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KODAI MATH. J.
40 (2017), 450–467

INSTABILITY OF SOLITARY WAVES FOR A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN A BORDERLINE CASE

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Abstract

We study the orbital instability of solitary waves for a derivative nonlinear Schrödinger equation with a general nonlinearity. We treat a borderline case between stability and instability, which is left as an open problem by Liu, Simpson and Sulem (2013). We give a sufficient condition for instability of a two-parameter family of solitary waves in a degenerate case by extending the results of Ohta (2011), and verify this condition for some cases.

1. Introduction

In this paper, we consider the following generalized derivative nonlinear Schrödinger equation.

$$\text{(gDNLS)} \qquad i\partial_t u = -\partial_x^2 u - i|u|^{2\sigma}\partial_x u, \quad (t,x) \in \mathbf{R}\times\mathbf{R},$$

where u is a complex-valued function of $(t,x) \in \mathbf{R}\times\mathbf{R}$ and $\sigma > 0$. When $\sigma = 1$, (gDNLS) appears in plasma physics, nonlinear optics, and so on (see, e.g., [18, 16, 17, 22, 25]).

It is known that (gDNLS) has a two-parameter family of solitary waves

$$u_\omega(t,x) = e^{i\omega_0 t}\phi_\omega(x-\omega_1 t),$$

where $\omega = (\omega_0,\omega_1) \in \Omega := \{(\omega_0,\omega_1)\in\mathbf{R}^2 \mid \omega_1^2 < 4\omega_0\}$,

$$\phi_\omega(x) = \varphi_\omega(x)\exp i\left(\frac{\omega_1}{2}x - \frac{1}{2\sigma+2}\int_{-\infty}^x \varphi_\omega(y)^{2\sigma}\,dy\right),$$

$$\varphi_\omega(x) = \left\{\frac{(\sigma+1)(4\omega_0-\omega_1^2)}{2\sqrt{\omega_0}\cosh(\sigma\sqrt{4\omega_0-\omega_1^2}\,x)-\omega_1}\right\}^{1/2\sigma}.$$

2010 *Mathematics Subject Classification*. 35Q55, 35B35.
Key words and phrases. Solitary wave, Orbital instability, DNLS.
Received June 27, 2016; revised December 7, 2016.

Note that φ_ω is a solution of

$$-\partial_x^2\varphi + \left(\omega_0 - \frac{\omega_1^2}{4}\right)\varphi + \frac{\omega_1}{2}|\varphi|^{2\sigma}\varphi - \frac{2\sigma+1}{(2\sigma+2)^2}|\varphi|^{4\sigma}\varphi = 0, \quad x \in \mathbf{R},$$

and that ϕ_ω is a solution of

$$-\partial_x^2\phi + \omega_0\phi + \omega_1 i\partial_x\phi - i|\phi|^{2\sigma}\partial_x\phi = 0, \quad x \in \mathbf{R}. \tag{1}$$

We regard $L^2(\mathbf{R}) := L^2(\mathbf{R},\mathbf{C})$ and $H^1(\mathbf{R}) := H^1(\mathbf{R},\mathbf{C})$ as real Hilbert spaces with inner products

$$(v,w)_{L^2} := \Re\int_{-\infty}^{\infty} v(x)\overline{w(x)}\,dx, \quad (v,w)_{H^1} := (v,w)_{L^2} + (\partial_x v, \partial_x w)_{L^2},$$

respectively.

Recently, Hayashi and Ozawa [8] proved that the Cauchy problem for (gDNLS) is locally well-posed in the energy space $H^1(\mathbf{R})$ for all $\sigma \geq 1$ (see also [1, 9, 10, 11, 12, 21, 23]). Moreover, (gDNLS) has three conserved quantities

$$E(u) := \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{2\sigma+2}(i|u|^{2\sigma}\partial_x u, u)_{L^2},$$

$$Q_0(u) := \frac{1}{2}\|u\|_{L^2}^2, \quad Q_1(u) := \frac{1}{2}(i\partial_x u, u)_{L^2}.$$

Note that (gDNLS) can be written in Hamiltonian form $i\partial_t u(t) = E'(u(t))$, and that Q_0 and Q_1 arise from the gauge and translation invariances of E, respectively.

For $\omega \in \Omega$, we define the action

$$S_\omega(u) := E(u) + \sum_{j=0}^{1}\omega_j Q_j(u), \quad u \in H^1(\mathbf{R}).$$

Then (1) is equivalent to $S'_\omega(\phi) = 0$. We define

$$d(\omega) := S_\omega(\phi_\omega), \quad \omega \in \Omega.$$

Then we have

$$d'(\omega) = (\partial_{\omega_0}d(\omega), \partial_{\omega_1}d(\omega)) = (Q_0(\phi_\omega), Q_1(\phi_\omega)),$$

and

$$d''(\omega) = \begin{bmatrix} \partial_{\omega_0}^2 d(\omega) & \partial_{\omega_0}\partial_{\omega_1}d(\omega) \\ \partial_{\omega_1}\partial_{\omega_0}d(\omega) & \partial_{\omega_1}^2 d(\omega) \end{bmatrix} = \begin{bmatrix} \langle Q_0'(\phi_\omega), \partial_{\omega_0}\phi_\omega\rangle & \langle Q_1'(\phi_\omega), \partial_{\omega_0}\phi_\omega\rangle \\ \langle Q_0'(\phi_\omega), \partial_{\omega_1}\phi_\omega\rangle & \langle Q_1'(\phi_\omega), \partial_{\omega_1}\phi_\omega\rangle \end{bmatrix}. \tag{2}$$

The stability of solitary waves is defined as follows.

Definition 1. The solitary wave $e^{i\omega_0 t}\phi_\omega(\cdot - \omega_1 t)$ is said to be *stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property. For $u_0 \in B_\delta(\phi_\omega)$, the solution $u(t)$ of (gDNLS) with $u(0) = u_0$ exists globally in time, and $u(t) \in U_\varepsilon(\phi_\omega)$ for all $t \geq 0$, where

$$B_\delta(\phi) := \{v \in H^1(\mathbf{R}) \mid \|v - \phi\|_{H^1} < \delta\},$$

$$U_\varepsilon(\phi) := \left\{ v \in H^1(\mathbf{R}) \;\middle|\; \inf_{(s_0, s_1) \in \mathbf{R}^2} \|v - e^{is_0}\phi(\cdot - s_1)\|_{H^1} < \varepsilon \right\}.$$

Otherwise, $e^{i\omega_0 t}\phi_\omega(\cdot - \omega_1 t)$ is said to be *unstable*.

For the case $\sigma = 1$, Guo and Wu [7] proved that the solitary wave $e^{i\omega_0 t}\phi_\omega(\cdot - \omega_1 t)$ is stable for $\omega \in \Omega$ with $\omega_1 < 0$, and Colin and Ohta [2] proved that the solitary wave is stable for all $\omega \in \Omega$.

In [14], Liu, Simpson and Sulem proved that when $0 < \sigma < 1$, the solitary wave is stable for all $\omega \in \Omega$, and when $\sigma \geq 2$, the solitary wave is unstable for all $\omega \in \Omega$. They also proved that for $1 < \sigma < 2$, the solitary wave is stable if $-2\sqrt{\omega_0} < \omega_1 < 2z_0\sqrt{\omega_0}$, and unstable if $2z_0\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$, where the constant $z_0 = z_0(\sigma) \in (-1, 1)$ is the solution of

$$F_\sigma(z) := (\sigma - 1)^2 \left[\int_0^\infty (\cosh y - z)^{-1/\sigma} \, dy \right]^2 - \left[\int_0^\infty (\cosh y - z)^{-1/\sigma - 1}(z \cosh y - 1) \, dy \right]^2 = 0.$$

The authors [14] shows by numerical computation that when $1 < \sigma < 2$, the function F_σ is monotonically increasing, $F_\sigma(-1) < 0$ and $\lim_{z \uparrow 1} F_\sigma(z) = +\infty$. Therefore F_σ has exactly one root z_0 in the interval $(-1, 1)$. Note that $\det[d''(\omega)]$ has the same sign as $F_\sigma(\omega_1/2\sqrt{\omega_0})$ (see [14, Lemma 4.2]).

The proofs in [7, 14] are based on the spectral analysis of the linearized operator $S''_\omega(\phi_\omega)$ and the Hessian matrix $d''(\omega)$, and on the general theory of Grillakis, Shatah and Strauss [6]. The proof in [2] is based on the variational methods as in Shatah [24]. Every proof of [2, 7, 14] requires that the Hessian matrix $d''(\omega)$ is not degenerate. In the borderline case $\omega_1 = 2z_0\sqrt{\omega_0}$, however, we cannot apply their methods because the Hessian matrix $d''(\omega)$ has a zero eigenvalue and a negative eigenvalue. In [14], the authors conjectured that if $\omega_1 = 2z_0\sqrt{\omega_0}$, the solitary wave is unstable, but left the stability problem in that case as an open problem. Although there are several papers treating the stability and instability of a one-parameter family of solitary waves in degenerate cases (see [4, 3, 13, 15, 19, 26]), to the best of our knowledge, there are none for a two-parameter family of solitary waves.

In this paper, we consider the borderline case $\omega_1 = 2z_0\sqrt{\omega_0}$ and prove the following theorem, which verify the conjecture of Liu, Simpson and Sulem [14] for $7/6 < \sigma < 2$.

THEOREM 1. *Let* $7/6 < \sigma < 2$ *and* $z_0 = z_0(\sigma) \in (-1, 1)$ *satisfy* $F_\sigma(z_0) = 0$. *Then the solitary wave* $e^{i\omega_0 t}\phi_\omega(\cdot - \omega_1 t)$ *is unstable if* $\omega_1 = 2z_0\sqrt{\omega_0}$.

Remark 1. Our proof requires a certain amount of regularity of E (see Proposition 1 (iii) below). Therefore, the stability problem in the case $1 < \sigma \le 7/6$ and $\omega_1 = 2z_0\sqrt{\omega_0}$ still remains open.

Our proof of Theorem 1 is based on the Lyapunov functional methods as in Ohta [19, 20] and Maeda [15]. In [19], Ohta gave a sufficient condition for instability of a one-parameter family of solitary waves $e^{i\omega t}\tilde{\phi}_\omega$ for the following abstract Hamiltonian system in a degenerate case.

$$\frac{du}{dt}(t) = J\tilde{E}'(u(t)). \tag{3}$$

Moreover, he proved that this condition holds if $\tilde{d}''(\omega) = 0$ and $\tilde{d}'''(\omega) \ne 0$ under a certain spectral assumption of $\tilde{S}''_\omega(\phi_\omega)$ (see [19, (B2a)]), where $\tilde{S}_\omega$ is the action corresponding to (3), and $\tilde{d}(\omega) := \tilde{S}_\omega(\tilde{\phi}_\omega)$. In [15], Maeda treated the more degenerate cases $\tilde{d}''(\omega) = \tilde{d}'''(\omega) = 0$. In [20], Ohta proved instability of a two-parameter family of solitary waves for the following nonlinear Schrödinger equation of derivative type in non-degenerate cases.

$$i\partial_t u = -\partial_x^2 u - i|u|^2\partial_x u - b|u|^4 u, \quad (t, x) \in \mathbf{R} \times \mathbf{R}, \tag{4}$$

where $b > 0$. Combining the idea of [15, 19] with that of [20], we extend the results of Ohta [19] to two-parameter cases, and obtain a sufficient condition for instability (see Proposition 1 below). Moreover, we prove that if $d''(\omega)$ has a zero eigenvalue with an eigenvector $\xi \in \mathbf{R}^2$, and $\frac{d^3}{d\lambda^3} d(\omega + \lambda\xi)|_{\lambda=0} \ne 0$, then this condition holds under a certain spectral condition (see Lemma 2 below).

Remark 2. Our method can be formulated as an abstract theory such as [4, 5, 6, 15, 19].

Remark 3. The equation (4) has a similar situation to (gDNLS), but our method is not applicable to (4). Indeed, (4) has a two-parameter family of solitary waves $e^{i\omega_0 t}\hat{\phi}_\omega(\cdot - \omega_1 t)$, where $\omega = (\omega_0, \omega_1) \in \Omega$,

$$\hat{\phi}_\omega(x) = \hat{\varphi}_\omega(x) \exp i\left(\frac{\omega_1}{2}x - \frac{1}{4}\int_{-\infty}^x |\hat{\varphi}_\omega(y)|^2\, dy\right),$$

$$\hat{\varphi}_\omega(x) = \left\{\frac{2(4\omega_0 - \omega_1^2)}{-\omega_1 + \sqrt{\omega_1^2 + (1 + 16b/3)(4\omega_0 - \omega_1^2)}\cosh(\sqrt{4\omega_0 - \omega_1^2}\,x)}\right\}^{1/2}.$$

Ohta [20] proved that there exists $\kappa = \kappa(b) \in (0, 1)$ such that the solitary wave $e^{i\omega_0 t}\hat{\phi}_\omega(\cdot - \omega_1 t)$ is stable if $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$, and unstable if $2\kappa\sqrt{\omega_0} <$

$\omega_1 < 2\sqrt{\omega_0}$, and left the case $\omega_1 = 2\kappa\sqrt{\omega_0}$ as an open problem. In the case $\omega_1 = 2\kappa\sqrt{\omega_0}$, however, the Hessian matrix $\hat{d}''(\omega)$ has a zero eigenvalue with an eigenvector $\hat{\xi}$, and $\frac{d^3}{d\lambda^3}\hat{d}(\omega+\lambda\hat{\xi})|_{\lambda=0} = 0$, where $\hat{d}(\omega) := \hat{S}_\omega(\hat{\phi}_\omega)$, and $\hat{S}_\omega$ is the action corresponding to (4). In fact, we see that $\frac{d^4}{d\lambda^4}\hat{d}(\omega+\lambda\hat{\xi})|_{\lambda=0} < 0$. Therefore we may conjecture from the instability result of Maeda [15, Theorem 3] that the solitary wave is unstable. However, we do not know whether the results of [15] can be extended to two-parameter cases.

The rest of this paper is organized as follows. In Section 2, we give a sufficient condition for instability of the solitary wave $e^{i\omega_0 t}\phi_\omega(\cdot - \omega_1 t)$ in a degenerate case, and show that this condition holds when $7/6 < \sigma < 2$ and $\omega_1 = 2z_0\sqrt{\omega_1}$. In Section 3, we prove that this condition implies instability.

2. Sufficient condition for instability

In this section, we give a sufficient condition for instability of solitary waves in a degenerate case. For convenience, we give some notations. For $s = (s_0, s_1) \in \mathbf{R}$, we define

$$T(s)v := e^{is_0}v(\cdot - s_1).$$

Then the generator $T_0'(0)$ of $\{T((s,0))\}_{s\in\mathbf{R}}$ and $T_1'(0)$ of $\{T((0,s))\}_{s\in\mathbf{R}}$ are given by

$$T_0'(0)v = iv, \quad T_1'(0)v = -\partial_x v, \quad v \in H^1(\mathbf{R})$$

respectively. For $j = 0, 1$, we define the bounded linear operator B_j from $H^1(\mathbf{R})$ to $L^2(\mathbf{R})$ by

$$B_j v := -iT_j'(0)v.$$

Then we have $Q_j'(v) = B_j v$. Note that E and Q_j are invariant under T, that is,

$$E(T(s)v) = E(v), \quad Q_j(T(s)v) = Q_j(v), \quad v \in H^1(\mathbf{R}),\ s \in \mathbf{R}^2.$$

By differentiating $S_\omega'(T(s)\phi_\omega) = 0$ at $s = 0$, we obtain

$$S_\omega''(\phi_\omega)T_j'(0)\phi_\omega = 0, \quad \omega \in \Omega,\ j = 0, 1. \tag{5}$$

For $\xi = (\xi_0, \xi_1) \in \mathbf{R}^2$, let

$$B_\xi v := \sum_{j=0}^{1} \xi_j B_j v, \quad Q_\xi(v) := \frac{1}{2}(B_\xi v, v)_{L^2}, \quad v \in H^1(\mathbf{R}).$$

The aim of this section is to prove the following proposition, which gives a sufficient condition for instability (cf. [19, Theorem 2]).

Proposition 1. *Let $1 < \sigma < 2$ and $z_0 = z_0(\sigma) \in (-1, 1)$ satisfy $F_\sigma(z_0) = 0$. Let $\omega_1 = 2z_0\sqrt{\omega_0}$ and $\xi \in \mathbf{R}^2$ be an eigenvector of the Hessian matrix $d''(\omega)$*

corresponding to the zero eigenvalue. Then there exists $\psi \in H^1(\mathbf{R})$ *with the following properties.*

(i) $(B_j\phi_\omega, \psi)_{L^2} = (T_j'(0)\phi_\omega, \psi)_{L^2} = 0$ *for all* $j = 0, 1$, $S_\omega''(\phi_\omega)\psi = -B_\xi\phi_\omega$, *and*

$$S_\omega(\phi_\omega + \lambda\psi) = S_\omega(\phi_\omega) + \frac{\lambda^3}{6}\gamma + o(\lambda^3), \quad \gamma \neq -6Q_\xi(\psi). \tag{6}$$

(ii) *There exists* $k_0 > 0$ *such that* $\langle S_\omega''(\phi_\omega)w, w\rangle \geq k_0\|w\|_{H^1}^2$ *for all* $w \in W$, *where*

$$W := \{w \in H^1(\mathbf{R}) \mid (w, \psi)_{L^2} = (w, B_\xi\phi_\omega)_{L^2} = (w, T_j'(0)\phi_\omega)_{L^2} = 0, j = 0, 1\}.$$

Moreover, if $7/6 < \sigma < 2$, *then*

(iii) *there exists an open neighborhood* $V \subset H^1(\mathbf{R})$ *of* ϕ_ω *such that*

$$\|S_\omega''(v) - S_\omega''(w)\|_{\mathscr{L}(H^1, H^{-1})} = o(\|v - w\|_{H^1}^{1/3}) \quad \text{as } v, w \in V,\ \|v - w\|_{H^1} \to 0. \tag{7}$$

Remark 4. If $3/2 \leq \sigma < 2$, then $E \in C^3(H^1(\mathbf{R}), \mathbf{R})$. Therefore, (6) is equivalent to $\langle S_\omega'''(\phi_\omega)(\psi, \psi), \psi\rangle \neq -6Q_\xi(\psi)$, and (7) is naturally satisfied. The property (7) is only used in the proof of Lemma 9 below.

We will show in Section 3 that Proposition 1 implies Theorem 1. Proposition 1 (i) follows from the next lemma.

Lemma 1. *Let* $1 < \sigma < 2$ *and* $z_0 = z_0(\sigma) \in (-1, 1)$ *satisfy* $F_\sigma(z_0) = 0$. *Let* $\omega_1 = 2z_0\sqrt{\omega_0}$ *and* $\xi \in \mathbf{R}^2$ *be an eigenvector of the Hessian matrix* $d''(\omega)$ *corresponding to the zero eigenvalue. Then* $\dfrac{d^3}{d\lambda^3}d(\omega + \lambda\xi)|_{\lambda=0} \neq 0$.

The proof of Lemma 1 is given in Appendix A. To prove Proposition 1 (ii), we use the spectral property of the linearized operator $S_\omega''(\phi_\omega)$. Here, note that

$$S_\omega''(v)f = (-\partial_x^2 - i\sigma|v|^{2\sigma-2}\bar{v}\partial_x v - i|v|^{2\sigma}\partial_x + \omega_0 + \omega_1 i\partial_x)f - i\sigma|v|^{2\sigma-2}v\partial_x v\bar{f}, \tag{8}$$
$$v, f \in H^1(\mathbf{R}).$$

The following result is due to [14].

Lemma 2 ([14, Theorem 3.1]). *For* $\sigma \geq 1$ *and* $\omega \in \Omega$, *there exist* $\chi_\omega \in H^1(\mathbf{R})\backslash\{0\}$, $\lambda_\omega < 0$ *and* $k_1 > 0$ *such that* $S_\omega''(\phi_\omega)\chi_\omega = \lambda_\omega\chi_\omega$ *and* $\langle S_\omega''(\phi_\omega)p, p\rangle \geq k_1\|p\|_{L^2}^2$ *for all* $p \in H^1(\mathbf{R})$ *satisfying*

$$(p, \chi_\omega)_{L^2} = (p, T_j'(0)\phi_\omega)_{L^2} = 0, \quad j = 0, 1.$$

Now, we verify Proposition 1.

Proof of Proposition 1. First, we show that $\psi := \partial_\lambda\phi_{\omega+\lambda\xi}|_{\lambda=0} + \sum_{j=0}^1 \mu_j T_j'(0)\phi_\omega$ satisfies (i), where $(\mu_0, \mu_1) \in \mathbf{R}^2$ is taken so that

$$(T_j'(0)\phi_\omega, \psi)_{L^2} = 0, \quad j = 0, 1. \tag{9}$$

Since ξ is an eigenvector of $d''(\omega)$ corresponding to the zero eigenvalue, by (2), we deduce

$$0 = d''(\omega)\xi = \begin{bmatrix} \langle Q_0'(\phi_\omega), \xi_0\partial_{\omega_0}\phi_\omega + \xi_1\partial_{\omega_1}\phi_\omega\rangle \\ \langle Q_1'(\phi_\omega), \xi_0\partial_{\omega_0}\phi_\omega + \xi_1\partial_{\omega_1}\phi_\omega\rangle \end{bmatrix} = \begin{bmatrix} (B_0\phi_\omega, \partial_\lambda\phi_{\omega+\lambda\xi}|_{\lambda=0})_{L^2} \\ (B_1\phi_\omega, \partial_\lambda\phi_{\omega+\lambda\xi}|_{\lambda=0})_{L^2} \end{bmatrix}.$$

By differentiating $S'_{\omega+\lambda\xi}(\phi_{\omega+\lambda\xi}) = 0$ at $\lambda = 0$, we have

$$S''_\omega(\phi_\omega)\partial_\lambda\phi_{\omega+\lambda\xi}|_{\lambda=0} = -B_\xi\phi_\omega.$$

Since $(B_j\phi_\omega, T_k'(0)\phi_\omega)_{L^2} = 0$ for $j,k = 0,1$, we have $(B_j\phi_\omega, \psi)_{L^2} = 0$ for $j = 0,1$. Moreover, by (5), we have $S''_\omega(\phi_\omega)\psi = -B_\xi\phi_\omega$. Next, we check (6). By differentiating $d(\omega+\lambda\xi) = S_{\omega+\lambda\xi}(\phi_{\omega+\lambda\xi})$ with respect to λ, we obtain

$$\frac{d}{d\lambda}d(\omega+\lambda\xi) = Q_\xi(\phi_{\omega+\lambda\xi}).$$

By Taylor's expansion, we have

$$\begin{aligned} (10)\quad S_{\omega+\lambda\xi}(\phi_{\omega+\lambda\xi}) &= d(\omega+\lambda\xi) \\ &= d(\omega) + \lambda\frac{d}{d\eta}d(\omega+\eta\xi)\Big|_{\eta=0} + \frac{\lambda^3}{6}\frac{d^3}{d\eta^3}d(\omega+\eta\xi)\Big|_{\eta=0} + o(\lambda^3) \\ &= S_\omega(\phi_\omega) + \lambda Q_\xi(\phi_\omega) + \frac{\lambda^3}{6}\frac{d^3}{d\eta^3}d(\omega+\eta\xi)\Big|_{\eta=0} + o(\lambda^3), \end{aligned}$$

where we used $(d^2/d\eta^2)d(\omega+\eta\xi)|_{\eta=0} = \langle d''(\omega)\xi, \xi\rangle = 0$. Put $\Phi_\lambda := T(\mu_0\lambda, \mu_1\lambda)\phi_{\omega+\lambda\xi}$, where (μ_0, μ_1) is given in (9). Then we have $\Phi_0 = \phi_\omega$ and $\partial_\lambda\Phi_0 = \psi$, which implies that $R_\lambda := \phi_\omega + \lambda\psi - \Phi_\lambda$ satisfies $\|R_\lambda\|_{H^1} = O(\lambda^2)$. By Taylor's expansion, therefore, we deduce from (10) that

$$\begin{aligned} S_\omega(\phi_\omega + \lambda\psi) &= S_{\omega+\lambda\xi}(\phi_\omega + \lambda\psi) - \lambda Q_\xi(\phi_\omega + \lambda\psi) \\ &= S_{\omega+\lambda\xi}(\Phi_\lambda + R_\lambda) - \lambda Q_\xi(\phi_\omega) - \lambda^3 Q_\xi(\psi) \\ &= S_{\omega+\lambda\xi}(\Phi_\lambda) - \lambda Q_\xi(\phi_\omega) - \lambda^3 Q_\xi(\psi) + o(\lambda^3) \\ &= S_\omega(\phi_\omega) + \frac{\lambda^3}{6}\left(\frac{d^3}{d\eta^3}d(\omega+\eta\xi)\Big|_{\eta=0} - 6Q_\xi(\psi)\right) + o(\lambda^3), \end{aligned}$$

where we used $S_\omega = S_{\omega+\lambda\xi} - \lambda Q_\xi$, $S'_{\omega+\lambda\xi}(\Phi_\lambda) = 0$ and $S_{\omega+\lambda\xi}(\Phi_\lambda) = S_{\omega+\lambda\xi}(\phi_{\omega+\lambda\xi})$. By Lemma 1, we have

$$\gamma := \frac{d^3}{d\eta^3}d(\omega+\eta\xi)\Big|_{\eta=0} - 6Q_\xi(\psi) \neq -6Q_\xi(\psi).$$

Next, we show that ψ satisfies (ii). Since $\psi \neq 0$, $(\psi, T_j'(0)\phi_\omega)_{L^2} = 0$ for $j = 0,1$, and $\langle S''_\omega(\phi_\omega)\psi, \psi\rangle = 0$, it follows from Lemma 2 that $(\psi, \chi_\omega)_{L^2} \neq 0$.

Let $w \in W$, and put

$$a := -\frac{(w, \chi_\omega)_{L^2}}{(\psi, \chi_\omega)_{L^2}}, \quad p := w + a\psi.$$

Then we have $(p, \chi_\omega)_{L^2} = (p, T_j'(0)\phi_\omega)_{L^2} = 0$ for $j = 0, 1$. By Lemma 2 and $(w, \psi)_{L^2} = 0$, we obtain

$$\langle S_\omega''(\phi_\omega)p, p\rangle \geq k_1\|w + a\psi\|_{L^2}^2 \geq k_1\|w\|_{L^2}^2.$$

On the other hand, by $\langle S_\omega''(\phi_\omega)\psi, \psi\rangle = 0$, $S_\omega''(\phi_\omega)\psi = -B_\xi\phi_\omega$ and $(w, B_\xi\phi_\omega)_{L^2} = 0$, we have $\langle S_\omega''(\phi_\omega)p, p\rangle = \langle S_\omega''(\phi_\omega)w, w\rangle$, and therefore,

$$\langle S_\omega''(\phi_\omega)w, w\rangle \geq k_1\|w\|_{L^2}^2, \quad w \in W. \tag{11}$$

Moreover, since ϕ_ω, $\partial_x\phi_\omega \in L^\infty(\mathbf{R})$, by (8), we see that there exist positive constants c and C such that

$$c\|v\|_{H^1}^2 \leq \langle S_\omega''(\phi_\omega)v, v\rangle + C\|v\|_{L^2}^2, \quad v \in H^1(\mathbf{R}).$$

This inequality and (11) imply (ii).

Finally, (iii) follows from (8) and a direct calculation. This completes the proof. □

3. Proof of Theorem 1

In this section, we prove Theorem 1 by using Proposition 1. Throughout this section, let $7/6 < \sigma < 2$, $z_0 = z_0(\sigma) \in (-1, 1)$ satisfy $F_\sigma(z_0) = 0$, $\omega_1 = 2z_0\sqrt{\omega_0}$ and $\xi = (\xi_0, \xi_1) \in \mathbf{R}^2$ be an eigenvector of the Hessian matrix $d''(\omega)$ corresponding to the zero eigenvalue.

LEMMA 3. *There exist $\lambda_0 > 0$ and a C^∞-mapping $\rho : (-\lambda_0, \lambda_0) \to \mathbf{R}$ such that*

$$Q_\xi(\phi_\omega + \lambda\psi + \rho(\lambda)B_\xi\phi_\omega) = Q_\xi(\phi_\omega) \tag{12}$$

for all $\lambda \in (-\lambda_0, \lambda_0)$, and

$$\rho(\lambda) = -\frac{Q_\xi(\psi)}{\|B_\xi\phi_\omega\|_{L^2}^2}\lambda^2 + o(\lambda^2) \tag{13}$$

as $\lambda \to 0$.

Proof. We define

$$F(\lambda, \rho) := Q_\xi(\phi_\omega + \lambda\psi + \rho B_\xi\phi_\omega) - Q_\xi(\phi_\omega), \quad (\lambda, \rho) \in \mathbf{R}^2.$$

Then we have $F(0, 0) = 0$ and

$$\partial_\rho F(0, 0) = \langle Q_\xi'(\phi_\omega), B_\xi\phi_\omega\rangle = \|B_\xi\phi_\omega\|_{L^2}^2 \neq 0.$$

By the implicit function theorem, there exist $\lambda_0 > 0$ and a C^∞-mapping $\rho : (-\lambda_0, \lambda_0) \to \mathbf{R}$ such that $F(\lambda, \rho(\lambda)) = 0$ for all $\lambda \in (-\lambda_0, \lambda_0)$.

Moreover, by differentiating $F(\lambda, \rho(\lambda)) = 0$ at $\lambda = 0$, we obtain

$$\rho'(0) = 0, \quad \rho''(0) = -\frac{2Q_\xi(\psi)}{\|B_\xi \phi_\omega\|_{L^2}^2}.$$

This completes the proof. □

We define

$$\Psi(\lambda) := \phi_\omega + \lambda\psi + \rho(\lambda) B_\xi \phi_\omega, \quad \lambda \in (-\lambda_0, \lambda_0).$$

LEMMA 4. *There exist* $\varepsilon_0 > 0$ *and* C^3*-mappings* $\alpha = (\alpha_0, \alpha_1) : U_{\varepsilon_0}(\phi_\omega) \to \mathbf{R}^2$, $\Lambda : U_{\varepsilon_0}(\phi_\omega) \to (-\lambda_0, \lambda_0)$, $\beta : U_{\varepsilon_0}(\phi_\omega) \to \mathbf{R}$, $w : U_{\varepsilon_0}(\phi_\omega) \to W$ *such that*

$$T(\alpha(u))u = \Psi(\Lambda(u)) + \beta(u) B_\xi \phi_\omega + w(u) \tag{14}$$

for all $u \in U_{\varepsilon_0}(\phi_\omega)$. *Moreover,*

$$\alpha(T(s)u) = \alpha(u) - s, \quad \Lambda(T(s)u) = \Lambda(u), \quad \beta(T(s)u) = \beta(u), \quad w(T(s)u) = w(u)$$

for all $u \in U_{\varepsilon_0}(\phi_\omega)$ *and* $s \in \mathbf{R}^2$.

Proof. We define

$$G(u, \alpha, \Lambda, \beta) := \begin{bmatrix} (T(\alpha)u - \Psi(\Lambda) - \beta B_\xi \phi_\omega, T_0'(0)\phi_\omega)_{L^2} \\ (T(\alpha)u - \Psi(\Lambda) - \beta B_\xi \phi_\omega, T_1'(0)\phi_\omega)_{L^2} \\ (T(\alpha)u - \Psi(\Lambda) - \beta B_\xi \phi_\omega, \psi)_{L^2} \\ (T(\alpha)u - \Psi(\Lambda) - \beta B_\xi \phi_\omega, B_\xi \phi_\omega)_{L^2} \end{bmatrix}$$

for $(u, \alpha, \Lambda, \beta) \in H^1(\mathbf{R}) \times \mathbf{R}^2 \times \mathbf{R} \times \mathbf{R}$. Then we have $G(\phi_\omega, 0, 0, 0) = 0$ and

$$\frac{\partial G}{\partial(\alpha, \Lambda, \beta)}(\phi_\omega, 0, 0, 0)$$
$$= \begin{bmatrix} \|T_0'(0)\phi_\omega\|_{L^2}^2 & (T_1'(0)\phi_\omega, T_0'(0)\phi_\omega)_{L^2} & 0 & 0 \\ (T_0'(0)\phi_\omega, T_1'(0)\phi_\omega)_{L^2} & \|T_1'(0)\phi_\omega\|_{L^2}^2 & 0 & 0 \\ 0 & 0 & -\|\psi\|_{L^2}^2 & 0 \\ 0 & 0 & 0 & -\|B_\xi \phi_\omega\|_{L^2}^2 \end{bmatrix}.$$

Since $T_0'(0)\phi_\omega$, $T_1'(0)\phi_\omega$ are linearly independent, we see that $\dfrac{\partial G}{\partial(\alpha, \Lambda, \beta)}(\phi_\omega, 0, 0, 0)$ is invertible. Thus by the implicit function theorem, there exist $\varepsilon_0 > 0$, $\alpha = (\alpha_0, \alpha_1) : B_{\varepsilon_0}(\phi_\omega) \to \mathbf{R}^2$, $\Lambda : B_{\varepsilon_0}(\phi_\omega) \to (-\lambda_0, \lambda_0)$ and $\beta : B_{\varepsilon_0}(\phi_\omega) \to \mathbf{R}$ such that $G(u, \alpha(u), \Lambda(u), \beta(u)) = 0$ for all $u \in B_{\varepsilon_0}(\phi_\omega)$. We extend α, Λ and β to the mappings on $U_{\varepsilon_0}(\phi_\omega)$ (see [5, Lemma 3.2]). Finally, we define

$$w(u) := T(\alpha(u))u - \Psi(\Lambda(u)) - \beta(u) B_\xi \phi_\omega, \quad u \in U_{\varepsilon_0}(\phi_\omega).$$

Then we have the conclusion. □

Remark 5. By the uniqueness of the solution of $G = 0$, we have

$$\alpha(\Psi(\lambda)) = 0, \quad \Lambda(\Psi(\lambda)) = \lambda, \quad \beta(\Psi(\lambda)) = 0, \quad w(\Psi(\lambda)) = 0$$

for all $\lambda \in (-\lambda_0, \lambda_0)$.

Lemma 5. *$\alpha_j'(u)$, $\Lambda'(u)$, $\alpha_j''(u)v \in H^1(\mathbf{R})$ for all $u \in U_{\varepsilon_0}(\phi_\omega)$ and $v \in H^1(\mathbf{R})$.*

Proof. By differentiating $G(u, \alpha(u), \Lambda(u), \beta(u)) = 0$ with respect to u, we have

$$\begin{bmatrix} \alpha_0'(u) \\ \alpha_1'(u) \\ \Lambda'(u) \\ \beta'(u) \end{bmatrix} = -\left[\frac{\partial G}{\partial(\alpha, \Lambda, \mu)}(u, \alpha(u), \Lambda(u), \beta(u))\right]^{-1} \begin{bmatrix} T(-\alpha(u))T_0'(0)\phi_\omega \\ T(-\alpha(u))T_1'(0)\phi_\omega \\ T(-\alpha(u))\psi \\ T(-\alpha(u))B_\xi\phi_\omega \end{bmatrix} \in H^1(\mathbf{R})^4,$$

where we used the fact $\phi_\omega \in H^2(\mathbf{R})$. Similarly, we also see that $\alpha_j''(u)v \in H^1(\mathbf{R})$. This completes the proof. □

Lemma 6. *For $u \in U_{\varepsilon_0}(\phi_\omega)$ satisfying $Q_\xi(u) = Q_\xi(\phi_\omega)$,*

$$\beta(u) = O(|\Lambda(u)|\, \|w(u)\|_{H^1} + \|w(u)\|_{H^1}^2)$$

as $\inf_{s \in \mathbf{R}^2} \|u - T(s)\phi_\omega\|_{H^1} \to 0$.

Proof. For $u \in U_{\varepsilon_0}(\phi_\omega)$ satisfying $Q_\xi(u) = Q_\xi(\phi_\omega)$, by (14), (12) and $(B_\xi\phi_\omega, w(u))_{L^2} = 0$, we have

$$\begin{aligned} 0 &= Q_\xi(u) - Q_\xi(\phi_\omega) = Q_\xi(T(\alpha(u))u) - Q_\xi(\phi_\omega) \\ &= \beta(u)^2 Q_\xi(B_\xi\phi_\omega) + Q_\xi(w(u)) + \beta(u)(B_\xi\Psi(\Lambda(u)), B_\xi\phi_\omega)_{L^2} \\ &\quad + \beta(u)(B_\xi^2\phi_\omega, w(u))_{L^2} + (B_\xi\Psi(\Lambda(u)), w(u))_{L^2} \\ &= \beta(u)[\|B_\xi\phi_\omega\|_{L^2}^2 + o(1)] + O(|\Lambda(u)|\, \|w(u)\|_{L^2} + \|w(u)\|_{H^1}^2). \end{aligned}$$

This implies the conclusion. □

We define

$$M(u) := T(\alpha(u))u, \quad A(u) := -(M(u), i\psi)_{L^2}, \quad u \in U_{\varepsilon_0}(\phi_\omega).$$

Then

$$\begin{aligned} A'(u) &= -\sum_{j=0}^{1} (T_j'(0)M(u), i\psi)_{L^2}\alpha_j'(u) - iT(-\alpha(u))\psi \\ &= -\sum_{j=0}^{1} (B_j M(u), \psi)_{L^2}\alpha_j'(u) - iT(-\alpha(u))\psi. \end{aligned} \tag{15}$$

By Lemma 5, we see that $A'(u)$, $A''(u)v \in H^1(\mathbf{R})$ for all $u \in U_{\varepsilon_0}(\phi_\omega)$ and $v \in H^1(\mathbf{R})$. Moreover, we have

$$iA'(\phi_\omega) = \psi. \tag{16}$$

Since M and A are invariant under T, it follows that

$$0 = \partial_{s_j} A(T(s)u)|_{s=0} = \langle A'(u), T_j'(0)u \rangle = -\langle Q_j'(u), iA'(u) \rangle. \tag{17}$$

We define

$$P(u) := \langle E'(u), iA'(u) \rangle, \quad u \in U_{\varepsilon_0}(\phi_\omega).$$

Then by (17), we have $P(u) = \langle S_\omega'(u), iA'(u) \rangle$. By $S_\omega'(\phi_\omega) = 0$, (16) and $S_\omega''(\phi_\omega)\psi = -B_\xi \phi_\omega$, we obtain

$$P'(\phi_\omega) = -B_\xi \phi_\omega. \tag{18}$$

Note that P is invariant under T.

Lemma 7. *Let I be an interval of $\mathbf{R}$. Let $u \in C(I, H^1(\mathbf{R})) \cap C^1(I, H^{-1}(\mathbf{R}))$ be a solution of* (gDNLS), *and assume that $u(t) \in U_{\varepsilon_0}(\phi_\omega)$ for all $t \in I$. Then*

$$\frac{d}{dt} A(u(t)) = P(u(t))$$

for all $t \in I$.

Proof. By [5, Lemma 4.6], we see that $t \mapsto A(u(t))$ is C^1 on I, and

$$\frac{d}{dt} A(u(t)) = \langle \partial_t u(t), A'(u(t)) \rangle = \langle E'(u(t)), iA'(u(t)) \rangle = P(u(t))$$

for all $t \in I$. This completes the proof. □

Put

$$v := \gamma + 6Q_\xi(\psi).$$

Then $v \neq 0$ by Proposition 1 (i).

Lemma 8. *For $\lambda \in (-\lambda_0, \lambda_0)$,*

$$S_\omega(\Psi(\lambda)) - S_\omega(\phi_\omega) = \frac{\lambda^3}{6} v + o(\lambda^3), \tag{19}$$

$$P(\Psi(\lambda)) = \frac{\lambda^2}{2} v + o(\lambda^2) \tag{20}$$

as $\lambda \to 0$.

Proof. First, we show that (19). Since $S_\omega'(\phi_\omega) = 0$ and $S_\omega''(\phi_\omega)\psi = -B_\xi\phi_\omega$, by Taylor's expansion, we have

$$S_\omega'(\phi_\omega + \lambda\psi) = -\lambda B_\xi\phi_\omega + o(\lambda). \tag{21}$$

By (6) and (13), we obtain

$$\begin{aligned} S_\omega(\Psi(\lambda)) &= S_\omega(\phi_\omega + \lambda\xi) + \rho(\lambda)\langle S_\omega'(\phi_\omega + \lambda\psi), B_\xi\phi_\omega\rangle + o(\lambda^3) \\ &= S_\omega(\phi_\omega) + \frac{\lambda^3}{6}[\gamma + 6Q_\xi(\psi)] + o(\lambda^3). \end{aligned}$$

Next, we show that (20). By Taylor's expansion, we have

$$S_\omega'(\Psi(\lambda)) = S_\omega'(\phi_\omega + \lambda\psi) + \rho(\lambda)S_\omega''(\phi_\omega)B_\xi\phi_\omega + o(\lambda^2).$$

By (15), Remark 5 and $(B_j\phi_\omega, \psi)_{L^2} = 0$ for $j = 0, 1$, we have

$$\begin{aligned} iA'(\Psi(\lambda)) &= \psi - \sum_{j=0}^{1}(B_j\Psi(\lambda), \psi)_{L^2} i\alpha_j'(\Psi(\lambda)) \\ &= \psi - \lambda\sum_{j=0}^{1}(B_j\psi, \psi)_{L^2} i\alpha_j'(\phi_\omega) + O(\lambda^2). \end{aligned}$$

Therefore, by (13) and (21), we obtain

$$\begin{aligned} P(\Psi(\lambda)) &= \langle S_\omega'(\Psi(\lambda)), iA'(\Psi(\lambda))\rangle \\ &= \langle S_\omega'(\phi_\omega + \lambda\psi), \psi\rangle - \lambda\sum_{j=0}^{1}(B_j\psi, \psi)_{L^2}\langle S_\omega'(\phi_\omega + \lambda\psi), i\alpha_j'(\phi_\omega)\rangle \\ &\quad + \rho(\lambda)\langle S_\omega''(\phi_\omega)B_\xi\phi_\omega, \psi\rangle + o(\lambda^2) \\ &= \langle S_\omega'(\phi_\omega + \lambda\psi), \psi\rangle - \lambda^2\sum_{j=0}^{1}(B_j\psi, \psi)_{L^2}\sum_{k=0}^{1}\xi_k(T_k'(0)\phi_\omega, \alpha_j'(\phi_\omega))_{L^2} \\ &\quad + \lambda^2 Q_\xi(\psi) + o(\lambda^2). \end{aligned}$$

Here, it follows from (6) that

$$\langle S_\omega'(\phi_\omega + \lambda\psi), \psi\rangle = \frac{d}{d\lambda}S(\phi_\omega + \lambda\psi) = \frac{\lambda^2}{2}\gamma + o(\lambda^2).$$

Moreover, by differentiating $\alpha(T(s)u) = \alpha(u) - s$ at $s = 0$, we have

$$(T_k'(0)u, \alpha_j'(u))_{L^2} = -\delta_{j,k}, \quad u \in U_{\varepsilon_0}(\phi_\omega),\ j, k = 0, 1.$$

Thus, we deduce

$$P(\Psi(\lambda)) = \frac{\lambda^2}{2}[\gamma + 6Q_\xi(\psi)] + o(\lambda^2).$$

This completes the proof. □

LEMMA 9. *For* $u \in U_{\varepsilon_0}(\phi_\omega)$ *satisfying* $Q_\xi(u) = Q_\xi(\phi_\omega)$,

$$S_\omega(u) - S_\omega(\phi_\omega) = \frac{\Lambda(u)^3}{6}\nu + \frac{1}{2}\langle S_\omega''(\phi_\omega)w(u), w(u)\rangle + o(|\Lambda(u)|^3 + \|w(u)\|_{H^1}^2),$$

$$\Lambda(u)P(u) = \frac{\Lambda(u)^3}{2}\nu + o(|\Lambda(u)|^3 + \|w(u)\|_{H^1}^2)$$

as $\inf_{s\in\mathbf{R}^2}\|u - T(s)\phi_\omega\|_{H^1} \to 0$.

Proof. Since

$$S_\omega'(\Psi(\Lambda(u))) = -\Lambda(u)B_\xi\phi_\omega + \rho(\Lambda(u))S_\omega''(\phi_\omega)B_\xi\phi_\omega + o(\Lambda(u)^2),$$

by Lemmas 4, 6 and (19), we have

$$\begin{aligned}
S_\omega(u) - S_\omega(\phi_\omega) &= S_\omega(M(u)) - S_\omega(\phi_\omega)\\
&= S_\omega(\Psi(\Lambda(u))) - S_\omega(\phi_\omega) + \langle S_\omega'(\Psi(\Lambda(u))), \beta(u)B_\xi\phi_\omega + w(u)\rangle\\
&\quad + \frac{1}{2}\langle S_\omega''(\Psi(\Lambda(u)))(\beta(u)B_\xi\phi_\omega + w(u)), \beta(u)B_\xi\phi_\omega + w(u)\rangle\\
&\quad + o(\|\beta(u)B_\xi\phi_\omega + w(u)\|_{H^1}^2)\\
&= \frac{\Lambda(u)^3}{6}\nu + \frac{1}{2}\langle S_\omega''(\phi_\omega)w(u), w(u)\rangle + o(|\Lambda(u)|^3 + \|w(u)\|_{H^1}^2).
\end{aligned}$$

On the other hand, by Lemmas 4, 6 and (20), we deduce

$$\begin{aligned}
P(u) &= P(M(u))\\
&= P(\Psi(\Lambda(u)) + w(u)) + O(|\Lambda(u)|\,\|w(u)\|_{H^1} + \|w(u)\|_{H^1}^2)\\
&= P(\Psi(\Lambda(u))) + \langle P'(\Psi(\Lambda(u))), w(u)\rangle + O(\Lambda(u)\|w(u)\|_{H^1}) + o(\|w(u)\|_{H^1}^{4/3})\\
&= \frac{\Lambda(u)^2}{2}\nu + \langle P'(\phi_\omega), w(u)\rangle + O(\Lambda(u)\|w(u)\|_{H^1}) + o(|\Lambda(u)|^2 + \|w(u)\|_{H^1}^{4/3})\\
&= \frac{\Lambda(u)^2}{2}\nu + O(\Lambda(u)\|w(u)\|_{H^1}) + o(|\Lambda(u)|^2 + \|w(u)\|_{H^1}^{4/3}),
\end{aligned}$$

where we used (7) and (18). This implies the conclusion. □

Proof of Theorem 1. By Lemma 9 and Proposition 1 (ii), we see that there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and $c > 0$ such that

$$S_\omega(u) - S_\omega(\phi_\omega) - \Lambda(u)P(u) \geq -c[v\Lambda(u)^3]_+ \tag{22}$$

for all $u \in U_{\varepsilon_1}(\phi_\omega)$, where $a_+ := \max\{a, 0\}$ for $a \in \mathbf{R}$.

Without loss of generality, we may assume that $v > 0$. Suppose that $T(\omega t)\phi_\omega$ is stable. Let $u_\lambda(t)$ be the solution of (gDNLS) with $u_\lambda(0) = \Psi(\lambda)$. Then by (19), there exists $\lambda_1 \in (0, \lambda_0)$ such that $S_\omega(\phi_\omega) - S_\omega(\Psi(\lambda)) > 0$ for all $\lambda \in (-\lambda_1, 0)$. Since $T(\omega t)\phi_\omega$ is stable, there exists $\lambda_2 \in (0, \lambda_1)$ such that $u_\lambda(t) \in U_{\varepsilon_1}(\phi_\omega)$ for all $\lambda \in (-\lambda_2, \lambda_2)$ and $t \geq 0$. Let $\lambda \in (-\lambda_2, 0)$. Then by the conservation of S_ω and (22), we have

$$\begin{aligned} 0 < \delta_\lambda &:= S_\omega(\phi_\omega) - S_\omega(u_\lambda(0)) = S_\omega(\phi_\omega) - S_\omega(u_\lambda(t)) \\ &\leq C\Lambda(u_\lambda(t))_+^3 - \Lambda(u_\lambda(t))P(u_\lambda(t)) \end{aligned}$$

for all $t \geq 0$. By this inequality, $\Lambda(u_\lambda(0)) = \lambda < 0$ and the continuity of $t \mapsto \Lambda(u_\lambda(t))$, we see that $\Lambda(u_\lambda(t)) < 0$ for all $t \geq 0$. Thus, we have $\delta_\lambda < \lambda_0 P(u_\lambda(t))$ for all $t \geq 0$. Moreover, by Lemma 7, we have

$$\frac{d}{dt}A(u_\lambda(t)) = P(u_\lambda(t)) > \frac{\delta_\lambda}{\lambda_0}$$

for all $t \geq 0$, which implies $A(u_\lambda(t)) \to \infty$ as $t \to +\infty$. This contradicts the fact that there exists $C > 0$ such that $|A(u)| \leq C$ for all $u \in U_{\varepsilon_0}(\phi_\omega)$. Hence, $T(\omega t)\phi_\omega$ is unstable. □

Appendix A. Proof of Lemma 1

In this section, we prove Lemma 1. Throughout this section, let $1 < \sigma < 2$ and $z_0 = z_0(\sigma) \in (-1, 1)$ satisfy $F_\sigma(z_0) = 0$. For $\omega \in \Omega$, we define

$$\kappa_\omega := \sqrt{4\omega_0 - \omega_1^2}, \quad \tilde{\kappa}_\omega := 2^{1/\sigma-2}\sigma^{-1}(\sigma+1)^{1/\sigma}\omega_0^{-1/2\sigma-1/2}\kappa_\omega^{2/\sigma-2}.$$

Then we have

$$\partial_{\omega_0}\kappa_\omega = \frac{2}{\kappa_\omega}, \quad \partial_{\omega_1}\kappa_\omega = -\frac{\omega_1}{\kappa_\omega}, \tag{23}$$

and

$$\partial_{\omega_0}\tilde{\kappa}_\omega = -\frac{\tilde{\kappa}_\omega}{\sigma}\left[\frac{4(\sigma-1)}{\kappa_\omega^2} + \frac{\sigma+1}{2\omega_0}\right], \quad \partial_{\omega_1}\tilde{\kappa}_\omega = \tilde{\kappa}_\omega \frac{2(\sigma-1)\omega_1}{\sigma\kappa_\omega^2}. \tag{24}$$

For $\omega \in \Omega$ and $n \in \mathbf{Z}_+$, we define

$$\alpha_{n,\omega} := \int_0^\infty \left(\cosh(\sigma\kappa_\omega x) - \frac{\omega_1}{2\sqrt{\omega_0}}\right)^{-1/\sigma-n} dx.$$

Then it follows from [14, Lemmas A.1 and A.2] that

(25) $$\partial_{\omega_0}\alpha_{0,\omega} = -\frac{2}{\kappa_\omega^2}\alpha_{0,\omega} - \frac{\omega_1}{4\sigma\omega_0^{3/2}}\alpha_{1,\omega},$$

(26) $$\partial_{\omega_1}\alpha_{0,\omega} = \frac{\omega_1}{\kappa_\omega^2}\alpha_{0,\omega} + \frac{1}{2\sigma\sqrt{\omega_0}}\alpha_{1,\omega},$$

(27) $$\partial_{\omega_0}\alpha_{1,\omega} = -\frac{\omega_1}{\sigma\sqrt{\omega_0}\kappa_\omega^2}\alpha_{0,\omega} - \frac{(2+\sigma)\omega_1^2 + 4\sigma\omega_0}{2\sigma\omega_0\kappa_\omega^2}\alpha_{1,\omega},$$

(28) $$\partial_{\omega_1}\alpha_{1,\omega} = \frac{2\sqrt{\omega_0}}{\sigma\kappa_\omega^2}\alpha_{0,\omega} + \frac{2(\sigma+1)\omega_1}{\sigma\kappa_\omega^2}\alpha_{1,\omega}.$$

By [14, Lemma A.3] and (2), we obtain

(29) $$\partial_{\omega_0}^2 d(\omega) = \frac{\tilde{\kappa}_\omega}{\sqrt{\omega_0}}(2\omega_1^2 - 8(\sigma-1)\omega_0)\alpha_{0,\omega} - \frac{\tilde{\kappa}_\omega}{\omega_0}\kappa_\omega^2\omega_1\alpha_{1,\omega} = \frac{\partial_{\omega_1}^2 d(\omega)}{\omega_0},$$

(30) $$\partial_{\omega_1}\partial_{\omega_0} d(\omega) = -4\tilde{\kappa}_\omega\sqrt{\omega_0}\omega_1(2-\sigma)\alpha_{0,\omega} + 2\tilde{\kappa}_\omega\kappa_\omega^2\alpha_{1,\omega} = \partial_{\omega_0}\partial_{\omega_1} d(\omega).$$

By differentiating (29) with respect to ω_j $(j = 0, 1)$, we have

(31) $$\omega_0\partial_{\omega_0}^3 d(\omega) = \partial_{\omega_0}\partial_{\omega_1}^2 d(\omega) - \partial_{\omega_0}^2 d(\omega), \quad \partial_{\omega_1}^3 d(\omega) = \omega_0\partial_{\omega_0}^2\partial_{\omega_1} d(\omega).$$

On the other hand, by differentiating (30), it follows from (23)–(28) that

(32) $$\partial_{\omega_0}^2\partial_{\omega_1} d(\omega) = \frac{2\omega_1\tilde{\kappa}_\omega\alpha_{0,\omega}}{\sigma\kappa_\omega^2\sqrt{\omega_0}}[4(3\sigma-2)(2-\sigma)\omega_0 - (\sigma-1)\kappa_\omega^2] + \frac{\tilde{\kappa}_\omega\alpha_{1,\omega}}{\sigma\omega_0}[4(2-\sigma)\omega_0 - 2\sigma\omega_1^2 - (\sigma+1)\kappa_\omega^2],$$

(33) $$\partial_{\omega_0}\partial_{\omega_1}^2 d(\omega) = \frac{4\sqrt{\omega_0}\tilde{\kappa}_\omega\alpha_{0,\omega}}{\sigma\kappa_\omega^2}[-(3\sigma-2)(2-\sigma)\omega_1^2 + (\sigma-1)^2\kappa_\omega^2] + \frac{2(3\sigma-2)\omega_1\tilde{\kappa}_\omega\alpha_{1,\omega}}{\sigma}.$$

Let $\omega_1 = 2z_0\sqrt{\omega_0}$. Then by $\det[d''(\omega)] = 0$ and (29), we have

(34) $$(\partial_{\omega_0}\partial_{\omega_1} d(\omega))^2 = \partial_{\omega_0}^2 d(\omega)\partial_{\omega_1}^2 d(\omega) = \omega_0(\partial_{\omega_0}^2 d(\omega))^2.$$

Let

$$\xi = (\xi_0, \xi_1) = (-\omega_0\partial_{\omega_0}^2 d(\omega), \partial_{\omega_0}\partial_{\omega_1} d(\omega)).$$

Then ξ is an eigenvector of $d''(\omega)$ corresponding to the zero eigenvalue.

LEMMA 10. *Let* $\omega_1 = 2z_0\sqrt{\omega_0}$.

- *If* $\partial_{\omega_0}\partial_{\omega_1} d(\omega) = -\sqrt{\omega_0}\partial_{\omega_0}^2 d(\omega)$, *then*

$$\frac{d^3}{d\lambda^3} d(\omega + \lambda\xi)\bigg|_{\lambda=0} = \omega_0^2(\partial_{\omega_0}^2 d(\omega))^3[-4\partial_{\omega_0}\partial_{\omega_1}^2 d(\omega) - 4\sqrt{\omega_0}\partial_{\omega_0}^2\partial_{\omega_1} d(\omega) + \partial_{\omega_0}^2 d(\omega)].$$

• *If* $\partial_{\omega_0}\partial_{\omega_1}d(\omega) = \sqrt{\omega_0}\partial^2_{\omega_0}d(\omega)$, *then*

$$\frac{d^3}{d\lambda^3}d(\omega+\lambda\xi)\bigg|_{\lambda=0} = \omega_0^2(\partial^2_{\omega_0}d(\omega))^3[-4\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) + 4\sqrt{\omega_0}\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) + \partial^2_{\omega_0}d(\omega)].$$

Proof. By (31) and (34), we have

$$\begin{aligned}
\xi_0^3\partial^3_{\omega_0}d(\omega) &= -\omega_0^2(\partial^2_{\omega_0}d(\omega))^3\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) + \omega_0^2(\partial^2_{\omega_0}d(\omega))^4,\\
\xi_0^2\xi_1\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) &= \omega_0^2(\partial^2_{\omega_0}d(\omega))^2\partial_{\omega_0}\partial_{\omega_1}d(\omega)\partial^2_{\omega_0}\partial_{\omega_1}d(\omega),\\
\xi_0\xi_1^2\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) &= -\omega_0^2(\partial^2_{\omega_0}d(\omega))^4 - \omega_0^2(\partial^2_{\omega}d(\omega))^3\partial^3_{\omega_0}d(\omega),\\
\xi_1^3\partial^3_{\omega_1}d(\omega) &= \omega_0^2(\partial^2_{\omega_0}d(\omega))^2\partial_{\omega_0}\partial_{\omega_1}d(\omega)\partial^2_{\omega_0}\partial_{\omega_1}d(\omega).
\end{aligned}$$

These imply that

$$\begin{aligned}
\frac{d^3}{d\lambda^3}d(\omega+\lambda\xi)\bigg|_{\lambda=0} &= \xi_0^3\partial^3_{\omega_0}d(\omega) + 3\xi_0^2\xi_1\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) + 3\xi_0\xi_1^2\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) + \xi_1^3\partial^3_{\omega_1}d(\omega)\\
&= \omega_0^2(\partial^2_{\omega_0}d(\omega))^2[-4\partial^2_{\omega_0}d(\omega)\partial_{\omega_0}\partial^2_{\omega_1}d(\omega)\\
&\qquad + 4\partial_{\omega_0}\partial_{\omega_1}d(\omega)\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) + (\partial^2_{\omega_0}d(\omega))^2].
\end{aligned}$$

Then we obtain the conclusion. □

Proof of Lemma 1. Let $\omega_1 = 2z_0\sqrt{\omega_0}$. Then by (29), (30), (32) and (33), we have

$$\begin{aligned}
\partial^2_{\omega_0}d(\omega) &= 8\tilde{\kappa}_\omega\sqrt{\omega_0}\alpha_{0,\omega}(z_0^2-\sigma+1) - 8\tilde{\kappa}_\omega\sqrt{\omega_0}\alpha_{1,\omega}z_0(1-z_0^2),\\
\partial_{\omega_0}\partial_{\omega_1}d(\omega) &= -8\tilde{\kappa}_\omega\omega_0\alpha_{0,\omega}z_0(2-\sigma) + 8\tilde{\kappa}_\omega\omega_0\alpha_{1,\omega}(1-z_0^2),\\
\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) &= \frac{4\sqrt{\omega_0}\tilde{\kappa}_\omega\alpha_{0,\omega}}{\sigma(1-z_0^2)}[-(3\sigma-2)(2-\sigma)z_0^2 + (\sigma-1)^2(1-z_0^2)]\\
&\quad + \frac{4\sqrt{\omega_0}\tilde{\kappa}_\omega\alpha_{1,\omega}(3\sigma-2)z_0}{\sigma},\\
\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) &= \frac{4z_0\tilde{\kappa}_\omega\alpha_{0,\omega}}{\sigma(1-z_0^2)}[(3\sigma-2)(2-\sigma) - (\sigma-1)(1-z_0^2)]\\
&\quad + \frac{4\tilde{\kappa}_\omega\alpha_{1,\omega}}{\sigma}(-\sigma z_0^2 + z_0^2 - 2\sigma + 1),
\end{aligned}$$

If $\partial_{\omega_0}\partial_{\omega_1}d(\omega) = -\omega^{1/2}\partial^2_{\omega_0}d(\omega)$, we have $-(1-z_0-\sigma)\alpha_{0,\omega} = (1-z_0^2)\alpha_{1,\omega}$. This implies that

$$-4\partial_{\omega_0}\partial^2_{\omega_1}d(\omega) - 4\sqrt{\omega_0}\partial^2_{\omega_0}\partial_{\omega_1}d(\omega) + \partial^2_{\omega_0}d(\omega) = 8\sqrt{\omega_0}\tilde{\kappa}_\omega\alpha_{0,\omega}(\sigma-1)(1-z_0) \neq 0.$$

Similarly, if $\partial_{\omega_0}\partial_{\omega_1}d(\omega) = \omega^{1/2}\partial_{\omega_0}^2 d(\omega)$, we obtain

$$-4\partial_{\omega_0}\partial_{\omega_1}^2 d(\omega) + 4\sqrt{\omega_0}\partial_{\omega_0}^2\partial_{\omega_1}d(\omega) + \partial_{\omega_0}^2 d(\omega) = -8\sqrt{\omega_0}\tilde{\kappa}_\omega\alpha_{0,\omega}(\sigma-1)(1+z_0) \neq 0.$$

By Lemma 10, we conclude $\dfrac{d^3}{d\lambda^3}d(\omega+\lambda\xi)|_{\lambda=0} \neq 0$. □

Acknowledgements. The author would like to express his gratitude to Professor Masahito Ohta for helpful advices and encouragements. The author would also like to thank Professor Masaya Maeda for valuable discussion.

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