G. LE
KODAI MATH. J.
41 (2018), 52–69

AN EFFECTIVE SCHMIDT'S SUBSPACE THEOREM FOR HYPERSURFACES IN SUBGENERAL POSITION IN PROJECTIVE VARIETIES OVER FUNCTION FIELDS

Giang Le

Abstract

We established an effective version of Schmidt's subspace theorem on a smooth projective variety $\mathscr{X}$ over function fields of characteristic zero for hypersurfaces located in m-subgeneral position with respect to $\mathscr{X}$.

1. Introduction

One of the cornerstones of modern Diophantine Approximation is the Schmidt Subspace Theorem. In the number field case, there is still no effective version of this theorem. On the other hand, with techniques from Nevanlinna theory it has become possible to obtain effective version of several important results in Diophantine approximation over algebraic function fields. In [1], An and Wang obtained an effective Schmidt's subspace theorem for non-linear forms over function fields. In [9], Ru and Wang extended such effective results to divisors of a projective variety $\mathscr{X} \subset \mathbf{P}^M$ coming from hypersurfaces in $\mathbf{P}^M$ in general position with respect to $\mathscr{X}$. Our purpose is to generalize the above results to the case in which hypersurfaces are located in m-subgeneral position with respect to $\mathscr{X}$.

Here let $\mathscr{X}$ be a n-dimensional projective subvariety of $\mathbf{P}^M$ defined over K and m, q be positive integers with $m \geq n$ and $q \geq m+1$. Recall that homogeneous polynomials $Q_1, \dots, Q_q \in K[X_0, \dots, X_M]$ are said to be in *m-subgeneral position with respect to* $\mathscr{X}$ if $\bigcap_{j=1}^{m+1}(\{Q_{i_j} = 0\}) \cap \mathscr{X}(\overline{K}) = \emptyset$ for any distinct $i_1, \dots, i_{m+1} \in \{1, \dots, q\}$, where $\overline{K}$ is the algebraic closure of K. When $m = n$, they are said to be in *general position with respect to* $\mathscr{X}$.

Recently, Chen, Ru, Yan (see [4]) and Levin (see [7], Theorem 5.1) established Schmidt's subspace theorem for hypersurfaces located in m-subgeneral position over number fields and showed the analogous result for the case of holomorphic curves. This paper is inspired by these works.

2010 *Mathematics Subject Classification*. 11J97, 11J61.

Key words and phrases. Schmidt's subspace theorem, Function fields, Diophantine approximation.
Received March 29, 2016; revised March 7, 2017.

To state our results, we will recall some definitions and basic facts from algebraic geometry.

Let k be an algebraically closed field of characteristic 0 and let V be a projective variety (always assumed irreducible), non-singular in codimension 1 and defined over k. For the rest of paper, we shall fix an embedding of V such that $V \subset \mathbf{P}^{M_0}$ for some positive integer M_0.

Denote by $K = k(V)$ the function field of V. Let M_K be the set of discrete absolute values of the function field K obtained from the prime divisors of V. Let $\mathfrak{p}$ be a prime divisor of V over k. Such a prime divisor determines its local ring in the function field $k(V)$ and this local ring is a discrete valuation ring. Thus, we have the notion of order at $\mathfrak{p}$ of a function $x \in K$, $x \neq 0$, noted $\mathrm{ord}_{\mathfrak{p}}\, x$. We can associate to x its divisors

$$(x) = \sum_{\mathfrak{p} \in M_K} \mathrm{ord}_{\mathfrak{p}}(x)\mathfrak{p}.$$

By the *degree* of $\mathfrak{p}$, noted $\deg \mathfrak{p}$, we shall mean the projective degree, i.e. the number of points of intersection with a generic linear variety of complementary dimension in the given projective embedding. Then we have the sum formula

$$\deg(x) = \sum_{\mathfrak{p} \in M_K} \mathrm{ord}_{\mathfrak{p}}(x) \deg \mathfrak{p} = 0$$

for all $x \in K^*$.

Let $\mathbf{x} = [x_0 : x_1 : \cdots : x_M] \in \mathbf{P}^M(K)$ and define

$$e_{\mathfrak{p}}(\mathbf{x}) := \min_{0 \leq i \leq M} \{\mathrm{ord}_{\mathfrak{p}}(x_i)\}.$$

We define the (logarithmic) height of $\mathbf{x}$ by:

$$h(\mathbf{x}) = -\sum_{\mathfrak{p} \in M_K} e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p}.$$

By the sum formula, the height function is well-defined on $\mathbf{P}^M(K)$.

Let $Q = \sum_I a_I \mathbf{x}^I$ be a homogeneous polynomial of degree d in $K[X_0, \ldots, X_M]$, where $\mathbf{x}^I = x_0^{i_0} \cdots x_M^{i_M}$ and the sum is taken over all index sets $I = \{i_0, \ldots, i_M\}$ such that $i_j \geq 0$ and $\sum_{j=0}^M i_j = d$. For each $\mathfrak{p} \in M_K$, we set

$$e_{\mathfrak{p}}(Q) := \min_I \{\mathrm{ord}_{\mathfrak{p}}(a_I)\}.$$

The height of a homogeneous polynomial Q of degree d in $K[X_0, \ldots, X_M]$ is defined by the height of coefficients:

$$h(Q) = \sum_{\mathfrak{p} \in M_K} -e_{\mathfrak{p}}(Q) \deg \mathfrak{p}.$$

From the sum formula, we have $h(\alpha Q) = h(Q)$ for all $\alpha \in K^*$. Since we may assume that one of the non-zero coefficient of Q is 1, it follows that $h(Q) \geq 0$.

The Weil function $\lambda_{\mathfrak{p},Q}$ is defined by

$$\lambda_{\mathfrak{p},Q}(\mathbf{x}) := (\mathrm{ord}_{\mathfrak{p}}(Q(\mathbf{x})) - de_{\mathfrak{p}}(\mathbf{x}) - e_{\mathfrak{p}}(Q)) \deg \mathfrak{p} \geq 0$$

for $\mathbf{x} \in \mathbf{P}^M(K) \backslash \{Q = 0\}$.

Let $Q_1, Q_2, \ldots, Q_q$ be homogeneous polynomials of degree d in $K[X_0, \ldots, X_M]$. We define

$$e_{\mathfrak{p}}(Q_1, \ldots, Q_q) = \min\{e_{\mathfrak{p}}(Q_1), \ldots, e_{\mathfrak{p}}(Q_q)\}$$

and

$$h(Q_1, \ldots, Q_q) = -\sum_{\mathfrak{p} \in M_K} e_{\mathfrak{p}}(Q_1, \ldots, Q_q) \deg \mathfrak{p}.$$

Let $\mathscr{X}$ be a n-dimensional projective subvariety of $\mathbf{P}^M$ defined over K. The height of $\mathscr{X}$ is defined by

$$h(\mathscr{X}) := h(F_{\mathscr{X}}),$$

where $F_{\mathscr{X}}$ is the Chow form of $\mathscr{X}$.

In this paper, we will prove the following effective version of the generalized Schmidt's subspace theorem over K which corresponds to Chen-Ru-Yan's result [4] in number field case.

Main Theorem. *Let K be the function field of a nonsingular projective variety V defined over an algebraically closed field of characteristic 0 and let S be a finite set of prime divisors of V. Let $\mathscr{X}$ be a smooth n-dimensional projective subvariety of $\mathbf{P}^N$ defined over K with projective degree $\triangle_{\mathscr{X}}$. Let m, q be integers with $m \geq n$ and $q \geq m+1$. For all $i = 1, \ldots, q$, let Q_i be homogeneous polynomials of degree d_i in $K[X_0, \ldots, X_N]$ in m-subgeneral position with respect to $\mathscr{X}$. Then for any given $\varepsilon > 0$, there exists an effectively computable finite union $\mathscr{W}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^N(K)$ not containing $\mathscr{X}$ and effectively computable constants C_ε, C'_ε such that for any $\mathbf{x} \in \mathscr{X}(K) \backslash \mathscr{W}_\varepsilon$ either*

$$h(\mathbf{x}) \leq C_\varepsilon$$

or

$$\sum_{i=1}^{q} \sum_{\mathfrak{p} \in S} d_i^{-1} \lambda_{\mathfrak{p},Q_i}(\mathbf{x}) \leq (m(n+1) + \varepsilon) h(\mathbf{x}) + C'_\varepsilon.$$

The algebraic subsets in $\mathscr{W}_\varepsilon$ and the constants C_ε, C'_ε depend on ε and N, q, m, K, S, $\mathscr{X}$ and the Q_i. Furthermore, the degrees of the algebraic subsets in $\mathscr{W}_\varepsilon$ can be bounded above by

$$2(2n+1) d^{n+1} \triangle_{\mathscr{X}} \left(\binom{d+N}{N} + q + 1 \right) \varepsilon^{-1} + d,$$

where $d = lcm(d_1, \ldots, d_q)$.

Remark 1.1. The constants C_ε, C'_ε will be given in (5.17) and (5.18). They may depend on ε, the degree of the canonical divisor class of V, the projective degree of V, the degree of S (i.e. $\sum_{\mathfrak{p}\in S}\deg\mathfrak{p}$), the projective degree of $\mathscr{X}$, the dimension of $\mathscr{X}$, the height of $\mathscr{X}$ and the Q_i, q and m, N.

We would like to notice that Levin's result (Theorem 5.1) gives us a hope to improve the constant in front of $h(\mathbf{x})$ to $\dfrac{m(m-1)(n+1)}{m+n-2}$. However, in order to have an effective version, we need to make everything explicit and effective. The complexity of Levin's method (using 'lcm' of each pair of divisors instead of individual divisor, applying Riemann-Roch's theorem, e.t.c) causes us some difficulty to do this task.

2. Chow forms, Chow weights and Hilbert weights

2.1. Let $\mathscr{Y}$ be a n-dimensional projective subvariety of $\mathbf{P}^M$ defined over K of degree $\triangle_{\mathscr{Y}}$. To $\mathscr{Y}$, we can associate, up to a constant scalar, a unique polynomial

$$F_{\mathscr{Y}}(\mathbf{u}_0,\dots,\mathbf{u}_n)=F_{\mathscr{Y}}(u_{00},\dots,u_{0M};\dots;u_{n0},\dots,u_{nM})$$

in $(n+1)$ blocks of variables $\mathbf{u}_i=(u_{i0},\dots,u_{iM})$, $i=0,\dots,n$, which is called the *Chow form* of $\mathscr{Y}$, with the following properties:

$F_{\mathscr{Y}}$ is irreducible,

$F_{\mathscr{Y}}$ is homogeneous in each block u_i, $i=0,\dots,n$,

$F_{\mathscr{Y}}(\mathbf{u}_0,\dots,\mathbf{u}_n)=0$ if and only if $\mathscr{Y}\cap H_{\mathbf{u}_0}\cap\cdots\cap H_{\mathbf{u}_n}$ contains a $\overline{K}$-rational point, where $H_{\mathbf{u}_i}$, $i=0,\dots,n$ are hyperplanes given by $\mathbf{u}_i\cdot\mathbf{x}=u_{i0}x_0+\cdots+u_{iM}x_M=0$. It is well-known that the degree of $F_{\mathscr{Y}}$ in each block $\mathbf{u}_i$ is $\triangle_{\mathscr{Y}}$.

Let $\mathbf{c}=(c_0,\dots,c_M)$ be a tuple of reals. Let t be an auxiliary variable. We consider the decomposition

$$\begin{aligned}&F_{\mathscr{Y}}(t^{c_0}u_{00},\dots,t^{c_M}u_{0M},\dots,t^{c_0}u_{n0},\dots,t^{c_M}u_{nM})\\&\quad=t^{e_0}G_0(\mathbf{u}_0,\dots,\mathbf{u}_n)+\cdots+t^{e_r}G_r(\mathbf{u}_0,\dots,\mathbf{u}_n),\end{aligned}$$

with $G_0,\dots,G_r\in K[u_{00},\dots,u_{0M};\dots;u_{n0},\dots,u_{nM}]$ and $e_0>\cdots>e_r$. Now, we define the *Chow weight of $\mathscr{Y}$ with respect to* $\mathbf{c}$ by

$$e_{\mathscr{Y}}(\mathbf{c}):=e_0.$$

2.2. Let $\mathscr{Y}$ be a projective algebraic variety of $\mathbf{P}^M$, defined over K of dimension n and degree $\triangle_{\mathscr{Y}}$. Denote by $I_{\mathscr{Y}}$ the ideal of $\overline{K}[y_0,\dots,y_M]$ consisting of all polynomials vanishing identically on $\mathscr{Y}$. For a positive integer m, let $\overline{K}[y_0,\dots,y_M]_m$ denote the vector space of homogeneous polynomials in $\overline{K}[y_0,\dots,y_M]$ of degree m (together with the zero polynomial) and put

$$(I_{\mathscr{Y}})_m=\overline{K}[y_0,\dots,y_M]_m\cap I_{\mathscr{Y}}.$$

Then the Hilbert function of $\mathscr{Y}$ is defined by

$$H_{\mathscr{Y}}(m) := \dim_{\bar{K}}(\bar{K}[y_0, \dots, y_M]_m/(I_{\mathscr{Y}})_m)$$

for each $m \geq 1$.

By the usual theory of Hilbert polynomials, we have

$$H_{\mathscr{Y}}(m) = \triangle_{\mathscr{Y}} \cdot \frac{m^n}{n!} + O(m^{n-1}) \quad \text{as } m \to \infty. \tag{2.1}$$

We define the *m^{th}-Hilbert weight $s_{\mathscr{Y}}(m, \mathbf{c})$ of $\mathscr{Y}$ with respect to a tuple* $\mathbf{c} = (c_0, \dots, c_M) \in \mathbf{R}^{M+1}$ by

$$s_{\mathscr{Y}}(m, \mathbf{c}) = \max(\mathbf{a}_1 + \dots + \mathbf{a}_{H_{\mathscr{Y}}(m)}) \cdot \mathbf{c},$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{H_{\mathscr{Y}}(m)}}$ whose residue classes modulo $(I_{\mathscr{Y}})_m$ form a basis of the $\bar{K}$-vector space $\bar{K}[y_0, \dots, y_M]_m/(I_{\mathscr{Y}})_m$.

According to Mumford [8], proposition 2.11 we have

$$s_{\mathscr{Y}}(m, \mathbf{c}) = e_{\mathscr{Y}}(\mathbf{c}) \cdot \frac{m^{n+1}}{(n+1)!} + O(m^n).$$

Together with (2.1), this implies that

$$\lim_{m \to \infty} \frac{1}{mH_{\mathscr{Y}}(m)} \cdot s_{\mathscr{Y}}(m, \mathbf{c}) = \frac{1}{\triangle_{\mathscr{Y}} \cdot (n+1)} \cdot e_{\mathscr{Y}}(\mathbf{c}). \tag{2.2}$$

We call $\frac{1}{mH_{\mathscr{Y}}(m)} \cdot s_{\mathscr{Y}}(m, \mathbf{c})$ the *m-th normalized Hilbert weight* and $\frac{1}{\triangle_{\mathscr{Y}} \cdot (n+1)} \cdot e_{\mathscr{Y}}(\mathbf{c})$ the *normalized Chow weight of $\mathscr{Y}$ with respect to* $\mathbf{c}$.

2.3. The estimate on Chow weight of a projective variety $\mathscr{Y}$ [9, Lemma 3] plays an essential role in the proof of Ru-Wang's main theorem [9]. However, Ru-Wang Lemma 3 only studies the case of $\mathscr{Y}$ satisfying $\mathscr{Y} \cap \bigcap_{i \in I} H_i = \emptyset$ where $\#I = \dim \mathscr{Y} + 1$ and H_i, $i \in I$ be distinct coordinate hyperplanes. Thus, this lemma is not suitable for our need. To deal with the m-subgeneral position case, we need to give a lower bound for the Chow weight of a projective variety $\mathscr{Y}$ which may not satisfy the above-mentioned condition.

Proposition 2.1. *Let $\mathscr{Y}$ be a n-dimensional projective algebraic subvariety of $\mathbf{P}^M$ defined over K of degree $\triangle_{\mathscr{Y}}$. Let $\mathbf{c} = (c_0, \dots, c_M) \in \mathbf{R}_+^{M+1}$. Let I be a subset of $\{0, \dots, M\}$ such that $\mathscr{Y}$ is not contained in any coordinate hyperplane $H_i := \{y_i = 0\}$ for all $i \in I$. Then,*

$$e_{\mathscr{Y}}(\mathbf{c}) \geq \triangle_{\mathscr{Y}} \cdot \max_{i \in I} c_i.$$

Proof. Without loss of generality, we can assume that $0 \in I$ and $c_0 = \max_{i \in I} c_i$. Then, it is sufficient to prove that

$$e_{\mathscr{Y}}(\mathbf{c}) \geq \triangle_{\mathscr{Y}} \cdot c_0.$$

For each positive integer m, we consider the following filtration on the vector space $\overline{K}[y_0, \ldots, y_M]_m/(I_{\mathscr{Y}})_m$ with respect to y_0: The filtration

$$\overline{K}[y_0, \ldots, y_M]_m/(I_{\mathscr{Y}})_m = W_0 \supset W_1 \supset \cdots \supset W_m$$

is defined by

$$W_i = \{g^* \mid g \in \overline{K}[y_0, \ldots, y_M]_m \text{ and } y_0^i \mid g\},$$

where g^* is the projection of g to $\overline{K}[y_0, \ldots, y_M]_m/(I_{\mathscr{Y}})_m$. Take a basis $\psi_1^*, \ldots, \psi_{H_{\mathscr{Y}}(m)}^*$, ($\psi_j \in \overline{K}[y_0, \ldots, y_M]_m, j = 1, \ldots, H_{\mathscr{Y}}(m)$) of the vector space W_0 in the following way:

Since $\mathscr{Y} \not\subset H_0 := \{y_0 = 0\}$, we have $(y_0^m)^* \neq 0$. Then, $\{(y_0^m)^*\}$ is a basis for W_m. Choose $\psi_1 = y_0^m$.

The finite set of vectors $\{(y_0^m)^*, (y_0^{m-1}y_1)^*, \ldots, (y_0^{m-1}y_M)^*\}$ generates $\overline{K}$-vector space W_{m-1}. Then, there exists a finite set $I_{m-1} \subset \{1, \ldots, M\}$ such that $\{\psi_1^*, (y_0^{m-1}y_i)^* : i \in I_{m-1}\}$ form a basis for W_{m-1}. Choose ψ_k for $k = 2, \ldots, \dim W_{m-1}$ so that

$$\{\psi_k \mid k = 2, \ldots, \dim W_{m-1}\} = \{y_0^{m-1}y_i \mid i \in I_{m-1}\}.$$

Similarly, for each $j = m-1, \ldots, 1$, the finite set of vectors

$$\left\{(y_0^m)^*, (y_0^{m-1}y_1)^*, \ldots, (y_0^{m-1}y_M)^*, \ldots, (y_0^{j-1}y^{\mathbf{b}})^* : \mathbf{b} = (0, b_1, \ldots, b_M), \right.$$
$$\left. \sum_{i=1}^{M} b_i = m - j + 1\right\}$$

spans $\overline{K}$-vector space W_{j-1}. Then, there exists a finite subset $I_{j-1} \subset \{(0, b_1, \ldots, b_M) \mid \sum_{i=1}^M b_i = m - j + 1\}$ such that $\{\psi_1^*, \ldots, \psi_{\dim W_j}^*, (y_0^{j-1}y^{\mathbf{b}})^* : \mathbf{b} \in I_{j-1}\}$ form a basis for W_{j-1}. Choose

$$\{\psi_k \mid k = \dim W_j + 1, \ldots, \dim W_{j-1}\} = \{y_0^{j-1}y^{\mathbf{b}} \mid \mathbf{b} \in I_{j-1}\}.$$

The basis $\psi_1^*, \ldots, \psi_{H_{\mathscr{Y}}(m)}^*$ of the vector space W_0 compatible with the filtration W_i, i.e., for each $i = 0, \ldots, m$, it contains a basis of W_i.

For each $j = 1, \ldots, H_{\mathscr{Y}}(m)$, we can represent ψ_j in the form

$$\psi_j = y_0^{i_j} \cdot \mathbf{y}^{\mathbf{a}_j}, \tag{2.3}$$

where $\mathbf{a}_j = (0, a_{j1}, \ldots, a_{jM}) \in \mathbf{N}^{M+1}$. Notice that, there are exactly $\dim(W_i/W_{i+1})$ elements ψ_j with $i_j = i$ in the set $\psi_1, \ldots, \psi_{H_{\mathscr{Y}}(m)}$.

Now, we estimate the sum $\sum_{j=1}^{H_{\mathscr{Y}}(m)} i_j$. To do it, we need a lemma from Chen-Ru-Yan [4], Lemma 2.2. We have included a proof of this lemma for the sake of completeness.

LEMMA 2.2.

$$\sum_{j=1}^{H_{\mathscr{Y}}(m)} i_j = \frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{(n+1)!}(1+o(1)),$$

where the function $o(1)$ *depends only on the variety* $\mathscr{Y}$.

Proof. It is clear that there are exactly $\dim(W_i/W_{i+1})$ elements ψ_j with $i_j = i$ in the set $\psi_1, \ldots, \psi_{H_{\mathscr{Y}}(m)}$. Hence,

$$\sum_{j=1}^{H_{\mathscr{Y}}(m)} i_j = \sum_{i=1}^{m} i \cdot \dim(W_i/W_{i+1}), \tag{2.4}$$

in which $W_{m+1} = \{\vec{0}\}$.

Next, we claim that $\dim W_i = \dim \overline{K}[y_0, \ldots, y_M]_{m-i}/(I_{\mathscr{Y}})_{m-i}$. To see it, notice that each element ψ of W_i can be represented as $\psi = y_0^i.g$ with $g \in \overline{K}[y_0, \ldots, y_M]_{m-i}$. Furthermore, two polynomials g_1, g_2 such that $y_0^i.g_1 = y_0^i.g_2$ in W_i iff $y_0^i(g_1 - g_2)$ vanishes identically in $\mathscr{Y}$, that means, $g_1 - g_2$ vanishes identically in $\mathscr{Y}$. Therefore $\dim W_i = H_{\mathscr{Y}}(m-i)$. In view of (2.1), for each positive integer L,

$$\dim \overline{K}[y_0, \ldots, y_M]_L/(I_{\mathscr{Y}})_L = \triangle_{\mathscr{Y}} \cdot \frac{L^n}{n!} + O(L^{n-1}).$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{m} i \cdot \dim(W_i/W_{i+1}) &= \sum_{i=1}^{m} i(\dim W_i - \dim W_{i+1}) \\
&= \sum_{i=1}^{m} i \cdot \dim W_i - \sum_{i=1}^{m}((i+1)\dim W_{i+1} - \dim W_{i+1}) \\
&= \sum_{i=1}^{m} \dim W_i = \frac{\triangle_{\mathscr{Y}}}{n!}\sum_{i=1}^{m}((m-i)^n + O(m^{n-1})) \\
&= \frac{\triangle_{\mathscr{Y}} \cdot m^n}{n!}\sum_{i=1}^{m}\left(1 - \frac{i}{m}\right)^n + O(m^n) \\
&= \frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{n!}\left(\int_0^1 (1-t)^n \cdot dt + o(1)\right) + O(m^n).
\end{aligned}$$

Therefore

$$\sum_{i=1}^{m} i \cdot \dim(W_i/W_{i+1}) = \frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{(n+1)!}(1+o(1)). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain the desired result. □

Now, we continue with the proof of Proposition 2.1. Set $\mathbf{e} = (1, 0, \dots, 0) \in \mathbf{Z}_{\geq 0}^{M+1}$, in which all coordinates of $\mathbf{e}$ are 0 except the first one. In view of (2.3), the monomials $y^{\mathbf{a}_j + i_j \mathbf{e}}$, $j = 1, \dots, H_{\mathscr{Y}}(m)$ form a basis of W_0. Now, we consider the sum

$$\sum_{j=1}^{H_{\mathscr{Y}}(m)} (\mathbf{a}_j + i_j \mathbf{e}) \cdot \mathbf{c}.$$

We have

$$\begin{aligned}\sum_{j=1}^{H_{\mathscr{Y}}(m)} (\mathbf{a}_j + i_j \mathbf{e}) \cdot \mathbf{c} &\geq \sum_{j=1}^{H_{\mathscr{Y}}(m)} i_j \mathbf{e} \cdot \mathbf{c} \\ &\geq \left(\sum_{j=1}^{H_{\mathscr{Y}}(m)} i_j \right) c_0 = \frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{(n+1)!} (1 + o(1)) \cdot c_0.\end{aligned}$$

By definition of the m^{th}-Hilbert weight $s_{\mathscr{Y}}(m, \mathbf{c})$, we have

$$s_{\mathscr{Y}}(m, \mathbf{c}) \geq \frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{(n+1)!} (1 + o(1)) \cdot c_0.$$

In view of (2.1), it implies that

$$\lim_{m \to \infty} \frac{1}{m H_{\mathscr{Y}}(m)} s_{\mathscr{Y}}(m, \mathbf{c}) \geq \lim_{m \to \infty} \frac{\frac{\triangle_{\mathscr{Y}} \cdot m^{n+1}}{(n+1)!} (1 + o(1)) \cdot c_0}{m \frac{\triangle_{\mathscr{Y}} \cdot m^n}{n!} (1 + o(1))} = \frac{1}{n+1} \cdot c_0.$$

Together with (2.2), this implies that

$$\frac{1}{\triangle_{\mathscr{Y}} \cdot (n+1)} e_{\mathscr{Y}}(\mathbf{c}) \geq \frac{1}{n+1} \cdot c_0.$$

Hence, we have

$$e_{\mathscr{Y}}(\mathbf{c}) \geq \triangle_{\mathscr{Y}} \cdot c_0.$$

This completes the proof. □

2.4. We recall an estimate on heights of Chow forms, due to Ru-Wang [9, Lemma 8].

LEMMA 2.3. *Let $\mathscr{X}$ be a projective variety of $\mathbf{P}^M$ defined over K with dimension $n \geq 1$ and degree $\triangle_{\mathscr{X}}$. Let $\psi : \mathscr{X} \to \mathbf{P}^R$ be a finite morphism given by $\psi(\mathbf{x}) = [g_0(\mathbf{x}) : \cdots : g_R(\mathbf{x})]$, where $g_0, \dots, g_R$ are homogeneous polynomials of degree d in $K[X_0, \dots, X_M]$. Let $\mathscr{Y} = \psi(\mathscr{X})$. Then,*

$$h(F_{\mathscr{Y}}) \leq d^{n+1} h(F_{\mathscr{X}}) + (n+1) d^n \triangle_{\mathscr{X}} h(g_0, \dots, g_R).$$

3. Canonical polynomials from Chow form

Let $\mathcal{X}$ be a n-dimensional projective subvariety of $\mathbf{P}^M$ defined over K. Let $I_{\mathcal{X}}$ be the homogeneous prime ideal defining $\mathcal{X}$. Brownawell [2] has shown a canonical way to find polynomials from the Chow form of $\mathcal{X}$ which have the same zero set as $I_{\mathcal{X}}$. We now recall this construction from [2].

First of all, we note that a generic hyperplane passing through a given point $\mathbf{x}$ has the form $S\mathbf{x}$ for a skew-symmetric matrix.

Let $F_{\mathcal{X}}$ be the Chow form of $\mathcal{X}$. We now consider how closely the Chow form $F_{\mathcal{X}}$ determines $I_{\mathcal{X}}$.

Let $S^{(0)}, S^{(1)}, \ldots, S^{(n)}$ be $(n+1)$ generic skew symmetric $(M+1) \times (M+1)$ matrices, $S^{(i)} = (s_{jk}^{(i)})$, $0 \le i \le n$, and write

$$F_{\mathcal{X}}(S^{(0)}\mathbf{x}, \ldots, S^{(n)}\mathbf{x}) = \sum_{\sigma \in \mathcal{M}} P_\sigma(\mathbf{x})\sigma \tag{3.1}$$

where $\mathcal{M}$ be the set of all monomials in the $n+1$ blocks of variables $s^{(i)} = (s_{jk}^i : 0 \le j < k \le M)$, $(0 \le i \le n)$, which are homogeneous of degree $\triangle_{\mathcal{X}}$ in each block. We note that the coefficients of $F_{\mathcal{X}}$ are in K since $\mathcal{X}$ is defined over K. Therefore, the coefficients of $P_\sigma(\mathbf{x})$ are in K for all $\sigma \in \mathcal{M}$. We define P_σ, $\sigma \in \mathcal{M}$ as the *canonical polynomials.*

Since $S^{(0)}\mathbf{x}, \ldots, S^{(n)}\mathbf{x}$ are $(n+1)$ generic hyperplanes through $\mathbf{x}$, $F_{\mathcal{X}}(S^{(0)}\mathbf{x}, \ldots, S^{(n)}\mathbf{x}) = 0$ if and only if $\mathbf{x} \in \mathcal{X}(\overline{K})$.

But clearly from (3.1), we have $F_{\mathcal{X}}(S^{(0)}\mathbf{x}, \ldots, S^{(n)}\mathbf{x}) = 0$ if and only if $P_\sigma(\mathbf{x}) = 0$ for all $\sigma \in \mathcal{M}$.

Therefore, the ideal generated by $\{P_\sigma \mid \sigma \in \mathcal{M}\}$ determines $\mathcal{X}(\overline{K})$ set theoretically. By Hilbert's Nullstellensatz, we have that *$I_{\mathcal{X}}$ is the radical of the ideal generated by P_σ, $\sigma \in \mathcal{M}$.*

We also recall the following result of Catanese [3].

Theorem 3.1 (Catanese [3]). *If $\mathcal{X}$ is a smooth projective variety in $\mathbf{P}^M$, then the polynomials P_σ, $(\sigma \in \mathcal{M})$ cut out $\mathcal{X}$ scheme-theoretically. In other words, if $p_{\sigma,i}$ denotes the dehomogenization of P_σ in the affine piece $X_i \neq 0$ for $i = 0, \ldots, n$ the ideal generated by $p_{\sigma,i}$, $(\sigma \in \mathcal{M})$ equals to the ideal $I_{\mathcal{X} \cap U_i}$, where $U_i = \{X_i \neq 0\}$.*

We end this section by listing some information on P_σ. First, clearly from (3.1) that the degree of P_σ is $(n+1)\triangle_{\mathcal{X}}$. Moreover, the coefficients of P_σ are $\mathbf{Z}$-linear combinations of coefficients of the Chow form $F_{\mathcal{X}}$, hence

$$e_{\mathfrak{p}}(P_\sigma) \ge e_{\mathfrak{p}}(F_{\mathcal{X}}). \tag{3.2}$$

It is obvious that the number of generating polynomials P_σ is at most

$$\binom{(n+1)\triangle_{\mathcal{X}} + \dfrac{M(M-1)}{2}}{(n+1)\triangle_{\mathcal{X}}}^{n+1}. \tag{3.3}$$

4. Some effective results

Now, we recall the following version of an effective Hilbert's Nullstellensatz (See [5], [6]).

THEOREM 4.1 (Jenolek [5], Kollár [6]). *Let $P_0, \dots, P_l$ be homogeneous polynomials in $K[X_0, \dots, X_M]$ of total degree at most d such that P_0 vanishes at all common zeros (if any) of $P_1, \dots, P_l$ in $\overline{K}^{M+1}$. Then there exist a positive integer $u \le (4d)^{M+2}$ and homogeneous polynomials $A_1, \dots, A_l$ in $K[X_0, \dots, X_M]$ of total degree at most $(4d)^{M+2}$, such that*

$$\alpha P_0^u = A_1 P_1 + \cdots + A_l P_l$$

for some non-zero element α of K. Furthermore, there exists a positive integer

$$l_0 \le l(4(4d)^{M+2})^M$$

such that

$$\min\{\mathrm{ord}_{\mathfrak{p}}(\alpha), e_{\mathfrak{p}}(A_1), \dots, e_{\mathfrak{p}}(A_l)\} \ge l_0 \cdot \min_{1 \le i \le l}\{e_{\mathfrak{p}}(P_i)\}$$

for each $\mathfrak{p} \in M_K$.

By using the same method as in Ru-Wang [9, Lemma 16], we will prove a slight generalization of this result from general position to sub-general position.

LEMMA 4.2. *Let $\mathscr{X}$ be a smooth n-dimensional projective subvariety of $\mathbf{P}^M$ defined over K of degree $\triangle_{\mathscr{X}}$. Let m, q be integers with $m \ge n$ and $q \ge m+1$. Let $Q_1, \dots, Q_q$ be homogeneous polynomials in $K[X_0, \dots, X_M]$ of degree d, in m-subgeneral position with respect to $\mathscr{X}$. For given $\mathfrak{p} \in M_K$, and $\mathbf{x} \in \mathscr{X} \setminus \bigcup_{i=1}^{q}\{Q_i = 0\}$, we assume that*

$$\mathrm{ord}_{\mathfrak{p}}(Q_1(\mathbf{x})) \ge \cdots \ge \mathrm{ord}_{\mathfrak{p}}(Q_q(\mathbf{x})). \tag{4.1}$$

Then

$$\begin{aligned}&\mathrm{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) \deg \mathfrak{p} - d \cdot e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p} \\ &\quad \le (6 \max\{(m+1)\triangle_{\mathscr{X}}, d\})^{(n+1)(M^2+M)}(h(F_{\mathscr{X}}) + h(Q_1, \dots, Q_q))\end{aligned}$$

for $\mathfrak{p} \in M_K$ and $m+1 \le i \le q$.

Proof. As $h(F_{\mathscr{X}}) = h(\alpha F_{\mathscr{X}})$ for $\alpha \in K^*$, we may assume that one of coefficients of $F_{\mathscr{X}}$ is 1. Similarly, since $h(Q_1, \dots, Q_q) = h(\alpha Q_1, \dots, \alpha Q_q)$, we can make the same assumption for Q_1. Therefore, we have

$$e_{\mathfrak{p}}(F_{\mathscr{X}}) \le 0, \quad \min_{1 \le i \le q} e_{\mathfrak{p}}(Q_i) \le 0,$$

for each $\mathfrak{p} \in M_K$.

Let $P_1, \dots, P_r \in K[X_0, \dots, X_M]$ be the canonical polynomials from the Chow form $F_{\mathscr{X}}$ of $\mathscr{X}$ defined in (3.1). Let

$$d' = \max\{\deg P_1, \dots, \deg P_r, d\}.$$

Since $Q_1, \dots, Q_q$ are in m-subgeneral position with respect to $\mathscr{X} \subset \mathbf{P}^M$, then $P_1, \dots, P_r$, $Q_1, \dots, Q_{m+1}$ have no common zeros in $\mathbf{P}^M(\bar{K})$. Theorem 4.1 tell us that there exists a constant $u \le (4d')^{M+2}$ and polynomials $A_{j1}, \dots, A_{jr}, A_{j,r+1}, \dots, A_{j,r+m+1} \in K[X_0, \dots, X_M]$ of total degree at most $(4d')^{M+2}$ such that for $0 \le j \le M$, we have

$$\alpha_j X_j^u = A_{j1}P_1 + \cdots + A_{jr}P_r + A_{j,r+1}Q_1 + \cdots + A_{j,r+m+1}Q_{m+1}$$

for some non-zero elements α_j of K. Furthermore, there exists a positive integer

$$l_0 \le (r + m + 1)(4(4d')^{M+2})^M \tag{4.2}$$

such that

$$\begin{aligned} \min\{&\mathrm{ord}_{\mathfrak{p}}(\alpha_0), \dots, \mathrm{ord}_{\mathfrak{p}}(\alpha_M), e_{\mathfrak{p}}(A_{j1}), \dots, e_{\mathfrak{p}}(A_{j,r+m+1})\} \\ &\ge l_0 \cdot \min\{e_{\mathfrak{p}}(P_i), e_{\mathfrak{p}}(Q_i)\} \ge l_0 \cdot \left(e_{\mathfrak{p}}(F_{\mathscr{X}}) + \min_{1\le i\le q} e_{\mathfrak{p}}(Q_i)\right) \end{aligned} \tag{4.3}$$

for each $\mathfrak{p} \in M_K$.

We may assume that A_{ji}, $(1 \le i \le r + m + 1)$ are homogeneous polynomials and therefore the degrees of $A_{j,r+1}, \dots, A_{j,r+m+1}$ are $u - d$.

Let $\mathbf{x} \in \mathscr{X}(K) \setminus \bigcup_{i=1}^q \{Q_i = 0\}$. Then

$$\alpha_j x_j^u = A_{j,r+1}(\mathbf{x})Q_1(\mathbf{x}) + \cdots + A_{j,r+m+1}(\mathbf{x})Q_{m+1}(\mathbf{x})$$

and hence, for all j, we have

$$\begin{aligned} \mathrm{ord}_{\mathfrak{p}}(\alpha_j) + u \cdot \mathrm{ord}_{\mathfrak{p}}(x_j) &\ge \min_{1\le i\le m+1} \mathrm{ord}_{\mathfrak{p}}(A_{j,r+i}(\mathbf{x})Q_i(\mathbf{x})) \\ &\ge \min_{1\le i\le m+1} \mathrm{ord}_{\mathfrak{p}}(A_{j,r+i}(\mathbf{x})) + \min_{1\le i\le m+1} \mathrm{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) \\ &\ge (u-d)e_{\mathfrak{p}}(\mathbf{x}) + l_0 \cdot \left(e_{\mathfrak{p}}(F_{\mathscr{X}}) + \min_{1\le i\le q} e_{\mathfrak{p}}(Q_i)\right) \\ &\quad + \mathrm{ord}_{\mathfrak{p}}(Q_{m+1}(\mathbf{x})). \end{aligned}$$

(Here, the last inequality follows from (4.3) and (4.1)). Hence

$$\mathrm{ord}_{\mathfrak{p}}(Q_{m+1}(\mathbf{x})) \le de_{\mathfrak{p}}(\mathbf{x}) + \max_{0\le j\le M}\{\mathrm{ord}_{\mathfrak{p}}(\alpha_j)\} - l_0 \cdot \left(e_{\mathfrak{p}}(F_{\mathscr{X}}) + \min_{1\le i\le q} e_{\mathfrak{p}}(Q_i)\right) \tag{4.4}$$

for $\mathfrak{p} \in M_K$. Since $\alpha_j \ne 0$, $(0 \le j \le M)$, from the sum formula and (4.3) we have

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}(\alpha_j) \deg \mathfrak{p} &= -\sum_{\mathfrak{q}\in M_K\setminus\{\mathfrak{p}\}} \mathrm{ord}_{\mathfrak{q}}(\alpha_j) \deg \mathfrak{q} \\ &\leq -\sum_{\mathfrak{q}\in M_K\setminus\{\mathfrak{p}\}} l_0 \cdot \left(e_{\mathfrak{q}}(F_{\mathscr{X}}) + \min_{1\leq i\leq q} e_{\mathfrak{q}}(Q_i)\right) \deg \mathfrak{q}.\end{aligned}$$

Combining with (4.4), we have

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}(Q_{m+1}) \deg \mathfrak{p} \leq d \cdot e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p} &- \sum_{\mathfrak{q}\in M_K\setminus\{\mathfrak{p}\}} l_0 \cdot \left(e_{\mathfrak{q}}(F_{\mathscr{X}}) + \min_{1\leq i\leq q} e_{\mathfrak{q}}(Q_i)\right) \deg \mathfrak{q} \\ &- l_0 \cdot \left(e_{\mathfrak{p}}(F_{\mathscr{X}}) + \min_{1\leq i\leq q} e_{\mathfrak{p}}(Q_i)\right) \deg \mathfrak{p} \\ &= d \cdot e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p} + l_0 \cdot (h(F_{\mathscr{X}}) + h(Q_1, \ldots, Q_q)).\end{aligned}$$

Now, we estimate l_0 introduced in (4.2).

We first estimate the number r introduced in (3.3). This number r can be bounded by

$$r \leq \binom{(n+1)\triangle_{\mathscr{X}} + \frac{M(M-1)}{2}}{(n+1)\triangle_{\mathscr{X}}}^{n+1} \leq (5(n+1)\triangle_{\mathscr{X}})^{(n+1)M(M-1)/2}. \tag{4.5}$$

Here, we use the following inequality

$$\binom{A+B}{A} \leq \frac{(A+B)^{A+B}}{A^A B^B} = \left(1+\frac{B}{A}\right)^A \cdot \left(1+\frac{A}{B}\right)^B \leq e^B\left(1+\frac{A}{B}\right)^B,$$

where A, B are positive integers and e is the natural exponential number.

Since the degree of P_σ is $(n+1)\triangle_{\mathscr{X}}$, we have $d' \leq \max\{(n+1)\triangle_{\mathscr{X}}, d\}$. Therefore,

$$4(4d')^{M+2} \leq (6d')^{M+2} \leq [6 \max\{(m+1)\triangle_{\mathscr{X}}, d\}]^{M+2}. \tag{4.6}$$

By (4.5), we have

$$\begin{aligned}r+m+1 &\leq (5(n+1)\triangle_{\mathscr{X}})^{(n+1)M(M-1)/2} + (m+1) \\ &\leq (6(m+1)\triangle_{\mathscr{X}})^{(n+1)M(M-1)/2} \\ &\leq [6 \max\{(m+1)\triangle_{\mathscr{X}}, d\}]^{(n+1)M(M-1)/2}.\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) and (4.2), we have

$$l_0 \leq (6 \max\{(m+1)\triangle_{\mathscr{X}}, d\})^{(n+1)(M^2+M)}. \qquad \square$$

5. Proof of main theorem

We first recall the following theorem, due to Ru-Wang [9, Theorem 23].

Theorem 5.1 (Ru-Wang [9]). *Let K be the function field of a nonsingular projective variety V defined over an algebraically closed field of characteristic* 0. *Let S be a finite set of prime divisors of V. Let $\mathcal{Y}$ be an n-dimensional smooth projective subvariety of $\mathbf{P}^M$ defined over K. For every $\mathfrak{p} \in M_K$ and $\mathbf{y} = [y_0 : \cdots : y_M]$, we let $c_{\mathfrak{p},i}(\mathbf{y}) = (\mathrm{ord}_{\mathfrak{p}}(y_i) - e_{\mathfrak{p}}(\mathbf{y})) \cdot \deg \mathfrak{p}$ $(0 \le i \le M)$ and $\mathbf{c}_{\mathfrak{p}}(\mathbf{y}) = (c_{\mathfrak{p},0}(\mathbf{y}), \ldots, c_{\mathfrak{p},M}(\mathbf{y}))$. Then for a given $\varepsilon > 0$, there exists an effectively computable finite union $\mathcal{Z}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^M(K)$ not containing $\mathcal{Y}$ and effectively computable constants a_ε, a'_ε such that for any $\mathbf{y} \in \mathcal{Y}(K) \backslash \mathcal{Z}_\varepsilon$ either*

$$h(\mathbf{y}) \le a_\varepsilon(h(F_{\mathcal{Y}}) + 1)$$

or

$$\sum_{\mathfrak{p} \in S} e_{\mathcal{Y}}(\mathbf{c}_{\mathfrak{p}}(\mathbf{y})) \le (n + 1 + \varepsilon)\triangle_{\mathcal{Y}} \cdot h(\mathbf{y}) + a'_\varepsilon(h(F_{\mathcal{Y}}) + 1).$$

The algebraic subsets in $\mathcal{Z}_\varepsilon$ and the constants a_ε, a'_ε depend on ε and M, K, S and $\mathcal{Y}$. Furthermore, the degrees of the algebraic subsets in $\mathcal{Z}_\varepsilon$ can be bounded above by $1 + 2(2n+1)\triangle_{\mathcal{Y}}(M+1)\varepsilon^{-1}$.

We first use Theorem 5.1 and Proposition 2.1 to prove Theorem 5.2. Then, we will show that the main theorem is an implication of Theorem 5.2.

Theorem 5.2. *Let K be the function field of a nonsingular projective variety V defined over an algebraically closed field of characteristic* 0. *Let S be a finite set of prime divisors of V. Let $\mathcal{Y}$ be an n-dimensional smooth projective subvariety of $\mathbf{P}^M$ defined over K. Denote by I_0 the subset of $\{0, \ldots, M\}$ consisting of all $i \in \{0, \ldots, M\}$ such that $\mathcal{Y}$ is not contained in coordinate hyperplane Y_i. Let m_0 be a positive integer with $m_0 \le |I_0|$ and $\varepsilon > 0$. Then there exists an effectively computable finite union $\mathcal{R}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^M(K)$ not containing $\mathcal{Y}$ and effectively computable constants b_ε, b'_ε such that for any $\mathbf{y} \in \mathcal{Y} \backslash \mathcal{R}_\varepsilon$ either*

$$h(\mathbf{y}) \le b_\varepsilon(h(F_{\mathcal{Y}}) + 1)$$

or

$$\sum_{\mathfrak{p} \in S} \max_I \sum_{i \in I} \lambda_{\mathfrak{p}, Y_i}(\mathbf{y}) \le (m_0(n+1) + \varepsilon)\triangle_{\mathcal{Y}} \cdot h(\mathbf{y}) + b'_\varepsilon(h(F_{\mathcal{Y}}) + 1).$$

Here the maximum is taken over all subsets I of I_0 with cardinality m_0.

The algebraic subsets in $\mathcal{R}_\varepsilon$ and the constants b_ε, b'_ε depend on ε and M, K, S and $\mathcal{Y}$. Furthermore, the degrees of the algebraic subsets in $\mathcal{R}_\varepsilon$ can be bounded above by $1 + 2(2n+1)\triangle_{\mathcal{Y}}(M+1)\varepsilon^{-1}$.

Proof. For every $\mathfrak{p} \in M_K$ and $\mathbf{y} = [y_0 : \cdots : y_M]$, we let

$$c_{\mathfrak{p},i}(\mathbf{y}) = (\mathrm{ord}_{\mathfrak{p}}(y_i) - e_{\mathfrak{p}}(\mathbf{y})) \cdot \deg \mathfrak{p} \quad (0 \le i \le M)$$

and $\mathbf{c}_{\mathfrak{p}}(\mathbf{y}) = (c_{\mathfrak{p},0}(\mathbf{y}), \ldots, c_{\mathfrak{p},M}(\mathbf{y}))$.

Theorem 5.1 implies that for a given $\varepsilon > 0$, there exists an effectively computable finite union $\mathscr{Z}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^M(K)$ not containing $\mathscr{Y}$ and effectively computable constants a_{ε/m_0}, a'_{ε/m_0} such that for any $\mathbf{y} \in \mathscr{Y}(K)\backslash\mathscr{Z}_\varepsilon$ either

$$h(\mathbf{y}) \leq a_{\varepsilon/m_0}(h(F_{\mathscr{Y}}) + 1)$$

or

$$\sum_{\mathfrak{p}\in S} e_{\mathscr{Y}}(\mathbf{c}_{\mathfrak{p}}(\mathbf{y})) \leq \left(n + 1 + \frac{\varepsilon}{m_0}\right)\triangle_{\mathscr{Y}} \cdot h(\mathbf{y}) + a'_{\varepsilon/m_0}(h(F_{\mathscr{Y}}) + 1). \tag{5.1}$$

Let I be an arbitrary subset of $\{0, \dots, M\}$ with cardinality m_0 such that $\mathscr{Y}$ is not contained in coordinate hyperplanes Y_i for all $i \in I$. It follows from Proposition 2.1 that

$$\sum_{i\in I} c_{\mathfrak{p},i}(\mathbf{y}) \leq \frac{m_0}{\triangle_{\mathscr{Y}}} e_{\mathscr{Y}}(\mathbf{c}_{\mathfrak{p}}(\mathbf{y})). \tag{5.2}$$

On the other hand, by the definition, $\lambda_{\mathfrak{p},Y_i}(\mathbf{y}) = c_{\mathfrak{p},i}(\mathbf{y})$. Hence,

$$\sum_{i\in I} \lambda_{\mathfrak{p},Y_i}(\mathbf{y}) \leq \frac{m_0}{\triangle_{\mathscr{Y}}} e_{\mathscr{Y}}(\mathbf{c}_{\mathfrak{p}}(\mathbf{y})). \tag{5.3}$$

Therefore,

$$\sum_{\mathfrak{p}\in S} \max_I \sum_{i\in I} \lambda_{\mathfrak{p},Y_i}(\mathbf{y}) \leq \frac{m_0}{\triangle_{\mathscr{Y}}} \sum_{\mathfrak{p}\in S} e_{\mathscr{Y}}(\mathbf{c}_{\mathfrak{p}}(\mathbf{y})). \tag{5.4}$$

Set $\mathscr{R}_\varepsilon = \mathscr{Z}_\varepsilon$. By combining (5.1) and (5.4), we have

$$\sum_{\mathfrak{p}\in S} \max_I \sum_{i\in I} \lambda_{\mathfrak{p},Y_i}(\mathbf{y}) \leq (m_0(n+1) + \varepsilon) \cdot h(\mathbf{y}) + a'_{\varepsilon/m_0} \frac{m_0}{\triangle_{\mathscr{Y}}}(h(F_{\mathscr{Y}}) + 1).$$

for all $\mathbf{y} \in \mathscr{Y}\backslash\mathscr{R}_\varepsilon$.

The constants b_ε and b'_ε in the assertion can be given by

$$b_\varepsilon = a_{\varepsilon/m_0}; \quad b'_\varepsilon = m_0 \cdot a'_{\varepsilon/m_0}, \tag{5.5}$$

where a_ε and a'_ε are constants from Theorem 5.1. This completes the proof of Theorem 5.2. □

Now, we will show that Theorem 5.2 implies the main theorem.

Proof of the main theorem. Let d is the l.c.m of d'_i, $1 \leq i \leq q$, and let $M_0, \dots, M_{N_1}$ be all the monomials in $X_0, \dots, X_N$ of degree d. We define the map

$$\psi : \mathscr{X} \to \mathbf{P}^{N_1+q}, \quad \psi(\mathbf{x}) = [M_0(\mathbf{x}) : \cdots : M_{N_1}(\mathbf{x}) : Q_1^{d/d_1}(\mathbf{x}) : \cdots : Q_q^{d/d_q}(\mathbf{x})]. \tag{5.6}$$

Let $\mathcal{Y} = \psi(\mathcal{X})$. Then this map is an embedding and $\mathcal{Y}$ is a smooth projective subvariety of $\mathbf{P}^{N_1+q}$ defined over K with $\dim \mathcal{Y} = n$ and $\deg \mathcal{Y} =: \triangle_{\mathcal{Y}} \le d^n \triangle_{\mathcal{X}}$. It follows from Lemma 2.3 that

$$h(F_{\mathcal{Y}}) \le d^{n+1} h(F_{\mathcal{X}}) + (n+1)\triangle_{\mathcal{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q}).$$

Since $\mathcal{X} \not\subseteq \{Q_i = 0\}$ then we have $\mathcal{Y} \not\subseteq Y_{N_1+i}$, $i = 1, \dots, q$. We apply Theorem 5.2 to $\mathcal{Y} \in \mathbf{P}^{N_1+q}$ and the coordinate hyperplanes $Y_{N_1+1}, \dots, Y_{N_1+q}$ and $m_0 = m$. Then, for a given $\varepsilon > 0$, there exists an effectively computable finite union $\mathcal{R}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^{N_1+q}(K)$ not containing $\mathcal{Y}$ and effectively computable constants b_ε, b'_ε such that for any $\psi(\mathbf{x}) \in \mathcal{Y} \backslash \mathcal{R}_\varepsilon$ either

$$\begin{aligned} (5.7) \qquad h(\psi(\mathbf{x})) &\le b_\varepsilon(h(F_{\mathcal{Y}}) + 1) \\ &\le b_\varepsilon(1 + d^{n+1} h(F_{\mathcal{X}}) + (n+1)\triangle_{\mathcal{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q})) \\ &\le \tilde{b}_\varepsilon(h(F_{\mathcal{X}}) + 1) \end{aligned}$$

or

$$\begin{aligned} (5.8) \qquad \sum_{\mathfrak{p} \in S} \max_I \sum_{i \in I} \lambda_{\mathfrak{p}, Y_i}(\mathbf{y}) &\le (m(n+1) + \varepsilon) h(\psi(\mathbf{x})) + b'_\varepsilon(h(F_{\mathcal{Y}}) + 1) \\ &\le (m(n+1) + \varepsilon) h(\psi(\mathbf{x})) + \tilde{b}'_\varepsilon(h(F_{\mathcal{X}}) + 1). \end{aligned}$$

Here the maximum is taken over all subsets I of $\{N_1 + 1, \dots, N_1 + q\}$ with cardinality m and the constants $\tilde{b}_\varepsilon$ and $\tilde{b}'_\varepsilon$ are given by

$$(5.9) \qquad \tilde{b}_\varepsilon = b_\varepsilon \cdot (d^{n+1} + (n+1)\triangle_{\mathcal{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q}))$$

and

$$(5.10) \qquad \tilde{b}'_\varepsilon = b'_\varepsilon \cdot (d^{n+1} + (n+1)\triangle_{\mathcal{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q})).$$

Here b_ε and b'_ε are the constants from Theorem 5.2 with $M = N_1 + q$. Notice that the degrees of the algebraic subsets in $\mathcal{R}_\varepsilon$ can be bounded by

$$(5.11) \quad 2(2n+1)\triangle_{\mathcal{Y}}(M+1)\varepsilon^{-1} + 1 \le 2(2n+1) d^n \triangle_{\mathcal{X}} \left(\binom{d+N}{N} + q + 1 \right) \varepsilon^{-1} + 1.$$

For a given $\mathbf{x} \in \mathcal{X} \backslash \bigcup_{i=1}^q \{Q_i = 0\}$ and a fixed $\mathfrak{p} \in S$, we may reindex the Q_i so that $(d/d_1) \operatorname{ord}_{\mathfrak{p}}(Q_1(\mathbf{x})) \ge \cdots \ge (d/d_q) \operatorname{ord}_{\mathfrak{p}}(Q_q(\mathbf{x}))$. Since $Q_1, \dots, Q_q$ are in m-subgeneral position with respect to $\mathcal{X}$ we can apply Lemma 4.2 to $Q_1^{d/d_1}, \dots, Q_q^{d/d_q}$. Then, for all $m + 1 \le j \le q$, we have

$$(5.12) \qquad \frac{d}{d_j} \operatorname{ord}_{\mathfrak{p}}(Q_j(\mathbf{x})) \deg \mathfrak{p} \le d \cdot e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p} + c_2,$$

where

$$(5.13) \qquad c_2 = (6 \max\{(m+1)\triangle, d\})^{(n+1)(N^2+N)} (h(F_{\mathcal{X}}) + h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q})).$$

Notice that $c_2 \geq 0$. Thus,

$$e_{\mathfrak{p}}(\psi(\mathbf{x})) \deg \mathfrak{p} = \min\left\{d \cdot e_{\mathfrak{p}}(\mathbf{x}), \frac{d}{d_1} \operatorname{ord}_{\mathfrak{p}}(Q_1(\mathbf{x})), \ldots, \frac{d}{d_q} \operatorname{ord}_{\mathfrak{p}}(Q_q(\mathbf{x}))\right\} \deg \mathfrak{p}$$
$$\leq d \cdot e_{\mathfrak{p}}(\mathbf{x}) \deg \mathfrak{p}.$$

Hence, for $1 \leq i \leq q$,

$$\lambda_{\mathfrak{p}, Y_{N_1+i}}(\psi(\mathbf{x})) = \left(\frac{d}{d_i} \operatorname{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) - e_{\mathfrak{p}}(\psi(\mathbf{x}))\right) \deg \mathfrak{p} \tag{5.14}$$
$$\geq \left(\frac{d}{d_i} \operatorname{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) - d \cdot e_{\mathfrak{p}}(\mathbf{x})\right) \deg \mathfrak{p}.$$

On the other hand,

$$\sum_{i=1}^{q} \frac{d}{d_i} \lambda_{\mathfrak{p}, Q_i}(\mathbf{x}) = \sum_{i=1}^{q} \left(\frac{d}{d_i} \operatorname{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) - d \cdot e_{\mathfrak{p}}(\mathbf{x}) - \frac{d}{d_i} e_{\mathfrak{p}}(Q_i)\right) \deg \mathfrak{p}$$

is smaller than

$$\sum_{i=1}^{q} \left[\left(\frac{d}{d_i} \operatorname{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) - d \cdot e_{\mathfrak{p}}(\mathbf{x})\right) \deg \mathfrak{p} - c_2\right] - q \min_{1 \leq i \leq q} \frac{d}{d_i} e_{\mathfrak{p}}(Q_i) \deg \mathfrak{p} + q \cdot c_2,$$

which by (5.12) does not exceed

$$\sum_{i=1}^{m} \left[\left(\frac{d}{d_i} \operatorname{ord}_{\mathfrak{p}}(Q_i(\mathbf{x})) - d \cdot e_{\mathfrak{p}}(\mathbf{x})\right) \deg \mathfrak{p} - c_2\right] - q \min_{1 \leq i \leq q} \frac{d}{d_i} e_{\mathfrak{p}}(Q_i) \deg \mathfrak{p} + q \cdot c_2.$$

Combining with (5.14), we have

$$\sum_{i=1}^{q} \frac{d}{d_i} \lambda_{\mathfrak{p}, Q_i}(\mathbf{x}) \tag{5.15}$$
$$\leq \sum_{i=1}^{m} \lambda_{\mathfrak{p}, Y_{N_1+i}}(\psi(\mathbf{x})) - mc_2 - q \min_{1 \leq i \leq q} \frac{d}{d_i} e_{\mathfrak{p}}(Q_i) \deg \mathfrak{p} + q \cdot c_2$$
$$\leq \max_{I} \sum_{i \in I} \lambda_{\mathfrak{p}, Y_i}(\psi(\mathbf{x})) - q \min_{1 \leq i \leq q} \frac{d}{d_i} e_{\mathfrak{p}}(Q_i) \deg \mathfrak{p} + (q-m) \cdot c_2.$$

Here the maximum is taken over all subsets I of $\{N_1 + 1, \ldots, N_1 + q\}$ with cardinality m. Combining with (5.8), we have

$$\sum_{\mathfrak{p} \in S} \sum_{i=1}^{q} \frac{d}{d_i} \lambda_{\mathfrak{p}, Q_i}(\mathbf{x}) \leq (m(n+1) + \varepsilon) h(\psi(\mathbf{x})) + \tilde{b}'_{\varepsilon}(h(F_{\mathscr{X}}) + 1) \tag{5.16}$$
$$+ q \cdot h(Q_1^{d/d_1}, \ldots, Q_q^{d/d_q}) + (q-m)|S| \cdot c_2.$$

We may conclude the proof of the theorem by the following facts. Firstly, if P is one of the homogeneous polynomials in $K[Y_0, \dots, Y_{N_1+q}]$ defining $\mathscr{R}_\varepsilon$, then $G = P \circ \psi$ is a homogeneous polynomials of degree $d \cdot \deg P$ in $K[X_0, \dots, X_N]$ and all such G form an effectively computable finite union $\mathscr{W}_\varepsilon$ of proper algebraic subsets of $\mathbf{P}^N(K)$ with degree bounded above by

$$2(2n+1)d^{n+1}\triangle_{\mathscr{X}}\left(\binom{d+N}{N} + q + 1\right)\varepsilon^{-1} + d$$

by (5.11). Secondly, it is easy to check that

$$dh(\mathbf{x}) \leq h(\psi(\mathbf{x})) \leq dh(\mathbf{x}) + h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q}).$$

Hence, (5.16) becomes

$$\sum_{\mathfrak{p} \in S} \sum_{i=1}^{q} \frac{d}{d_i} \lambda_{\mathfrak{p}, Q_i}(\mathbf{x}) \leq (m(n+1) + \varepsilon) dh(\mathbf{x}) + \tilde{b}'_\varepsilon(h(F_{\mathscr{X}}) + 1)$$
$$+ (q + (m(n+1) + \varepsilon)) \cdot h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q}) + (q - m)|S| \cdot c_2$$

and (5.7) becomes

$$h(\mathbf{x}) \leq \frac{1}{d} \tilde{b}_\varepsilon(h(F_{\mathscr{X}}) + 1).$$

Combining with (5.5) and (5.9), (5.10) the constants C_ε, C'_ε in the assertion can be given by

$$C_\varepsilon = \frac{1}{d} a_{\varepsilon/m} \cdot (d^{n+1} + (n+1)\triangle_{\mathscr{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q})) \cdot (h(F_{\mathscr{X}}) + 1) \tag{5.17}$$

and

$$C'_\varepsilon = \frac{1}{d} a'_{\varepsilon/m} \cdot m \cdot (d^{n+1} + (n+1)\triangle_{\mathscr{X}} d^n h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q})) \cdot (h(F_{\mathscr{X}}) + 1)$$
$$+ \frac{1}{d} \cdot (q + (m(n+1) + \varepsilon)) \cdot h(Q_1^{d/d_1}, \dots, Q_q^{d/d_q}) + \frac{1}{d} \cdot (q - m)|S| \cdot c_2, \tag{5.18}$$

where $a_{\varepsilon/m}$ and $a'_{\varepsilon/m}$ are the constants from Theorem 5.1.

Acknowledgements. The author would like to thank the reviewers for their helpful comments and suggestions. This work was done during a stay of the author at the Vietnam Institute for Advanced Study in Mathematics (VIASM). We would like to thank the staff here, in particular the support of VIASM. Research is supported by Hanoi National University of Education Grant No SPHN17-01.

References

[1] T. An and J. T. Y. Wang, An effective Schmidt's subspace theorem for non-linear forms over function fields, J. Number Theory **125** (2007), 210–228.
[2] W. D. Brownawell, Applications of Cayley-Chow forms, H. P. Schlickewei, E. Wirsing (eds.), Number theory, Lecture notes in math **1380**, Springer, Berlin, Heidelberg, 1989, 1–18.
[3] F. Catanese, Chow varieties, Hilbert schemes, and moduli spaces of surfaces of general type, J. Algebraic Geometry **1** (1992), 561–595.
[4] Z. Chen, M. Ru and Q. Yan, The degenerated second main theorem and Schmidt's subspace theorem, Science China Math. **55** (2012), 1367–1380.
[5] Z. Jelonek, On effective Nullstellensats, Invent. Math. **162** (2005), 1–17.
[6] J. Kollár, Sharp effective Nullstellensats, J. Am. Math. Soc. **1** (1988), 963–975.
[7] A. Levin, On the Schmidt subspace theorem for algebraic points, Duke. Math. J. **163** (2014), 2841–2885.
[8] D. Mumford, Stability of projective varieties, L'Enseignement Mathematique **23**(1–2) (1977), 39–110.
[9] M. Ru and J. T. Y. Wang, An effective Schmidt's subspace theorem for projective varieties over function fields, Int. Math. Res. Not. **3** (2012), 651–684.
[10] J. T. Y. Wang, An effective Roth's theorem for function fields, Rocky Mountain J. Math. **26** (1996), 1225–1234.
[11] J. T. Y. Wang, An effective Schmidt's subspace theorem over function fields, Math. Z. **246** (2004), 811–844.

Giang Le
Department of Mathematics
Hanoi National University of Education
136-Xuan Thuy, Cau Giay, Hanoi
Vietnam
E-mail: legiang01@yahoo.com