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DEHN TWISTS ON KAUFFMAN BRACKET SKEIN ALGEBRAS

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Abstract

We give an explicit formula for the action of the Dehn twist along a simple closed curve in a compact connected oriented surface on the completion of the filtered skein modules of the surface. To do this, we introduce filtrations of the Kauffman bracket skein algebra and the Kauffman bracket skein modules of the surface.

1. Introduction

Recently it has come to light that the Goldman Lie algebra of a surface plays an important role in the study of the mapping class group of the surface. See [3], [4] and [7] for details. Before that, Turaev [13] drew an analogy between the Goldman Lie algebra and some skein algebra. Hence it is important to establish some explicit connection between the Kauffman bracket skein algebra and the mapping class group. This new connection motivates much of the interest in theory of the mapping class group of a surface and one of knots and links. In fact, skein algebras give us a new way of studying the mapping class group. Furthermore, we expect that this connection will bring us some information about 3-manifolds including the Casson invariant.

The aim of this paper is to explain a new relationship between the Kauffman bracket skein algebra and the mapping class group. Let Σ be a compact connected oriented surface with non-empty boundary. Kawazumi-Kuno [4] [3] defined an action σ of the Goldman Lie algebra on the group ring of the fundamental group of Σ. Using this action, Kawazumi-Kuno [4] [3] and Massuyeau-Turaev [7] obtained a formula for the action of the right handed Dehn twist t_c along a simple closed curve c

$$t_c = \exp\left(\sigma\left(\frac{1}{2}|(\log(c))^2|\right)\right) : \mathbf{Q}\widehat{\pi_1(\Sigma, *)} \to \mathbf{Q}\widehat{\pi_1(\Sigma, *)}, \tag{1}$$

where $\mathbf{Q}\widehat{\pi_1(\Sigma, *)}$ is the completed group ring of the fundamental group of Σ with base point $* \in \partial\Sigma$, $\mathbf{Q}\widehat{\hat{\pi}_1(\Sigma)}$ is the completed Goldman Lie algebra and

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$|\cdot| : \widehat{\mathbf{Q}\pi_1(\Sigma,*)} \to \widehat{\mathbf{Q}\hat{\pi}_1(\Sigma)}$ is the quotient map. Our goal in this paper is to establish a skein algebra version of this formula.

Let Σ be a compact connected oriented surface, I the closed interval $[0,1]$ and $\mathbf{Q}[A,A^{-1}]$ the ring of Laurent polynomials over $\mathbf{Q}$ in an indeterminate A. The Kauffman bracket skein algebra $\mathscr{S}(\Sigma)$ is defined to be the quotient of the free $\mathbf{Q}[A,A^{-1}]$-module with basis the set of unoriented framed links in $\Sigma \times I$ by the skein relation which defines the Kauffman bracket. Let J be a finite subset of $\partial\Sigma$. The Kauffman bracket skein module $\mathscr{S}(\Sigma,J)$ is defined to be the quotient of the free $\mathbf{Q}[A,A^{-1}]$-module with basis $\mathscr{T}(\Sigma,J)$ by the same skein relation, where we denote by $\mathscr{T}(\Sigma,J)$ the set of unoriented framed tangles with the base point set $J \times \{\frac{1}{2}\}$. For details, see Subsection 3.1. The Kauffman bracket skein algebra $\mathscr{S}(\Sigma)$ has a structure of an associative algebra and a Lie algebra over $\mathbf{Q}[A,A^{-1}]$. The Kauffman bracket skein module $\mathscr{S}(\Sigma,J)$ has a structure of an $\mathscr{S}(\Sigma)$-bimodule. Furthermore, we define an action σ of $\mathscr{S}(\Sigma)$ on $\mathscr{S}(\Sigma,J)$ such that $\mathscr{S}(\Sigma,J)$ is $\mathscr{S}(\Sigma)$-module under the action σ when we regard $\mathscr{S}(\Sigma)$ as a Lie algebra. For details, see Subsection 3.2. In this paper, we introduce a filtration $\{F^n\mathscr{S}(\Sigma)\}_{n\geq 0}$ of $\mathscr{S}(\Sigma)$ and a filtration $\{F^n\mathscr{S}(\Sigma,J)\}_{n\geq 0}$ of $\mathscr{S}(\Sigma,J)$ defined by an augmentation ideal $\ker\varepsilon$, where the augmentation map ε is defined by $\varepsilon(A) = -1$ and $\varepsilon(L) = (-2)^{\#\pi_0(L)}$ for any link L in $\Sigma \times I$. These operations are continuous in the topologies of $\mathscr{S}(\Sigma)$ and $\mathscr{S}(\Sigma,J)$ induced by these filtrations. We remark that there is some relationship between the completion of the group ring of the fundamental group of Σ and these filtrations of $\mathscr{S}(\Sigma)$ and $\mathscr{S}(\Sigma,J)$ which will appear in [10]. We denote the completions of $\mathscr{S}(\Sigma)$ and $\mathscr{S}(\Sigma,J)$ in these topologies by $\hat{\mathscr{S}}(\Sigma)$ and $\hat{\mathscr{S}}(\Sigma,J)$, respectively. For details, see Subsection 3.3. The main result of the paper is the formula for the action of the Dehn twist t_c along a simple closed curve c

$$t_c(\cdot) = \exp\left(\sigma\left(\frac{-A+A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{c}{2}\right)\right)^2\right)\right)(\cdot) : \hat{\mathscr{S}}(\Sigma,J) \to \hat{\mathscr{S}}(\Sigma,J).$$

which is a skein version of the formula (1). Here $\log(-A) = \sum_{i=1}^{\infty}\frac{-1}{i}(A+1)^i \in \mathbf{Q}[[A+1]]$ and $\left(\operatorname{arccosh}\left(\frac{-c}{2}\right)\right)^2 = \sum_{i=0}^{\infty}\frac{i!i!}{(i+1)(2i+1)!}\left(1-\frac{c^2}{4}\right)^{i+1} \in \mathbf{Q}[[c+2]]$. This skein version does not follow from the original one [3] [4] [7].

In Section 5, we prove the following three properties of the filtrations of $\mathscr{S}(\Sigma)$ and $\mathscr{S}(\Sigma,J)$.

(1) Let Σ be a compact connected oriented surface with non-empty boundary. The topology on $\mathscr{S}(\Sigma,J)$ introduced by the filtration is Hausdorff, in other words, we have $\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma,J) = 0$.

(2) Let Σ and Σ' be two oriented compact connected surfaces satisfying $\pi_1(\Sigma) \simeq \pi_1(\Sigma')$, J and J' finite subsets of $\partial\Sigma$ and $\partial\Sigma'$, respectively, satisfying $\#J = \#J'$. There exists a diffeomorphism $\xi : (\Sigma \times I, J \times I) \to (\Sigma' \times I, J' \times I)$. Then we have $\xi(F^n\mathscr{S}(\Sigma,J)) = F^n\mathscr{S}(\Sigma',J')$. But the induced map $\xi : \mathscr{S}(\Sigma) \to \mathscr{S}(\Sigma')$ does not seem to be an algebra homomorphism.

(3) We have

$$\sum_{L' \subset L} (-1)^{|L'|}(-2)^{-|L'|}[L'] \in (\ker \varepsilon)^n$$

for any link L in $\Sigma \times I$ having components more than n, where the sum is over all sublinks $\emptyset \subsetneqq L' \subsetneqq L$ and $|L|$ the number of components of L. In other words, for any link L in $\Sigma \times I$, $(-2)^{-|L'|}[L'] \bmod (\ker \varepsilon)^n$ is a finite type invariant of order n in the sense of Le [5] (3.2).

The second and third properties follow from Lemma 5.3. Using the second property and Lickorish's theorem [6] (Theorem 5.6), we prove the first property. In subsequent papers, we need all the above properties. In particular, we need the first property to prove the faithfulness of the action of the mapping class group of a compact connected oriented surface with non-empty boundary on the completed skein algebra of the surface.

In subsequent papers, using this formula of Dehn twists, we obtain an embedding of the Torelli group of a surface into the completed skein algebra of the surface defined in this paper. This embedding gives a construction of the first Johnson homomorphism and a new filtration consisting of normal subgroups in the mapping class group. Furthermore, it gives an invariant $z(M) \in \mathbf{Q}[[A+1]]$ for an integral homology 3-sphere M. The invariant induces $z(M) \bmod (A+1)^{n+1}$ which is a finite type invariant of order n. The details will appear elsewhere [11] and [12].

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Contents

2. Definition of tangles in $\Sigma \times I$

In this section, let Σ be a compact connected oriented surface. We define the set of tangles in $\Sigma \times I$.

Definition 2.1. Let J be a finite subset of $\partial\Sigma$. We define $\mathscr{E}(\Sigma, J)$ to be the set consisting of all injective maps $E = \coprod_i \tau_i \sqcup \coprod_j \upsilon_j$ from a domain D consisting of a finite collection of stripes $\coprod_i I \times (-\varepsilon, \varepsilon)$ and annuli $\coprod_j S^1 \times (-\varepsilon, \varepsilon)$ into $\Sigma \times (0,1)$ satisfying the following.

(1) Each υ_j is an embedding into $\Sigma \times (0,1)$.
(2) The restriction of each τ_i to $(0,1) \times (-\varepsilon, \varepsilon)$ is an embedding into $\Sigma \times (0,1)$.
(3) The restriction of each τ_i to $\{0,1\} \times (-\varepsilon, \varepsilon)$ is an orientation preserving embedding into $J \times I$.
(4) For $j \in J$, $E(D) \cap (j \times I)$ is not empty and is connected.

Two elements E_0 and E_1 of $\mathscr{E}(\Sigma, J)$ which have the same domain D are unoriented-isotopic if there exists a continuous map $H : D \times I \to \Sigma \times I$ such that $H(D \times \{0\}) = E_0(D)$, $H(D \times \{1\}) = E_1(D)$ and $H(\cdot, t) \in \mathscr{E}(\Sigma, J)$ for $t \in I$. We denote by $\mathscr{T}(\Sigma, J)$ the set of unoriented-isotopy classes of elements of $\mathscr{E}(\Sigma, J)$. We denote by $\langle \cdot \rangle$ the quotient map $\mathscr{E}(\Sigma, J) \to \mathscr{T}(\Sigma, J)$. If $J = \emptyset$, we simply denote $\mathscr{T}(\Sigma, J)$ and $\mathscr{E}(\Sigma, J)$ by $\mathscr{T}(\Sigma)$ and $\mathscr{E}(\Sigma)$. An element of $\mathscr{T}(\Sigma, J)$ is called a tangle.

The definition of 'tangles' is similar to the definition of 'links' of marked surfaces in [8]. But, a tangle in this definition has one arc on each point of J.

Definition 2.2. Let J be a finite subset of $\partial\Sigma$. An element E of $\mathscr{E}(\Sigma, J)$ is generic if $E : (\coprod_i I \sqcup \coprod_j S^1) \times (-\varepsilon, \varepsilon) \to \Sigma \times I$ satisfies the following.

(1) For $x \in \coprod_i I \sqcup \coprod_j S^1$, the map $(-\varepsilon, \varepsilon) \to I$, $t \mapsto p_2 \circ E(x, t)$ is an orientation preserving embedding map, where we denote by p_2 the projection $\Sigma \times I \to I$.
(2) The map $\coprod_i I \sqcup \coprod_j S^1 \to \Sigma$, $x \mapsto p_1 \circ E(x, 0)$ is an immersion such that the intersections of the image of the map consist of transverse double points, where we denote by p_1 the projection $\Sigma \times I \to \Sigma$.

It is convenient to present tangles in $\Sigma \times I$ by tangle diagrams on Σ in the same fashion in which links in $\mathbf{R}^3$ may be presented by planar link diagrams.

Definition 2.3. Let J be a finite subset of $\partial\Sigma$, T an element of $\mathscr{T}(\Sigma, J)$ and $E : (\coprod_i I \sqcup \coprod_j S^1) \times (-\varepsilon, \varepsilon) \to \Sigma \times I$ an element of $\mathscr{E}(\Sigma, J)$ representing T which is generic. The tangle diagram of T is $p_1 \circ E((\coprod_i I \sqcup \coprod_j S^1) \times \{0\})$ together with height-information, i.e., the choice of the upper branch of the curve at each crossing. The chosen branch is called an over crossing; the other branch is called an under crossing.

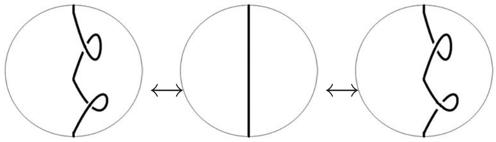

FIGURE 1. RI: Reidemester move I

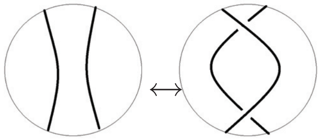

FIGURE 2. RII: Reidemeister move II

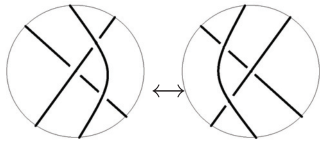

FIGURE 3. RIII: Reidemeister move III

PROPOSITION 2.4 (see, for example, [1]). *Let J be a finite subset of $\partial\Sigma$. Let T and T' be two elements of $\mathcal{T}(\Sigma, J)$ presented by tangle diagrams d and d', respectively. Then, T equals T' if and only if d can be transformed into d' by a sequence of isotopies of Σ and the RI, RII, RIII moves as shown in Figure* 1, 2, *and* 3.

Let J and J' be two finite subsets of $\partial\Sigma$ with $J \cap J' = \emptyset$. Here e_1 and e_2 denote the embedding maps from $\Sigma \times I$ to $\Sigma \times I$ defined by $e_1(x, t) = \left(x, \frac{t+1}{2}\right)$ and $e_2(x, t) = \left(x, \frac{t}{2}\right)$, respectively. We define $\boxtimes : \mathcal{T}(\Sigma, J) \times \mathcal{T}(\Sigma, J') \to \mathcal{T}(\Sigma, J \cup J')$ by

$$\langle E \rangle \boxtimes \langle E' \rangle \stackrel{\text{def.}}{=} \langle e_1 \circ E \sqcup e_2 \circ E' \rangle$$

for $E \in \mathcal{E}(\Sigma, J)$ and $E' \in \mathcal{E}(\Sigma, J')$.

Let J be a finite subset of $\partial\Sigma$, T an element of $\mathcal{T}(\Sigma, J)$ represented by $E \in \mathcal{E}(\Sigma, J)$ and ξ an element of $\mathcal{M}(\Sigma)$ represented by a diffeomorphism $\mathcal{X}_\xi$, where we denote by $\mathcal{M}(\Sigma)$ the mapping class group of Σ preserving the boundary pointwise. We denote by ξT an element of $\mathcal{T}(\Sigma, J)$ represented by $(\mathcal{X}_\xi \times \mathrm{id}_I) \circ E \in \mathcal{E}(\Sigma, J)$.

3. Kauffman bracket skein modules

Throughout this section, let Σ be a compact connected oriented surface.

3.1. Definition of Kauffman bracket skein modules. In this subsection, we define Kauffman bracket skein modules.

First of all, we define Kauffman triples.

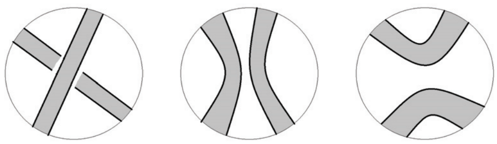

FIGURE 4. Kauffman triple

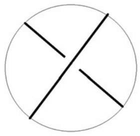

FIGURE 5

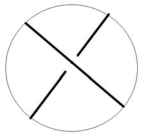

FIGURE 6

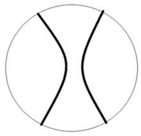

FIGURE 7

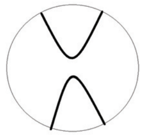

FIGURE 8

DEFINITION 3.1. Let J be a finite subset of $\partial\Sigma$. A triple of three tangles T_1, T_∞ and $T_0 \in \mathscr{T}(\Sigma, J)$ is a Kauffman triple if there exist E_1, E_∞ and $E_0 \in \mathscr{E}(\Sigma, J)$ whose domains are D_1, D_∞ and D_0 satisfying the following.

- We have $\langle E_1 \rangle = T_1$, $\langle E_\infty \rangle = T_\infty$ and $\langle E_0 \rangle = T_0$.
- The three images $E_1(D_1)$, $E_\infty(D_\infty)$ and $E_0(D_0)$ are identical except for some neighborhood of a point, where they differ as shown in Figure 4.

In other words, there exist three tangle diagrams d_1, d_∞ and d_0 presenting T_1, T_∞ and T_0, respectively, which are identical except for some neighborhood of a point, where they differ as shown in Figure 5, Figure 7 and Figure 8, respectively.

We define Kauffman bracket skein modules.

DEFINITION 3.2 (Kauffman bracket skein module). Let J be a finite subset of $\partial\Sigma$. We define $\mathscr{S}(\Sigma, J)$ to be the quotient of the free $\mathbf{Q}[A, A^{-1}]$-module $\mathbf{Q}[A, A^{-1}]\mathscr{T}(\Sigma, J)$ by the skein relation, i.e., by the submodule of $\mathbf{Q}[A, A^{-1}]\mathscr{T}(\Sigma, J)$ generated by

$$\{-T_1 + AT_\infty + A^{-1}T_0 \mid (T_1, T_\infty, T_0) \text{ is a Kauffman triple}\}$$
$$\cup \{T \boxtimes \mathcal{O} + (A^2 + A^{-2})T \mid T \in \mathscr{T}(\Sigma, J)\}$$

where $\mathcal{O} \in \mathscr{T}(\Sigma)$ is a trivial knot. Following [13], the element of $\mathscr{S}(\Sigma, J)$ represented by $T \in \mathscr{T}(\Sigma, J)$ is denoted by $[T]$. We simply denote $\mathscr{S}(\Sigma, \emptyset)$ by $\mathscr{S}(\Sigma)$.

In [8], Muller also defined skein modules for a surface with boundary. We, however, do not need 'the boundary skein relation' and 'the value of a contractible arc'.

Let J and J' be two finite subsets of $\partial\Sigma$ satisfying $J \cap J' = \emptyset$. The $\mathbf{Q}[A, A^{-1}]$-bilinear homomorphism $\boxtimes : \mathscr{S}(\Sigma, J) \times \mathscr{S}(\Sigma, J') \to \mathscr{S}(\Sigma, J \cup J')$ is defined by $[T] \boxtimes [T'] \stackrel{\text{def.}}{=} [T \boxtimes T']$ for $T \in \mathscr{T}(\Sigma, J)$ and $T' \in \mathscr{T}(\Sigma, J')$. The skein module $\mathscr{S}(\Sigma)$ is the associative algebra over $\mathbf{Q}[A, A^{-1}]$ with product defined by $ab = a \boxtimes b$ for a and $b \in \mathscr{S}(\Sigma)$. The skein module $\mathscr{S}(\Sigma, J)$ is the $\mathscr{S}(\Sigma)$-bimodule given by $av = a \boxtimes v$ and $va = v \boxtimes a$ for $a \in \mathscr{S}(\Sigma)$ and $v \in \mathscr{S}(\Sigma, J)$. For $v \in \mathscr{S}(\Sigma, J)$, $v' \in \mathscr{S}(\Sigma, J')$ and $a \in \mathscr{S}(\Sigma)$, we have $(va) \boxtimes v' = v \boxtimes (av')$.

3.2. Some Poisson-like structure on $\mathscr{S}(\Sigma)$. In this subsection, we define a Lie bracket of $\mathscr{S}(\Sigma)$ and an action σ of $\mathscr{S}(\Sigma)$ on $\mathscr{S}(\Sigma, J)$.

Let J be a finite subset of $\partial\Sigma$. We denote by $\mathscr{E}_0(\Sigma, J)$ the set of 1-dimensional submanifolds of Σ with boundary J and no inessential components. Here a connected 1-dimensional submanifold of Σ is inessential if it is a boundary of a disk in Σ. We denote the set of isotopy classes in $\mathscr{E}_0(\Sigma, J)$ by $\mathscr{T}_0(\Sigma, J)$.

THEOREM 3.3. *Let J be a finite subset of $\partial\Sigma$. The skein module $\mathscr{S}(\Sigma, J)$ is the free $\mathbf{Q}[A, A^{-1}]$-module with basis $\mathscr{T}_0(\Sigma, J)$.*

In the case when $J = \emptyset$, this is proved by Przytycky [9]. For the general case, it is proved in a similar way to [9].

COROLLARY 3.4. *We have $\mathscr{S}(S^1 \times I) = \mathbf{Q}[A, A^{-1}][l]$ where l is the element represented by the link whose diagram is $S^1 \times \{\frac{1}{2}\}$.*

COROLLARY 3.5. *Let J be a finite subset of $\partial\Sigma$. The $\mathbf{Q}[A, A^{-1}]$-module homomorphism $-A + A^{-1} : \mathscr{S}(\Sigma, J) \to \mathscr{S}(\Sigma, J)$, $x \mapsto (-A + A^{-1})x$ is an injective map.*

LEMMA 3.6. *Let J be a finite subset of $\partial\Sigma$. Let T_1, T_2, T_3 and T_4 be four elements of $\mathscr{T}(\Sigma, J)$ presented by four diagrams which are identical except for some neighborhood of a point, where they differ as shown in Figure* 5, *Figure* 6, *Figure* 7 *and Figure* 8, *respectively. Then we have $[T_1] - [T_2] = (A - A^{-1})([T_3] - [T_4])$.*

Proof. We have

$$[T_1] - [T_2] = (A[T_3] + A^{-1}[T_4]) - (A^{-1}[T_3] + A[T_4]) = (A - A^{-1})([T_3] - [T_4]). \qquad \square$$

In Definition 3.8, we introduce a Lie bracket in $\mathscr{S}(\Sigma)$ by using the following proposition and Corollary 3.5.

PROPOSITION 3.7. *Let J and J' be two finite subsets of $\partial\Sigma$ satisfying $J \cap J' = \emptyset$. We have $v \boxtimes v' - v' \boxtimes v \in (A - A^{-1})\mathscr{S}(\Sigma, J \cup J')$ for $v \in \mathscr{S}(\Sigma, J)$ and $v' \in \mathscr{S}(\Sigma, J')$.*

Proof. Let T be an element of $\mathscr{T}(\Sigma, J)$ and T' an element of $\mathscr{T}(\Sigma, J')$. Choose tangle diagrams d and d' presenting T and T', respectively, such that the

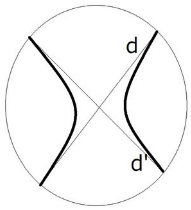

FIGURE 9

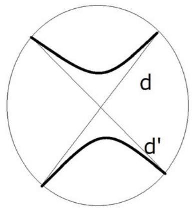

FIGURE 10

intersections of d and d' consist of transverse double points $P_1, P_2, \dots, P_m$. For $i = 1, 2, \dots, m$, let $d(1, i)$ and $d(-1, i)$ be two tangle diagrams satisfying the following.

- The two tangle diagrams $d(1, i)$ and $d(-1, i)$ equal $d \cup d'$ with the same height-information as d and d' except for the neighborhoods of the intersections of d and d'.
- The branches of $d(1, i)$ and $d(-1, i)$ in the neighborhood of P_j belonging to d' are over crossings for $j = 1, \dots, i-1$.
- The branches of $d(1, i)$ and $d(-1, i)$ in the neighborhood of P_j belonging to d are over crossings for $j = i+1, \dots, m$.
- The two tangle diagrams $d(1, i)$ and $d(-1, i)$ are as shown in Figure 9 and Figure 10, respectively, in the neighborhood of P_i.

We denote by $T(1, i)$ a tangle presented by $d(1, i)$ and by $T(-1, i)$ a tangle presented by $d(-1, i)$. Using Lemma 3.6, we have

$$[T] \boxtimes [T'] - [T'] \boxtimes [T] = (A - A^{-1}) \sum_{i=1}^{m} ([T(1, i)] - [T(-1, i)]). \tag{2}$$

This proves the proposition. □

DEFINITION 3.8. Let J be a finite subset of $\partial\Sigma$. We define a bracket $[\,,]$ of $\mathscr{S}(\Sigma)$ by

$$[x, y] \stackrel{\text{def.}}{=} \frac{1}{-A + A^{-1}} (xy - yx)$$

for x and $y \in \mathscr{S}(\Sigma)$. We define an action σ of $\mathscr{S}(\Sigma)$ on $\mathscr{S}(\Sigma, J)$ by

$$\sigma(x)(v) \stackrel{\text{def.}}{=} \frac{1}{-A + A^{-1}} (xv - vx)$$

for $x \in \mathscr{S}(\Sigma)$ and $v \in \mathscr{S}(\Sigma, J)$.

It is easy to prove the following proposition.

PROPOSITION 3.9. *Let J be a finite subset of $\partial\Sigma$. The bracket $[\,,] : \mathscr{S}(\Sigma) \times \mathscr{S}(\Sigma) \to \mathscr{S}(\Sigma)$ makes $\mathscr{S}(\Sigma)$ a Lie algebra. The action $\sigma : \mathscr{S}(\Sigma) \times \mathscr{S}(\Sigma, J) \to$*

$\mathscr{S}(\Sigma, J)$ makes $\mathscr{S}(\Sigma, J)$ an $\mathscr{S}(\Sigma)$-module when we regard $\mathscr{S}(\Sigma)$ as a Lie algebra. Furthermore, for x, y and $z \in \mathscr{S}(\Sigma)$ and $v \in \mathscr{S}(\Sigma, J)$, we have the Leibniz rules:

$$[xy, z] = x[y, z] + [x, z]y,$$
$$\sigma(xy)(v) = x\sigma(y)(v) + \sigma(x)(v)y,$$
$$\sigma(x)(yv) = [x, y]v + y\sigma(x)(v),$$
$$\sigma(x)(vy) = \sigma(x)(v)y + v[x, y].$$

Let J and J' be two finite subsets of $\partial\Sigma$ satisfying $J \cap J' = \emptyset$. We have

$$\sigma(x)(v \boxtimes v') = \sigma(x)(v) \boxtimes v' + v \boxtimes \sigma(x)(v')$$

for $x \in \mathscr{S}(\Sigma)$, $v \in \mathscr{S}(\Sigma, J)$ and $v' \in \mathscr{S}(\Sigma, J')$.

3.3. Filtrations and completions. We introduce filtrations of Kauffman bracket skein modules and define completed Kauffman bracket skein modules.

We define an augmentation map $\varepsilon : \mathscr{S}(\Sigma) \to \mathbf{Q}$ by $A \mapsto -1$ and $[L] \mapsto (-2)^{|L|}$ for $L \in \mathscr{T}(\Sigma)$ where $|L|$ is the number of components of L.

PROPOSITION 3.10. *The augmentation map ε is well-defined.*

Proof. Let T_1, T_∞ and T_0 be three elements of $\mathscr{T}(\Sigma)$ such that (T_1, T_∞, T_0) is a Kauffman triple. There are three cases,

$$|T_1| - 1 = |T_\infty| = |T_0|,$$
$$|T_1| = |T_\infty| - 1 = |T_0|,$$
$$|T_1| = |T_\infty| = |T_0| - 1.$$

In each case, we have $\varepsilon([T_1] - A[T_\infty] - A^{-1}[T_0]) = 0$. For $T \in \mathscr{T}(\Sigma)$, we have $\varepsilon([T \boxtimes \mathcal{O}] + (A^2 + A^{-2})[T]) = 0$. This proves the proposition. □

LEMMA 3.11. *We have $[\mathscr{S}(\Sigma), \mathscr{S}(\Sigma)] \subset \ker \varepsilon$.*

Proof. Since the algebra $\mathscr{S}(\Sigma)$ is generated by the set of elements represented by knots, it suffices to show that $[[T], [T']] \in \ker \varepsilon$ for any two elements T and T' of $\mathscr{T}(\Sigma)$ satisfying $|T| = 1$ and $|T'| = 1$. Using equation (2), we obtain

$$[[T], [T']] = -\sum_{i=1}^{m}([T(1, i)] - [T(-1, i)]),$$

where $T(1, i)$ and $T(-1, i)$ are some knots for $i \in \{1, \dots, m\}$. Then, we have $\varepsilon(-\sum_{i=1}^{m}([T(1, i)] - [T(-1, i)])) = 0$. This proves the proposition. □

Let J be a finite subset of $\partial\Sigma$. We define a filtration of $\mathscr{S}(\Sigma)$ by $F^n\mathscr{S}(\Sigma) = (\ker \varepsilon)^n$ and a filtration of $\mathscr{S}(\Sigma, J)$ by $F^n\mathscr{S}(\Sigma, J) = (F^n\mathscr{S}(\Sigma))\mathscr{S}(\Sigma, J)$.

Theorem 3.12. (1) *Let J be a finite subset of $\partial\Sigma$. We have*

$$F^n\mathscr{S}(\Sigma)F^m\mathscr{S}(\Sigma) \subset F^{n+m}\mathscr{S}(\Sigma),$$
$$F^n\mathscr{S}(\Sigma)F^m\mathscr{S}(\Sigma,J) \subset F^{n+m}\mathscr{S}(\Sigma,J),$$
$$F^n\mathscr{S}(\Sigma,J)F^m\mathscr{S}(\Sigma) \subset F^{n+m}\mathscr{S}(\Sigma,J),$$

for n and $m \in \mathbf{Z}_{\geq 0}$.

(2) *We have $[F^i\mathscr{S}(\Sigma), F^j\mathscr{S}(\Sigma)] \subset F^{\max(i+j-1,i,j)}\mathscr{S}(\Sigma)$ and $\sigma(F^i\mathscr{S}(\Sigma))(F^j\mathscr{S}(\Sigma,J)) \subset F^{\max(i+j-1,i-1,j)}\mathscr{S}(\Sigma,J)$ for i and $j \in \mathbf{Z}_{\geq 0}$.*

Proof. In order to show, for i and $j \in \mathbf{Z}_{\geq 0}$,

$$F^i\mathscr{S}(\Sigma)F^j\mathscr{S}(\Sigma,J) \subset F^{i+j}\mathscr{S}(\Sigma,J),$$
$$F^j\mathscr{S}(\Sigma,J)F^i\mathscr{S}(\Sigma) \subset F^{i+j}\mathscr{S}(\Sigma,J),$$

it suffices to prove

$$(\ker\varepsilon)\mathscr{S}(\Sigma,J) = \mathscr{S}(\Sigma,J)(\ker\varepsilon),$$

which is obvious by Proposition 3.7. This proves (1).

Using the Leibniz rule and Lemma 3.11, we obtain, for i and $j \in \mathbf{Z} \geq 0$,

$$[F^i\mathscr{S}(\Sigma), F^j\mathscr{S}(\Sigma)] \subset F^{\max(i+j-1,i,j)}\mathscr{S}(\Sigma),$$
$$\sigma(F^i\mathscr{S}(\Sigma))(F^j\mathscr{S}(\Sigma,J)) \subset F^{\max(i+j-1,i-1,j)}\mathscr{S}(\Sigma,J).$$

This proves (2). □

Let J be a finite subset of $\partial\Sigma$. We define an action of $\mathscr{M}(\Sigma)$ on $\mathscr{S}(\Sigma,J)$ by $\xi[T] = [\xi T]$ for $\xi \in \mathscr{M}(\Sigma)$ and $T \in \mathscr{T}(\Sigma,J)$. By definition, we have

$$\xi(F^n\mathscr{S}(\Sigma)) = F^n\mathscr{S}(\Sigma),$$
$$\xi(F^n\mathscr{S}(\Sigma,J)) = F^n\mathscr{S}(\Sigma,J)$$

for $\xi \in \mathscr{M}(\Sigma)$ and $n \in \mathbf{Z}_{\geq 0}$.

Remark 3.13. We have $\dim_{\mathbf{Q}}(F^n\mathscr{S}(\Sigma,J)/F^{n+1}\mathscr{S}(\Sigma,J)) < \infty$. The proof will appear in [10].

Let J be a finite subset of $\partial\Sigma$. We consider the topology on $\mathscr{S}(\Sigma)$ induced by the filtration $\{F^n\mathscr{S}(\Sigma)\}_{n\geq 0}$. By Theorem 3.12, the product and the bracket of $\mathscr{S}(\Sigma)$ are continuous in the topology. We denote its completion by $\hat{\mathscr{S}}(\Sigma) \overset{\text{def.}}{=} \varprojlim_{i\to\infty} \mathscr{S}(\Sigma)/F^i\mathscr{S}(\Sigma)$. We call $\hat{\mathscr{S}}(\Sigma)$ the completed skein algebra. We also consider the topology on $\mathscr{S}(\Sigma,J)$ induced by the filtration $\{F^n\mathscr{S}(\Sigma,J)\}_{n\geq 0}$. By Theorem 3.12, the right action, the left action and the Lie action σ of $\mathscr{S}(\Sigma)$ on $\mathscr{S}(\Sigma,J)$ are continuous in the topology. We denote its completion by $\hat{\mathscr{S}}(\Sigma,J) \overset{\text{def.}}{=} \varprojlim_{i\to\infty} \mathscr{S}(\Sigma,J)/F^i\mathscr{S}(\Sigma,J)$. We call $\hat{\mathscr{S}}(\Sigma,J)$ the completed skein module. The completed skein algebra $\hat{\mathscr{S}}(\Sigma)$ has a filtration $\hat{\mathscr{S}}(\Sigma) = F^0\hat{\mathscr{S}}(\Sigma) \supset F^1\hat{\mathscr{S}}(\Sigma) \supset F^2\hat{\mathscr{S}}(\Sigma) \supset \cdots$ such that $\hat{\mathscr{S}}(\Sigma)/F^n\hat{\mathscr{S}}(\Sigma) \simeq \mathscr{S}(\Sigma)/F^n\mathscr{S}(\Sigma)$ for $n \in \mathbf{Z}_{\geq 0}$. The completed skein module $\hat{\mathscr{S}}(\Sigma, J)$ also has a filtration $\hat{\mathscr{S}}(\Sigma, J) = F^0\hat{\mathscr{S}}(\Sigma, J) \supset F^1\hat{\mathscr{S}}(\Sigma, J) \supset F^2\hat{\mathscr{S}}(\Sigma, J) \supset \cdots$ such that $\hat{\mathscr{S}}(\Sigma, J)/F^n\hat{\mathscr{S}}(\Sigma, J) \simeq \mathscr{S}(\Sigma, J)/F^n\mathscr{S}(\Sigma, J)$ for $n \in \mathbf{Z}_{\geq 0}$. We remark that the completed skein algebra $\hat{\mathscr{S}}(\Sigma)$ is an associative $\mathbf{Q}[[A+1]]$-algebra and that the completed skein module $\hat{\mathscr{S}}(\Sigma, J)$ is a $\mathbf{Q}[[A+1]]$-module. The set $\{(\mathscr{S}(\Sigma, J), \{F^n\mathscr{S}(\Sigma, J)\}_{n \geq 0}) \mid J \subset \partial\Sigma, \#J < \infty\}$ is denoted by $\Theta(\Sigma)$.

We denote by $\check{\mathscr{M}}(\Sigma) \subset \mathscr{M}(\Sigma)$ the subset consisting of all elements ξ satisfying that, for any finite subset J of $\partial\Sigma$, any non-negative integer m and any element $v \in F^m\mathscr{S}(\Sigma, J)$, there exists a non-negative integer N such that $j \geq N \Rightarrow (\mathrm{id} - \xi)^j(v) \in F^{m+1}\mathscr{S}(\Sigma, J)$.

For $\xi \in \check{\mathscr{M}}(\Sigma)$ and a finite subset J of $\partial\Sigma$, a $\mathbf{Q}[[A+1]]$-module homomorphism $\log(\xi) : \hat{\mathscr{S}}(\Sigma, J) \to \hat{\mathscr{S}}(\Sigma, J)$ is defined by $\log(\xi)(v) = \sum_{i=1}^{\infty} \frac{-1}{i}(\mathrm{id} - \xi)^i(v)$. For $\xi \in \check{\mathscr{M}}(\Sigma)$, $x \in \hat{\mathscr{S}}(\Sigma)$ and $z \in \hat{\mathscr{S}}(\Sigma, J)$, since $\xi(xz) = \xi(x)\xi(z)$ and $\xi(zx) = \xi(z)\xi(x)$, $\log(\xi)$ satisfies the Leibniz rule

$$\log(\xi)(xz) = \log(\xi)(x)z + x\log(\xi)(z),$$
$$\log(\xi)(zx) = \log(\xi)(z)x + z\log(\xi)(x).$$

Definition 3.14. For $\xi \in \check{M}(\Sigma)$, an element $x_\xi \in \hat{\mathscr{S}}(\Sigma)$ is a skein representative of ξ by $((\mathscr{S}(\Sigma), \{F^n\mathscr{S}(\Sigma)\}_{n \geq 0}), \Theta(\Sigma))$ if we have

$$\log(\xi) = \sigma(x_\xi) : \hat{\mathscr{S}}(\Sigma, J) \to \hat{\mathscr{S}}(\Sigma, J),$$

in other words

$$\xi(\cdot) = \exp(\sigma(x_\xi)) : \hat{\mathscr{S}}(\Sigma, J) \to \hat{\mathscr{S}}(\Sigma, J),$$

for any finite subset J of $\partial\Sigma$.

4. Dehn twists

In this section, we show the following.

Theorem 4.1. *Let Σ be a compact connected oriented surface and c a simple closed curve. We also denote by c an element of $\mathscr{S}(\Sigma)$ represented by a knot presented by the simple closed curve c. Then we have $t_c \in \check{\mathscr{M}}(\Sigma)$, and $\frac{-A + A^{-1}}{4\log(-A)}\left(\mathrm{arccosh}\left(-\frac{c}{2}\right)\right)^2 \in \hat{\mathscr{S}}(\Sigma)$ is a skein representative of $t_c \in \check{\mathscr{M}}(\Sigma)$ by $((\mathscr{S}(\Sigma), \{F^n\mathscr{S}(\Sigma)\}_{n \geq 0}), \Theta(\Sigma))$ in the sense of Definition 3.14. Here $\frac{-A + A^{-1}}{4\log(-A)}$ is an element of $\mathbf{Q}[[A+1]]$ and $\left(\mathrm{arccosh}\left(-\frac{c}{2}\right)\right)^2 = \sum_{i=0}^{\infty} \frac{i!i!}{(i+1)(2i+1)!}\left(1 - \frac{c^2}{4}\right)^{i+1} \in \mathbf{Q}[[c+2]]$.*

Using the following lemma, we prove Theorem 4.1. The following lemma will be proved later.

LEMMA 4.2. *We denote by S^1 the quotient $\mathbf{R}/\mathbf{Z}$, by c_l a simple closed curve $S^1 \times \{\frac{1}{2}\}$ in $S^1 \times I$, by t the Dehn twist along c_l and by l an element of $\mathscr{S}(S^1 \times I)$ represented by the knot presented by c_l. Fix a positive integer m. Choose points $p_1 = \dfrac{1}{2m}, \dots, p_i = \dfrac{i}{2m}, \dots, p_m = \dfrac{m}{2m}$ in S^1. We denote by r_i^0 an element of $\mathscr{S}(S^1 \times I, \{(p_i, 0), (p_i, 1)\})$ represented by the tangle presented by $\{p_i\} \times I$. Then we have the following.*

- (1) *We have* $(t - \mathrm{id})^{2n+m}(r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0) \in F^n\mathscr{S}(S^1 \times I, \{p_1, \dots, p_m\} \times \{0, 1\})$.
- (2) *We have*

$$\log(t)(r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0)$$
$$= \sigma\left(\frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0).$$

Proof of Theorem 4.1. We fix an embedding $\imath : S^1 \times I \to \Sigma$ such that $\imath(c_l) = c$. In order to prove the theorem, it is sufficient to consider two cases: the simple closed curve is separating or not.

We assume that $\imath(S^1 \times I)$ is separating Σ into two surfaces Σ^1 and Σ^2. For a finite set $J' = \left\{\dfrac{1}{2m}, \dots, \dfrac{m}{2m}\right\} \subset S^1$, we consider the trilinear map

$$\varpi_{J'} : \mathscr{S}(\Sigma^1, (J \cap \partial\Sigma^1) \cup \imath(J' \times \{1\})) \times \mathscr{S}(S^1 \times I, J' \times \{0, 1\})$$
$$\times \mathscr{S}(\Sigma^2, (J \cap \partial\Sigma^2) \cup \imath(J' \times \{0\})) \to \mathscr{S}(\Sigma, J)$$

defined by $\varpi_{J'}([T_1], [T_2], [T_3]) = [T_1T_2T_3]$ for $T_1 \in \mathscr{T}(\Sigma^1, (J \cap \partial\Sigma^1) \cup \imath(J' \times \{1\}))$, $T_2 \in \mathscr{T}(S^1 \times I, J' \times \{0, 1\})$ and $T_3 \in \mathscr{T}(\Sigma^2, (J \cap \partial\Sigma^2) \cup \imath(J' \times \{0\}))$. Here we denote by $T_1T_2T_3$ the tangle presented by $d_1 \cup \imath(d_2) \cup d_3$, where d_1, d_2 and d_3 present T_1, T_2 and T_3, respectively. We remark that $d_1 \cup \imath(d_2) \cup d_3$ must be smoothed out in the neighborhood of $\imath(S^1 \times \{0, 1\})$. By Theorem 3.3, the set

$$\bigcup_{J'} \varpi_{J'}(\mathscr{S}(\Sigma^1, (J \cap \partial\Sigma^1) \cup \imath(J' \times \{1\})) \times \{\mathrm{id}_{J'}\}$$
$$\times \mathscr{S}(\Sigma^2, (J \cap \partial\Sigma^2) \cup \imath(J' \times \{0\})))$$

generates $\mathscr{S}(\Sigma, J)$ as a $\mathbf{Q}[A, A^{-1}]$-module, where we set $\mathrm{id}_{J'} \stackrel{\text{def.}}{=} r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0$. In order to show the theorem, we use the following.

- The map $\varpi_{J'}$ preserves the filtrations, in other words,

$$\varpi_{J'}(\mathscr{S}(\Sigma^1, (J \cap \partial\Sigma^1) \cup \imath(J' \times \{1\})) \times F^n\mathscr{S}(S^1 \times I, J' \times \{0, 1\})$$
$$\times \mathscr{S}(\Sigma^2, (J \cap \partial\Sigma^2) \cup \imath(J' \times \{0\}))) \subset F^n\mathscr{S}(\Sigma, J).$$

- We have $t_c \circ \varpi_{J'} = \varpi_{J'} \circ (\mathrm{id}, t, \mathrm{id})$ and $\sigma(\iota(x)) \circ \varpi_{J'} = \varpi_{J'} \circ (\mathrm{id}, \sigma(x), \mathrm{id})$ for $x \in \mathscr{S}(S^1 \times I)$.

By Lemma 4.2, for any $x \in \mathscr{S}(\Sigma^1, (J \cap \partial\Sigma^1) \cup \iota(J' \times \{1\}))$ and $y \in \mathscr{S}(\Sigma^2, (J \cap \partial\Sigma^2) \cup \iota(J' \times \{0\}))$, we obtain

$$(t_c - \mathrm{id})^{2n+m}(\varpi_{J'}(x, \mathrm{id}_{J'}, y)) = \varpi_{J'}(x, (t - \mathrm{id})^{2n+m}(\mathrm{id}_{J'}), y) \in F^n\mathscr{S}(\Sigma, J),$$

$$\begin{aligned}\sigma(L(c))(\varpi_{J'}(x, \mathrm{id}_{J'}, y)) &= \varpi_{J'}(x, \sigma(L(c_l))(\mathrm{id}_{J'}), y) \\ &= \varpi_{J'}(x, \log(t)(\mathrm{id}_{J'}), y) = \log(t_c)(\varpi_{J'}(x, \mathrm{id}_{J'}, y)),\end{aligned}$$

where $L(c) \stackrel{\text{def.}}{=} \dfrac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\dfrac{c}{2}\right)\right)^2$ for any simple closed curve c. This proves the theorem in the case that c is a separating simple closed curve.

We assume that $\Sigma \backslash \iota(S^1 \times (0,1))$ is a connected surface Σ^1. For a finite set $J' = \left\{\dfrac{1}{2m}, \dots, \dfrac{m}{2m}\right\} \subset S^1$, we consider the bilinear map

$$\varpi_{J'} : \mathscr{S}(\Sigma^1, J \cup \iota(J' \times \{0,1\})) \times \mathscr{S}(S^1 \times I, J' \times \{0,1\}) \to \mathscr{S}(\Sigma, J)$$

defined by $\varpi_{J'}([T_1],[T_2]) = [T_1 T_2]$ for $T_1 \in \mathscr{T}(\Sigma', J \cup \iota(J' \times \{0,1\}))$ and $T_2 \in \mathscr{T}(S^1 \times I, J' \times \{0,1\})$. Here we denote by $T_1 T_2$ the tangle presented by $d_1 \cup \iota(d_2)$, where d_1 and d_2 present T_1 and T_2, respectively. We remark that $d_1 \cup \iota(d_2)$ must be smoothed out in the neighborhood of $\iota(S^1 \times \{0,1\})$. By Theorem 3.3, the set

$$\varpi_{J'}(\mathscr{S}(\Sigma^1, J \cup \iota(J' \times \{0,1\})) \times \{\mathrm{id}_{J'}\})$$

generates $\mathscr{S}(\Sigma, J)$ as a $\mathbf{Q}[A, A^{-1}]$-module, where we set $\mathrm{id}_{J'} \stackrel{\text{def.}}{=} r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0$. In order to show the theorem, we use the following.

- The map $\varpi_{J'}$ preserves the filtrations, in other words,

$$\varpi_{J'}(\mathscr{S}(\Sigma^1, J \cup \iota(J' \times \{0,1\})) \times F^n\mathscr{S}(S^1 \times I, J' \times \{0,1\})) \subset F^n\mathscr{S}(\Sigma, J).$$

- We have $t_c \circ \varpi_{J'} = \varpi_{J'} \circ (\mathrm{id}, t)$ and $\sigma(\iota(x)) \circ \varpi_{J'} = \varpi_{J'} \circ (\mathrm{id}, \sigma(x))$ for $x \in \mathscr{S}(S^1 \times I)$.

By Lemma 4.2, for any $x \in \mathscr{S}(\Sigma^1, J \cup \iota(J' \times \{0,1\}))$, we obtain

$$(t_c - \mathrm{id})^{2n+m}(\varpi_{J'}(x, \mathrm{id}_{J'})) = \varpi_{J'}(x, (t - \mathrm{id})^{2n+m}(\mathrm{id}_{J'})) \in F^n\mathscr{S}(\Sigma, J),$$

$$\begin{aligned}\sigma(L(c))(\varpi_{J'}(x, \mathrm{id}_{J'})) &= \varpi_{J'}(x, \sigma(L(c_l))(\mathrm{id}_{J'})) \\ &= \varpi_{J'}(x, \log(t)(\mathrm{id}_{J'})) = \log(t_c)(\varpi_{J'}(x, \mathrm{id}_{J'})),\end{aligned}$$

where $L(c) \stackrel{\text{def.}}{=} \dfrac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\dfrac{c}{2}\right)\right)^2$ for any simple closed curve c. This proves the theorem in the case that c is not a separating simple closed curve.

This proves the theorem. $\square$

In order to prove Lemma 4.2, we need a $\mathbf{Q}[A, A^{-1}]$-bilinear map $(\cdot)(\cdot) : \mathscr{S}(S^1 \times I, J \times \{0,1\}) \times \mathscr{S}(S^1 \times I, J \times \{0,1\}) \to \mathscr{S}(S^1 \times I, J \times \{0,1\})$ defined by $[T_1][T_2] = [T_1 T_2]$ for any finite subset $J \subset S^1$. Here we denote by $T_1 T_2$ the tangle presented by $\mu_1(D_1) \cup \mu_2(D_2)$ where we choose tangle diagrams D_1 and D_2 presenting T_1 and T_2, respectively, and define embedding maps μ_1 and $\mu_2 : S^1 \times I \to S^1 \times I$ by $\mu_1(\theta, t) = \left(\theta, \frac{t+1}{2}\right)$ and $\mu_2(\theta, t) = \left(\theta, \frac{t}{2}\right)$. We remark that $\mu_1(D_1) \cup \mu_2(D_2)$ must be smoothed out in the neighborhood of c_l. By this bilinear map, we define the product of $\mathscr{S}(S^1 \times I, J \times \{0,1\})$. By definition we have $(F^k \mathscr{S}(S^1 \times I, J \times \{0,1\}))(F^l \mathscr{S}(S^1 \times I, J \times \{0,1\})) \subset F^{k+l}\mathscr{S}(S^1 \times I, J \times \{0,1\})$ for k and $l \in \mathbf{Z}_{\geq 0}$.

At first, we prove the part (1) of Lemma 4.2.

Proof of Lemma 4.2(1). For $i = 1, \dots, m$, we set $x_i \overset{\text{def.}}{=} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes t(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0$ and by $x_i^{-1} \overset{\text{def.}}{=} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes t^{-1}(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0$. We simply denote $\mathrm{id} \overset{\text{def.}}{=} r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0$. We remark that $x_i x_i^{-1} = \mathrm{id}$.

We have

$$(t-1)^{2n+m}(\mathrm{id}) = (x_1 x_2 \cdots x_m - \mathrm{id})^{2n+m} = \left(\sum_{i=1}^{m} x_1 x_2 \cdots x_{i-1}(x_i - \mathrm{id})\right)^{2n+m}.$$

Since $(t(r_i^0) - r_i^0)^2 = -(l+2)t(r_i^0) + (A+1)t^2(r_i^0) + (A^{-1}+1)r_i^0 \in F^1\mathscr{S}(S^1 \times I, p_i \times \{0,1\})$, we have $(t-1)^{2n+m}(\mathrm{id}) \in F^n \mathscr{S}(S^1 \times I, \{p_1, \dots, p_m\} \times \{0,1\})$. This proves the part (1) of the lemma. $\square$

To prove Lemma 4.2 (2), we need the following lemma.

Lemma 4.3. *We have* $\sigma\left(\frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_i^0) = \log(t)(r_i^0)$ *for* $i = 1, \dots, m$.

For $n = 0, 1, \dots,$ we define the Chebyshev polynomial $T_n(X) \in \mathbf{Z}[X]$ by setting $T_0(X) = 2$, $T_1(X) = X$ and $T_{n+1}(X) = XT_n(X) - T_{n-1}(X)$. We set $(T+1)_n(X) \overset{\text{def.}}{=} \sum_{i=0}^n \frac{n!}{i!(n-i)!} T_i(X)$. It is obvious that $(T+1)_n(q+q^{-1}) = (q+1)^n + (q^{-1}+1)^n$. Since

$$(T+1)_n(x) = (\sqrt{x+2})^n \left(\left(\frac{\sqrt{x+2} - \sqrt{x-2}}{2}\right)^n + \left(\frac{\sqrt{x+2} + \sqrt{x-2}}{2}\right)^n\right),$$

we have the following proposition.

Proposition 4.4. *We have*

$$(T+1)_{2n}(x) \in (x+2)^n \mathbf{Z}[x],$$
$$(T+1)_{2n+1}(x) \in (x+2)^n \mathbf{Z}[x].$$

We define a sequence $\{a_n\}_{n\geq 2}$ by $(\log(-x))^2 = \sum_{n=2}^{\infty} a_n(x+1)^n \in \mathbf{Q}[[x+1]]$. Since

$$\begin{aligned}\sum_{n=2}^{\infty} a_n(T+1)_n(q+q^{-1}) &= \sum_{n=2}^{\infty} a_n((q+1)^n + (q^{-1}+1)^n)\\ &= (\log(-q))^2 + (\log(-q^{-1}))^2 = 2(\log(-q))^2\\ &= 2\left(\operatorname{arccosh}\left(\frac{-q-q^{-1}}{2}\right)\right)^2,\end{aligned}$$

we obtain $2\left(\operatorname{arccosh}\left(-\frac{X}{2}\right)\right)^2 = (\log(-T))^2(X) \stackrel{\text{def.}}{=} \sum_{n=2}^{\infty} a_n(T+1)_n(X) \in \mathbf{Q}[[X+2]]$.

By Theorem 3.3, we have $\mathscr{S}(S^1 \times I, p_i \times \{0,1\}) = \mathbf{Q}[A^{\pm 1}, r^{\pm 1}]$ as a commutative algebra, where $1 \stackrel{\text{def.}}{=} r_i^0$ and $r^n \stackrel{\text{def.}}{=} t^n(r_i^0)$ for any $n \in \mathbf{Z}$. Since $(A+1)^i(r-1)^{2j+1} \subset F^{i+j}\mathscr{S}(S^1 \times I, p_i \times \{0,1\})$, we have $\hat{\mathscr{S}}(S^1 \times I, p_i \times \{0,1\}) = \mathbf{Q}[[A+1, r-1]]$.

Proof of Lemma 4.3. We have

$$\begin{aligned}&\sigma\left(\frac{-A+A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_i^0)\\ &\quad = \sigma\left(\frac{-A+A^{-1}}{8\log(-A)}(\log(-T))^2(l)\right)(r_i^0)\\ &\quad = \frac{1}{8\log(-A)}((\log(-T))^2(l)(r_i^0) - (r_i^0)(\log(-T))^2(l)).\\ &\quad = \frac{1}{8\log(-A)}\sum_{k=2}^{\infty}(a_k(T+1)_k(l)r_i^0 - a_k r_i^0(T+1)_k(l)).\end{aligned}$$

Since, for any $n \in \mathbf{Z}_{\geq 0}$,

$$l^n r_i^0 = (Ar + A^{-1}r^{-1})^n, \quad r_i^0 l^n = (A^{-1}r + Ar^{-1})^n$$

we have

$$\begin{aligned}&\frac{1}{8\log(-A)}\sum_{k=2}^{\infty}(a_k(T+1)_k(l)r^0 - a_k r^0(T+1)_k(l))\\ &\quad = \frac{1}{8\log(-A)}\sum_{k=2}^{\infty}(a_k(Ar^1+1)^k + a_k(A^{-1}r^{-1}+1)^k\\ &\qquad\qquad - a_k(A^{-1}r^1+1)^k - a_k(Ar^{-1}+1)^k).\end{aligned}$$

$$= \frac{1}{8\log(-A)}((\log(-Ar))^2 + (\log(-A^{-1}r^{-1}))^2 - (\log(-A^{-1}r))^2 - (\log(-Ar^{-1}))^2)$$
$$= \frac{1}{4\log(-A)}((\log(-Ar))^2 - (\log(-Ar^{-1}))^2)$$
$$= \frac{1}{4\log(-A)}((\log(-A) + \log(r))^2 - (-\log(-A) + \log(r))^2)$$
$$= \log(r) = \log(t)(r_i^0).$$

This proves the lemma. □

Proof of Lemma 4.2(2). We have

$$\sigma\left(\frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0)$$
$$= \sum_{i=1}^{m} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes \sigma\left(\frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0.$$

Using Lemma 4.3, we obtain

$$\sum_{i=1}^{m} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes \sigma\left(\frac{-A + A^{-1}}{4\log(-A)}\left(\operatorname{arccosh}\left(-\frac{l}{2}\right)\right)^2\right)(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0$$
$$= \sum_{i=1}^{m} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes \log(t)(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0 = \sum_{i=1}^{m} \log(x_i).$$

Since $x_i x_j = x_j x_i$, we obtain

$$\sum_{i=1}^{m} \log(x_i) = \log(x_1 x_2 \cdots x_m) = \log(t)(\mathrm{id}).$$

This proves the lemma. □

Remark 4.5. Let Σ be a compact connected oriented surface with non-empty connected boundary and let $\mathscr{I}(\Sigma) \subset \mathscr{M}(\Sigma)$ be the Torelli group of Σ. Then we have

(1) $\mathscr{I}(\Sigma) \subset \check{\mathscr{M}}(\Sigma)$.
(2) For any $\xi \in \mathscr{I}(\Sigma)$, there exists $x_\xi \in \hat{\mathscr{S}}(\Sigma)$ satisfying that x_ξ is a skein representative of $\xi \in \mathscr{I}(\Sigma) \subset \check{\mathscr{M}}(\Sigma)$ by $((\mathscr{S}(\Sigma), \{F^n\mathscr{S}(\Sigma)\}_{n\geq 0}), \Theta(\Sigma))$ in the sense of Definition 3.14.

The proof will appear in [11].

5. Filtrations

5.1. The filtrations depend only on the underlying 3-manifold. In this subsection, we prove the following theorem. The proof of the theorem is analogous to that of [2] Proposition 6.10.

THEOREM 5.1. *Let Σ and Σ' be two compact connected oriented surfaces, J a finite subset of $\partial\Sigma$ and J' a finite subset of $\partial\Sigma'$ such that there exists a diffeomorphism $\xi : (\Sigma \times I, J \times I) \to (\Sigma' \times I, J' \times I)$. Then we have $\xi(F^n\mathscr{S}(\Sigma, J)) = F^n\mathscr{S}(\Sigma', J')$ for $n \geq 0$.*

To prove it, we need new filtrations of the Kauffman bracket skein modules.

Let $\mathbf{Q}[A, A^{-1}]\mathscr{T}(\Sigma, J)$ be the free module with basis $\mathscr{T}(\Sigma, J)$ over $\mathbf{Q}[A, A^{-1}]$ and $\langle\cdot\rangle$ the natural surjection $\mathbf{Q}[A, A^{-1}]\mathscr{T}(\Sigma, J) \to \mathscr{S}(\Sigma, J)$. For a tangle $T \in \mathscr{T}(\Sigma, J)$ and closed components $L_1, L_2, \dots, L_m$ of T, we define

$$\left(T, \bigcup_{i=1}^{m} L_i\right) \stackrel{\text{def.}}{=} \sum_{j=0}^{m} \sum_{\{i_1, i_2, \dots, i_j\} \subset \{1, 2, \dots, m\}} 2^{m-j} T' \cup \bigcup_{h=1}^{j} L_{i_h} \in \mathbf{Q}[A, A^{-1}]\mathscr{T}(\Sigma, J)$$

where $T' = T \backslash \bigcup_{i=1}^{m} L_i$. We set $F^{\star 0}\mathscr{S}(\Sigma, J) \stackrel{\text{def.}}{=} \mathscr{S}(\Sigma, J)$ and denote by $F^{\star n}\mathscr{S}(\Sigma, J)$ the $\mathbf{Q}[A, A^{-1}]$ submodule generated by $(A+1)F^{\star(n-1)}\mathscr{S}(\Sigma, J)$ and the subset of $\mathscr{S}(\Sigma, J)$ consisting of all elements $\langle(T, \bigcup_{i=1}^{n} K_i)\rangle \in \mathscr{S}(\Sigma, J)$ for $T \in \mathscr{T}(\Sigma, J)$ and closed components $K_1, K_2, \dots, K_n$ of T for $n \geq 1$. Similarly, the filtration $\{F^{\star n}\mathscr{S}(\Sigma', J')\}_{n \geq 0}$ is defined as $\{F^{\star n}\mathscr{S}(\Sigma, J)\}_{n \geq 0}$.

LEMMA 5.2. *Let $E_1 : D' \times I \sqcup D \times I \to \Sigma \times I$ and $E_2 : D' \times I \sqcup D \times I \to \Sigma \times I$ be elements of $\mathscr{E}(\Sigma, J)$ satisfying the following, where D' and $D = \bigsqcup_{i=1}^{n} (S^1)_i$ are 1-dimensional manifolds with $\#(\partial D') = \#J$.*

- *The embeddings E_1 and E_2 are generic, which means satisfying the conditions in Definition 2.2.*
- *The images $E_1(D' \times I \sqcup D \times I)$ and $E_2(D' \times I \sqcup D \times I)$ are identical except for $\Delta \times I$, where they differ as shown in Figure 11 and Figure 12, respectively. Here, Δ is a closed disk in Σ.*

Then, we have

$$\langle\langle E_1\rangle, \langle E_{1|D\times I}\rangle\rangle - \langle\langle E_2\rangle, \langle E_{2|D\times I}\rangle\rangle \in (A+1)F^{\star(n-1)}\mathscr{S}(\Sigma, J).$$

Proof. There exists two cases.

(1) For some $j \neq k$, $\Delta \times I \cap E_1((S^1)_j \times I), \Delta \times I \cap E_1((S^1)_k \times I) \subsetneq \Delta \times I \cap E_1(D' \times I \sqcup D \times I) \subset E_1((S^1)_j \times I \cup (S^1)_k \times I)$.

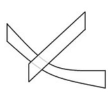

FIGURE 11. E_1

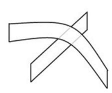

FIGURE 12. E_2

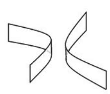

FIGURE 13. E_0

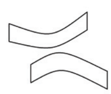

FIGURE 14. E_∞

(2) For some j, $\Delta \times I \cap E_1(D' \times I \sqcup D \times I) \subset E_1(D' \sqcup ((S^1)_j) \times I)$.

(1) Let $E_0 : D' \times I \sqcup D_0 \times I \to \Sigma \times I$ and $E_\infty : D' \times I \sqcup D_\infty \times I \to \Sigma \times I$ be two elements of $\mathscr{E}(\Sigma, J)$ satisfying the following.

- $E_{0|D' \times I} = E_{\infty|D' \times I} = E_{1|D' \times I} = E_{2|D' \times I}$.
- $E_0(D' \times I \sqcup D_0 \times I)$ and $E_\infty(D' \times I \sqcup D_\infty \times I)$ equal $E_1(D' \times I \sqcup D \times I)$ except $\Delta \times I$, where they are shown in Figure 13 and Figure 14, respectively.

Using Lemma 3.6, we obtain

$$\begin{aligned}&\langle\langle E_1\rangle, \langle E_{1|D\times I}\rangle\rangle - \langle\langle E_2\rangle, \langle E_{2|D\times I}\rangle\rangle \\ &\quad = (A - A^{-1})(\langle\langle E_0\rangle, \langle E_{0|D_0\times I}\rangle\rangle - \langle\langle E_\infty\rangle, \langle E_{\infty|D_\infty\times I}\rangle\rangle.\end{aligned}$$

Since $\#\pi_0(D_0) = \#\pi_0(D_\infty) = n - 1$, we have

$$\langle\langle E_1\rangle, \langle E_{1|D\times I}\rangle\rangle - \langle\langle E_2\rangle, \langle E_{2|D\times I}\rangle\rangle \in (A+1)F^{\star(n-1)}\mathscr{S}(\Sigma, J).$$

(2) We denote $D\backslash(S^1)_j$ by D''. Let $E_0 : D_0' \times I \cup D'' \to \Sigma \times I$ and $E_\infty : D_\infty' \times I \cup D'' \to \Sigma \times I$ be two elements of $\mathscr{E}(\Sigma, J)$ satisfying the following.

- $E_{0|D'' \times I} = E_{\infty|D'' \times I} = E_{1|D'' \times I} = E_{2|D'' \times I}$.
- $E_0(D_0' \times I \sqcup D'' \times I)$ and $E_\infty(D_\infty' \times I \sqcup D'' \times I)$ equal $E_1(D' \times I \sqcup D \times I)$ except $\Delta \times I$, where they are shown in Figure 13 and Figure 14, respectively.

Using Lemma 3.6, we obtain

$$\begin{aligned}&\langle\langle E_1\rangle, \langle E_{1|D\times I}\rangle\rangle - \langle\langle E_2\rangle, \langle E_{2|D\times I}\rangle\rangle \\ &\quad = (A - A^{-1})(\langle\langle E_0\rangle, \langle E_{0|D''\times I}\rangle\rangle - \langle\langle E_\infty\rangle, \langle E_{\infty|D''\times I}\rangle\rangle.\end{aligned}$$

Since $\#\pi_0(D'') = n - 1$, we have

$$\langle\langle E_1\rangle, \langle E_{1|D\times I}\rangle\rangle - \langle\langle E_2\rangle, \langle E_{2|D\times I}\rangle\rangle \in (A+1)F^{\star(n-1)}\mathscr{S}(\Sigma, J).$$

This proves the lemma. □

Lemma 5.3. *Let Σ be a compact connected oriented surface, and J a finite subset of $\partial\Sigma$. We have $F^n\mathscr{S}(\Sigma, J) = F^{\star n}\mathscr{S}(\Sigma, J)$ for any non-negative integer n. Furthermore, we have*

$$\sum_{L' \subset L} (-1)^{|L'|}(-2)^{-|L'|}[L'] \in (\ker \varepsilon)^n$$

for any link L in $\Sigma \times I$ having components more than n, where the sum is over all sublinks $L' \subset L$ including the empty link and we denote by $|L|$ the number of components of L. In other words, for any link L in $\Sigma \times I$, $(-1)^{|L'|}(-2)^{-|L'|}[L'] \bmod (\ker \varepsilon)^n$ is a finite type invariant of order $n+1$ in the sense of Le [5] (3.2).

Proof. We prove the lemma by induction on n. If $n = 0$, we have $F^{\star 0}\mathscr{S}(\Sigma, J) = F^0\mathscr{S}(\Sigma, J) = \mathscr{S}(\Sigma, J)$. We assume that $n > 0$ and $F^{n-1}\mathscr{S}(\Sigma, J) =$

$F^{\star(n-1)}\mathscr{S}(\Sigma, J)$. For any tangle $T \in \mathscr{T}(\Sigma, J)$ and knots $K_1, K_2, \dots, K_n \in \mathscr{T}(\Sigma)$, we have

$$(\langle K_1\rangle + 2)(\langle K_2\rangle + 2)\cdots(\langle K_n\rangle + 2)\langle T\rangle$$
$$= \langle(K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n \boxtimes T, K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n)\rangle.$$

Hence we have $F^{\star n}\mathscr{S}(\Sigma, J) \supset F^n\mathscr{S}(\Sigma, J)$. Using Lemma 5.2 repeatedly, for any tangle T and closed components $K_1, K_2, \dots, K_n$ of T, we have

$$\langle(K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n \boxtimes T', K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n)\rangle - \left\langle\left(T, \bigcup_{i=1}^{n} K_i\right)\right\rangle$$
$$\in (A - A^{-1})F^{\star(n-1)}\mathscr{S}(\Sigma, J) = (A - A^{-1})F^{n-1}\mathscr{S}(\Sigma, J) \subset F^n\mathscr{S}(\Sigma, J).$$

Here we set $T' \overset{\text{def.}}{=} T\backslash(\bigcup_{i=1}^{n} K_i)$. Since $\langle(K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n \boxtimes T', K_1 \boxtimes K_2 \boxtimes \cdots \boxtimes K_n)\rangle = (\langle K_1\rangle + 2)(\langle K_2\rangle + 2)\cdots(\langle K_n\rangle + 2)\langle T'\rangle \in F^n\mathscr{S}(\Sigma, J)$, we have $\langle(T, \bigcup_{i=1}^{n} K_i)\rangle \in F^n\mathscr{S}(\Sigma, J)$. If $J = \emptyset$, by definition, we have

$$\sum_{L' \subset L} (-1)^{|L'|}(-2)^{-|L'|}[L'] \in F^{\star n}\mathscr{S}(\Sigma) = F^n\mathscr{S}(\Sigma) = (\ker \varepsilon)^n.$$

This proves the theorem. □

Proof of Theorem 5.1. By definition, we have $\xi(F^{\star n}\mathscr{S}(\Sigma, J)) = F^{\star n}\mathscr{S}(\Sigma', J')$. Using Lemma 5.3, we have $\xi(F^n\mathscr{S}(\Sigma, J)) = \xi(F^{\star n}\mathscr{S}(\Sigma, J)) = F^{\star n}\mathscr{S}(\Sigma', J') = F^n\mathscr{S}(\Sigma', J')$. This proves the theorem. □

In this paper, we define the Kauffman bracket $\mathscr{K}$: {unoriented framed links in S^3} $\to \mathbf{Q}[A, A^{-1}]$ by $\langle L\rangle = \mathscr{K}(L)\langle\emptyset\rangle \in \mathscr{S}(I \times I)$ for $L \in$ {links in S^3} $= \mathscr{T}(I \times I)$. For an unoriented framed link L in S^3 and components $K_1, K_2, \dots, K_m$ of L, we define $\mathscr{K}(L, \bigcup_{i=1}^{m} K_i) \overset{\text{def.}}{=} \sum_{j=0}^{m} \sum_{\{i_1, i_2, \dots, i_j\} \subset \{1,2,\dots,m\}} 2^{m-j}\mathscr{K}(L' \cup \bigcup_{h=1}^{j} K_{i_h})$. Here we set $L' \overset{\text{def.}}{=} L\backslash \bigcup_{i=1}^{m} K_i$.
Using Lemma 5.3, we have the following corollary.

Corollary 5.4. *For an unoriented framed link L in S^3 and some components $K_1, K_2, \dots, K_m$ of L, we have $\mathscr{K}(L, \bigcup_{i=1}^{m} K_i) \in (A + 1)^m\mathbf{Q}[A, A^{-1}]$.*

Proof. Since $\ker \varepsilon = (A + 1)^m\mathbf{Q}[A, A^{-1}]\langle\emptyset\rangle$, we have $F^m\mathscr{S}(I \times I) = (A + 1)^m\mathbf{Q}[A, A^{-1}]\langle\emptyset\rangle$. By Lemma 5.3, we have $(A + 1)^m\mathbf{Q}[A, A^{-1}]\langle\emptyset\rangle = F^m\mathscr{S}(I \times I) = F^{\star m}\mathscr{S}(I \times I)$. We obtain $\langle(L, \bigcup_{i=1}^{m} K_i)\rangle = \mathscr{K}(L, \bigcup_{i=1}^{m} K_i)\langle\emptyset\rangle$. This proves the corollary. □

5.2. Filtrations are Hausdorff. In this subsection, we prove the following.

Theorem 5.5. *Let Σ be a compact connected oriented surface with non-empty boundary and J a finite subset of $\partial\Sigma$. We have $\bigcap_{n=1}^{\infty} F^n\mathscr{S}(\Sigma, J) = \{0\}$. In other words, the natural homomorphism $\mathscr{S}(\Sigma, J) \to \hat{\mathscr{S}}(\Sigma, J)$ is injective.*

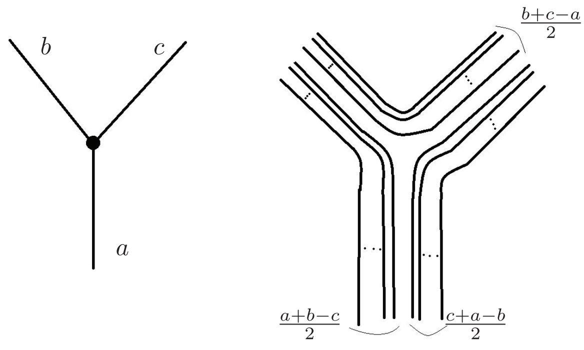

FIGURE 15. $(a,b,c) \in \mathbf{V}$

We denote by $\mathbf{V}$ the subset of $\mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0}$ consisting of all triples $(a,b,c) \in \mathbf{Z}^3_{\geq 0}$ satisfying $a+b+c \in 2\mathbf{Z}_{\geq 0}$ and $|b-c| \leq a \leq b+c$. For $(a,b,c) \in \mathbf{V}$, we denote the right figure in Figure 15 by the left figure in Figure 15.

Let $\Sigma_{0,g+1}$ be the surface $D^2 \backslash \coprod_{i=1}^{g} d_i$ where we denote by $D^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2+y^2 \leq 1\}$ and by d_i the open disk $\left\{(x,y) \in \mathbf{R}^2 \,\middle|\, x^2 + \left(y+1-\frac{i}{g+1}\right)^2 < \left(\frac{1}{4g+4}\right)^2\right\}$ for $1 \leq i \leq g$. We denote by $\mathbf{V}(g)$ the set consisting of all $(i_1, i_2, \dots, i_{3g-3})$ which satisfies

$$(i_{3j-3}, i_{3j-2}, i_{3j-1}), (i_{3j-1}, i_{3j}, i_{3j+1}) \in \mathbf{V}$$

for $j = 1, \dots, g-1$. Here we define $i_0 \stackrel{\text{def.}}{=} i_1$ and $i_{3g-2} \stackrel{\text{def.}}{=} i_{3g-3}$. We denote by $\lambda(g,0)(i_1, i_2, \dots, i_{3g-3})$ the element of $\mathscr{T}(\Sigma_{0,g+1})$ presented by Figure 16 for $(i_1, i_2, \dots, i_{3g-3}) \in \mathbf{V}(g)$.

Fix an orientation preserving embedding $e_3 : D^2 \times I \to S^3$ and a diffeomorphism $e_4 : \Sigma_{0,g+1} \times I \to \overline{S^3 \backslash e_3(\Sigma_{0,g+1} \times I)}$ where we denote the closure of $S^3 \backslash e_3(\Sigma_{0,g+1} \times I)$ by $\overline{S^3 \backslash e_3(\Sigma_{0,g+1} \times I)}$. Then we define a bilinear map $(\cdot,\cdot) : \mathscr{S}(\Sigma_{0,g+1}) \times \mathscr{S}(\Sigma_{0,g+1}) \to \mathbf{Q}[A, A^{-1}]$ by $(\langle L_1 \rangle, \langle L_2 \rangle) = \mathscr{K}(e_3(L_1) \cup e_4(L_2))$ for L_1 and $L_2 \in \mathscr{T}(\Sigma_{0,g+1})$. Here we simply denote $e_3 \stackrel{\text{def.}}{=} e_{3|\Sigma_{0,g+1} \times I}$. The bilinear map induces $(\cdot,\cdot) : \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \times \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \to \mathbf{C}[A, A^{-1}]$. Here we denote by $\mathbf{C}[A, A^{-1}]$ the ring of Laurent polynomials over $\mathbf{C}$. For a primitive $2r$-th root of unity γ, the bilinear map induces $(\cdot,\cdot) : \mathscr{S}^\gamma(\Sigma_{0,g+1}) \times \mathscr{S}^\gamma(\Sigma_{0,g+1}) \to \mathbf{C}$ where we set $\mathscr{S}^\gamma(\Sigma_{0,g+1}) \stackrel{\text{def.}}{=} \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) / (A-\gamma)\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1})$. The bilinear map induces the linear map $\psi : \mathscr{S}(\Sigma_{0,g+1}) \to \mathrm{Hom}_{\mathbf{Q}[A,A^{-1}]}(\mathscr{S}(\Sigma_{0,g+1}), \mathbf{Q}[A, A^{-1}])$ by $v \mapsto (u \mapsto (v,u))$. It induces the linear maps

$$\psi : \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \to \mathrm{Hom}_{\mathbf{C}[A,A^{-1}]}(\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}), \mathbf{C}[A, A^{-1}]),$$

$$\psi : \mathscr{S}^\gamma(\Sigma_{0,g+1}) \to \mathrm{Hom}_{\mathbf{C}}(\mathscr{S}^\gamma(\Sigma_{0,g+1}), \mathbf{C}).$$

We denote by $\cdot|_{A=\gamma}$ the quotient map $\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \to \mathscr{S}^\gamma(\Sigma_{0,g+1})$.

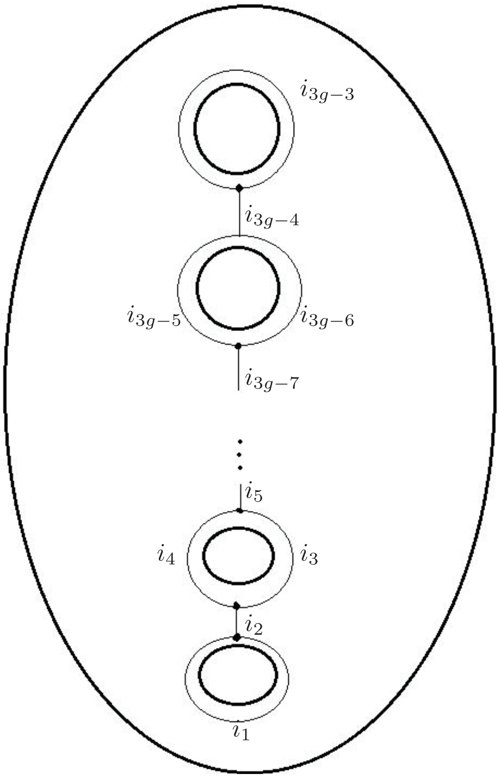

FIGURE 16. $\lambda(g,0)(i_1, i_2, \ldots, i_{3g-3})$

For a surface Σ and any finite subset $J \subset \partial\Sigma$, we recall that we denote by $\mathscr{T}_0(\Sigma, J)$ the set of isotopy classes of 1-dimensional submanifolds of Σ with boundary J and no inessential components and that $\mathscr{S}(\Sigma, J)$ is freely generated by $\mathscr{T}_0(\Sigma, J)$ as a $\mathbf{Q}[A, A^{-1}]$-module.

THEOREM 5.6 (Lickorish [6], P.347, Theorem). (1) *The map* $\lambda(g,0) : \mathbf{V}(g) \to \mathscr{T}_0(\Sigma_{0,g+1})$ *is bijective.*

(2) *For a primitive* $4r$*-th root of unity* γ, $\mathscr{S}^{\gamma}(\Sigma_{0,g+1})/\ker\psi$ *is a free* $\mathbf{C}$*-module with basis*

$$\{\lambda(g,0)(i_1, \ldots, i_{3g-3}) \mid (i_{3j-3}, i_{3j-2}, i_{3j-1}), (i_{3j-1}.i_{3j}, i_{3j+1}) \in \mathbf{V},$$
$$2r-4 \geq i_{3j-3} + i_{3j-2} + i_{3j-1}, 2r-4 \geq i_{3j-1} + i_{3j} + i_{3j+1}\}.$$

We remark that Lickorish gave another basis in [6]. Using Theorem in [6], we have $\lambda(g,0)$ is injective. It is proved in a similar way to the proof of Lemma 5.10 in this paper that $\lambda(g,0)$ is surjective.

LEMMA 5.7. (1) *The* $\mathbf{C}[A, A^{-1}]$*-module homomorphism* $\psi : \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \to \mathrm{Hom}_{\mathbf{C}[A,A^{-1}]}(\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}), \mathbf{C}[A, A^{-1}])$ *is injective.*

(2) *The* $\mathbf{Q}[A, A^{-1}]$*-module homomorphism*

$$\psi : \mathscr{S}(\Sigma_{0,g+1}) \to \mathrm{Hom}_{\mathbf{Q}[A,A^{-1}]}(\mathscr{S}(\Sigma_{0,g+1}), \mathbf{Q}[A, A^{-1}])$$

is injective.

Proof. Let x be an element of $\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1}) \backslash \{0\}$. Using Theorem 5.6, we have $\psi(x|_{A=\gamma}) \neq 0$ for some primitive $4r$-th root of unity γ. In other words, we have $(x|_{A=\gamma}, y) \neq 0$ for some $y \in \mathscr{S}^{\gamma}(\Sigma_{0,g+1})$. We regard y as an element of $\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1})$ by $\mathscr{S}^{\gamma}(\Sigma_{0,g+1}) = \mathbf{C}\mathscr{T}_0(\Sigma_{0,g+1}) \hookrightarrow \mathbf{C}[A, A^{-1}]\mathscr{T}_0(\Sigma_{0,g+1}) = \mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1})$. Since $(x, y)|_{A=\gamma} = (x|_{A=\gamma}, y) \neq 0$, we have $(x, y) \neq 0$. This proves (1).

Let x be an element of $\mathscr{S}(\Sigma_{0,g+1}) \backslash \{0\}$. We regard x as an element of $\mathbf{C} \otimes \mathscr{S}(\Sigma_{0,g+1})$. By (1), we have $(x, \sum_{j=1}^{m}(a_j + b_j\sqrt{-1})c_j) \neq 0$ for some a_j and $b_j \in \mathbf{R}$ and $c_j \in \mathscr{T}_0(\Sigma_{0,g+1})$. Let k be an integer satisfying that the coefficient of A^k in $(x, \sum_{j=1}^{m}(a_j + b_j\sqrt{-1})c_j)$ is not 0. We denote by $\omega(u_1, \ldots, u_m)$ the coefficient of A^k in $(x, \sum_{j=1}^{m} u_j c_j)$ for $u_j \in \mathbf{R}$. Then, $\omega : \mathbf{R}^m \to \mathbf{R}$, $(u_1, \ldots, u_m) \mapsto \omega(u_1, \ldots, u_m)$ is a linear map. Since ω is linear, ω is continuous. By definition, we have $\omega(a_1, \ldots, a_m) \neq 0$ or $\omega(b_1, \ldots, b_m) \neq 0$. Using the density of $\mathbf{Q}$ in $\mathbf{R}$, we have $\omega(q_1, \ldots, q_m) \neq 0$ for some $q_1, \ldots, q_m \in \mathbf{Q}$. Hence we obtain $(x, \sum_{j=1}^{m} q_j c_j) \neq 0$. This proves (2). □

To prove Theorem 5.5 in the case $J = \emptyset$, we need the following lemma.

LEMMA 5.8. *Let* Σ *be a compact connected oriented surface with non-empty boundary. We have*

$$\psi(F^k\mathscr{S}(\Sigma)) = \psi(F^{\star k}\mathscr{S}(\Sigma)) \subset (A+1)^k \,\mathrm{Hom}_{\mathbf{Q}[A,A^{-1}]}(\mathscr{S}(\Sigma), \mathbf{Q}[A, A^{-1}])$$

for $k \in \mathbf{Z}_{\geq 0}$.

Proof. By Theorem 5.1, it is sufficient to prove the lemma in the case $\Sigma = \Sigma_{0,g+1}$. Let L and L' be links in $\Sigma_{0,g+1} \times I$ and $K_1, \ldots, K_k$ components of L. By Corollary 5.4, we have $\mathscr{K}(e_3(L) \cup e_4(L'), \bigcup_{i=1}^{k} e_3(K_i)) \in (A+1)^k\mathbf{Q}[A, A^{-1}]$. This proves the lemma. □

LEMMA 5.9 (A special case of Theorem 5.5). *Let* Σ *be a compact connected oriented surface with non-empty boundary. We have* $\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma) = \{0\}$.

Proof. By Theorem 5.1, it is sufficient to prove the lemma in the case $\Sigma = \Sigma_{0,g+1}$. By Lemma 5.8, we have

$$\psi\left(\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma_{0,g+1})\right) \subset \bigcap_{n=0}^{\infty}(A+1)^n \,\mathrm{Hom}_{\mathbf{Q}[A,A^{-1}]}(\mathscr{S}(\Sigma), \mathbf{Q}[A, A^{-1}]) = \{0\}.$$

Since ψ is injective, we have $\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma) = \{0\}$. This proves the lemma. □

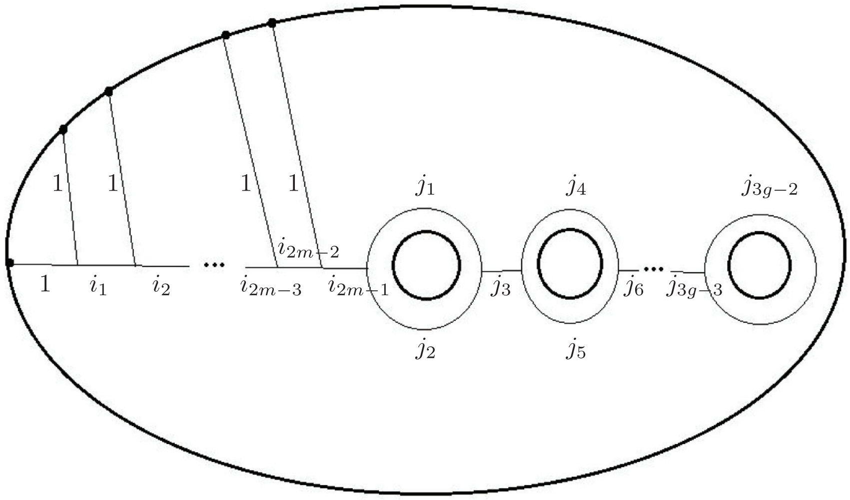

FIGURE 17. $\lambda(g,m)(i_1,i_2,\ldots,i_{2m-1},j_1,j_2,\ldots,j_{3g-2})$

For $g \geq 1$ and $m \geq 1$, we denote by $\mathbf{V}(g,m)$ the set consisting of all $(i_1, i_2, \ldots, i_{2m-1}, j_1, j_2, \ldots, j_{3g-2})$ which satisfy $(i_{k-1}, i_k, 1) \in \mathbf{V}$ for $k = 1, \ldots, 2m-1$, $(j_{3k-2}, j_{3k-1}, j_{3k}), (j_{3k}, j_{3k+1}, j_{3k+2}) \in \mathbf{V}$ for $k = 1, \ldots, g-1$ and $(i_{2m-1}, j_1, j_2) \in \mathbf{V}$, where we denote $i_0 \stackrel{\text{def.}}{=} 1$ and $i_{3g-1} \stackrel{\text{def.}}{=} i_{3g-2}$. Let J be a finite subset of $\partial D^2 \subset \partial \Sigma_{0,g+1}$ satisfying $\# J = 2m$. Let $\lambda(g,m)(i_1,i_2,\ldots,i_{2m-1},j_1,j_2,\ldots,j_{3g-2})$ be the element of $\mathscr{T}_0(\Sigma_{0,g+1}, J)$ represented by a submanifold of $\Sigma_{0,g+1}$ presented by the diagram as in Figure 17 for any $(i_1,i_2,\ldots,i_{2m-1},j_1,j_2,\ldots,j_{3g-2}) \in \mathbf{V}(g,m)$.

LEMMA 5.10. *For $g \geq 1$ and $m \geq 1$, $\lambda(g,m) : \mathbf{V}(g,m) \to \mathscr{T}_0(\Sigma_{0,g+1}, J)$ is surjective.*

Proof. We use the following proposition. For any $L \in \mathscr{T}_0(\Sigma_{0,g+1}, J)$, there exists $\tilde{L}$ representing L and satisfying the following conditions for some n in Proposition 5.11. This proves the lemma. □

Let $\mathbf{I}_1, \ldots, \mathbf{I}_{2m-1}, \mathbf{J}_1, \ldots, \mathbf{J}_{3g-2}$ be one-dimensional submanifolds of $\Sigma_{0,g+1}$ as in Figure 18. We set $\mathbf{L} \stackrel{\text{def.}}{=} (\bigcup_{q=1}^{2m-1} \mathbf{I}_q) \cup (\bigcup_{r=1}^{3g-2} \mathbf{J}_r)$.

We prove the following proposition by induction on n.

PROPOSITION 5.11 (n). *Let $\tilde{L}$ be a one-dimensional submanifold of $\Sigma_{0,g+1}$ satisfying the following.*

- *There is no closed disk d in $\Sigma_{0,g+1}$ such that $\partial d \subset \tilde{L}$.*
- *We have $\partial \tilde{L} = J$.*
- *The intersections $\tilde{L} \cap \mathbf{L}$ consist of transverse double points.*

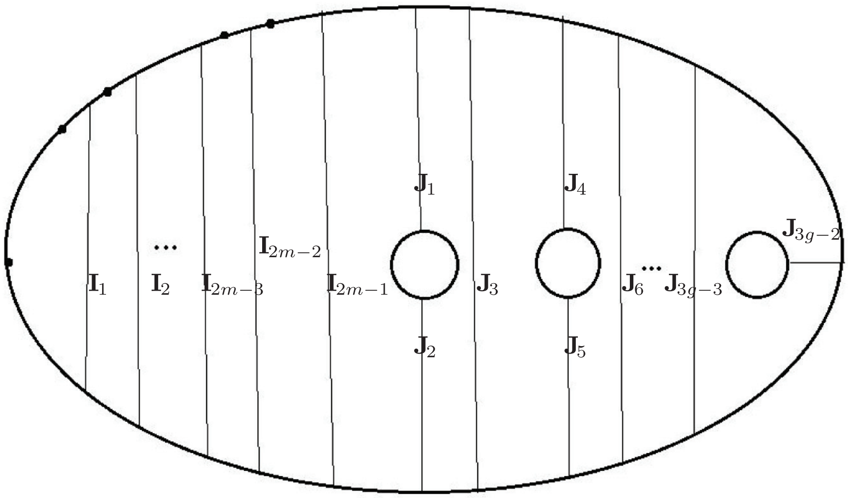

FIGURE 18. $\mathbf{L}$

We denote by $\alpha(\tilde{L})$ *the set consisting of all* $P \in \tilde{L} \cap \mathbf{L}$ *satisfying*

$$e(\{(x,y) \in \partial D^2 \mid x \geq 0\}) \subset \tilde{L},\ e(\{(x,y) \in \partial D^2 \mid x \leq 0\}) \subset \mathbf{L}, e(0.1) = P.$$

for some embedding $e : D^2 \to \Sigma_{0,g+1}$. *Then we have* $\#(\alpha(\tilde{L})) \leq n \Rightarrow L \in \lambda(g,m)(\mathbf{V}(g,m))$ *where we denote by* L *the isotopy class of* $\tilde{L}$.

Proof. By definition, we have Proposition 5.11 (0). We assume $n > 0$ and Proposition 5.11 $(n-1)$. Let $\tilde{L}$ be a one-dimensional submanifold of $\Sigma_{0,g+1}$ satisfying the above conditions and $\#\alpha(\tilde{L}) = n$. Since $\#\alpha(\tilde{L}) > 0$, there exists an embedding $e : D^2 \to \Sigma_{0,g+1}$ such that $e(\{(x,y) \in \partial D^2 \mid x \geq 0\}) \subset \tilde{L}$, $e(\{(x,y) \in \partial D^2 \mid x \leq 0\}) \subset \mathbf{L} \backslash \tilde{L}$ as in Figure 19. Choose $\tilde{L}'$ a one-dimensional submanifold of $\Sigma_{0,g+1}$ which is $\tilde{L}$ except for the neighborhood of $e(D)$, where it looks as shown in Figure 19.

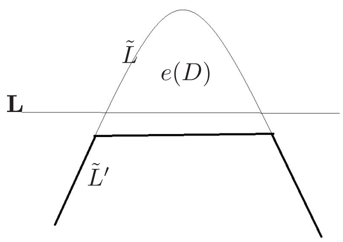

FIGURE 19. $\tilde{L}'$

Since $\tilde{L} \simeq \tilde{L}'$ and $\#(\alpha(\tilde{L}')) < n$, we have $L \in \lambda(g,m)(\mathbf{V}(g,m))$ where we denote by L the isotopy class of $\tilde{L}$. This proves Proposition 5.11 (n) for any $n \geq 0$. $\square$

We define an injective map

$$\begin{aligned} \imath(g,m) &: \mathbf{V}(g,m) \to \mathbf{V}(g+m) \\ &(i_1, i_2, \dots, i_{2m-1}, j_1, j_2, \dots, j_{3g-2}) \\ &\quad \mapsto (1, i_1, 1, i_2, i_3, 1, \dots, i_{2m-3}, 1, i_{2m-2}, i_{2m-1}, j_1, j_2, \dots, j_{3g-2}) \end{aligned}$$

for $m \geq 1$ and $g \geq 1$. Let J be a finite subset of $\partial D^2 \subset \partial \Sigma_{0,g+1}$ satisfying $\#J = 2m$. We define the $\mathbf{Q}[A, A^{-1}]$-module homomorphism $i(g,m) : \mathscr{S}(\Sigma_{0,g+1}, J) \to \mathscr{S}(\Sigma_{0,g+m+1})$ by $\langle \lambda(g,m)(v) \rangle \to \langle \lambda(g+m,0)(\imath(g,m)(v)) \rangle$. Using Theorem 5.6(1), we have the following proposition.

Proposition 5.12. *The $\mathbf{Q}[A, A^{-1}]$-module homomorphism $i(g,m) : \mathscr{S}(\Sigma_{0,g+1}, J) \to \mathscr{S}(\Sigma_{0,g+m+1})$ is well-defined and injective.*

Corollary 5.13. *The map $\lambda(g,m) : \mathbf{V}(g,m) \to \mathscr{T}_0(\Sigma_{0,g+1}, J)$ is bijective.*

Let Σ be a compact connected oriented surface with non-empty boundary, J a finite subset of $\partial\Sigma$ and P_1 and P_2 two points of J. We choose two orientation preserving embeddings $\delta_1, \delta_2 : I \to \partial\Sigma$ such that $\delta_1(I) \cap J = \delta_1(\frac{1}{2}) = P_1$ and that $\delta_2(I) \cap J = \delta_2(\frac{1}{2}) = P_2$. We define a surface $\Sigma(P_1, P_2)$ by gluing Σ and $I \times I$ by $(0, 1-t) = \delta_1(t)$ and $(1, t) = \delta_2(t)$. We introduce $i'(P_1, P_2) : \mathscr{T}_0(\Sigma, J) \to \mathscr{T}_0(\Sigma(P_1, P_2), J \backslash \{P_1, P_2\})$ such that $i'(P_1, P_2)(L)$ is the isotopy class of $\tilde{L} \cup \{(t, \frac{1}{2}) \in I \times I \mid t \in I\}$ where $\tilde{L}$ represents L. The map $i'(P_1, P_2)$ induces a $\mathbf{Q}[A, A^{-1}]$-module homomorphism $i(P_1, P_2) : \mathscr{S}(\Sigma, J) \to \mathscr{S}(\Sigma(P_1, P_2), J \backslash \{P_1, P_2\})$.

Lemma 5.14. *Let Σ be a compact connected oriented surface with non-empty boundary and $J = \{P_1, P_2, \dots, P_{2m-1}, P_{2m}\}$ a finite subset of $\partial\Sigma$. We set $\eta \overset{\text{def.}}{=} i(P_{2m-1}, P_{2m}) \circ \cdots \circ i(P_3, P_4) \circ i(P_1, P_2)$ and $\tilde{\Sigma} \overset{\text{def.}}{=} \Sigma(P_1, P_2)(P_3, P_4) \cdots (P_{2m-1}, P_{2m})$. Then η is injective. Since η is injective, $i'(P_1, P_2) : \mathscr{T}_0(\Sigma, J) \to \mathscr{T}_0(\Sigma(P_1, P_2), J \backslash \{P_1, P_2\})$ and $i(P_1, P_2) : \mathscr{S}(\Sigma, J) \to \mathscr{S}(\Sigma(P_1, P_2), J \backslash \{P_1, P_2\})$ are injective.*

Proof. For some integer g and some finite subset $J' \subset \partial D^2 \subset \partial\Sigma_{0,g+1}$, we choose a diffeomorphism $\chi : (\Sigma \times I, J \times I) \to (\Sigma_{0,g+1} \times I, J' \times I)$ and $\chi' : \tilde{\Sigma} \times I \to \Sigma_{0,g+m+1} \times I$ satisfying $(\chi_*)^{-1} \circ i(g,m) \circ \chi'_* = \eta$. Here we denote by $\chi_* : \mathscr{S}(\Sigma, J) \to \mathscr{S}(\Sigma_{0,g+1}, J')$ and $\chi'_* : \mathscr{S}(\tilde{\Sigma}) \to \mathscr{S}(\Sigma_{0,g+m+1})$ the $\mathbf{Q}[A, A^{-1}]$-module isomorphisms induced by χ and χ', respectively. Since $i(g,m)$ is injective, η is also injective. Hence $i(P_1, P_2)$ is injective. This proves the lemma. $\square$

Proof of Theorem 5.5 in general cases. We suppose $J \neq \emptyset$. Let J be $\{P_1, \dots, P_{2m}\}$. We set $\eta \overset{\text{def.}}{=} i(P_{2m-1}, P_{2m}) \circ \cdots \circ i(P_1, P_2)$ and $\tilde{\Sigma} \overset{\text{def.}}{=} \Sigma(P_1, P_2) \cdots$

(P_{2m-1}, P_{2m}). By definition, we have $\eta(F^n\mathscr{S}(\Sigma, J)) \subset F^n\mathscr{S}(\tilde{\Sigma})$ for any $n \in \mathbf{Z}_{\geq 0}$. Using Lemma 5.9, we have $\eta(\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma, J)) \subset \bigcap_{n=0}^{\infty} F^n\mathscr{S}(\tilde{\Sigma}) = \{0\}$. Here, by Lemma 5.14, η is injective. Hence, we have $\bigcap_{n=0}^{\infty} F^n\mathscr{S}(\Sigma, J) = \{0\}$. This proves the theorem. □

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