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KODAI MATH. J.
40 (2017), 577–583

A NOTE ON ATIYAH'S Γ-INDEX THEOREM IN HEISENBERG CALCULUS

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Abstract

In this note, we prove an index theorem on Galois coverings for Heisenberg elliptic (but not elliptic) differential operators, which is analogous to Atiyah's Γ-index theorem. This note also contains an example of Heisenberg differential operators with non-trivial Γ-index.

Introduction

Let $\tilde{M} \to M$ be a Galois covering over a closed manifold M with a deck transformation group Γ and D an elliptic differential opeator on M. M. F. Atiyah [1] introduced the notion of the Γ-index $\mathrm{index}_\Gamma(\tilde{D})$ for a lifted elliptic differential operator $\tilde{D}$ on $\tilde{M}$ and proved that the Γ-index $\mathrm{index}_\Gamma(\tilde{D})$ equals the Fredholm index $\mathrm{index}(D)$ of the original operator D. Atiyah [1] also investigated properties of a Γ-trace tr_Γ at the same time. The Γ-trace is a trace of the Γ-trace operators, so it induces a homomorphism $(\mathrm{tr}_\Gamma)_*$ from K_0-group of the Γ-compact operators to the real numbers: $(\mathrm{tr}_\Gamma)_* : K_0(\mathscr{K}_\Gamma) \to \mathbf{R}$. Out of a lifted elliptic differential operator $\tilde{D}$, we can define the Γ-index class $\mathrm{Ind}_\Gamma(\tilde{D}) \in K_0(\mathscr{K}_\Gamma)$ by using the Connes-Skandalis idempotent [3, II.9.α (p. 131)]. We send it by the induced homomorphism $(\mathrm{tr}_\Gamma)_*$, then the image $(\mathrm{tr}_\Gamma)_*(\mathrm{Ind}_\Gamma(\tilde{D})) \in \mathbf{R}$ equals the Γ-index $\mathrm{index}_\Gamma(\tilde{D})$ and thus the Fredholm index $\mathrm{index}(D)$ of the original operator D.

On the other hand, there is another pseudo-differential calculus on Heisenberg manifolds which is called the Heisenberg calculus; see, for instance [5]. Roughly speaking, Heisenberg calculus is "weighted" calculus and the product of the "Heisenberg principal symbols" is defined by convolution product. When the Heisenberg principal symbol of P is invertible, we call P a Heisenberg elliptic operator. Note that any Heisenberg elliptic operator is not elliptic. For a Heisenberg elliptic operator P, we can construct a parametrix by using its inverse, so P is a Fredholm operator if the base manifold is closed. Thus the Fredholm

2000 *Mathematics Subject Classification*. Primary 19K56; Secondary 46L87.

Key words and phrases. Index theory, Heisenberg calculus, Galois covering, Heisenberg structure, contact structure.

Received September 26, 2016; revised January 23, 2017.

index of P on a closed manifold is well defined, but a solution of an index problem of P does not obtained in general. However, index problems for Heisenberg elliptic operators on contact manifolds or foliated manifolds are solved by E. van Erp and P. F. Baum; see [2], [6], [7], [9].

In this note, we define the Γ-index $\mathrm{index}_\Gamma(\tilde{P})$ and the Γ-index class $\mathrm{Ind}_\Gamma(\tilde{P})$ for a lifted Heisenberg elliptic differential operator $\tilde{P}$. Once these ingredients are defined, the proof of the Γ-index theorem

$$\mathrm{index}_\Gamma(\tilde{P}) = (\mathrm{tr}_\Gamma)_*(\mathrm{Ind}_\Gamma(\tilde{P})) = \mathrm{index}(P)$$

is straightforward; see subsection 2.1. We also investigate an example of Heisenberg differential operators on a Galois covering over the 3-torus with non-trivial Γ-index by using the index formula in [2]; see subsection 2.2.

1. Short review of Atiyah's Γ-index theorem

In this section, we recall Atiyah's Γ-index theorem in ordinary pseudo-differential calculus. The main reference of this section is Atiyah's paper [1]. Let $\tilde{M} \to M$ be a Galois covering with a deck transformation group Γ over a closed manifold M with a smooth measure μ and $D : C^\infty(E) \to C^\infty(F)$ an elliptic differential operator on Hermitian vector bundles $E, F \to M$. We lift these ingredients on $\tilde{M}$ and denote by $\tilde{\mu}$ and $\tilde{D} : C^\infty(\tilde{E}) \to C^\infty(\tilde{F})$. Let $\mathrm{Ker}_{L^2}(\tilde{D})$ (resp. $\mathrm{Ker}_{L^2}(\tilde{D}^*)$) be the L^2-solutions of $\tilde{D}u = 0$ (resp. $\tilde{D}^*u = 0$) and denote by Π_0 (resp. Π_1) the orthogonal projection on a closed subspace $\mathrm{Ker}_{L^2}(\tilde{D})$ (resp. $\mathrm{Ker}_{L^2}(\tilde{D}^*)$) of the L^2-sections.

A Γ-invariant bounded operator T on the L^2-sections $L^2(\tilde{E})$ of $\tilde{E}$ is of Γ-*trace class* if $\phi T \psi \in L^2(\tilde{E})$ is of trace class for any compactly suppoted smooth functions ϕ, ψ on $\tilde{M}$. Denote by $\mathscr{L}^1_\Gamma$ the set of Γ-trace class operators and $\mathrm{tr}_\Gamma(T)$ the Γ-*trace* of a Γ-trace class operator T defined by

$$\mathrm{tr}_\Gamma(T) = \mathrm{Tr}(\phi T \psi) \in \mathbf{C} \quad \text{for} \ \sum_{\gamma \in \Gamma} \gamma^*(\phi\psi) = 1.$$

Here, the right hand side is the trace of a trace class operator $\phi T \psi$ and this quantity does not depend on the choice of functions ϕ and ψ. By using ellipticity of $\tilde{D}$, operators $\phi \Pi_0 \psi$ and $\phi \Pi_1 \psi$ are smoothing operators on compact sets. Thus Π_0 and Π_1 are of Γ-trace class and thus one obtains the Γ-index of $\tilde{D}$:

$$\mathrm{index}_\Gamma(\tilde{D}) = \mathrm{tr}_\Gamma(\Pi_0) - \mathrm{tr}_\Gamma(\Pi_1) \in \mathbf{R}.$$

In the context of the Γ-index theorem, the most important class of Γ-trace class operators is the lifts of almost local smoothing operators on M. Let S be an almost local smoothing operator with a smooth kernel k_S and $\tilde{S}$ a lift of S. Then $\tilde{S}$ is of Γ-trace class and its Γ-trace is calculated by the following:

$$(*) \qquad \mathrm{tr}_\Gamma(\tilde{S}) = \int_M \mathrm{tr}(k_S(x,x))\, d\mu = \mathrm{Tr}(S).$$

Denote by $\mathscr{K}_\Gamma$ the C^*-closure of $\mathscr{L}^1_\Gamma$ and $K_0(\mathscr{K}_\Gamma)$ the analytic K_0-group. Then tr_Γ induces a homomorphism of abelian groups by substitution:

$$(\mathrm{tr}_\Gamma)_* : K_0(\mathscr{K}_\Gamma) \to \mathbf{R}.$$

On the other hand, since D is an elliptic differential operator, there exist an almost local parametrix Q and almost local smoothing operators S_0, S_1 such that one has $QD = 1 - S_0$ and $DQ = 1 - S_1$. Denote by $\tilde{Q}$, $\widetilde{S_0}$ and $\widetilde{S_1}$ lifts of these operators and then one has the same relations $\tilde{Q}\tilde{D} = 1 - \widetilde{S_0}$ and $\tilde{D}\tilde{Q} = 1 - \widetilde{S_1}$. Set

$$e_{\tilde{D}} = \begin{bmatrix} \widetilde{S_0}^2 & \widetilde{S_0}(1+\widetilde{S_0})\tilde{Q} \\ \widetilde{S_1}\tilde{D} & 1-\widetilde{S_1}^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

By using $\tilde{Q}\widetilde{S_1} = \widetilde{S_0}\tilde{Q}$ and $\widetilde{S_1}\tilde{D} = \tilde{D}\widetilde{S_0}$, one has $e_{\tilde{D}}^2 = e_{\tilde{D}}$, that is, $e_{\tilde{D}}$ is an idempotent. Note that this idempotent $e_{\tilde{D}}$ is called the Connes-Skandalis idempotent; see, for instance [3, II.9.α (p. 131)]. Moreover, a difference $e_{\tilde{D}} - e_1$ is of Γ-trace class. Hence we can define a Γ-index class

$$\mathrm{Ind}_\Gamma(\tilde{D}) = [e_{\tilde{D}}] - [e_1] \in K_0(\mathscr{K}_\Gamma).$$

By the definition of the map $(\mathrm{tr}_\Gamma)_*$ and Atiyah's paper, one has the following:

THEOREM 1.1 (Atiyah's Γ-index theorem [1, Theorem 3.8]). *In the above settings, we have the following equality*:

$$\mathrm{index}_\Gamma(\tilde{D}) = (\mathrm{tr}_\Gamma)_*(\mathrm{Ind}_\Gamma(\tilde{D})) = \mathrm{index}(D) \in \mathbf{Z}.$$

As described in subsection 2.1, Atiyah's proof of the above equality does not essentially use ellipticity. Note that ellipticity of D and $\tilde{D}$ is only used in the definition of these ingredients.

2. Atiya's Γ-index theorem in Heisenberg calculus

Let (M, H) be a closed Heisenberg manifold, that is, M is a closed manifold and $H \subset TM$ is a hyperplane bundle. Let $P : C^\infty(E) \to C^\infty(F)$ be a Heisenberg elliptic differential operator on Hermitian vector bundles $E, F \to (M, H)$, that is, the Heisenberg principal symbol $\sigma_H(P)$ (see [5, Definition 3.2.3]) of P is an invertible element. Since our P is a differential operator, the Heisenberg principal symbol of P is a homogeneous polynomial in the weighted sense; see [5, Example 3.2.5]. In this section, we prove the Γ-index theorem for P, which is analogous to Atiyah's Γ-index theorem. Note that P is not elliptic in the sense of ordinary pseudo-differential calculus.

2.1. Statement and proof. By [5, Proposition 3.3.1], there exist a parametrix Q and smoothing operators S_0, S_1 such that one has $QP = 1 - S_0$ and $PQ = 1 - S_1$. Thus P is a Fredholm operator and one has the Fredholm index $\mathrm{index}(P) \in \mathbf{Z}$ of P by compactness of M. Moreover, since an integral kernel of

Q is smooth off the diagonal, we can choose Q as an almost local operator and then S_0 and S_1 are also almost local operators.

Let $\tilde{M} \to M$ be a Galois covering with a deck transformation group Γ over a closed manifold M with a smooth measure μ. We lift all structures on M to $\tilde{M}$. Then $(\tilde{M}, \tilde{H})$ is a Heisenberg manifold, $\tilde{P} : C^\infty(\tilde{E}) \to C^\infty(\tilde{F})$ is a Heisenberg elliptic differential operator and one has $\tilde{Q}\tilde{P} = 1 - \tilde{S}_0$ and $\tilde{P}\tilde{Q} = 1 - \tilde{S}_1$.

Since $\tilde{P}$ is a differential operator (in particular, $\tilde{P}$ is local), there exists a constant $C = C(\tilde{P}, \phi) > 0$ such that we have an inequality

$$\|\tilde{P}(\phi f)\|_{L^2} \leq C(\|\chi \tilde{P} f\|_{L^2} + \|\chi f\|_{L^2})$$

for any $f \in C^\infty(\tilde{E})$; see [5, Proposition 3.3.2]. Here, $\phi, \chi \in C_c^\infty(\tilde{M})$ are compactly supported smooth functions and one assumes $\chi = 1$ on the support of ϕ. Thus by using Atiyah's technique of the proof of [1, Proposition 3.1], we have the following:

Lemma 2.1. *The minimal domain of $\tilde{P}$ equals the maximal domain of $\tilde{P}$.*

By Lemma 2.1, $\tilde{P}$ has the unique closed extension denoted by the same letter $\tilde{P}$. Thus the closure of the formal adjoint of $\tilde{P}$ (the formal adjoint is also Heisenberg elliptic) equals the Hilbert space adjoint $\tilde{P}^*$.

On the other hand, any L^2-solutions of $\tilde{P}u = 0$ and $\tilde{P}^*u = 0$ are smooth by the existence of a parametrix. Thus the orthogonal projection Π_0 (resp. Π_1) onto a closed subspace $\mathrm{Ker}_{L^2}(\tilde{P})$ (resp. $\mathrm{Ker}_{L^2}(\tilde{P}^*)$) of the L^2-sections is of Γ-trace class since an operator $\phi\Pi_0\psi$ (resp. $\phi\Pi_1\psi$) is a smoothing operator on compact sets. Thus one obtains the well-defined Γ-index of $\tilde{P}$.

Definition 2.2. The Γ-index of $\tilde{P}$ is defined to be

$$\mathrm{index}_\Gamma(\tilde{P}) = \mathrm{tr}_\Gamma(\Pi_0) - \mathrm{tr}_\Gamma(\Pi_1) \in \mathbf{R}.$$

By using operators $\tilde{P}$, $\tilde{Q}$, $\tilde{S}_0$ and $\tilde{S}_1$, we define

$$e_{\tilde{P}} = \begin{bmatrix} \tilde{S}_0{}^2 & \tilde{S}_0(1+\tilde{S}_0)\tilde{Q} \\ \tilde{S}_1\tilde{P} & 1-\tilde{S}_1{}^2 \end{bmatrix} \quad \text{and} \quad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since a difference $e_{\tilde{P}} - e_1$ is of Γ-trace class, one can define a Γ-index class of $\tilde{P}$.

Definition 2.3. We define Γ-index class $\mathrm{Ind}_\Gamma(\tilde{P})$ of $\tilde{P}$ by

$$\mathrm{Ind}_\Gamma(\tilde{P}) = [e_{\tilde{P}}] - [e_1] \in K_0(\mathscr{K}_\Gamma).$$

By using a Γ-trace, one has the Γ-index theorem in Heisenberg calculus.

Theorem 2.4. *Let P be a Heisenberg elliptic differential operator on a closed Heisenberg manifold (M, H) and $\tilde{P}$ its lift as above. Then one has*

$$\mathrm{index}_\Gamma(\tilde{P}) = (\mathrm{tr}_\Gamma)_*(\mathrm{Ind}_\Gamma(\tilde{P})) = \mathrm{index}(P) \in \mathbf{Z}.$$

Proof. First, note that equalities

$$1 - S_0^2 = 1 - (1 - QP)^2 = (2Q - QPQ)P \quad \text{and}$$
$$1 - S_1^2 = 1 - (1 - PQ)^2 = P(2Q - QPQ),$$

and note that operators $2Q - QPQ$, S_0^2 and S_1^2 are almost local operators. Thus by Atiyah's technique in [1, Section 5], one has

$$\text{index}_\Gamma(\tilde{P}) = \text{tr}_\Gamma(\Pi_0) - \text{tr}_\Gamma(\Pi_1) = \text{tr}_\Gamma(\widetilde{S_0}^2) - \text{tr}_\Gamma(\widetilde{S_1}^2).$$

Next, by the definition of the map $(\text{tr}_\Gamma)_*$, one has

$$(\text{tr}_\Gamma)_*(\text{Ind}_\Gamma(\tilde{P})) = \text{tr}_\Gamma \begin{bmatrix} \widetilde{S_0}^2 & \widetilde{S_0}(1 + \widetilde{S_0})\tilde{Q} \\ \widetilde{S_1}\tilde{P} & -\widetilde{S_1}^2 \end{bmatrix} = \text{tr}_\Gamma(\widetilde{S_0}^2) - \text{tr}_\Gamma(\widetilde{S_1}^2).$$

Since operators $\tilde{S}_0^2$ and $\tilde{S}_1^2$ are lifts of almost local smoothing operators, one has

$$\text{index}(P) = \text{Tr}(S_0^2) - \text{Tr}(S_1^2) = \text{tr}_\Gamma(\widetilde{S_0}^2) - \text{tr}_\Gamma(\widetilde{S_1}^2)$$

by using (∗) in Section 1. This proves the equality in the theorem. □

Remark 2.5. As pointed out in [8, Section 4], the results in [5, Section 3] hold verbatim for arbitrary codimension. That is, we do not need to assume that a distribution H is of codimension 1.

2.2. Example. Index problems for Heisenberg elliptic operators on arbitrary closed Heisenberg manifolds are not solved yet. However, van Erp [6, 7] and Baum and van Erp [2] solved the index problem on contact manifolds, which are good examples of Heisenberg manifolds. In this subsection, we investigate an example of Heisenberg elliptic differential operators with non-trivial Γ-index on a Galois covering over a closed contact manifold. In order to check its non-triviality, we use the index formula in [2] for a subLaplacian twisted by a complex vector bundle.

Let $T^2 = S^1 \times S^1 = \{(e^{ix}, e^{iy})\}$ be the 2-torus and $E \to T^2$ a smooth complex line bundle with

$$\int_{T^2} c_1(E) = -1.$$

Here, $c_1(E)$ denotes the first Chern class of E. Such a line bundle E exists because the first Chern class induces a surjective homomorphism $H^1(T^2, \mathcal{O}^*) \to H^2(T^2) \cong \mathbf{Z}$. See also [4, Section I. 2] for another explicit construction of E in the context of Noncommutative geometry. We fix a smooth connection of E.

Let $T^3 = T^2 \times S^1 = \{(e^{ix}, e^{iy}, e^{iz})\}$ be the 3-torus and $q : T^3 \to T^2$ the projection onto T^2 of the first component. Set $\theta_k = \cos(kz)\,dx - \sin(kz)\,dy$ for a positive integer k, $H_k = \ker(\theta_k)$, $f_l(x, y, z) = e^{ilz} + 1$ for an integer l and $F = q^*E$. Then (T^3, H_k) is a contact manifold. This vector bundle H_k

is given by a projection $e_k : C^\infty(TT^3) \cong C^\infty(T^3;\mathbf{R}^3) \to C^\infty(H_k)$ defined by $e_k(\partial/\partial x) = \sin(kz)\{\sin(kz)\partial/\partial x + \cos(kz)\partial/\partial y\}$, $e_k(\partial/\partial y) = \cos(kz)\{\sin(kz)\partial/\partial x + \cos(kz)\partial/\partial y\}$ and $e_k(\partial/\partial z) = \partial/\partial z$. Thus this bundle H_k admits a flat connection $e_k de_k$ by simple computation.

Denote by T_k the Reeb vector field for θ_k, ∇^F the pull-back connection on F and $\Delta^F_{H_k}$ the sum of squares of F. The operator $\Delta^F_{H_k}$ is locally expressed by $\Delta^F_{H_k} = -\nabla^F_{X_k}\nabla^F_{X_k} - \nabla^F_{Y_k}\nabla^F_{Y_k}$, where $\{X_k, Y_k\}$ is a local orthonormal frame of H_k. We can constract it globally via a partition of unity. Set

$$P_{k,l} = \Delta^F_{H_k} + if_l\nabla^F_{T_k}.$$

Since the values of $f_l - n$ contained in $\mathbf{C}^\times$ for any odd integer n, an operator $P_{k,l} : C^\infty(F) \to C^\infty(F)$ is a Heisenberg elliptic differential operator of Heisenberg order 2. By the index formula for $P_{k,l}$ in [2, Example 6.5.3], one has

$$\mathrm{index}(P_{k,l}) = \int_{T^3} \frac{-1}{2\pi i} e^{-ilz}\, de^{ilz} \wedge c_1(F) = \frac{-1}{2\pi i}\int_{S^1} e^{-ilz}\, de^{ilz} \int_{T^2} c_1(E) = l.$$

Note that a contact structure H_k is a lift of H_1 by a k-fold cover $p_k : T^3 \to T^3$; $(e^{ix}, e^{iy}, e^{iz}) \mapsto (e^{ix}, e^{iy}, e^{ikz})$. Since the lift $\widetilde{P_{1,l}}$ of a subLaplacian $P_{1,l}$ by p_k equals $P_{k,kl}$, we have the $\Gamma(= \mathbf{Z}/k\mathbf{Z})$-index of $\widetilde{P_{1,l}}$:

$$\mathrm{index}_\Gamma(\widetilde{P_{1,l}}) = \frac{1}{k}\,\mathrm{index}(\widetilde{P_{1,l}}) = \frac{1}{k}\,\mathrm{index}(P_{k,kl}) = l = \mathrm{index}(P_{1,l}).$$

Next, we consider a general Galois covering of T^3. Let $X \to T^3$ be a Galois covering with a deck transformation group Γ, which is a quotient of $\pi_1(T^3) = \mathbf{Z}^3$, for example, $X = \mathbf{R}^3$ and $\Gamma = \mathbf{Z}^3$ the universal covering. By Theorem 2.4, we have non-trivial Γ-index as follows:

$$\mathrm{index}_\Gamma(\widetilde{P_{k,l}}) = \mathrm{index}(P_{k,l}) = l.$$

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