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APPARENT CONTOURS OF STABLE MAPS BETWEEN CLOSED SURFACES

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Abstract

Let M and N be connected and orientable, closed surfaces. For a stable map $\varphi : M \to N$, denote by $c(\varphi)$ and $n(\varphi)$ the numbers of cusps and nodes of φ respectively. In this paper, we determine the minimal number $c(\varphi) + n(\varphi)$ among the apparent contours of degree d stable maps $M \to N$ whose singular points set consists of one component.

1. Introduction

Let M be a closed and connected surface, N a connected surface, and $\varphi : M \to N$ be a C^∞ map. Define the set of singular points of φ as

$$S(\varphi) = \{p \in M \mid \operatorname{rank} d\varphi_p < 2\}.$$

We call $\varphi(S(\varphi))$ the *apparent contour* (or *contour* for short) of φ and denote it by $\gamma(\varphi)$.

A C^∞ map $\varphi : M \to N$ is said to be *stable* if it satisfies the following two properties.

(1) For each $p \in M$, the map germ of φ at $p \in M$ is C^∞ right-left equivalent to one of the map germs at $0 \in \mathbf{R}^2$ below:
- $(a, x) \mapsto (a, x)$: a regular point,
- $(a, x) \mapsto (a, x^2)$: a fold point,
- $(a, x) \mapsto (a, x^3 + ax)$: a cusp point.

Hence, $S(\varphi)$ is a finite disjoint union of circles.

(2) For each $q \in \gamma(\varphi)$, the map germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is right-left equivalent to one of the three multi-germs as depicted in Figure 1.

According to a classical result of Whitney [12], stable maps form an open dense subset in the space of all C^∞ maps $M \to N$ with respect to the Whitney C^∞ topology.

For a stable map $\varphi : M \to N$, the numbers of connected components of $S(\varphi)$, cusps and nodes on $\gamma(\varphi)$ are denoted by $i(\varphi)$, $c(\varphi)$ and $n(\varphi)$ respectively.

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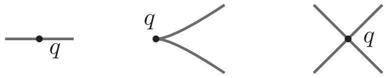

FIGURE 1. The multi-germs of $\varphi|_{S(\varphi)}$.

In this paper, we study a stable map with singular points.

An oriented and closed surface of genus g is denoted by Σ_g. The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbf{R}^2$ respectively.

Pignoni [8] introduced the notion of a minimal contour of a C^∞ map between surfaces and studied that of a C^∞ map $M \to \mathbf{R}^2$ of a closed surface into the plane: Let $\varphi_0 : M \to N$ be a C^∞ map and $\varphi : M \to N$ a stable map which is homotopic to φ_0 and whose singular points set consists of one component. The contour $\gamma(\varphi)$ is called a *minimal contour* of φ_0 if $c(\varphi) + n(\varphi)$ is the smallest among the contours of stable maps which are homotopic to φ_0 and whose singular points set consists of one component. Then, Demoto [1], Kamenosono and the author [4] studied minimal contours of C^∞ maps $M \to S^2$ of closed surfaces into the sphere. Furthermore, for each integer $i \geq 1$, the apparent contours of stable maps of connected and closed, orientable surfaces into the plane or the sphere whose singular points set consists i components were studied by Fukuda and the author [3, 14]. The author [15] also studied a 5-tuple of integers (g, d, i, c, n) such that there exists a degree d stable map $\Sigma_g \to N$, $N = \mathbf{R}^2$ or S^2, whose singular points set consists of i components and whose contour has c cusps and n nodes.

In this paper, we study the apparent contour of a stable map $\Sigma_g \to \Sigma_h$, $h \geq 1$, whose singular points set consists of one component. To study apparent contours, we generalize the formula obtained by Pignoni [8], Kamenosono and the author [4]. By using the generalized formula, Proposition 2.6, we study a minimal contour of degree d: Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d stable map whose singular points set consists of one component. Then, the contour $\gamma(\varphi)$ is called a *minimal contour of degree* d if the number $c(\varphi) + n(\varphi)$ is the smallest among the contours of degree d stable maps $\Sigma_g \to \Sigma_h$ whose singular points set consists of one component.

The purpose of this paper is to determine the number $c + n$ of minimal contour of degree d for each $d \geq 0$. Note that the contour of a minimal contour of degree d is not unique.

Recall that two C^∞ maps $\Sigma_g \to \Sigma_h$ are homotopic, then their degrees coincide. Recall also that two C^∞ maps $\Sigma_g \to S^2$ are homotopic if and only if their degrees coincide, see [7] for example. Thus, for any degree d C^∞ map $\varphi_0 : \Sigma_g \to \Sigma_h$, the notion of minimal contour of degree d and that of minimal contour of φ_0 coincide if $h = 0$.

Note that the following proposition was obtained by Kneser–Edmonds's theorem ([5, 6, 2]), see [13] for example.

PROPOSITION 1.1 ([5, 6, 2, 13]). *For integers $g \geq 0$ and $h \geq 1$, we define*

$$r(g,h) = \begin{cases} 0 & \text{if } g = 0,\ h \geq 1, \\ \infty & \text{if } g \geq 1,\ h = 1, \\ \left[\dfrac{g-1}{h-1}\right] & \text{otherwise,} \end{cases}$$

where $\left[\dfrac{g-1}{h-1}\right]$ *is the maximal integer which dose not exceed $(g-1)/(h-1)$.*

If $h \geq 1$ and $|d| > r(g,h)$, then there is no C^∞ map $\varphi : \Sigma_g \to \Sigma_h$ of degree d.

The main theorem of this paper is the following.

THEOREM 1.2. *Let g and h be non-negative integers with $h \geq 1$, and d a non-negative integer satisfying $d \leq r(g,h)$, $\varphi : \Sigma_g \to \Sigma_h$ a stable map whose singular points set consists of one component. Then, the contour $\gamma(\varphi)$ is a minimal contour of degree d if and only if the pair $(c(\varphi)), n(\varphi))$ is one of the items below*;

$h = 1$:

$$(c,n) = \begin{cases} (1)\ (0,0) & \text{if } d = 0 \text{ and } g = 0, \\ (2)\ (2,2) & \text{if } d = 0 \text{ and } g = 1, \\ (3)\ (2,0) & \text{if } d \neq 0 \text{ and } g = 1, \\ (4)\ (0, g-2) & \text{if } g \geq 2 \text{ is an even number and for any } d, \\ (5)\ (2, g-3) & \text{if } g \geq 3 \text{ is an odd number and for any } d, \end{cases}$$

$h \geq 2$:

$$(c,n) = \begin{cases} (6)\ (2,2) & \text{if } d = 0 \text{ and } g \text{ is an odd number satisfying } 1 \leq g \leq 2h-1, \\ (7)\ (2,0) & \text{if } d = 0 \text{ and } g \text{ is an odd number satisfying } g \geq 2h+1 \text{ or,} \\ & \text{if } d \geq 1 \text{ and } g > d(h-1),\ g \not\equiv d(h-1) \pmod 2, \\ (8)\ (0,0) & \text{otherwise.} \end{cases}$$

Figures 2 (1), (2), (3), (4) and (5) show examples of minimal contours of the cases (1), (2), (3), (4) and (5) respectively.

Theorem 1.2 shows the following corollary.

COROLLARY 1.3. *Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d stable map whose contour is a minimal contour of degree d. Then, the number of nodes on $\gamma(\varphi)$ is an even number.*

Remark that for a stable map $\varphi : M \to \Sigma_h$, the number of cusps $c(\varphi)$ and the Euler characteristic $\chi(M)$ have the same parity by a classical result of Thom [10]. Remark that for a C^∞ map $\varphi_0 : \Sigma_g \to \mathbf{R}^2$ or $\varphi_0 : \Sigma_g \to S^2$, the number of nodes on a minimal contour of φ_0 is an even number for each g, see [8] and [4] for the details. Note that there is a stable map $\Sigma_g \to \Sigma_h$ whose singular points set

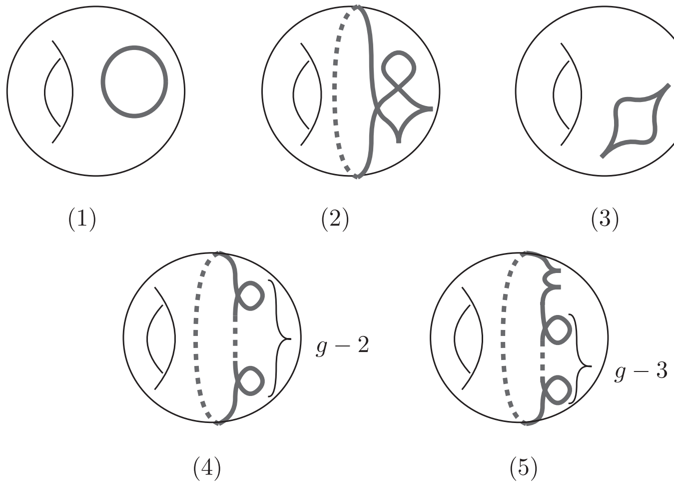

FIGURE 2. Minimal contours of degree d for C^∞ maps $\Sigma_g \to \Sigma_1$.

consists of one component and whose contour has odd number of nodes for each $g \geq 0$ and $h \geq 0$.

Remark 1.4. Theorem 1.2 makes the very first step toward classifying generic C^∞ maps between closed surfaces up to right-left equivalence.

This paper is organized as follows. In §2, we prepare some notions concerning to stable maps $\varphi : M \to \Sigma_h$, $(h \geq 0)$, and generalize the formula obtained by Pignoni [8], and Kamenosono and the author [4]. In §3, we construct stable maps $\Sigma_g \to \Sigma_h$, $(g \geq 0,\ h \geq 1)$, which are in the list of Theorem 1.2. In §4, we show the contours of stable maps constructed in §3 are minimal contours of degree d. In §5, we study the case of the apparent contours of fold maps $\varphi : \Sigma_g \to \Sigma_h$, $(g \geq 0,\ h \geq 1)$. In §6, we pose five problems which concern the apparent contours of stable maps $\varphi : M \to \Sigma_h$, $(h \geq 1)$.

Throughout this paper, all surfaces are connected and smooth of class C^∞, and all maps are smooth of class C^∞ unless stated otherwise. The symbols $d, g \geq 0$, $h \geq 0$ denote integers. For a topological space X, id_X denotes the identity map of X. For a closed surface M, $(M)_\ell$ denotes the surface obtained by removing ℓ open disks from M. The symbol D^2 denotes the closed disk in $\mathbf{R}^2$.

2. Preliminaries

In the following, for a closed surface M, we prepare some notions concerning the apparent contour of a stable map $\varphi : M \to \Sigma_h$, $(h \geq 0)$.

Let M be a closed surface and $\varphi : M \to \Sigma_h$ a stable map with singular points. Let $S(\varphi) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(\varphi)$ into the connected components and set $\gamma_i = \varphi(S_i)$, $(i = 1, \dots, \ell)$. Note that $\gamma(\varphi) = \gamma_1 \cup \cdots \cup \gamma_\ell$. Let $m(\varphi)$ be the smallest number of elements in the set $\varphi^{-1}(y)$, where $y \in \Sigma_h$ runs over all regular values of φ. Fix a regular value ∞ such that $\varphi^{-1}(\infty)$ consists of $m(\varphi)$ points. For each γ_i, denote by U_i the component of $\Sigma_h \backslash \gamma_i$ which contains ∞. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left hand side". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node, choose a normal vector v of γ_i at y such that $\varphi^{-1}(y')$ contains more elements than $\varphi^{-1}(y)$, where y' is a regular value of φ close to y in the direction of v. Let τ be a tangent vector of γ_i at y. It is easy to see that τ gives a well-defined orientation for γ_i.

Definition 2.1. A point $y \in \partial U_i \backslash \{\text{cusps}, \text{nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i. Otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \backslash \{\text{cusps}, \text{nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The number of positive and negative components is denoted by i^+ and i^- respectively.

Note that if $h = 0$, then there is at least one negative component. Note also that if $h \geq 1$ and $S(\varphi)$ consists of one component, then $\gamma(\varphi)$ is the negative component.

Definition 2.2. A point $y \in \partial U_i \backslash \{\text{cusps}, \text{nodes}\}$ is called an *admissible starting point* if y is a positive (or negative) point of a positive (resp. negative) component γ_i. Note that for each i, there always exists an admissible starting point on γ_i.

Definition 2.3. Let $y \in \gamma_i$ be an admissible starting point and $Q \in \gamma_i$ is a node. Let $\alpha : [0, 1] \to \gamma_i$ be a parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $0 < t_1 < t_2 < 1$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of Σ_h at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of Σ_h at Q; *negative*, otherwise, see Figure 3 for the details.

The number of positive and negative nodes on γ_i is denoted by $N_i^+(\varphi)$ and $N_i^-(\varphi)$ respectively. The definition of a positive (or negative) node on γ_i depends on the choice of an admissible starting point y. However, it is known that the algebraic number $N_i^+(\varphi) - N_i^-(\varphi)$ does not depend on the choice of y, see [11] for the details. Thus, the algebraic number $N^+(\varphi) - N^-(\varphi) = \sum_{i=1}^{\ell}(N_i^+(\varphi) - N_i^-(\varphi))$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

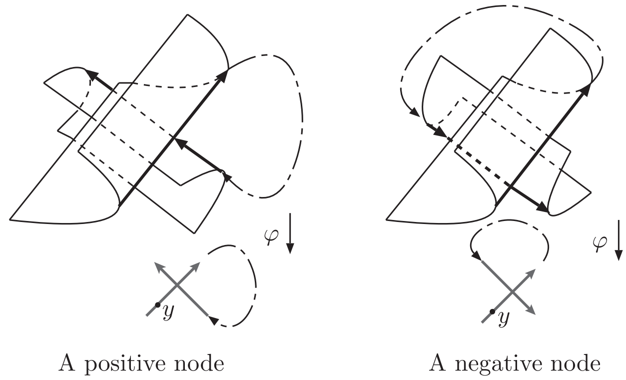

FIGURE 3. A positive node and a negative node.

Then, the following formula was obtained by Pignoni [8], Kamenosono and the author [4]

PROPOSITION 2.4 ([8, 4]). *For a stable map $\varphi : M \to S^2$ of a closed surface of genus g, we have*

$$g = \varepsilon(M)\left((N^+(\varphi) - N^-(\varphi)) + \frac{c(\varphi)}{2} + (1 + i^+ - i^-) - m(\varphi)\right) \tag{2.1}$$

where $\varepsilon(M)$ is equal to one if M is orientable, two otherwise.

In the following, we generalize formula (2.1) for a stable map $\varphi : M \to \Sigma_h$, $(h \geq 1)$, such that $i(\varphi) = 1$.

LEMMA 2.5. *Let $\varphi : M \to \Sigma_h$, $(h \geq 1)$ be a stable map such that $i(\varphi) = 1$. Then, there is a degree one stable map $\pi : \Sigma_h \to S^2$ such that it satisfies the following conditions*:

(1) *$\pi(\infty) \in S^2$ is a regular value of π, and $\pi^{-1}(\pi(\infty)) = \{\infty\}$,*
(2) *$S(\pi)$ consists of h components $S_1^\pi, \dots, S_h^\pi$, and $\gamma(\pi)$ has $4h$ cusps and no nodes, and*
(3) *$\gamma(\varphi) \cap S(\pi) = \emptyset$.*

Proof. Let us consider Σ_h is the sphere S with h 1-handles $H_1, \dots, H_h$. By modifying φ by a C^∞ homotopy if necessry, we assume that S contains ∞. Define a stable map $\pi_0 : \Sigma_h \to S^2$ by the projection of the 1-handles H_i $(i = 1, \dots, h)$ into S. Note that $S(\pi_0)$ consists of h components and $\gamma(\pi_0)$ has $4h$ cusps and no nodes. Let $S(\pi_0) = S_1^{\pi_0} \cup \cdots S_h^{\pi_0}$ be the decomposition of $S(\pi_0)$ into the connected components. Note that each $S_i^{\pi_0}$, $(i = 1, \dots, h)$, bounds a disk. Then, by shrinking $S_i^{\pi_0}$, $(i = 1, \dots, h)$, there exists a diffeomorphism $\Psi : \Sigma_h \to \Sigma_h$ such that $\Psi(S(\pi)) \cap \gamma(\varphi) = \emptyset$. Then, define $\pi = \pi_0 \circ \Psi^{-1}$. The map $\pi : \Sigma_h \to S^2$ satisfies the conditions (1), (2) and (3). $\square$

For a stable map $\varphi : M \to \Sigma_h$, $(h \geq 1)$, such that $i(\varphi) = 1$ and a stable map $\pi : \Sigma_h \to S^2$ as in Lemma 2.5, denote by $\xi_i(\varphi, \pi)$, $(i = \dots, h)$, the number of inverse image of $S_i^\pi \subset S(\pi)$, $(i = 1, \dots, h)$, by φ. Then, the following formula is obtained as an application of Proposition 2.4.

Proposition 2.6. *Let $\varphi : M \to \Sigma_h$, $(h \geq 1)$, be a stable map such that $i(\varphi) = 1$ and $\pi : \Sigma_h \to S^2$ be a stable map as in Lemma 2.5. Then, we have*

$$g = \varepsilon(M)\left((N^+(\varphi) - N^-(\varphi)) + \frac{c(\varphi)}{2} + \sum_{i=1}^{h} \xi_i(\varphi, \pi) - m(\varphi)\right) \tag{2.2}$$

where g is the genus of M, $\varepsilon(M)$ is equal to one if M is orientable, or two otherwise.

Proof. Let us consider the C^∞ map $\pi \circ \varphi : M \to S^2$, see Figure 4. The right bottom of Figure 4 shows the contour $\gamma(\pi \circ \varphi)$. Note that in the right bottom of Figure 4, the square $\gamma(\pi)$ overlaps $\xi_i(\varphi, \pi)$-fold. Then, by perturbing $\pi \circ \varphi$ as the $\xi_i(\varphi, \pi)$-fold square does not intersect each other, see the left bottom of Figure 4, we obtain a stable map $\Phi : M \to S^2$. Then, for a fixed regular value $\pi(\infty) \in S^2$, we obtain the folloiwng.

$$c(\Phi) = c(\varphi) + \sum_{i=1}^{h} 4\xi_i(\varphi, \pi), \quad i^+(\Phi) = 0, \quad i^-(\Phi) = 1 + \sum_{i=1}^{h} \xi_i(\varphi, \pi).$$

Let us consider the number of nodes $n(\Phi)$.

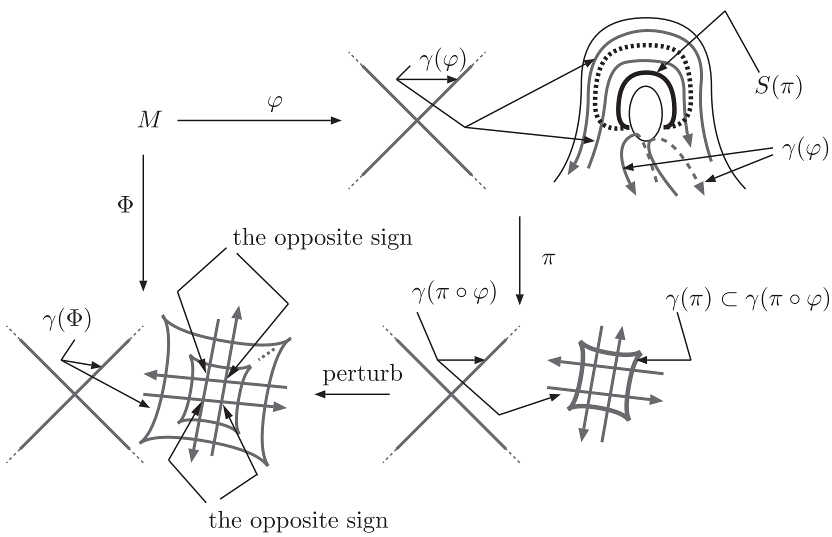

Figure 4

LEMMA 2.7. *Let us go along the contour $\gamma(\varphi)$ following the canonical orientation. If the contour $\gamma(\varphi)$ intersects transversely a longitude (or meridian) circle, then $\gamma(\varphi)$ intersects again the longitude (resp. meridian) circle in the opposite direction.*

Proof. For any point $q \in \gamma(\varphi)$, the difference between the number of inverse-image of a regular value in the left-hand side of q and that of a regular value in the right-hand side of q is two. It yeilds the conclusion. □

Lemma 2.7 yields that $n(\Phi)$ is of the form $n(\Phi) = n(\varphi) + 4k$, $(k \geq 0)$. Futhermore, we obtain the following lemma.

LEMMA 2.8. (1) *If a node $q \in \gamma(\varphi)$ is positive (or negative), then the corresponding node $q' \in \gamma(\Phi)$ is also positive (resp. negative).*
(2) *The algebraic number $N^+ - N^-$ of new nodes which are born by composing with π and perturbing $\pi \circ \varphi$ is zero.*

Proof. (1) It is trivial.
(2) New nodes appear as a pair of two positive nodes and two negative nodes, see the left bottom of Figure 4. □

Thus, we have

$$N^+(\varphi) - N^-(\varphi) = N^+(\Phi) - N^-(\Phi).$$

By applying formula (2.1) to Φ, we obtain formula (2.2). □

Let $\varphi : M \to \Sigma_h$, $(h \geq 1)$ be a stable map such that $i(\varphi) = 1$

LEMMA 2.9. *If $\gamma(\varphi)$ has a node, then it has at least one negative node.*

Proof. Let us go along $\gamma(\varphi)$ starting from an admissible starting point y, following the canonical orientation of $\gamma(\varphi)$. When we pass through a positive node on $\gamma(\varphi)$ for the first time, the number of points in the inverse-image decreases by two. This is a contradiction. □

In general, if the negative component $\gamma(\varphi)$ passes a positive (or negative) node, then the number of points in the inverse-image decrease (resp. increase) by two. Thus, we obtain the followin lemma.

LEMMA 2.10. *If there exists a point $q \in \Sigma_h$ such that $\varphi^{-1}(q)$ consists at least $m(\varphi) + 4$ points, then $\gamma(\varphi)$ has at least $(\#\varphi^{-1}(q) - m(\varphi) - 2)/2$ negative nodes, where $\#\varphi^{-1}(q)$ denote the number of inverse image of q by φ.*

3. Stable maps $\Sigma_g \to \Sigma_h$ in Theorem 1.2

In this section, we construct stable maps which are in the list of Theorem 1.2. Note that to construct such stable maps is a part of a proof of Theorem 1.2.

For any nonnegative integers d, g and h, we construct a degree d stable map $\Sigma_g \to \Sigma_h$ whose singular points set consists of one component. This map will be denoted by $\varphi_{d,g}^h$. In Figures, the numbers in the components of $\Sigma_h \backslash \gamma$ denote the numbers of inverse-images of a point in the components respectively.

3.1. Stable maps into Σ_1. For each odd number g and each even number $g \geq 2$, degree one stable maps $\varphi_{1,g}^0 : \Sigma_g \to S^2$ whose contours are in Figure 5 (1) with $u = 1$, $\ell_1 = g + 3$ if g is odd and Figure 5 (2) with $u = 1$, $\ell_2 = g + 2$ if $g \geq 2$ is even were obtained in [4] respectively. Then, by attaching five 1-handles and one 1-handle to the source surface and the target surface of $\varphi_{1,g}^0$ respectively, we obtain $\varphi_{1,g+5}^1 : \Sigma_{g+5} \to \Sigma_1$ whose contour is in Figure 2 (4) if g is odd, that is in Figure 2 (5) if $g \geq 2$ is even respectively. Figure 6 shows the procedure of $\varphi_{1,6}^1$.

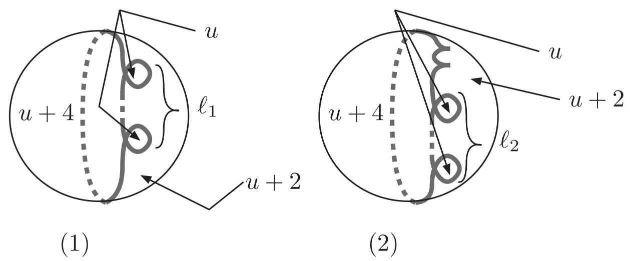

FIGURE 5. Apparent contours of stable maps $\Sigma_g \to S^2$.

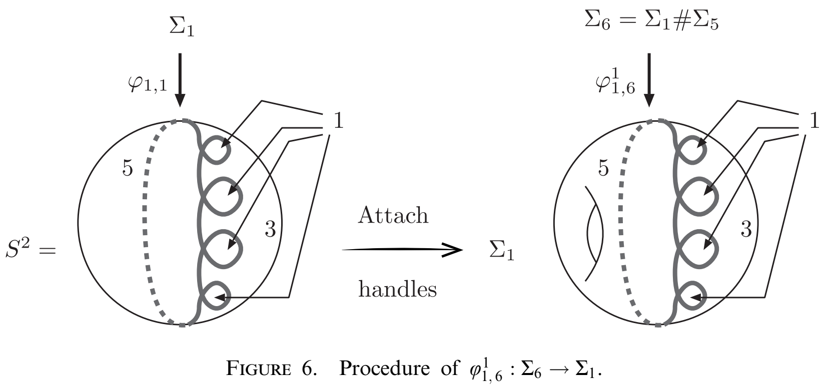

FIGURE 6. Procedure of $\varphi_{1,6}^1 : \Sigma_6 \to \Sigma_1$.

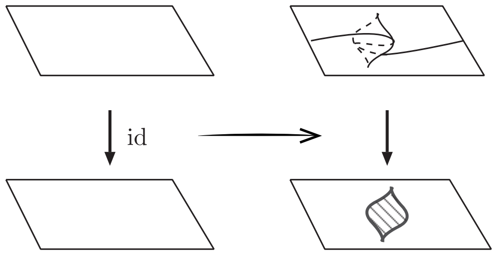

FIGURE 7. Making a pleat.

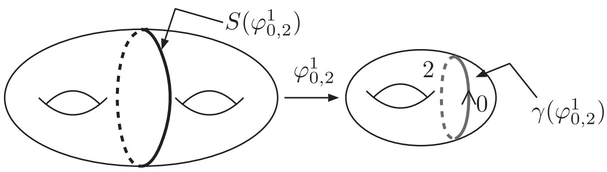

FIGURE 8. Stable map $\varphi^1_{0,2}$.

By making a pleat to $\mathrm{id}_{\Sigma_1} : \Sigma_1 \to \Sigma_1$ and $\mathrm{id}_{S^2} : S^2 \to S^2$, we obtain $\varphi^1_{1,1} : \Sigma_1 \to \Sigma_1$ and $\varphi^0_{1,0} : S^2 \to S^2$ whose pairs (c, n) are equal to $(2, 0)$ respectively. See Figure 7. By attaching three 1-handles and one 1-handle to the source source and the target surface of $\varphi^0_{1,0}$ respectively, we obtain $\varphi^1_{1,3} : \Sigma_3 \to \Sigma_1$ whose contour is in Figure 2 (5).

A degree zero stable map $\Sigma_1 \to S^2$ whose contour is in Figure 5 (2) with $u = 0$, $\ell_2 = 2$ was obtained in [4]. Then, by attaching four 1-handles and one 1-handle to the source surface and the target surface of the degree zero stable map $\Sigma_1 \to S^2$ respectively, we obtain $\varphi^1_{0,5} : \Sigma_5 \to \Sigma_1$ whose contour is in Figure 2 (5).

By combining the projection $S^2 \to D^2$ and the inclusion $D^2 \hookrightarrow \Sigma_1$, we obtain $\varphi^1_{0,0} : S^2 \to \Sigma_1$ whose contour is in Figure 2 (1). Similarly, by combining $\Sigma_1 \to D^2$ whose contour is minimal in [4] and the inclusion $D^2 \hookrightarrow \Sigma_1$, we obtain $\varphi^1_{0,1} : \Sigma_1 \to \Sigma_1$ whose contour is in Figure 2 (2).

Figures 8 and 9 define degree zero stable maps $\varphi^1_{0,2}$ and $\varphi^1_{0,4}$ respectively.

Thus, we obtained the following maps:

PROPOSITION 3.1. *There are degree zero stable maps $\varphi^0_{0,0}$, $\varphi^1_{0,1}$, $\varphi^1_{0,2}$, $\varphi^1_{0,4}$ and $\varphi^1_{0,5}$ and degree one stable maps $\varphi^1_{1,g} : \Sigma_g \to \Sigma_1$ with $g = 1, 3$ and $g \geq 6$ whose singular points set consist of one component and whose pairs (c, n) are one of the items below*:

$$(c,n) = \begin{cases} (0,0) & \textit{for } \varphi^1_{0,0} \textit{ and } \varphi^1_{0,2}, \\ (2,0) & \textit{for } \varphi^1_{1,1} \textit{ and } \varphi^1_{1,3}, \\ (2,2) & \textit{for } \varphi^1_{0,1} \textit{ and } \varphi^1_{0,5}, \\ (0,2) & \textit{for } \varphi^1_{0,4}, \\ (0,g-2) & \textit{for } \varphi^1_{1,g},\ g \geq 6 \textit{ is an even number}, \\ (2,g-3) & \textit{for } \varphi^1_{1,g},\ g \geq 7 \textit{ is an odd number}. \end{cases}$$

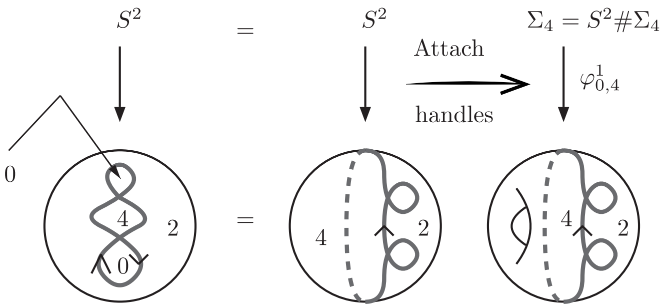

FIGURE 9. Procedure of $\varphi^1_{0,4}$.

Furthermore, by applying the following modification to $\varphi^1_{1,1}$, $\varphi^1_{0,2}$, $\varphi^1_{1,3}$, $\varphi^1_{0,4}$, $\varphi^1_{0,5}$ and $\varphi^1_{1,g}$ ($g \geq 6$), we obtain degree d stable maps $\varphi^1_{d,g} : \Sigma_g \to \Sigma_1$ whose contours are the same as $\gamma(\varphi^1_{1,1})$, $\gamma(\varphi^1_{0,2})$, $\gamma(\varphi^1_{1,3})$, $\gamma(\varphi^1_{0,4})$, $\gamma(\varphi^1_{0,5})$ and $\gamma(\varphi^1_{1,g})$ respectively for each $d \geq 1$ and $g \geq 1$: For a stable map $\varphi : \Sigma_g \to \Sigma_1$, $g \geq 1$, as the above, let μ be a meridian circle in $\varphi((\Sigma_g)_+) \subset \Sigma_1$, and $C \subset (\Sigma_g)_+$ a connected component of $\varphi^{-1}(\mu)$, where $(\Sigma_g)_+$ denotes the closure of the set of regular points whose neighborhoods are orientation preserved by the map. Then, by cutting Σ_g along C, we obtain two meridian kerfs A and B in $(\Sigma_g)_+$. Denote by N_B a sufficiently small neighborhood of $B \subset (\Sigma_g)_+$. Then, define $\varphi'|_{N_B}$ and $\varphi'|_{\Sigma_g \setminus N_B}$ by coiling N_B $d-1$ times along Σ_1 and $\varphi'|_{\Sigma_g \setminus N_B} = \varphi|_{\Sigma_g \setminus N_B}$ respectively. Finally stick the meridian kerfs A and B. Thus, we obtain a degree d stable map $\varphi' : \Sigma_g \to \Sigma_1$ whose contour is the same as $\gamma(\varphi)$. See Figure 10.

Similarly, for degree one stable maps $\varphi^1_{1,3}$ and $\varphi^1_{1,g}$ ($g \geq 6$) in Proposition 3.1, we obtain degree zero stable map $\varphi^1_{0,3}$ and $\varphi^1_{0,g}$ whose contours $\gamma(\varphi^1_{0,3})$ and $\gamma(\varphi^1_{0,g})$ are the same as $\gamma(\varphi^1_{1,3})$ and $\gamma(\varphi^1_{1,g})$ respectively. Thus, we obtain the following maps.

PROPOSITION 3.2. *For each $d \geq 0$, there is a degree d stable map $\varphi^1_{d,g} : \Sigma_g \to \Sigma_1$ whose singular points set consists of one component and whose pair (c,n) is in the list of Theorem* 1.2, *namely (c,n) is one of the following*:

$$(c,n)=\begin{cases}(1)\ \ (0,0) & \textit{if}\ \ d=0\ \ \textit{and}\ \ g=0,\\ (2)\ \ (2,2) & \textit{if}\ \ d=0\ \ \textit{and}\ \ g=1,\\ (3)\ \ (2,0) & \textit{if}\ \ d\neq 0\ \ \textit{and}\ \ g=1,\\ (4)\ \ (0,g-2) & \textit{if}\ \ g\geq 2\ \textit{is an even number and for any}\ d,\\ (5)\ \ (2,g-3) & \textit{if}\ \ g\geq 3\ \textit{is an odd number and for any}\ d.\end{cases}$$

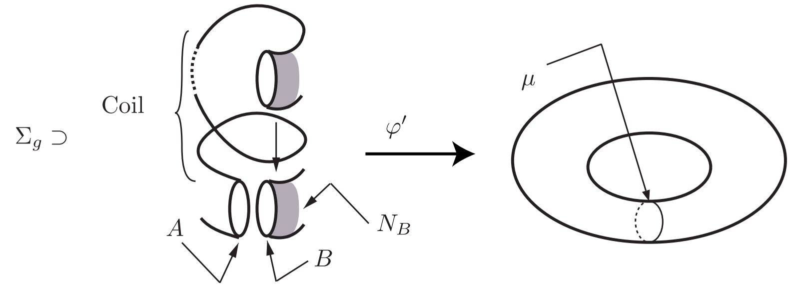

FIGURE 10

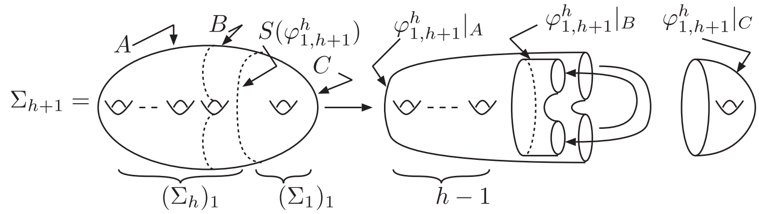

FIGURE 11

3.2. Stable maps into Σ_h, $(h \geq 2)$. In the following, assume that triples (d,g,h) satisfy the condition $d \leq r(g,h)$.

Denote by $\varphi^g_{1,g}$ the degree one stable map $\Sigma_g \to \Sigma_g$ which is obtained by making a pleat to id_{Σ_g}.

Figure 11 defines a degree one stable map $\Sigma_{h+1} \to \Sigma_h$ whose singular points set consists of one component and whose contour has no cusps and no nodes. Denote by $\varphi^h_{1,h+1}$ this degree one stable map. Note that, in Figure 11, Σ_{h+1} is divided into three parts A, B and C such that $A \cup B = (\Sigma_h)_1$ and $C = (\Sigma_1)_1$. Similarly, we define a stable map $\varphi^h_{1,g} : \Sigma_g \to \Sigma_h$ for each integers g and h satisfying $g \geq h+1$ and $g \equiv h+1 \pmod 2$. More precisely, for an integer g satisfying $g \geq h+1$ and $g \equiv h+1 \pmod 2$, put $g_+ = (g+h-1)/2$, $g_- = (g-h+1)/2$. Then, note that there are integers λ and μ satisfying $g_+ = h + \lambda(h-1) + \mu$ and $g_- = 1 + \lambda(h-1) + \mu$ where $\lambda \geq 0$ and $0 \leq \mu \leq h-2$. Then, define the map from $(\Sigma_{g_+})_1$ (or $(\Sigma_{g_-})_1$) into Σ_h as the similar way as

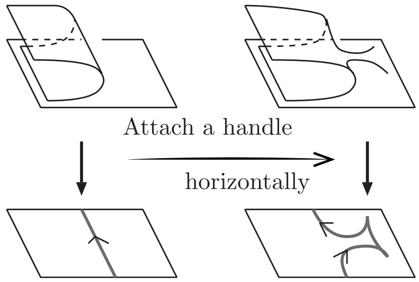

FIGURE 12. Attaching of a handle horizontally. The map is obtained when one projects these surfaces to the horizontal plane.

$\varphi^h_{1,h+1}|_{A\cup B}$ (resp. $\varphi^h_{1,h+1}|_C$). Note that $S(\varphi^h_{1,g})$ consists of one component and $\gamma(\varphi^h_{1,g})$ has no cusps and no nodes.

Then, by attaching a 1-handle horizontally to the source surface, see Figure 12 for the details, of $\varphi^h_{1,g}$ with $g \geq h+1$ and $g \equiv h+1 \pmod 2$, we obtain a degree one stable map $\Sigma_{g+1} \to \Sigma_h$ whose singular points set consists of one component and whose contour has two cusps, no nodes. Denote by $\varphi^h_{1,g+1}$ this degree one stable map. Thus, we obtain the following:

PROPOSITION 3.3. *For each $h \geq 2$ and each $g \geq h+1$, there is a degree one stable map $\varphi^h_{1,g} : \Sigma_g \to \Sigma_h$ whose singular points set consists of one component and the pair (c,n) is one of the following items*:

$$(c,n) = \begin{cases} (2,0) & \textit{if } \ g \equiv h \bmod 2, \\ (0,0) & \textit{otherwise}. \end{cases}$$

LEMMA 3.4. *Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d stable map whose singular points set consists of one component such that the genus of $(\Sigma_g)_+$ is greater than or equal to one. Then, there is a degree $d+1$ stable map $\varphi' : \Sigma_{g+h-1} \to \Sigma_h$ whose contour is diffeomorphic*[1] *to $\gamma(\varphi)$.*

Proof. For such C^∞ map $\varphi : \Sigma_g \to \Sigma_h$, divide Σ_g into $(\Sigma_1)_1 \cup (\Sigma_{g-1})_1$ such that $(\Sigma_1)_1 \subset (\Sigma_g)_+$. Then, insert $(\Sigma_{h-1})_2$ between $(\Sigma_1)_1$ and $(\Sigma_{g-1})_1$, and define $\varphi' : \Sigma_{g+h-1} \to \Sigma_h$ by $\varphi' = \varphi$ on $(\Sigma_1)_1$, $\varphi'|_{(\Sigma_{h-1})_2}$ is defined by Figure 13 and $\varphi'|_{(\Sigma_{g-1})_1} = r \circ \varphi|_{(\Sigma_{g-1})_1}$, where $r : \Sigma_h \to \Sigma_h$ denotes the diffeomorphism which is defined by the half-turn of Σ_h. See Figures 13 and 14. Thus, we obtain the desired degree $d+1$ stable map $\Sigma_{g+h-1} \to \Sigma_h$. $\square$

[1] Let M_i be smooth manifolds and $A_i \subset M_i$ be subsets, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every $q \in A_0$, there exists a smooth map $\tilde{g} : V \to M_1$ defined on a neighborhood V of $q \in M_0$ such that $\tilde{g}|_{V\cap A_0} = g|_{V\cap A_0}$. Furthermore, a smooth map $g : A_0 \to A_1$ is called a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth. When there exists a diffeomorphism between A_0 and A_1, we say that they are *diffeomorphic*.

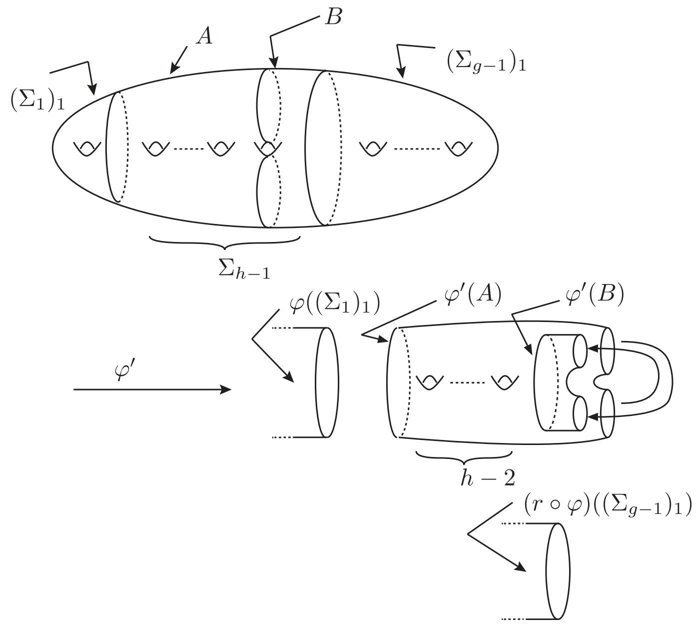

FIGURE 13. Construction $\varphi' : \Sigma_{g+h-1} \to \Sigma_h$.

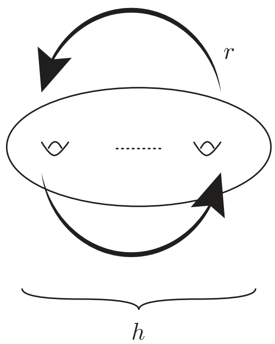

FIGURE 14. A diffeomorphism $r : \Sigma_h \to \Sigma_h$.

By applying Lemma 3.4 inductively to $\varphi^h_{1,g}$ in Proposition 3.3, we obtain a degree $d \geq 2$ stable maps $\varphi^h_{d,g+(d-1)(h-1)}$ whose contour is diffeomorphic to $\gamma(\varphi^h_{1,g})$.

For a stable map $\varphi^h_{1,g}$ in Proposition 3.3 with $g \not\equiv h \pmod 2$, by connecting Σ_h and Σ_g by a horizontal 1-handle, we define $\varphi^h_{0,g+h} : \Sigma_{g+h} \to \Sigma_h$ by $\varphi^h_{0,g+h}|_{\Sigma_h} = -\mathrm{id}_{\Sigma_h}$ and $\varphi^h_{0,g+h}|_{\Sigma_g} = \varphi$. Note that $S(\varphi^h_{0,g+h})$ consists of one component and $\gamma(\varphi^h_{0,g+h})$ has two cusps, no nodes. See Figure 15.

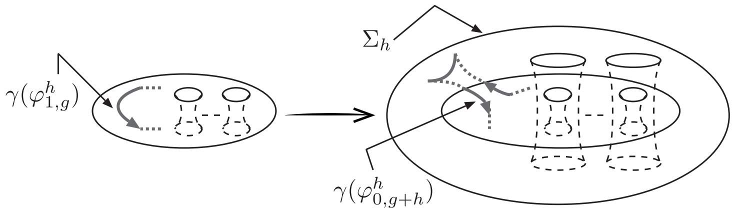

FIGURE 15. Attaching Σ_h horizontally.

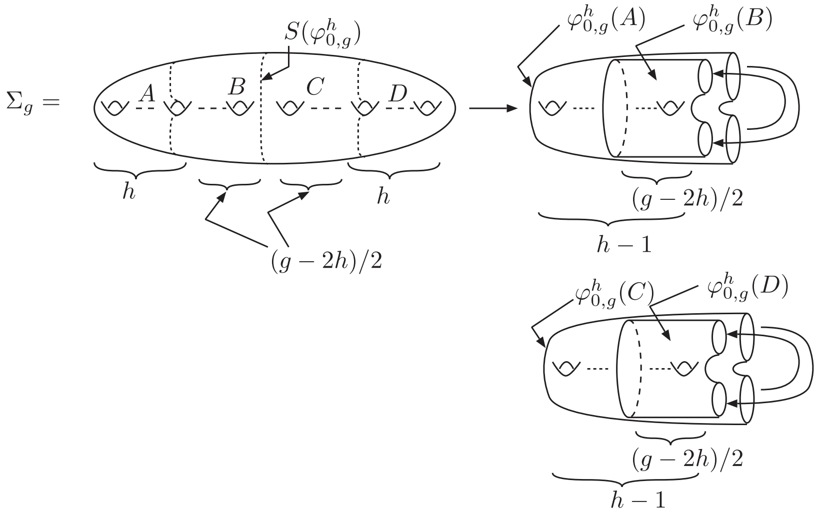

FIGURE 16

Figure 16 explains a construction of a degree zero stable map $\Sigma_g \to \Sigma_h$ whose singular points set consists of one component and the contour has no cusps, no nodes for each even number g satisfying $2h \leq g \leq 4h-4$. Denote by $\varphi^h_{0,g}$ the degree zero stable map. Similarly, for an even number g with $0 \leq g < 2h$, we have a degree zero stable map $\varphi^h_{0,g} : \Sigma_g \to \Sigma_h$ whose singular points set consists of one component and $\gamma(\varphi^h_{0,g})$ has no cusps and no nodes. Furthermore, we define $\varphi^h_{1,g} : \Sigma_g \to \Sigma_h$ for an even number g with $g > 4h-4$. More precisely, for an even number g which satisfies $g > 4h-4$, there are integers λ and μ which satisfy $g/2 = h + \lambda(h-1) + \mu$ where $\lambda \geq 0$ and $0 \leq \mu \leq h-2$. Then, define the map from $(\Sigma_{g/2})_1$ into Σ_h as $\varphi^h_{0,g}|_{A\cup B}$ in Figure 16. Note that $S(\varphi^h_{0,g})$ consists of one component and $\gamma(\varphi^h_{0,g})$ has no cusps and no nodes.

For each integer $h \geq 2$ and each odd number g in $1 \leq g \leq 2h-1$, put $(\ell_1, \ell_2, \ell_3) = s(1,-2,1) + 1/2(2h-g+1, g-1, 0)$, where s is an integer in $0 \leq s \leq (g-1)/4$. Note that $g = 2\ell_1 + 4\ell_3 + 1$ and $h = \ell_1 + \ell_2 + \ell_3$. Then, we have a

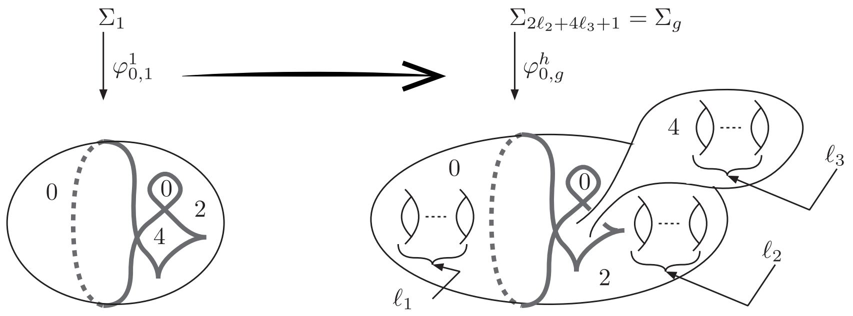

FIGURE 17

degree zero stable map $\varphi^h_{0,g} : \Sigma_g \to \Sigma_h$ as in Figure 17. This construction shows that stable maps whose singular points set consists of one component and whose pairs (c, n) are in the list of Theorem 1.2 are not unique.

Thus we obtain the following maps.

PROPOSITION 3.5. *For each $d \geq 0$ and $h \geq 2$ which satisfies $d \leq r(g,h)$, there is a degree d stable map $\varphi^h_{d,g} : \Sigma_g \to \Sigma_h$ whose singular points set consists of one component and whose pair (c,n) is in the list of Theorem* 1.2, *namely (c,n) is one of the following*:

$$(c,n) = \begin{cases} (6)\ (2,2) & \text{if } d = 0 \text{ and } g \text{ is an odd number satisfying } 1 \leq g \leq 2h-1, \\ (7)\ (2,0) & \text{if } d = 0 \text{ and } g \text{ is an odd number satisfying } g \geq 2h+1 \text{ or,} \\ & \text{if } d \geq 1 \text{ and } g > d(h-1),\ g \not\equiv d(h-1) \pmod 2, \\ (8)\ (0,0) & \text{otherwise.} \end{cases}$$

4. Proof of minimum of $c + n$ in Theorem 1.2

In this section, we prove that the contours $\gamma(\varphi^h_{d,g})$ constructed in §3 are minimal contours of degree d, for each integers $d \geq 0$ and $g \geq 0$, $h \geq 1$.

The contours $\gamma(\varphi^1_{d,g})$ in Proposition 3.2 (1), (4) with $g = 2$, and the contour $\gamma(\varphi^h_{d,g})$ in Proposition 3.5 (8) are trivially minimal contours of degree d respectively. Thus, the cases Theorem 1.2 (1), (4) with $g = 2$ and (8) are proved.

LEMMA 4.1. *Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d stable map such that $i(\varphi) = 1$. If the number $d - dh + g$ is odd, then $\gamma(\varphi)$ has at least two cusps.*

Proof. To prove this lemma, apply a result of Quine [9]: for a stable map $\varphi : M \to N$ between oriented surfaces, we have

$$\chi(M) - 2\chi(M_-) + \sum_{q_k:\text{cusp}} \operatorname{sign}(q_k) = (\deg \varphi)\chi(N),$$

where M_{-} denotes the closure of the set of regular points at which φ reverses orientation, and $\mathrm{sign}(q_k) = \pm 1$ the sign of a cusp q_k, see [9] for details.

Apply our situation to the Quine's formula:

$$\sum_{q_k:\mathrm{cusp}} \mathrm{sign}(q_k) = 2(d - dh + g - 2\mu),$$

where $(\Sigma_g)_{-}$ is homeomorphic to $(\Sigma_\mu)_1$, $(\mu = 0, 1, \ldots, g)$. Then, this follows immediately. $\square$

Lemma 4.1 shows the contours of stable maps $\varphi^1_{d,1}$ $(d \neq 0)$ and $\varphi^1_{d,3}$ $(d \neq 0)$ in Proposition 3.2 and the contours of stable maps $\varphi^h_{d,g}$ in Proposition 3.5 (7) are minimal contours of degree d respectively. Thus, the cases Theorem 1.2 (3), (5) with $g = 3$ and (7) are proved.

The rest cases of Theorem 1.2 (2) and (4) with $g \geq 4$, (5) with $g \geq 5$, (6) are proved by each case $h = 1$ and $h \geq 2$.

4.1. The case of $h = 1$**.** Let us consider the case (4) with $g \geq 4$ and (5) with $g \geq 5$. Let $\varphi : \Sigma_g \to \Sigma_1$ be a stable map such that $i(\varphi) = 1$. Then, formula (2.2) implies that

$$g = (N^+ - N^-) + \frac{c(\varphi)}{2} + \xi_1(\varphi, \pi) - m(\varphi). \tag{4.1}$$

Note that $\xi_1(\varphi, \pi) - m(\varphi) \geq 0$ is an even number. Let us divide the cases for the value $\xi_1(\varphi, \pi) - m(\varphi)$.

(i1) $\xi_1(\varphi, \pi) - m(\varphi) \leq 2$: In this case, formula (4.1) implies that if $\gamma(\varphi)$ has no node, then

$$c(\varphi) \geq 2(g - 2). \tag{4.2}$$

If $\gamma(\varphi)$ has a node, then formula (4.1) and Lemma 2.9 imply that $c(\varphi) + n(\varphi) \geq c(\varphi)/2 + g$. This inequality and Lemma 4.1 show that if $g \geq 4$ is even, then

$$c(\varphi) + n(\varphi) \geq g \tag{4.3}$$

otherwise,

$$c(\varphi) + n(\varphi) \geq g + 1. \tag{4.4}$$

(i2) $\xi_1(\varphi, \pi) - m(\varphi) \geq 4$: In this case, formula (4.1) implies that $c(\varphi) + n(\varphi) = c(\varphi)/2 + g + 2N^- - (\xi_1(\varphi, \pi) - m(\varphi))$. Then, Lemma 2.10 implies that $N^- \geq (\xi_1(\varphi, \pi) - m(\varphi) - 2)/2$. It yields that $c(\varphi) + n(\varphi) \geq c(\varphi)/2 + g - 2$. This inequality and Lemma 4.1 shows that if $g \geq 4$ is even, then

$$c(\varphi) + n(\varphi) \geq g - 2 \tag{4.5}$$

otherwise

$$c(\varphi) + n(\varphi) \geq g - 1. \tag{4.6}$$

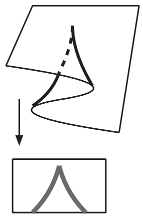

FIGURE 18. Geometrical condition of cusps.

Thus, from the inequalities (4.2), (4.3) and (4.5), Theorem 1.2 (4) with $g \geq 4$ is proved. Similarly, from the inequalities (4.2), (4.4), and (4.6), Theorem 1.2 (5) with $g \geq 5$ are proved.

Finally, let us consider the case (2). We will show that the contour $\gamma(\varphi_{0,1}^1)$ in Proposition 3.2 is a minimal contour of degree zero. Recall that the pair (c, n) of $\gamma(\varphi_{0,1}^1)$ is equal to $(2, 2)$. Let $\varphi : \Sigma_1 \to \Sigma_1$ be a degree zero stable map such that $i(\varphi) = 1$.

LEMMA 4.2. *Assume $h \geq 1$ and $0 \leq g \leq 2h - 1$. Then, there is no degree zero C^∞ map $\varphi : \Sigma_g \to \Sigma_h$ whose singular points set consists of one component such that $m(\varphi) > 0$.*

Proof. If there is a such C^∞ map $\varphi : \Sigma_g \to \Sigma_h$, then $(\Sigma_g)_+$ and $(\Sigma_g)_-$ contain $(\Sigma_h)_1$. It is a contradiction. □

Lemma 4.1 implies that the contour $\gamma(\varphi)$ has at least two cusps. Then, Lemma 4.2 and the geometrical condition for cusps, the contour $\gamma(\varphi)$ has at least one negative node, see Figure 18 for details. Thus, we have $c(\varphi) + n(\varphi) \geq 3$.

LEMMA 4.3. *There is no degree zero stable map $\varphi : \Sigma_1 \to \Sigma_1$ whose singular points set consists of one component and has two cusps and one node.*

Proof. Assume that there exists such stable map $\varphi : \Sigma_1 \to \Sigma_1$. Then, formula (4.1) implies that

$$1 = (0 - 1) + \frac{2}{2} + \xi_1(\varphi, \pi) - m(\varphi).$$

Note that $\xi_1(\varphi, \pi) - m(\varphi)$ is an even number. It is a contradiction. □

Thus, we have $c(\varphi) + n(\varphi) \geq 4$. It implies that the contour $\gamma(\varphi_{0,1}^1)$ is a minimal contour of degree zero.

Therefore, we have completed the proof of Theorem 1.2 with $h = 1$.

4.2. The case of $h \geq 2$. Let us prove the case (6).

By Lemma 4.2, for each $h \geq 2$ and each odd number g with $0 \leq g \leq 2h-1$, a degree zero C^∞ map $\varphi : \Sigma_g \to \Sigma_h$ satisfies $m(\varphi) = 0$. Then, we can prove the contours $\gamma(\varphi_{0,g}^h)$ in Proposition 3.5 (6) are minimal contour of degree zero by the similarly way as the case of Theorem 1.2 (2). We omit the proof here.

Therefore, we have completed the proof of Theorem 1.2 with $h \geq 2$.

It completes a proof of Theorem 1.2.

5. Fold map case

Let M be a connected and closed surface, and N be a connected surface. A stable map $\varphi : M \to N$ which has no cusp is called a *fold map*.

Let $\varphi : M \to N$ be a degree d fold map such that $i(\varphi) = 1$. Then, call the contour $\gamma(\varphi)$ an *$\mathscr{F}$-minimal contour* of degree d if the number $n(\varphi)$ is the smallest among the contours of degree d fold maps whose singular points set consists of one component.

Note that by Lemma 4.1, if $d - dh + g$ is odd, then there is no degree d fold map $\Sigma_g \to \Sigma_h$ whose singular points set consists of one component.

Then, as a corollary of Theorem 1.2, we obtain the following.

Theorem 5.1. *Assume $d - dh + g$ be an even number and $h \geq 1$. Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d fold map such that $i(\varphi) = 1$. Then, $\gamma(\varphi)$ is an $\mathscr{F}$-minimal contour of degree d if and only if the number of nodes $n(\varphi)$ is one of the items below*:

$h = 1$:

$$n(\varphi) = \begin{cases} 0 & \text{if } d = 0 \text{ and } g = 0, \\ g - 2 & \text{if } g \text{ is an even number and for any } d, \end{cases}$$

$h \geq 2$:

$$n(\varphi) = 0.$$

Corollary 5.2. *Assume $d - dh + g$ be an even number. Let $\varphi : \Sigma_g \to \Sigma_h$ be a degree d fold map such that $i(\varphi) = 1$. If the contour $\gamma(\varphi)$ is an $\mathscr{F}$-minimal contour of degree d, then it is a minimal contour of degree d.*

6. Problems

In this section, we pose some problems with respect to the apparent contour of a stable map $M \to N$ between closed surfaces.

Problem 6.1. For a degree d C^∞ map $\varphi_0 : \Sigma_g \to \Sigma_h$, study the relation between a minimal contour of φ_0 and a minimal contour of degree d.

Pignoni [8], Kamenosono and the author [4] studied a minimal contour of a C^∞ map $F \to S^2$ of a non-orientable surface.

Problem 6.2. Let $d_2 = 0$ or 1, and F a non-orientable closed surface. Study a minimal contour of modulo two degree d_2 for a C^∞ map $F \to \Sigma_h$. Furthermore, for a non-orientable closed surface F with even genus, study an $\mathscr{F}$-minimal contour of modulo two degree d_2 for a C^∞ map $F \to \Sigma_h$.

Let $\varphi : M \to N$ be a degree d stable map whose singular points set consists of one component. Then, the contour $\gamma(\varphi)$ is an *essential contour* of degree d if the pair (c, n) is the smallest with respect to the lexicographic order among the stable maps $M \to N$ whose degrees are d and whose singular points set consists of one component. Then, Theorem 1.2 yields the following theorem.

Theorem 6.3. *Let $\varphi : \Sigma_g \to \Sigma_h$, $(h \geq 1)$ be a degree d stable map whose singular points set consists of one component. Then, $\gamma(\varphi)$ is a minimal contour of degree d if and only if it is an essential contour of degree d.*

Note that for C^∞ map $h_0 : \mathbf{R}P^2 \to \mathbf{R}^2$ or $h_0 : \mathbf{R}P^2 \to S^2$ of modulo two degree one, a minimal (or an essential) contour of h_0 is not essential (resp. minimal), see [8, 4] for the details. Thus, we pose the following problem.

Problem 6.4. Study essential contours of C^∞ maps $h_0 : F \to \Sigma_h$, $(h \geq 1)$, of non-orientable surfaces F. Then, compare minimal contours of h_0 and essential contours of h_0.

The author [15] determined 5-tuples of integers (g, d, i, c, n) such that there exists a degree d stable map $\Sigma_g \to N$, $N = \mathbf{R}^2$ or S^2, whose singular points set consists of i components and whose contour has c cusps and n nodes.

Problem 6.5. Determine a 6-tuples of integers (g, h, d, i, c, n) such that there exists a degree d stable map $\Sigma_g \to \Sigma_h$ whose singular points set consists of i components and whose contour has c cusps and n nodes.

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