real variable. In this paper, we present some results on value distribution theory of tropical difference polynomials and uniqueness theory of tropical entire functions. Application to some ultra-discrete equations is also given.

1. Introduction

For a general background concerning tropical mathematics, see [13]. Recently, Halburd and Southall [5] described continuous piecewise linear functions of a real variable with one-side integer derivatives as tropical meromorphic functions, and established tropical versions of Nevanlinna's first main theorem, the lemma on the logarithmic derivative and Clunie's lemma. Laine and Tohge [9] took an extended point of view to tropical meromorphic functions, showed that tropical Nevanlinna theory also holds to piecewise linear functions with arbitrary real slopes, and obtained a tropical version of Nevanlinna's second main theorem.

Recalling the standard one-dimensional tropical structure, the max-plus (or tropical) semi-ring is the set $\mathbf{R} \cup \{-\infty\}$ with (tropical) addition and (tropical) multiplication defined by

$$x \oplus y := \max(x, y)$$

and

$$x \otimes y := x + y.$$

2010 *Mathematics Subject Classification*. 39A12, 30D35.

Key words and phrases. Tropical Nevanlinna theory, Tropical entire functions, Value distribution, Uniqueness, Ultra-discrete equations.

This research was supported by the National Natural Science Foundation of China (No. 11371225) and by the Fundamental Research Funds for the Central Universities.

† Corresponding author.

Received January 19, 2017; revised May 22, 2017.

We also use $x \oslash y := x - y$ and $x^{\otimes \alpha} := \alpha x$, for all $\alpha \in \mathbf{R}$. The identity elements $0_\circ$ for tropical addition is $0_\circ = -\infty$ and the identity elements $1_\circ$ for tropical multiplication is $1_\circ = 0$. This structure fails to be a ring, since not all elements have tropical additive inverses. In particular, the equation $x \oplus 2 = 1$ has no solution.

We assume that the reader is familiar with the notations and results of the classical Nevanlinna theory [1, 6, 8] and the tropical Nevanlinna theory [5, 7, 9]. Now, we recall some basic notations as follows.

Definition 1.1 ([5]). Let $f(x)$ be a tropical meromorphic function, $x \in \mathbf{R}$ and

$$\omega_f(x) = \lim_{\varepsilon \to 0^+} (f'(x+\varepsilon) - f'(x-\varepsilon)).$$

If $\omega_f(x) > 0$, then x is called a root (zero) of $f(x)$ with multiplicity $\omega_f(x)$. If $\omega_f(x) < 0$, then x is called a pole of $f(x)$ with multiplicity $-\omega_f(x)$.

The tropical proximity function for meromorphic functions is defined to be

$$m(r, f) := \frac{f^+(r) + f^+(-r)}{2},$$

where $f^+(x) := \max\{f(x), 0\}$ for $x \in \mathbf{R}$. The integrated tropical counting function for poles in $(-r, r)$ is defined to be

$$N(r, f) := \frac{1}{2} \int_0^r n(t, f)\, dt = \frac{1}{2} \sum_{|b_\nu| < r} \tau_f(b_\nu)(r - |b_\nu|),$$

where tropical counting function $n(t, f)$ gives the number of distinct poles of $f(x)$ in the interval $(-r, r)$, each pole multiplied by its multiplicity τ_f. Defining the tropical characteristic function $T(r, f)$ as usual,

$$T(r, f) := m(r, f) + N(r, f).$$

The order of $f(x)$ is defined as

$$\rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}$$

and the hyper-order of $f(x)$ is defined as

$$\rho_2(f) := \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

Recall that tropical polynomials admit finitely many roots and no poles. And tropical polynomials may be defined in the form

$$f(x) = \bigoplus_{i=0}^{n} (a_i \otimes x^{\otimes k_i}) = a_n \otimes x^{\otimes k_n} \oplus a_{n-1} \otimes x^{\otimes k_{n-1}} \oplus \cdots \oplus a_0 \otimes x^{\otimes k_0},$$

that is,

$$f(x) = \max\{a_n + k_n x, a_{n-1} + k_{n-1}x, \dots, a_0 + k_0 x\},$$

where the coefficients a_i are real constants and the exponents k_i are real numbers, $i = 0, 1, \dots, n,$ and $k_0 < k_1 < \cdots < k_n$.

Equivalently, tropical rational functions admit finitely many roots and poles. A natural definition of a tropical rational function is a function of the form

$$\begin{aligned} R(x) &= \left(\bigoplus_{i=0}^{n}(a_i \otimes x^{\otimes k_i})\right) \oslash \left(\bigoplus_{j=0}^{m}(b_j \otimes x^{\otimes l_j})\right) \\ &= \{a_n \otimes x^{\otimes k_n} \oplus a_{n-1} \otimes x^{\otimes k_{n-1}} \oplus \cdots \oplus a_0 \otimes x^{\otimes k_0}\} \\ &\quad \oslash \{b_m \otimes x^{\otimes l_m} \oplus b_{m-1} \otimes x^{\otimes l_{m-1}} \oplus \cdots \oplus b_0 \otimes x^{\otimes l_0}\} \\ &= \max\{a_n + k_n x, a_{n-1} + k_{n-1}x, \dots, a_0 + k_0 x\} \\ &\quad - \max\{b_m + l_m x, b_{m-1} + l_{m-1}x, \dots, b_0 + l_0 x\}, \end{aligned}$$

where the coefficients a_i, b_j are real constants and the exponents k_i, l_j are real numbers, $i = 0, 1, \dots, n,$ $j = 0, 1, \dots, m,$ and $k_0 < k_1 < \cdots < k_n$, $l_0 < l_1 < \cdots < l_m$, see [7, Chapter 2].

Concerning tropical exponential functions, recall their definitions as follows, see [9, Definition 8.1].

Definition 1.2. Let α be a real number with $|\alpha| > 1$. Define a function $e_\alpha(x)$ on $\mathbf{R}$ by

$$e_\alpha(x) := \alpha^{[x]}(x - [x]) + \sum_{j=-\infty}^{[x]-1} \alpha^j = \alpha^{[x]}\left(x - [x] + \frac{1}{\alpha - 1}\right).$$

If $\alpha > 1$, $e_\alpha(x)$ is strictly increasing, and $e_\alpha(x)$ is a tropical entire function, since it has no poles. If $\alpha < -1$, then $e_\alpha(x)$ is tropical meromorphic, but not tropical entire. For a real number β with $|\beta| < 1$, $\beta \neq 0$. Define a function $e_\beta(x)$ on $\mathbf{R}$ by

$$e_\beta(x) := \sum_{j=[x]}^{\infty} \beta^j - \beta^{[x]}(x - [x]) = \beta^{[x]}\left(\frac{1}{1-\beta} - x + [x]\right).$$

If $0 < \beta < 1$, $e_\beta(x)$ is strictly increasing, and $e_\beta(x)$ is a tropical entire function, since it has no poles. If $-1 < \beta < 0$, then $e_\beta(x)$ is tropical meromorphic, but not tropical entire.

In this paper, we frequently use the notations $\Pi(x)$, $\tilde{\Pi}(x)$ etc. for tropical meromorphic of 1-periodic, possibly meaning different functions at different occasions. We also use the notations $\Xi(x)$, $\tilde{\Xi}(x)$ etc. for tropical meromorphic of 2-periodic and anti-1-periodic, again possibly meaning different functions at different occasions. The remaining of this paper is now being organized as follows. In section 2, we present several preliminary lemmas that are needed in subsequent considerations. Section 3 is devoted to considering value distribution theory of tropical difference polynomials. Section 4 is treating the uniqueness theory of tropical entire functions. Section 5 will concentrate on applications to ultra-discrete equations.

2. Some preliminary lemmas

LEMMA 2.1 (See [9, Theorem 9.1]). *The equation*

$$y(x+1) = y(x)^{\otimes c} \tag{2.1}$$

with $c \in \mathbf{R} \backslash \{0\}$ *admits a non-constant tropical meromorphic solution on* $\mathbf{R}$ *of hyper-order* $\rho_2 < 1$ *if and only if* $c = \pm 1$. *Suppose that* $f(x)$ *is a non-constant tropical meromorphic solution to* (2.1), *then it can be extended onto* $\mathbf{R}$ *and*

(i) *if* $c = 1$, *then* $f(x)$ *is* 1*-periodic and* $f(x) = \Pi(x)$;

(ii) *if* $c = -1$, *then* $f(x)$ *is* 2*-periodic, anti-*1*-periodic and* $f(x) = \Xi(x)$;

(iii) *if* $c \neq 0, \pm 1$, *then* $f(x) = L_b(e_c(x-b))$, *where the notation* $L_b(e_c(x-b))$ *is finite linear combinations of* $\sum_{j=1}^{q} \beta_j e_c(x-b_j)$, $b = \{b_1, \dots, b_q\} \subset [0,1)$, *the coefficients* β_j *are constants.*

Remark 2.1. According to the Definition 1.2, we see that $e_c(x+1-b) = ce_c(x-b)$. In particular, if A is a non-zero constant, then $L_b(Ae_c(x-b)) = L_b(e_c(x-b))$, $b \in [0,1)$.

LEMMA 2.2 (See [10, Theorem 4.1]). *Let* α, β *be non-zero real numbers. Suppose that* $f(x)$ *is a non-constant tropical meromorphic solution to*

$$\alpha f(x) + \beta f(x+1) = 1, \tag{2.2}$$

then it can be extended onto $\mathbf{R}$ *and*

(i) *if* $\alpha = \beta$, *then* $f(x) = \Xi(x) + \dfrac{1}{2\beta}$;

(ii) *if* $\alpha = -\beta$, *then* $f(x) = \Pi(x) + \dfrac{1}{\beta}x$;

(iii) *if* $\alpha \neq \pm\beta$, *then* $f(x) = L_b(e_{-\alpha/\beta}(x-b)) + \dfrac{1}{\alpha+\beta}$, $b \in [0,1)$.

LEMMA 2.3 (See [10, Proposition 3.3]). *Suppose that* $f(x)$ *is a non-constant tropical meromorphic solution to*

$$f(x+1) - f(x) = c, \quad c \in \mathbf{R},$$

then it can be extended onto $\mathbf{R}$ *and* $f(x) = \Pi(x) + cx$.

Lemma 2.4 (See [10, Proposition 3.23]). *Suppose that $f(x)$ is a non-constant tropical meromorphic solution to*

$$f(x+1) - f(x) = cx, \quad c \in \mathbf{R},$$

then it can be extended onto $\mathbf{R}$ *and* $f(x) = \Pi(x) + c(\psi(x) - x)$, *where*

$$\psi(x) = ([x]+1)\left(x - \frac{1}{2}[x]\right).$$

Remark 2.2. $\psi(x) = ([x]+1)\left(x - \frac{1}{2}[x]\right)$ is a tropical entire function of order $\rho(\psi) = 2$ and satisfies the difference equation $\psi(x) - \psi(x-1) = x$, see [10, Proposition 3.22].

Lemma 2.5. *Let* α, β *be non-zero real numbers and* A, B *be real constants. Suppose that $f(x)$ is a non-constant tropical meromorphic solution to*

$$\alpha f(x) + \beta f(x+1) = Ax + B, \tag{2.3}$$

then it can be extended onto $\mathbf{R}$ *and*

(i) *if* $\alpha = \beta$, *then* $f(x) = \Xi(x) + \dfrac{A}{2\beta}x + \dfrac{B - A/2}{2\beta}$;

(ii) *if* $\alpha = -\beta$, *then* $f(x) = \Pi(x) + \dfrac{A}{\beta}\psi(x) - \dfrac{A-B}{\beta}x$, *where* $\psi(x) = ([x]+1)\left(x - \frac{1}{2}[x]\right)$;

(iii) *if* $\alpha \neq \pm\beta$, *then* $f(x) = L_b(e_{-\alpha/\beta}(x-b)) + \dfrac{A}{\alpha+\beta}x + \dfrac{B - \frac{A\beta}{\alpha+\beta}}{\alpha+\beta}$, $b \in [0,1)$.

Proof. If $\alpha = \beta$, then (2.3) can be rewritten as

$$f(x+1) + f(x) = \frac{Ax+B}{\beta}. \tag{2.4}$$

Clearly, by Lemma 2.1(ii), the solution to $f(x+1) + f(x) = 0$ are 2-periodic, anti-1-periodic functions, then $f(x) = \Xi(x)$. It is a trivial computation to verify that $f_0(x) = \dfrac{A}{2\beta}x + \dfrac{B - A/2}{2\beta}$ is a special solution to (2.4).

If $\alpha = -\beta$, then equation (2.3) now is

$$f(x+1) - f(x) = \frac{Ax+B}{\beta}. \tag{2.5}$$

It follows from Lemmas 2.3 and 2.4 that all tropical meromorphic solutions to (2.5) may be written in the form

$$f(x) = \Pi(x) + \frac{A}{\beta}\psi(x) - \frac{A-B}{\beta}x,$$

where $\psi(x) = ([x]+1)\left(x - \frac{1}{2}[x]\right)$.

If $\alpha \neq \pm\beta$, then equation (2.3) is

$$f(x+1) + \frac{\alpha}{\beta} f(x) = \frac{Ax+B}{\beta}. \tag{2.6}$$

The general solution $L_b(e_{-\alpha/\beta}(x-b))$ to the homogeneous equation $f(x+1) + \frac{\alpha}{\beta} f(x) = 0$ follows from Lemma 2.1(iii). Obviously, $f_0(x) = \frac{A}{\alpha+\beta} x + \frac{B - \frac{A\beta}{\alpha+\beta}}{\alpha+\beta}$ is a solution of (2.6).

Remark 2.3. Clearly, we see that Lemmas 2.1 and 2.2 are special cases of Lemma 2.5, when $A = B = 0$ and $A = 0$, respectively.

Lemma 2.6 (See [9, Proposition 4.2]). *Let f be a meromorphic function of hyper-order $\rho_2 < 1$. Given $\delta \in (0, 1-\rho_2)$ and $c \in \mathbf{R}$, then f satisfies*

$$m(r, f(x+c) \oslash f(x)) = o(T(r,f)/r^{\delta}),$$

as r approaches to infinity outside of a set of finite logarithmic measure.

3. Value distribution theory of tropical difference polynomials

In this section, we will consider value distribution of tropical difference polynomial of type $f(x)^{\otimes\alpha} \otimes P(x,f)$, for $\alpha > 0$. Yang and Laine [16] posed the following conjecture in 2010.

Conjecture. *Let f be an entire function of infinite order and $n \geq 2$ be an integer. Then a differential-difference polynomial of the form $f^n + P_{n-1}(z,f)$ cannot be a non-constant entire function of finite order. Here $P_{n-1}(z,f)$ is a differential-difference polynomial in f of total degree at most $n-1$ in f, its derivatives and its shifts, with entire functions of finite order as coefficients. Moreover, we assume that all terms of $P_{n-1}(z,f)$ have total degree ≥ 1.*

Li and Yang [11] studied the conjecture and obtained the following result.

Theorem A. *Let f be an entire function of infinite order and $n \geq 2$ be an integer. Suppose that $\rho_2(f) < 1$, then a differential-difference polynomial of the form $f^n + P_{n-1}(z,f)$ cannot be a non-constant entire function of finite order, where $P_{n-1}(z,f)$ is as defined in the conjecture above.*

Next, we proceed to consider tropical version of the conjecture. In fact, we investigate the growth of tropical difference polynomial of type $f(x)^{\otimes\alpha} \otimes P(x,f)$ for $\alpha > 0$. Before proceeding to formulate this result, we need to define tropical difference *Laurent* polynomials in a tropical function and its shifts, see [9, p. 899].

Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m)$ be a multi-index of real numbers, and consider

$$f(x \uplus c)^{\otimes\lambda} := \bigotimes_{j=0}^{m} f(x + c_j)^{\otimes\lambda_j}$$
$$= \lambda_0 f(x) + \lambda_1 f(x + c_1) + \cdots + \lambda_m f(x + c_m)$$

with the shifts $(0, c_1, c_2, \dots, c_m) \in \mathbf{R}^{m+1}$. Then an expression of the form

$$P(x, f) = \bigoplus_{\lambda \in \Lambda[P]} a_\lambda(x) \otimes f(x \uplus c)^{\otimes\lambda} = \max_{\lambda \in \Lambda[P]} \left\{ a_\lambda(x) + \sum_{j=0}^{m} \lambda_j f(x + c_j) \right\}, \quad c_0 = 0$$

with tropical meromorphic coefficients $a_\lambda(x)$ $(\lambda \in \Lambda[P])$, over a finite set $\Lambda[P]$ of real indices, is called a tropical difference *Laurent* polynomial of total degree

$$\deg(P) := \max_{\lambda \in \Lambda[P]} \|\lambda\| (\in \mathbf{R})$$

in f and its shifts, with $\|\lambda\| := \lambda_0 + \lambda_1 + \cdots + \lambda_m$.

In what follows, we state a tropical counterpart to the conjecture.

THEOREM 3.1. *Let $\alpha > 0$, $P(x, f)$ be a tropical difference Laurent polynomial with tropical meromorphic functions of finite order as coefficients and* $\deg(P) > 0$. *If $f(x)$ is a tropical entire function of infinite order such that $\rho_2(f) < 1$, then $f(x)^{\otimes\alpha} \otimes P(x, f)$ cannot be a non-constant tropical meromorphic function of finite order.*

Proof. Suppose $G(x) := f(x)^{\otimes\alpha} \otimes P(x, f)$ is of finite order. Given $r > 0$, let

$$S_+ := \{s : f(s) \geq 0, |s| = r\} \quad \text{and} \quad S_- := \{s : f(s) < 0, |s| = r\},$$

such that $S_+ \cup S_- = \{\pm r\}$. Then

$$m(r, P(x, f)) = \frac{1}{2}\left(\sum_{s \in S_+} P(s, f)^+ + \sum_{s \in S_-} P(s, f)^+ \right). \tag{3.1}$$

Let

$$P(x, f) = \bigoplus_{\lambda \in \Lambda[P]} a_\lambda(x) \otimes f(x \uplus c)^{\otimes\lambda}.$$

For any $x \in S_-$, $f(x) < 0$, $\deg(P) > 0$,

$$P(x, f) = \max_{\lambda \in \Lambda[P]} \left\{ a_\lambda(x) + \sum_{j=0}^{m} \lambda_j (f(x + c_j) - f(x)) + \|\lambda\| f(x) \right\}$$
$$\leq \max_{\lambda \in \Lambda[P]} \{a_\lambda(x)\} + \max_{\lambda \in \Lambda[P]} (f(x \uplus c) \oslash f(x))^{\otimes\lambda} + \deg(P) f(x)$$
$$\leq \max_{\lambda \in \Lambda[P]} \{a_\lambda(x)\} + \max_{\lambda \in \Lambda[P]} (f(x \uplus c) \oslash f(x))^{\otimes\lambda}.$$

So for $s \in S_-$, using Lemma 2.6, we see that

$$(3.2) \qquad \begin{aligned} P(s,f)^+ &\le \left(\max_{\lambda \in \Lambda[P]}\{a_\lambda(s)\}\right)^+ + \left(\max_{\lambda \in \Lambda[P]}(f(x \uplus c) \oslash f(x))^{\otimes \lambda}\right)^+ \\ &\le 2m\left(r, \max_{\lambda \in \Lambda[P]}\{a_\lambda(s)\}\right) + 2m\left(r, \max_{\lambda \in \Lambda[P]}(f(x \uplus c) \oslash f(x))^{\otimes \lambda}\right) \\ &= o(T(r,f)/r^\delta) + O(r^\kappa) \end{aligned}$$

for some constant $\kappa > 0$, outside of an exceptional set of finite logarithmic measure, where $0 < \delta < 1 - \rho_2$.

For any $x \in S_+$, $f(x) \ge 0$, $\alpha > 0$,

$$G(x) := f(x)^{\otimes \alpha} \otimes P(x,f) = \alpha f(x) + P(x,f) \ge P(x,f).$$

When $s \in S_+$, again using Lemma 2.6, we have

$$(3.3) \qquad P(s,f)^+ \le (G(s))^+ \le 2m(r, G(x)) = O(r^\iota)$$

for some constant $\iota > 0$ as $r \to \infty$.

It follows from (3.1)–(3.3) that

$$(3.4) \qquad m(r, P(x,f)) = o(T(r,f)/r^\delta) + O(r^M)$$

for some constant $M = \max\{\kappa, \iota\} > 0$, outside of an exceptional set of finite logarithmic measure, where $0 < \delta < 1 - \rho_2$.

An application of [9, Theorem 6.2] yields

$$(3.5) \qquad m(r, P(x,f)) = \deg(P) m(r,f) + o(T(r,f)/r^\delta) + O(r^N)$$

for some constant $N > 0$, outside of an exceptional set of finite logarithmic measure, where $0 < \delta < 1 - \rho_2$.

Since $f(x)$ is tropical entire and $\deg(P) > 0$, by combining (3.4) and (3.5), it follows that

$$T(r,f) = m(r,f) = o(T(r,f)/r^\delta) + O(r^L)$$

for some constant $L = \max\{M, N\} > 0$, a contradiction.

Remark 3.2. Why we choose the form $f(x)^{\otimes \alpha} \otimes P(x,f)$ rather than $f(x)^{\otimes \alpha} \oplus P(x,f)$, that is because $f(x)^{\otimes \alpha} \oplus P(x,f) \ge \alpha f(x)$ always holds for $\alpha > 0$, and then the conclusion of Theorem 3.1 is trivial.

Laine, Liu and Tohge [10] investigate the value distribution of a tropical entire function of type $f(x)^{\otimes \alpha} \otimes f(x+c)$, where $\alpha > 0$, and obtained the following Theorems.

THEOREM B. *If $f(x)$ is a non-linear tropical entire function and $\alpha > 0$, then $f(x)^{\otimes \alpha} \otimes f(x+c)$ must have at least one root.*

Theorem C. *If $f(x)$ is a tropical transcendental entire function and $\alpha > 0$, then $f(x)^{\otimes\alpha} \otimes f(x+c)$ must have infinitely many roots.*

We improve Theorems B and C, and prove the results as follows.

Theorem 3.3. *Let $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_m)$, where the λ_js are non-negative real numbers, be a multi-index with respect to the shifts $(0, c_1, \dots, c_m) \in \mathbf{R}^{m+1}$. Let*

$$f(x \uplus c)^{\otimes\lambda} := \lambda_0 f(x) + \lambda_1 f(x+c_1) + \cdots + \lambda_m f(x+c_m).$$

If $f(x)$ is a non-linear tropical entire function and $\alpha > 0$, then $f(x)^{\otimes\alpha} \otimes f(x \uplus c)^{\otimes\lambda}$ must have at least one root.

Proof. Let

$$\begin{aligned} F(x) &:= f(x)^{\otimes\alpha} \otimes f(x \uplus c)^{\otimes\lambda} \\ &= (\alpha+\lambda_0) f(x) + \lambda_1 f(x+c_1) + \cdots + \lambda_m f(x+c_m). \end{aligned}$$

Suppose $F(x)$ has a pole at x_0, say. If $f(x)$ has no root at x_0, then $f(x)$ must have a pole at $x_0 + c_j$ $(j = 1, \dots, m)$, a contradiction. If $(\alpha+\lambda_0) f(x)$ has a root at x_0, then $-(\alpha+\lambda_0) f(x)$ has a pole at x_0, thus, $\lambda_1 f(x+c_1) + \cdots + \lambda_m f(x+c_m) = F(x) - (\alpha+\lambda_0) f(x)$ has a pole at x_0, then $f(x)$ must have a pole at $x_0 + c_j$ $(j = 1, \dots, m)$, a contradiction. Hence, $F(x)$ is tropical entire as well. Assume that $F(x)$ has no roots. Then $F(x)$ should be a linear function $Ax + B$, thus

$$(\alpha+\lambda_0) f(x) + \lambda_1 f(x+c_1) + \cdots + \lambda_m f(x+c_m) = Ax + B,$$

where A, B are real constants. Since $f(x)$ is a non-linear tropical entire function, which implies that $(\alpha+\lambda_0) f(x)$ has at least one root, say at x_0. Then $f(x)$ must have a pole at $x_0 + c_j$ $(j = 1, \dots, m)$, a contradiction.

Theorem 3.4. *If $f(x)$ is a tropical transcendental entire function and $\alpha > 0$, then $f(x)^{\otimes\alpha} \otimes f(x \uplus c)^{\otimes\lambda}$ must have infinitely many roots, where $f(x \uplus c)^{\otimes\lambda}$ is defined as Theorem 3.3.*

Proof. As proof of Theorem 3.3, we see that $F(x) = f(x)^{\otimes\alpha} \otimes f(x \uplus c)^{\otimes\lambda}$ is tropical entire. Contrary to the assertion, if $F(x)$ has only finitely many roots, then $F(x)$ is a tropical polynomial. Let $x_1, \dots, x_n$ be its roots. If $(\alpha+\lambda_0) f(x)$ has a root at x such that $x_j < x < x_{j+1}$, then $F(x)$ is linear around x, hence $f(x)$ must have a pole at $x + c_j$ $(j = 1, \dots, m)$, a contradiction. Therefore, the only possible roots of $f(x)$ are at $\{x_1, \dots, x_n, x_1 + c_1, \dots, x_n + c_1, \dots, x_1 + c_m, \dots, x_n + c_m\}$, implying that $f(x)$ is a tropical polynomial, a contradiction.

Remark 3.5. If $\alpha < 0$, Theorem 3.4 is not true. For example, if $\alpha = -3$, then the tropical exponential function $e_{1/2}(x)$ satisfies $-3e_{1/2}(x) + e_{1/2}(x) +$

$2e_{1/2}(x+1) + 4e_{1/2}(x+2) = 0$. This implies that $e_{1/2}(x)^{\otimes(-3)} \otimes (e_{1/2}(x) + 2e_{1/2}(x+1) + 4e_{1/2}(x+2))$ has no roots.

If $\lambda_0, \lambda_1, \dots, \lambda_m < 0$, Theorem 3.4 is also not true. For example, if $\alpha = 3$, the tropical exponential function $e_2(x)$ satisfies $3e_2(x) - e_2(x) - \frac{1}{2}e_2(x+1) - \frac{1}{4}e_2(x+2) = 0$. This implies that $e_2(x)^{\otimes 3} \otimes \left(-e_2(x) - \frac{1}{2}e_2(x+1) - \frac{1}{4}e_2(x+2)\right)$ has no roots.

If there are different signs among $\lambda_0, \lambda_1, \dots, \lambda_m$, Theorem 3.4 is still not true. For example, if $\alpha = 1$, then the tropical entire function $\psi(x)$ (see [10, Proposition 3.22]) satisfies $\psi(x) + \psi(x+1) - 2\psi(x-1) = 3x + 1$. This implies that $\psi(x)^{\otimes 1} \otimes (\psi(x+1) - 2\psi(x-1))$ has no roots.

The above three examples are also available for Theorem 3.3 and the functions $e_2(x)$, $e_{1/2}(x)$ may be added by a linear function to obtain a non-vanishing and linear $f(x)^{\otimes \alpha} \otimes f(x \uplus c)^{\otimes \lambda}$, too.

4. Uniqueness theory of tropical entire functions

In this section, we study the uniqueness theory of tropical entire functions. As regards the uniqueness problems for entire functions, Fang and Hua [2] and also Yang and Hua [15] obtained some results. We now recall the following result.

Theorem D. *Let f and g be non-constant entire functions, and let $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

Qi, Yang and Liu [12], Theorem 1.2, proposed a difference analogue to the Theorem D, proving

Theorem E. *Let f and g be transcendental entire functions of finite order, and c be a non-zero complex constant. Let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 that satisfy $t_i^{n+1} = 1$, $i = 1, 2$.*

We consider a tropical counterpart to the preceding Theorem E, and obtain the result as follows.

Theorem 4.1. *Let $f(x)$ and $g(x)$ be tropical entire functions, $a \in \mathbf{R}$ be fixed and suppose that* $\max\{f(x)^{\otimes \alpha} \otimes f(x+1), a\}$ *and* $\max\{g(x)^{\otimes \alpha} \otimes g(x+1), a\}$ *have the same roots with the same multiplicity for all $x \in \mathbf{R}$, $\alpha > 0$. Then one of the following three cases holds.*

(i) *If $\alpha = 1$, then*

$$f(x) = \Xi(x) + \frac{A}{2}x + \frac{a + B - A/2}{2} \quad (A < 0)$$

or

$$f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2};$$

if $\alpha \neq 1$, *then*

$$f(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{a + B - \dfrac{A}{\alpha+1}}{\alpha+1} \quad (A < 0)$$

or

$$f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1},$$

$b \in [0, 1)$, *for all* x *approach to* $-\infty$.

(ii) *If* $\alpha = 1$, *then*

$$f(x) = \Xi(x) + \frac{A}{2}x + \frac{a + B - A/2}{2} \quad (A > 0)$$

or

$$f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2};$$

if $\alpha \neq 1$, *then*

$$f(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{a + B - \dfrac{A}{\alpha+1}}{\alpha+1} \quad (A > 0)$$

or

$$f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1},$$

$b \in [0, 1)$, *for all* x *approach to* $+\infty$.

(iii) *If* $\alpha = 1$, *then*

$$f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2};$$

if $\alpha \neq 1$,

$$f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1},$$

$b \in [0, 1)$, *for all* x *such that* $|x|$ *is large enough. Here* A, B *are real numbers,* $\Xi(x)$ *is a tropical* 2*-periodic, anti-*1*-periodic function,* $L_b(e_{-\alpha}(x-b))$ *is as defined in Lemma* 2.1.

Proof. An application of [9, Proposition 7.3] implies, $\max\{f(x)^{\otimes\alpha} \otimes f(x+1), a\} \oslash \max\{g(x)^{\otimes\alpha} \otimes g(x+1), a\}$ is linear, then

$$\max\{f(x)^{\otimes\alpha} \otimes f(x+1), a\} = \max\{g(x)^{\otimes\alpha} \otimes g(x+1), a\} + Ax + B, \tag{4.1}$$

for some real numbers A, B. As $f(x)$ and $g(x)$ are tropical entire, they cannot be upper bounded. We consider three cases: case 1, $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$; case 2, $\lim_{x\to-\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$; case 3, when $x \to \pm\infty$, $f(x) \to +\infty$.

Case 1. $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$. In this case, we discuss three subcases: subcase 1.1, $\lim_{x\to+\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$; subcase 1.2, $\lim_{x\to-\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$; subcase 1.3, when $x \to \pm\infty$, $g(x) \to +\infty$.

Subcase 1.1. $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$ and $\lim_{x\to+\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$. When $x \to +\infty$, it follows from (4.1) that $A = 0$. Then for x approaches to $-\infty$, (4.1) may be written in the form

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + B.$$

Set $h(x) = f(x) - g(x)$. Rewrite the equation above as

$$\alpha h(x) + h(x+1) = B.$$

If $\alpha = 1$, it follows from Lemma 2.2(i) that

$$h(x) = f(x) - g(x) = \Xi(x) + \frac{B}{2}.$$

If $\alpha \neq 1$, it follows from Lemma 2.2(iii) that

$$h(x) = f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{B}{\alpha+1}, \quad b \in [0, 1).$$

Subcase 1.2. $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$ and $\lim_{x\to-\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$. When $x \to +\infty$, it follows from (4.1) that $A < 0$. For x approaches to $-\infty$, $f(x) \to +\infty$ and if $g(x) \to -\infty$, then (4.1) takes the form

$$\alpha f(x) + f(x+1) = Ax + a + B.$$

If $\alpha = 1$, it follows from Lemma 2.5(i) that

$$f(x) = \Xi(x) + \frac{A}{2}x + \frac{a + B - A/2}{2}.$$

If $\alpha \neq 1$, it follows from Lemma 2.5(iii) that

$$f(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{a + B - \dfrac{A}{\alpha+1}}{\alpha+1}, \quad b \in [0,1).$$

For x approaches to $-\infty$, $f(x) \to +\infty$ and if $\lim_{x\to-\infty} g(x) \in \mathbf{R}$ and also if $\alpha g(x) + g(x+1) \leq a$, the same to the conclusion above. If $\alpha g(x) + g(x+1) > a$, then (4.1) takes the form

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + Ax + B.$$

Denote $h(x) = f(x) - g(x)$ and rewrite the equation above as

$$\alpha h(x) + h(x+1) = Ax + B.$$

If $\alpha = 1$, it follows from Lemma 2.5(i) that

$$h(x) = f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2}.$$

If $\alpha \neq 1$, it follows from Lemma 2.5(iii) that

$$h(x) = f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1}, \quad b \in [0,1).$$

Subcase 1.3. $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$ and $g(x) \to +\infty$ for $x \to \pm\infty$. When $x \to +\infty$, $\lim_{x\to+\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$ and $g(x) \to +\infty$, it follows from (4.1) that $A < 0$. For x approaches to $-\infty$, $f(x) \to +\infty$ and if $g(x) \to +\infty$, rewrite (4.1) as

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + Ax + B.$$

Similarly, we can conclude that

$$f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2}, \quad \text{for } \alpha = 1,$$

and

$$f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1}, \quad b \in [0,1), \text{ for } \alpha \neq 1.$$

Case 2. Similar reasoning applies to the case of $\lim_{x\to-\infty} f(x) \in \mathbf{R} \cup \{-\infty\}$.

Case 3. When $x \to \pm\infty$, $f(x) \to +\infty$. In this case, we also take into account three subcases: subcase 3.1, $\lim_{x\to+\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$; subcase 3.2, $\lim_{x\to-\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$; subcase 3.3, when $x \to \pm\infty$, $g(x) \to +\infty$.

SUBCASE 3.1. When $x \to +\infty$, $f(x) \to +\infty$ and $\lim_{x\to+\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$, it follows from (4.1) that $A > 0$. For x approaches to $-\infty$, $f(x) \to +\infty$ and if $g(x) \to +\infty$, then (4.1) takes the form

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + Ax + B.$$

Solving this equation above, if $\alpha = 1$,

$$f(x) - g(x) = \Xi(x) + \frac{A}{2}x + \frac{B - A/2}{2},$$

and if $\alpha \neq 1$,

$$f(x) - g(x) = L_b(e_{-\alpha}(x-b)) + \frac{A}{\alpha+1}x + \frac{B - \dfrac{A}{\alpha+1}}{\alpha+1}, \quad b \in [0,1).$$

SUBCASE 3.2. When $x \to -\infty$, $f(x) \to +\infty$ and $\lim_{x\to-\infty} g(x) \in \mathbf{R} \cup \{-\infty\}$, it follows from (4.1) that $A < 0$. For x approaches to $+\infty$, $f(x) \to +\infty$ and if $g(x) \to +\infty$, then (4.1) takes the form

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + Ax + B.$$

We can obtain similar conclusion of subcase 1.3.

SUBCASE 3.3. When $x \to \pm\infty$, $f(x) \to +\infty$ and $g(x) \to +\infty$, it follows from (4.1) that $A \in \mathbf{R}$ and

$$\alpha f(x) + f(x+1) = \alpha g(x) + g(x+1) + Ax + B.$$

We also obtain similar conclusion as above.

5. Applications to ultra-discrete equations

In the final section, we study some ultra-discrete equations. Yanagihara [14, Theorem 1′] investigated the difference equation

$$y(z+1) = R(z, y(z)), \tag{5.1}$$

where $R(z, y(z))$ is a rational function in z and y. If equation (5.1) possesses a transcendental meromorphic solution $y(z)$ of finite order, then $\deg_y R(z, y(z)) \leq 1$.

Halburd and Korhonen [4, p. 197] are concerned with the difference Riccati equation of the form

$$y(z+1) = \frac{A + \varepsilon y(z)}{\varepsilon - y(z)}, \tag{5.2}$$

where A is a polynomial, $\varepsilon = \pm 1$.

Grammatics et al. [3, p. 3817] considered $d - P_{IV}$ equation

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = \frac{P(y_n)}{Q(y_n)}, \tag{5.3}$$

where $P(y_n)$ and $Q(y_n)$ are polynomials in y_n.

In this paper, we consider the ultra-discrete equations

$$y(x+1) = P(y(x)) \oslash Q(y(x)), \tag{5.4}$$

$$y(x+1) = \max\{A, y(x)\} \oslash \max\{0, -y(x)\}, \quad (A \neq 0) \tag{5.5}$$

and

$$\max\{y(x+1), y(x)\} + \max\{y(x), y(x-1)\} = P(y(x)) \oslash Q(y(x)), \tag{5.6}$$

where $P(y(x))$ and $Q(y(x))$ are tropical polynomials in $y(x)$ with no common roots. We mainly investigate the tropical meromorphic solutions of ultra-discrete equations (5.4)–(5.6) and obtain the results as follows.

Theorem 5.1. *Let $P(y) = \max\{a_0 + k_0 y, a_1 + k_1 y, \dots, a_p + k_p y\}$ and $Q(y) = \max\{b_0 + l_0 y, b_1 + l_1 y, \dots, b_q + l_q y\}$ be two tropical polynomials with no common roots, and a_i, b_j, k_i, l_j $(i = 0, 1, \dots, p,\ j = 0, 1, \dots, q)$ be real numbers such that $0 < k_0 < k_1 < \cdots < k_p$ and $0 < l_0 < l_1 < \cdots < l_q$. If equation (5.4) admits such a tropical meromorphic solution $f(x)$ that is sufficiently large for all x larger than some number κ, then $f(x)$ may be represented in the following forms for all $x > \kappa$:*

if $k_p - l_q = 1$, then

$$f(x) = \Pi(x) + (a_p - b_q)x \quad \text{when } a_p > b_q \text{ and}$$

if $k_p - l_q > 1$, then

$$f(x) = \sum_{m=1}^{r} \beta_m e_{k_p - l_q}(x - b_m) + \frac{a_p - b_q}{1 - (k_p - l_q)}, \quad b_m \in [0, 1) \text{ and } \beta_m > 0,\ m = 1, \dots, r.$$

If equation (5.4) admits such a tropical meromorphic solution $f(x)$ that is sufficiently large for all x less than some number ι, then $f(x)$ may be represented in the following forms for all $x < \iota$:

if $k_p - l_q = 1$, then

$$f(x) = \Pi(x) + (a_p - b_q)x \quad \text{when } a_p < b_q \text{ and}$$

if $0 < k_p - l_q < 1$, then

$$f(x) = \sum_{n=1}^{s} \gamma_n e_{k_p - l_q}(x - c_n) + \frac{a_p - b_q}{1 - (k_p - l_q)}, \quad c_n \in [0, 1) \text{ and } \gamma_n > 0,\ n = 1, \dots, s.$$

Here $\Pi(x)$ is a tropical 1-periodic function.

Proof. If y is sufficiently large for all x larger than some number κ, then (5.4) reduces to

$$y(x+1) = (k_p - l_q)y(x) + a_p - b_q,$$

for all $x > \kappa$, and $k_p > l_q$, otherwise, which yields a contradiction. If $k_p - l_q = 1$, it follows from Lemma 2.2(ii) that

$$f(x) = \Pi(x) + (a_p - b_q)x.$$

When $a_p > b_q$, then $f(x)$ can be a tropical meromorphic solution of equation (5.4). If $k_p - l_q > 1$, it follows from Lemma 2.2(iii) that

$$f(x) = \sum_{m=1}^{r} \beta_m e_{k_p - l_q}(x - b_m) + \frac{a_p - b_q}{1 - (k_p - l_q)}, \quad b_m \in [0,1),\ m = 1, \dots, r.$$

Since $f(x)$ is sufficiently large for all $x > \kappa$, then $\beta_m > 0$.

Similarly, if y is sufficiently large for all x less than some number ι, then (5.4) reduces to

$$y(x+1) = (k_p - l_q)y(x) + a_p - b_q,$$

for all $x < \iota$, and $k_p > l_q$, otherwise, which yields a contradiction. If $k_p - l_q = 1$, then

$$f(x) = \Pi(x) + (a_p - b_q)x, \quad a_p < b_q.$$

If $0 < k_p - l_q < 1$, then

$$f(x) = \sum_{n=1}^{s} \gamma_n e_{k_p - l_q}(x - c_n) + \frac{a_p - b_q}{1 - (k_p - l_q)}, \quad c_n \in [0,1),\ \gamma_n > 0,\ n = 1, \dots, s.$$

THEOREM 5.2. *Let A be a non-zero real number. Then the equation* (5.5) *admits a non-constant tropical meromorphic solution f of the form $f(x) = \Pi(x)$, when and only when f satisfies $f(x) \geq \max(A, 0)$ on $\mathbf{R}$. Here $\Pi(x)$ is a tropical* 1*-periodic function.*

Proof. If f is a non-constant tropical meromorphic solution of (5.5) and satisfies $f(x) \geq \max(A, 0)$ on $\mathbf{R}$, then $f(x) \geq A(\neq 0)$, $f(x) \geq 0$ and $-f(x) \leq 0$. Rewrite (5.5) as $f(x+1) = f(x)$, by Lemma 2.1(i), we have $f(x) = \Pi(x)$.

If (5.5) admits a non-constant tropical meromorphic solution f of the form $f(x) = \Pi(x)$, then $f(x+1) = \Pi(x+1) = \Pi(x)$. It follows from (5.5) that

$$\max\{A, \Pi(x)\} = \Pi(x) + \max\{0, -\Pi(x)\} = \max\{\Pi(x), 0\}.$$

Since $A \neq 0$, from the equality above, we have $\Pi(x) \geq \max(A, 0)$ on $\mathbf{R}$.

THEOREM 5.3. *Let $P(y) = \max\{a_0 + k_0 y, a_1 + k_1 y, \dots, a_p + k_p y\}$ and $Q(y) = \max\{b_0 + l_0 y, b_1 + l_1 y, \dots, b_q + l_q y\}$ be two tropical polynomials with no common* *roots, and* a_i, b_j, k_i, l_j $(i = 0, 1, \dots, p, j = 0, 1, \dots, q)$ *be real numbers such that* $0 < k_0 < k_1 < \cdots < k_p$ *and* $0 < l_0 < l_1 < \cdots < l_q$. *If equation* (5.6) *admits such a tropical meromorphic solution* $f(x)$ *that is sufficiently large for all* x *larger than some number* κ, *then* $f(x)$ *may be represented in the following forms for all* $x > \kappa$:

if $k_p - l_q = 2$, *then*

$$f(x) = \Pi(x) + (a_p - b_q)x \quad \text{when } a_p > b_q \text{ and}$$

if $k_p - l_q > 2$, *then*

$$f(x) = \sum_{m=1}^{r} \beta_m e_{-(1-(k_p-l_q))}(x - b_m) + \frac{a_p - b_q}{2 - (k_p - l_q)}, \quad b_m \in [0, 1),\ \beta_m > 0,\ m = 1, \dots, r.$$

If equation (5.6) *admits such a tropical meromorphic solution* $f(x)$ *that is sufficiently large for all* x *less than some number* ι, *then* $f(x)$ *may be represented in the following forms for all* $x < \iota$:

if $k_p - l_q = 2$, *then*

$$f(x) = \Pi(x) - (a_p - b_q)x \quad \text{when } a_p > b_q \text{ and}$$

if $k_p - l_q > 2$, *then*

$$f(x) = \sum_{n=1}^{s} \gamma_n e_{-1/(1-(k_p-l_q))}(x - c_n) + \frac{a_p - b_q}{2 - (k_p - l_q)}, \quad c_n \in [0, 1),\ \gamma_n > 0,\ n = 1, \dots, s.$$

If equation (5.6) *admits such a tropical meromorphic solution* $f(x)$ *that is sufficiently large for all* x *larger than some number* κ *or less than some number* ι, *then* $f(x)$ *may be represented in the following forms for all* $x > \kappa$ *or* $x < \iota$:

if $k_p - l_q = 2$, *then*

$$f(x) = \tilde{\Pi}(x) + \Phi(x, \Pi) + \Pi(0)x + (a_p - b_q)(\psi(x) - x) \quad \text{when } a_p > b_q,$$

where

$$\Phi(x, \Pi) := [x](\Pi(x) - \Pi(0)) \quad \text{and} \quad \psi(x) = ([x] + 1)\left(x - \frac{1}{2}[x]\right),$$

if $k_p - l_q > 2$, *then*

$$f(x) = \sum_{m=1}^{r} \mu_m e_\alpha(x - b_m) + \sum_{n=1}^{s} \nu_n e_\beta(x - c_n) + \frac{a_p - b_q}{2 - (k_p - l_q)},$$

$$b_m \in [0, 1),\ \mu_m > 0,\ m = 1, \dots, r,\ c_n \in [0, 1),\ \nu_n > 0,\ n = 1, \dots, s,$$

where α, β *are the roots of* $\lambda^2 - (k_p - l_q)\lambda + 1 = 0$. *Here* $\Pi(x)$, $\tilde{\Pi}(x)$ *are tropical* 1-*periodic functions.*

Proof. If y is sufficiently large for all x larger than some number κ, then (5.6) reduces to $y(x+1) + y(x) = (k_p - l_q)y(x) + a_p - b_q$ for all $x > \kappa$, and $k_p > l_q$, otherwise, which yields a contradiction.

When $y(x+1) + y(x) = (k_p - l_q)y(x) + a_p - b_q$, $k_p - l_q \neq 1$, or else $y(x)$ is a constant. If $k_p - l_q = 2$, then $y(x+1) - y(x) = a_p - b_q$, it follows from Lemma 2.2(ii) that

$$f(x) = \Pi(x) + (a_p - b_q)x.$$

When $a_p > b_q$, then $f(x)$ can be a tropical meromorphic solution of equation (5.6). If $k_p - l_q \neq 1, 2$, by Lemma 2.2(iii), we have

$$f(x) = \sum_{m=1}^{r} \beta_m e_{-(1-(k_p-l_q))}(x - b_m) + \frac{a_p - b_q}{2 - (k_p - l_q)}, \quad b_m \in [0, 1), \beta_m > 0, m = 1, \dots, r.$$

Since $f(x)$ is sufficiently large for all $x > \kappa$, then $-(1 - (k_p - l_q)) > 1$, i.e. $k_p - l_q > 2$.

Similarly, if y is sufficiently large for all x less than some number ι, then (5.6) reduces to $y(x) + y(x-1) = (k_p - l_q)y(x) + a_p - b_q$ for all $x < \iota$, and $k_p > l_q$, otherwise, which yields a contradiction.

When $y(x) + y(x-1) = (k_p - l_q)y(x) + a_p - b_q$, $k_p - l_q \neq 1$, or else $y(x-1)$ is a constant. If $k_p - l_q = 2$, then

$$f(x) = \Pi(x) - (a_p - b_q)x \quad \text{when } a_p > b_q \text{ and}$$

if $k_p - l_q \neq 1, 2$, then

$$f(x) = \sum_{n=1}^{s} \gamma_n e_{-1/(1-(k_p-l_q))}(x - c_n) + \frac{a_p - b_q}{2 - (k_p - l_q)}, \quad c_n \in [0, 1), \gamma_n > 0, n = 1, \dots, s.$$

Since $f(x)$ is sufficiently large for all $x < \iota$, then $0 < -\dfrac{1}{1 - (k_p - l_q)} < 1$, i.e. $k_p - l_q > 2$.

If y is sufficiently large for all x larger than some number κ or less than some number ι, then (5.6) reduces to $y(x+1) + y(x-1) = (k_p - l_q)y(x) + a_p - b_q$ for all $x > \kappa$ or $x < \iota$, and $k_p > l_q$, otherwise, which yields a contradiction. When

$$y(x+1) + y(x-1) = (k_p - l_q)y(x) + a_p - b_q, \tag{5.7}$$

if $k_p - l_q = 2$, following the ideas in [10, pp. 934–935], then

$$f(x) = \tilde{\Pi}(x) + \Phi(x, \Pi) + \Pi(0)x + (a_p - b_q)(\psi(x) - x),$$

where

$$\Phi(x, \Pi) := [x](\Pi(x) - \Pi(0)) \quad \text{and} \quad \psi(x) = ([x] + 1)\left(x - \frac{1}{2}[x]\right).$$

When $a_p > b_q$, then $f(x)$ can be a tropical meromorphic solution of equation (5.6).

Let $f_1(x) = f(x) - \frac{a_p - b_q}{2 - (k_p - l_q)}$, then (5.7) can be rewritten as

$$f_1(x+1) - (k_p - l_q) f_1(x) + f_1(x-1) = 0.$$

If $k_p - l_q > 2$, an application of [9, Theorem 10.1(iii)] yields

$$f_1(x) = \sum_{m=1}^{r} \mu_m e_\alpha(x - b_m) + \sum_{n=1}^{s} \nu_n e_\beta(x - c_n) = f(x) - \frac{a_p - b_q}{2 - (k_p - l_q)},$$

$$b_m \in [0,1),\ \mu_m > 0,\ m = 1, \dots, r,\ c_n \in [0,1),\ \nu_n > 0,\ n = 1, \dots, s,$$

then

$$f(x) = \sum_{m=1}^{r} \mu_m e_\alpha(x - b_m) + \sum_{n=1}^{s} \nu_n e_\beta(x - c_n) + \frac{a_p - b_q}{2 - (k_p - l_q)},$$

where α, β are the roots of $\lambda^2 - (k_p - l_q)\lambda + 1 = 0$. If $0 < k_p - l_q < 2$, following the ideas in [9, p. 918], we have

$$f_1(x) = \cos(\theta[x])(x - [x]) + \frac{\cos(\theta[x])(\cos\theta - 1) + \sin(\theta[x])\sin\theta}{2(1 - \cos\theta)},$$

or

$$f_1(x) = \sin(\theta[x])(x - [x]) + \frac{\sin(\theta[x])(\cos\theta - 1) - \cos(\theta[x])\sin\theta}{2(1 - \cos\theta)},$$

which implies $f_1(x)$ is kept bounded as $x > \kappa$ or $x < \iota$, then $f(x) = f_1(x) + \frac{a_p - b_q}{2 - (k_p - l_q)}$ is also kept bounded as $x > \kappa$ or $x < \iota$, a contradiction.

Acknowledgements. The authors would like to thank the referee for his/her thorough reviewing with constructive suggestions and comments to the paper.

References

[1] W. Cherry and Z. Ye, Nevanlinna's theory of value distribution, Springer, Berlin, 2001.
[2] M. L. Fang and X. H. Hua, Entire functions that share one value, Nanjing Daxue Xuebao Shuxue Bannian Kan. **13** (1996), 44–48.
[3] B. Grammaticos, T. Tamizhmani, A. Ramani and K. M. Tamizhmani, Growth and integrability in discrete systems, J. Phys. A: Math. Gen. **34** (2001), 3811–3821.
[4] R. G. Halburd and R. J. Korhonen, Existence of finite-order meromorphic solutions as a detector of integrability in difference equations, Physica. D. **218** (2006), 191–203.
[5] R. G. Halburd and N. Southall, Tropical Nevanlinna theory and ultra-discrete equations, Int. Math. Res. Not. **2009** (2009), 887–911.
[6] W. K. Hayman, Meromorphic function, Clarendon Press, Oxford, 1964.

[7] R. Korhonen, I. Laine and K. Tohge, Tropical value distribution theory and ultra-discrete equations, World Scientific, Singapore, 2015.
[8] I. Laine, Nevanlinna theory and complex differential equations, W. de Gruyter, Berlin, 1993.
[9] I. Laine and K. Tohge, Tropical Nevanlinna theory and second main theorem, Proc. London Math. Soc. **102** (2011), 883–922.
[10] I. Laine, K. Liu and K. Tohge, Tropical variants of some complex analysis results, Ann. Acad. Sci. Fenn. Math. **41** (2016), 923–946.
[11] N. Li and L. Z. Yang, Some results related to complex linear and nonlinear difference equations, J. Differ. Equ. Appl. **20** (2014), 237–250.
[12] X. G. Qi, L. Z. Yang and K. Liu, Uniqueness and periodicity of meromorphic functions concerning the difference operator, Comp. Math. Appl. **60** (2010), 1739–1746.
[13] D. Speyer and B. Sturmfels, Tropical mathematics, Math. Mag. **82** (2009), 163–173.
[14] N. Yanagihara, Meromorphic solutions of some difference equations, Funkcial. Ekvac. **23** (1980), 309–326.
[15] C. C. Yang and X. H. Hua, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math. **22** (1997), 395–406.
[16] C. C. Yang and I. Laine, On analogies between nonlinear difference and differential equations, Proc. Japan Acad., Ser. A. **86** (2010), 10–14.

Min-Feng Chen
LMIB & School of Mathematics and Systems Science
Beihang University
Beijing, 100191
P. R. China
Email: chenminfeng198710@126.com

Zong-Sheng Gao
LMIB & School of Mathematics and Systems Science
Beihang University
Beijing, 100191
P. R. China
E-mail: zshgao@buaa.edu.cn

Ji-Long Zhang
LMIB & School of Mathematics and Systems Science
Beihang University
Beijing, 100191
P. R. China
E-mail: jilongzhang2007@gmail.com