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RATE FUNCTIONS FOR RANDOM WALKS ON RANDOM CONDUCTANCE MODELS AND RELATED TOPICS

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Abstract

We consider laws of the iterated logarithm and the rate function for sample paths of random walks on random conductance models under the assumption that the random walks enjoy long time sub-Gaussian heat kernel estimates.

1. Introduction

The random conductance model (RCM) is a pair of a graph and a family of non-negative random variables (random conductances) which are indexed by edges of the graph. The RCM includes various important examples such as the supercritical percolation cluster, whose random conductances are i.i.d. Bernoulli random variables. In the recent progress on the RCM, various asymptotic behaviors of random walks are obtained on a class of RCM such as invariance principle, functional CLT, local CLT and long time heat kernel estimates. Here is a partial list of examples of the RCM;

1. Uniform elliptic case [14],
2. The supercritical percolation cluster [3],
3. I.i.d. unbounded conductance bounded from below [5],
4. I.i.d. bounded conductance under some tail conditions near 0 [10],
5. The level sets of Gaussian free field and the random interlacements [34].

We refer to [8], [32], [36] for the invariance principle for random walks on the supercritical percolation cluster, [6] for the local limit theorem for random walks on the supercritical percolation cluster, [1] for the invariance principle on general i.i.d. RCMs, [2] for the Gaussian heat kernel upper bound on the possibly degenerate RCMs. We also refer to [9] and [29] for more details about the RCM.

In [31], we discussed the laws of the iterated logarithms (LILs) for discrete time random walks on a class of RCM under the assumption on long time heat kernel estimates. The aims of this paper are to establish the laws of the iterated

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logarithm and to describe the rate functions for the sample paths of continuous time random walks on the RCM.

The LILs describe the fluctuation of stochastic processes, which was originally obtained by Khinchin [24] for a random walk. We establish the LIL w.r.t. both $\sup_{0 \le s \le t} d(Y_0^\omega, Y_s^\omega)$ and $d(Y_0^\omega, Y_t^\omega)$, and another LIL, which describes liminf behavior of $\sup_{0 \le s \le t} d(Y_0^\omega, Y_s^\omega)$, where $\{Y_t^\omega\}_{t \ge 0}$ is a continuous time random walk on the random environment ω.

The rate function describes the sample path ranges of stochastic processes. For d-dimensional Brownian motion $B = \{B_t\}_{t \ge 0}$, the Kolmogorov test tells us that

$$\mathbf{P}(|B_t| \ge t^{1/2} h(t) \text{ for sufficiently large } t) = \begin{cases} 1 \\ 0, \end{cases} \text{according as } \int_1^\infty \frac{1}{t} h(t)^d e^{-h(t)^2/2}\, dt \begin{cases} < \infty \\ = \infty, \end{cases}$$

where $h(t)$ is a positive function such that $h(t) \nearrow \infty$ as $t \to \infty$. For $d \ge 3$, the Dvoretzky and Erdős test tells us that

$$\mathbf{P}(|B_t| \ge t^{1/2} h(t) \text{ for sufficiently large } t) = \begin{cases} 1 \\ 0, \end{cases} \text{according as } \int_1^\infty \frac{1}{t} h(t)^{d-2}\, dt \begin{cases} < \infty \\ = \infty, \end{cases} \tag{1.1}$$

where $h(t)$ is a positive function such that $h(t) \searrow 0$ as $t \to \infty$. These results were extended to various frameworks such as symmetric stable processes on $\mathbf{R}^d$, Brownian motions on Riemannian manifolds, symmetric Markov chains on weighted graphs and β stable like processes ($\beta \ge 2$). We refer to [21], [25], [26], [39], [41] for stable processes on $\mathbf{R}^d$, [18], [19] for Brownian motions on Riemannian manifolds, [22], [23] for symmetric Markov chains on weighted graphs, [35] for β stable like processes. We establish an analogue of (1.1) w.r.t. random walks on the RCM.

Our approach is as follows; We assume quenched heat kernel estimates and establish both quenched LILs and an analogue of the Dvoretzky and Erdős test. As we will see in Section 1.2, our results are applicable for various models since heat kernel estimates are obtained for random walks on various RCMs. The concrete examples are given in Section 1.2.

The organization of this paper is as follows. First, we give the framework and main results of this paper in Section 1.1 and examples in Section 1.2. In Section 2 we establish some preliminary results. In Section 3 we give the proof of the LILs. In Section 4 we establish an analogue of (1.1). Finally in Section 5 we discuss the case where $G = \mathbf{Z}^d$ and the media is ergodic.

In this paper, we use the following notation.

Notation. (1) *We use $c, C, c_1, c_2, \ldots$ as the deterministic positive constants. These constants do not depend on the random environment ω, time parameters $t, s \ldots$, distance parameters $r, \ldots$, and vertices of graphs.*

(2) *We define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.*

1.1. Framework and main results

Let $G = (V, E) = (V(G), E(G))$ be a countable and connected graph of bounded degree, i.e. $M := \sup_{x \in V(G)} \deg x < \infty$. We write $x \sim y$ iff $(x, y) \in E(G)$. A sequence $\ell_{xy} : x = x_0,\ x_1, \dots, x_n = y$ on G is called a path from x to y if $x_i \sim x_{i+1}$ for all $i = 0, 1, \dots, n-1$. We write $d(\cdot,\cdot)$ as the usual graph distance, that is, the length of a shortest path in G, and denote $B(x, r) = \{y \in V(G) \mid d(x, y) \le r\}$.

Throughout this paper we assume that there exist $\alpha \ge 1$, $c_1, c_2 > 0$ such that

$$c_1 r^{\alpha} \le \# B(x, r) \le c_2 r^{\alpha} \tag{1.2}$$

for any $x \in V(G)$ and $r \ge 1$.

We introduce the random conductance model below. Let $\omega = \{\omega_e = \omega_{xy}\}_{e=(x,y) \in E(G)}$ be a family of non-negative weight which is defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We call ω the random conductance. For non-negative weights $\omega = \{\omega_e\}_e$, we define $\pi^{\omega}(x) = \sum_{y; y \sim x} \omega_{xy}$ and $\nu^{\omega}(x) = 1$. We fix a base point $x_0 \in V(G)$, and define graphs $G^{\omega} = (V(G^{\omega}), E(G^{\omega}))$ as

$$V(G^{\omega}) = \left\{ y \in V(G) \,\middle|\, \begin{array}{l} \text{There exists a path } \ell_{x_0 y} : x_0, x_1, \dots, x_n = y \text{ such that} \\ \omega_{x_i x_{i+1}} > 0 \text{ for all } i = 0, 1, \dots, n-1. \end{array} \right\},$$

$$E(G^{\omega}) = \{e = (x, y) \in E(G) \mid x, y \in V(G^{\omega}) \text{ and } \omega_{xy} > 0\}.$$

We denote $d^{\omega}(\cdot,\cdot)$ as the graph distance of G^{ω}. Note that $G^{\omega} = G$ and $d^{\omega} = d$ if conductance ω is strictly positive.

We will consider two types of random walks, constant speed random walk (CSRW) and variable speed random walk (VSRW) associated with $\omega \in \Omega$. Both CSRW and VSRW are continuous time random walks whose transition probability is given by $P^{\omega}(x, y) = \dfrac{\omega_{xy}}{\pi^{\omega}(x)}$. For the CSRW, the holding time distribution at $x \in V(G^{\omega})$ is Exp (1), whereas for the VSRW, the holding time distribution at $x \in V(G^{\omega})$ is Exp $(\pi^{\omega}(x))$. We write $\mathcal{L}_{\theta}^{\omega}$ for the generator which is given by

$$\mathcal{L}_{\theta}^{\omega} f(x) = \frac{1}{\theta^{\omega}(x)} \sum_{y; y \sim x} (f(y) - f(x)) \omega_{xy},$$

and we also write the corresponding heat kernel as

$$q_t^{\omega}(x, y) = \frac{P^{\omega}(x, y)}{\theta^{\omega}(y)},$$

where $\theta^{\omega} = \pi^{\omega}$ for the CSRW case and $\theta^{\omega} \equiv 1$ for the VSRW case. We write $Y^{\omega} = \{Y_t^{\omega}\}_{t \ge 0}$ as either the CSRW or the VSRW, P_x^{ω} as the law of the random walk Y^{ω} which starts at x, and

$$\begin{aligned} \tau_F &= \tau_F^{\omega} = \inf\{t \ge 0 \mid Y_t^{\omega} \notin F\}, \quad \sigma_F = \sigma_F^{\omega} = \inf\{t \ge 0 \mid Y_t^{\omega} \in F\}, \\ \sigma_F^{+} &= \sigma_F^{+\omega} = \inf\{t > 0 \mid Y_t^{\omega} \in F\}. \end{aligned} \tag{1.3}$$

We denote $F^{\omega} = F \cap V(G^{\omega})$, $V^{\omega}(F) = \sum_{y \in F \cap V^{\omega}(G)} \theta^{\omega}(y)$ for $F \subset V(G)$ and $V^{\omega}(x, r) = V^{\omega}(B(x, r))$. We write $T_0^{\omega} = 0$ and $T_{n+1}^{\omega} = \inf\{t > T_n^{\omega} \mid Y_t^{\omega} \neq Y_{T_n^{\omega}}^{\omega}\}$, and introduce a discrete time random walk $\{X_n^{\omega} := Y_{T_n^{\omega}}^{\omega}\}_{n \geq 0}$.

First, we state the results about the LILs. To do this, we need the following assumptions.

ASSUMPTION 1.1. *There exist positive constants ε, β such that $\varepsilon + 1 < \beta$ and a family of non-negative random variables $\{N_x = N_{x,\varepsilon}\}_{x \in V(G)}$ such that the following hold*;

(1) *There exist positive constants $c_{1.1}$, $c_{1.2}$, $c_{1.3}$, $c_{1.4}$ such that*

$$q_t^{\omega}(x, y) \leq \begin{cases} \dfrac{c_{1.1}}{t^{\alpha/\beta}} \exp\left(-c_{1.2}\left(\dfrac{d(x, y)^{\beta}}{t}\right)^{1/(\beta-1)}\right), & \text{if } t \geq d(x, y), \\ c_{1.3} \exp\left(-c_{1.4} d(x, y)\left(1 \vee \log \dfrac{d(x, y)}{t}\right)\right), & \text{if } t \leq d(x, y), \end{cases} \tag{1.4}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^{\omega})$ and $t \geq N_x(\omega)$.

(2) *There exist positive constants $c_{2.1}$, $c_{2.2}$ such that*

$$q_t^{\omega}(x, y) \geq \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left(-c_{2.2}\left(\frac{d(x, y)^{\beta}}{t}\right)^{1/(\beta-1)}\right) \tag{1.5}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^{\omega})$ and $t \geq 0$ with $d(x, y)^{1+\varepsilon} \vee N_x(\omega) \leq t$.

(3) *There exist positive constants $c_{3.1}$, $c_{3.2}$ such that*

$$c_{3.1} r^{\alpha} \leq V^{\omega}(x, r) \leq c_{3.2} r^{\alpha} \tag{1.6}$$

for almost all $\omega \in \Omega$, all $x \in V(G^{\omega})$ and $r \geq N_x(\omega)$.

(4) *There exist positive constants $c_{4.1}$, $c_{4.2}$, $c_{4.3}$, $c_{4.4}$, $c_{4.5}$ such that*

$$q_t^{\omega}(x, y) \leq \begin{cases} \dfrac{c_{4.1}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}} \exp\left(-c_{4.2}\dfrac{d(x, y)^2}{t}\right), & \text{if } t \geq c_{4.3} d(x, y), \\ \dfrac{c_{4.4}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}} \exp\left(-c_{4.5} d(x, y)\left(1 \vee \log \dfrac{d(x, y)}{t}\right)\right), & \text{if } t \leq c_{4.3} d(x, y), \end{cases} \tag{1.7}$$

for almost all $\omega \in \Omega$, all $t > 0$ and $x, y \in V(G^{\omega})$ with $d(x, y) \geq N_x(\omega) \wedge N_y(\omega)$.

Note that (1.4) holds for $t \geq N_x(\omega)$ while (1.7) holds for all $t > 0$. (1.7) is called the Carne-Varopoulos bound. This type of bound was originally obtained by [11], [43]. It is known that (1.7) holds under general conditions which will be described in the following Proposition (see [17, Theorems 2.1 and 2.2]).

Proposition 1.2. *Let* $\{N_x\}$ *be as in Assumption* 1.1 *and* $d_\theta^\omega(\cdot,\cdot)$ *be a metric on* $G^\omega = (V(G^\omega), E(G^\omega))$ *which satisfies*

$$\frac{1}{\theta^\omega(x)} \sum_{y \in V(G^\omega)} d_\theta^\omega(x,y)^2 \omega_{xy} \le 1. \tag{1.8}$$

If there exists a positive constant c *such that* $d_\theta^\omega(x,y) \ge c d(x,y)$ *for all* $x, y \in V(G^\omega)$ *with* $d(x,y) \ge N_x(\omega) \wedge N_y(\omega)$, *then* (1.7) *holds.*

Next we assume the following three types of integrability conditions.

Assumption 1.3. *Let* $\{N_x\}_{x \in V(G)}$ *be as in Assumption* 1.1 *and define* $f(t) = f_\varepsilon(t) = \mathbf{P}(N_x \ge t)$. *We impose one of the following three types of integrability conditions on* $f(t)$.

(1) $\sum_{n\ge 1} n^\alpha f(n) < \infty$,
(2) $\sum_{n\ge 1} n^{\alpha\beta} f(n) < \infty$,
(3) *For positive and non-increasing function* $h(t)$, $\sum_n n^\alpha f(nh(n^\beta)) < \infty$.

We now state the main results of this paper.

Theorem 1.4. (1) *Under Assumptions* 1.1 (1) (2) (3) *and* 1.3 (1), *for almost all* $\omega \in \Omega$ *there exists positive numbers* $c_1 = c_1^\omega$, $c_2 = c_2^\omega$ *such that*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = c_1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \\ &\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = c_2, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega). \end{aligned} \tag{1.9}$$

(2) *Under Assumptions* 1.1 (1) (2) (3) *and* 1.3 (2), *for almost all* $\omega \in \Omega$ *there exist a positive number* $c_3 = c_3^\omega$ *such that*

$$\liminf_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = c_3, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega). \tag{1.10}$$

Theorem 1.5. *Suppose Assumption* 1.1 (1) (2) (3) (4) *and* $\alpha/\beta > 1$. *In addition* $\theta^\omega(x) = \pi^\omega(x) \ge c$ *for a positive constant* $c > 0$ *in the case of CSRW. Let* $h : (1,\infty) \to (0,\infty)$ *be a function such that* $h(t) \searrow 0$ *as* $t \to \infty$ *and the function* $\varphi(t) := t^{1/\beta}h(t)$ *is increasing. If* $h(t)$ *satisfies Assumption* 1.3 (3), *then*

$$P_x^\omega(d(x, Y_t^\omega) \ge t^{1/\beta}h(t) \text{ for all sufficiently large } t) = 1,$$
$$\text{for almost all } \omega \in \Omega \text{ and all } x \in V(G^\omega),$$

or

$$P_x^{\omega}(d(x, Y_t^{\omega}) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t) = 0,$$
$$\text{for almost all } \omega \in \Omega \text{ and all } x \in V(G^{\omega}),$$

according as $\int_1^{\infty} \frac{1}{t} h(t)^{\alpha-\beta}\, dt < \infty$ *or* $= \infty$ *respectively.*

Note that the condition $\alpha/\beta > 1$ implies the transience of $\{Y_t^{\omega}\}_{t \geq 0}$.

Finally we discuss the constants c_1, c_2, c_3 in (1.9) and (1.10). When we consider a case of $G = \mathbf{Z}^d$, we can take c_1, c_2 as deterministic constants under some appropriate assumptions. To state this, we take the base point $x_0 = 0 \in \mathbf{Z}^d$ and we write shift operators as τ_x, $(x \in \mathbf{Z}^d)$, where τ_x is given by

$$(\tau_x \omega)_{yz} = \omega_{x+y,x+z}. \tag{1.11}$$

We assume the following conditions.

Assumption 1.6. *Assume that* $(\Omega, \mathcal{F}, \mathbf{P})$ *satisfies the following conditions;*

(1) $\mathbf{P}$ *is ergodic with respect to the translation operators* τ_x, *namely* $\mathbf{P} \circ \tau_x = \mathbf{P}$ *and if* $\tau_x(A) = A$ *for all* $x \in \mathbf{Z}^d$ *and for all* $A \in \mathcal{F}$ *then* $\mathbf{P}(A) = 0$ *or* 1.
(2) *For almost all environment* ω, $V(G^{\omega})$ *contains a unique infinite connected component.*
(3) (*VSRW case*) $\mathbf{E}\left[\frac{1}{\pi^{\omega}(0)}\right] \in (0, \infty)$.

Theorem 1.7. *Suppose that the same assumptions as in Theorem* 1.4 *are fulfilled and suppose in addition Assumption* 1.6. *Then we can take* c_1, c_2, c_3 *in* (1.9) *and* (1.10) *as deterministic constants* (*i.e. do not depend on* ω).

1.2. Example

In this subsection, we give some examples for which our results are applicable.

Example 1.8 (Bernoulli supercritical percolation cluster). Let $G = (\mathbf{Z}^d, E_d)$ be a graph, where $E_d = \{\{x, y\} \mid x, y \in \mathbf{Z}^d, |x - y|_1 = 1\}$. Put a Bernoulli random variable ω_e with $\mathbf{P}(\eta_e = 1) = p$ on each edge. This model is called the bond percolation. We write $p_c(d)$ as the critical probability. It is known that there exists a unique infinite connected component when $p > p_c(d)$. See [20] for more details about the percolation.

Barlow [3] proved that heat kernels of CSRWs on the super-critical percolation cluster (that is, when $p > p_c(d)$) on $\mathbf{Z}^d$, $d \geq 2$ satisfy Assumptions 1.1 (1) (2) (3) (4) and 1.3 (1) (2) with $\alpha = d$, $\beta = 2$ and $f_{\varepsilon}(t) = c\exp(-c't^{\delta})$ for some $c, c', \delta > 0$. Since the media is i.i.d. and there exists a unique infinite connected component, we can obtain Theorem 1.4 with deterministic constants by Theorem 1.7.

In addition, we can easily check that $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa > 0$ satisfy the conditions in Assumption 1.3 (3) and the assumptions of Theorem 1.5 in the case of $d > 2$. Thus $P_x^{\omega}(d(x, Y_t^{\omega}) \geq t^{1/\beta}h(t)$ for all sufficiently large $t) = 1, 0$ according as $\kappa > d-2$, $\leq d-2$ respectively by Theorem 1.5.

Note that (1.9) for the supercritical percolation cluster was already obtained by [16, Theorem 1.1].

Example 1.9 (Gaussian free fields and random interlacements). The Gaussian free field on a graph $G = (V, E)$ is a family of centered Gaussian variables $\{\varphi_x\}_{x \in G}$ with covariance $E[\varphi_x \varphi_y] = g(x, y)$, where $g(x, y)$ is the Green function of a random walk on G. Here we are interested in the level sets of the Gaussian free field $E_h = \{x \in V \mid \varphi_x \geq h\}$. We can regard the level sets as one of the percolation models which has correlation among the vertices in V. See [38] for the details.

The random interlacements concern geometries of random walk trajectories, e.g. how many random walk trajectories are needed to make the underlying graph disconnected? Sznitman [37] formulated the model of random interlacements. Although the model of random interlacements is defined through Poisson point process on a trajectory space, we can also regard this model as the percolation model with long range correlation. From the viewpoint of the RCM, we can regard the model of random interlacements as one of the RCM whose conductances take the value 0 or 1 and the conductances are not independent. See [15] for the details.

Sapozhnikov [34, Theorem 1.15] proved that for $\mathbf{Z}^d$, $d \geq 3$, the CSRWs on (i) certain level sets of the Gaussian free fields; (ii) random interlacements at level $u > 0$; (iii) vacant sets of random interlacements for suitable level sets, satisfy our Assumption 1.1 (1) (2) (3) with $\alpha = d$, $\beta = 2$ and the tail estimates of $N_x(\omega)$ as $f_\varepsilon(t) = c\exp(-c'(\log t)^{1+\delta})$ for some $c, c', \delta > 0$. As the same reason with the case of Bernoulli supercritical percolation cluster, Assumption 1.1 (3) is also satisfied in these models. This subexponential tail estimate is sufficient for Assumption 1.3 (3) with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa > 0$. Since the media is ergodic and there is a unique infinite connected components (see [33], [37, Corollary 2.3] and [42, Theorem 1.1]), Theorem 1.4 holds with deterministic constants by Theorem 1.7, and Theorem 1.5 holds with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa \geq d-2$, $< d-2$ respectively.

Example 1.10 (Uniform elliptic case). Suppose that a graph $G = (V, E)$ is endowed with weight 1 on each edge and satisfies (1.2) and the scaled Poincaré inequalities. Take c_1, c_2 as positive constants and put random conductances on all edges so that $c_1 \leq \omega(e) \leq c_2$ for all $e \in E$ and for almost all ω. Delmotte [14] obtained Gaussian heat kernel estimates for CSRWs in this framework. Thus

Assumption 1.1 (1) (2) (3) hold with $\beta = 2$ and $N_x \equiv 1$. Hence Theorem 1.4 holds.

In addition, this model satisfies Assumption 1.1 by [13, Corollaries 11 and 12]. (See also Proposition 1.2, note that the graph distance satisfies (1.8) for CSRW case.) Thus Theorem 1.5 holds with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d-2$, $< d-2$ respectively).

Example 1.11 (Unbounded conductance bounded from below). Let $G = \mathbf{Z}^d$ ($d \geq 2$) and put random conductances $\omega = \{\omega_{xy}\}_{xy \in E}$ which take the value $[1, \infty)$. Barlow and Deuschel [5, Theorem 1.2] proved that the heat kernels of VSRW satisfy Assumptions 1.1 (1) (2) and 1.3 (1) (2) with $\alpha = d$, $\beta = 2$ and $f_\varepsilon(t) = c_1 \exp(-c_2 t^\delta)$ for some $c_1, c_2, \delta > 0$. (Note that Assumption 1.1 (3) is trivial since $V^\omega(x, r) = \# B(x, r)$ for the VSRW case.) Hence Theorem 1.4 holds.

In addition, this model satisfies Assumption 1.1 (4) by either [5, Theorems 2.3 and 4.3 (b)] or [17, Theorems 2.1 and 2.2]. Thus Theorem 1.5 for the VSRW holds with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d-2$, $< d-2$ respectively).

Moreover, if the conductances $\{\omega_e\}_e$ satisfy Assumption 1.6 (3) then Theorem 1.4 holds with deterministic constants.

2. Consequences of Assumption 1.1

In this section we give some preliminary results of our assumptions.

2.1. Consequences of heat kernel estimates

In this subsection, we give preliminary results of Assumption 1.1 (1) (2) (3). Recall the notations in (1.4).

Lemma 2.1. *Suppose Assumption* 1.1 (1) (3). *For all* $\delta \in (0, c_{1.2} \wedge c_{1.4})$ *there exist positive constants* $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, $c_3 = c_3(\delta)$ *such that*

$$P_x^\omega(d(x, Y_t^\omega) \geq r) \leq c_1 \exp\left[-(c_{1.2} - \delta)\left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2 \exp(-c_3 t) \tag{2.1}$$

for almost all $\omega \in \Omega$, *all* $x \in V(G^\omega)$, $r \geq N_x(\omega)$ *and* $t \geq N_x(\omega)$.

This lemma is standard except for the part of estimates of Poissonian regime (the bottom line of (1.4)). For the sake of completeness we give the proof here.

Proof. We first prepare some preliminary facts to estimate $P_x^\omega(d(x, Y_t^\omega) \geq r)$. Set $h_1(\eta, s) = \exp[-\eta s^{\beta/(\beta-1)}]$ and $h_2(\eta, s) = \exp[-\eta s]$. For $h_1(\eta, s)$, we can easily see that there exists a constant $\zeta_0 > 1$ such that

$$h_1(\eta, \zeta s) \leq h_1(\eta, 1) h_1(\eta, s) \tag{2.2}$$

for all $\zeta \geq \zeta_0$, $\eta > 0$ and $s \geq 1$. (We can take ζ_0 as the positive number which satisfies $\zeta_0^{\beta/(\beta-1)} - 1 = 1$.) For $h_2(\eta, s)$, we can easily see that

$$h_2(\eta, \zeta s) \leq h_2(\eta, 1) h_2(\eta, s) \tag{2.3}$$

for all $\zeta \geq 2$, $\eta > 0$ and $s \geq 1$. Next, we easily see that for all $\zeta > 1$ there exists $c_1 = c_1(\zeta)$ such that for almost all $\omega \in \Omega$

$$V^\omega(x, r\zeta) \leq c_1 V^\omega(x, r) \tag{2.4}$$

for all $x \in V(G)$ and for all $r \geq N_x(\omega)$. (Use (1.6) and take $c_1 = \frac{c_{3.2}\zeta^\alpha}{c_{3.1}}$.) Thirdly, it is also easy to see that for all $\delta \in (0, c_{1.2})$ there exists $c_2(\delta)$ such that

$$s^\alpha \exp[-c_{1.2} s^{\beta/(\beta-1)}] \leq c_2(\delta) \exp[-(c_{1.2} - \delta) s^{\beta/(\beta-1)}] \tag{2.5}$$

for all $s \geq 1$, where $c_{1.2}$ is the same constant as in (1.4). We can also see that for all $\delta \in (0, c_{1.4})$ there exists a positive constant $c_3 = c_3(\delta)$ such that

$$s^\alpha \exp[-c_{1.4} s] \leq c_3(\delta) \exp[-(c_{1.4} - \delta) s] \tag{2.6}$$

for all $s \geq 1$. Using (2.5), we can see that for $d(x, z) \geq s \geq t^{1/\beta}$ and $\delta \in (0, c_{1.2})$

$$\begin{aligned} &\frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \\ &= \frac{c_{1.1}}{d(x,z)^\alpha}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^\alpha \exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \\ &\leq \frac{c_4(\delta)}{d(x,z)^\alpha} \exp\left[-(c_{1.2} - \delta)\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \quad \text{(use (2.5))} \\ &\leq \frac{c_4(\delta)}{s^\alpha} \exp\left[-(c_{1.2} - \delta)\left(\frac{s}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right], \quad \text{(use } d(x,z) \geq s). \end{aligned} \tag{2.7}$$

Now we estimate $P_x^\omega(d(x, Y_t^\omega) \geq r)$. We first consider the case $r \leq t^{1/\beta}$. Since $s \mapsto h_1(\eta, s)$, $(\eta > 0)$ is non-increasing, we have

$$P_x^\omega(d(x, Y_t^\omega) \geq r) \leq 1 \leq \frac{h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right)}{h_1(c_{1.2}, 1)} = c_5 h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right), \tag{2.8}$$

where we set $c_5 = 1/h(c_{1.2}, 1)$. So we may and do assume $r \geq t^{1/\beta}$. Take $\zeta \geq \zeta_0 \vee 2$ so that (2.2), (2.3) and (2.4) hold. We divide $P_x^\omega(d(x, Y_t^\omega) \geq r)$ into

$$(2.9)\qquad \begin{aligned}&\left(\sum_{k=0}^{K-1}\sum_{z\in B^\omega(x,r\zeta^{k+1})\setminus B^\omega(x,r\zeta^k)}+\sum_{z\in B^\omega(x,\lfloor t\rfloor)\setminus B^\omega(x,r\zeta^K)}\right)q_t^\omega(x,z)\theta^\omega(z),\\&\left(\sum_{z\in B^\omega(x,r\zeta^{K+1})\setminus B^\omega(x,\lfloor t\rfloor)}+\sum_{k=K+1}^{\infty}\sum_{z\in B^\omega(x,r\zeta^{k+1})\setminus B^\omega(x,r\zeta^k)}\right)q_t^\omega(x,z)\theta^\omega(z),\end{aligned}$$

where K is the positive integer which satisfies $r\zeta^K\le t<r\zeta^{K+1}$ and $\lfloor t\rfloor$ is the greatest integer which is less than or equal to t. We have for $t\ge N_x(\omega)$, $r\ge N_x(\omega)$ and using (1.4)

(2.10) (The first term of (2.9))

$$\begin{aligned}&\le\sum_{k=0}^{K}\sum_{z\in B^\omega(x,r\zeta^{k+1})\setminus B^\omega(x,r\zeta^k)}\frac{c_{1.1}}{t^{\alpha/\beta}}\exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right]\theta^\omega(z)\\&\le\sum_{k=0}^{K}\frac{c_6(\delta)}{(r\zeta^k)^\alpha}\exp\left[-(c_{1.2}-\delta)\left(\frac{r\zeta^k}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right](r\zeta^{k+1})^\alpha\\&\qquad\text{(use (2.7) and (1.6))}\\&\le\sum_{k=0}^{K}c_7(\delta,\zeta)h_1\left(c_{1.2}-\delta,\frac{r\zeta^k}{t^{1/\beta}}\right)\\&\le c_7(\delta,\zeta)h_1\left(c_{1.2}-\delta,\frac{r}{t^{1/\beta}}\right)\sum_{k=0}^{K}h_1(c_{1.2}-\delta,1)^k\quad\text{(use (2.2))}\\&\le c_8(\delta,\zeta)\exp\left[-(c_{1.2}-\delta)\left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right],\quad\text{(since }h_1(c_{1.2}-\delta,1)<1).\end{aligned}$$

For the second term of (2.9), using (1.4), $t\ge N_x(\omega)$ and $r\ge N_x(\omega)$ we have

(2.11) (The second term of (2.9))

$$\begin{aligned}&\le\sum_{k=K}^{\infty}\sum_{z\in B^\omega(x,r\zeta^{k+1})\setminus B^\omega(x,r\zeta^k)}c_{1.3}\exp\left[-c_{1.4}d(x,z)\left(1\vee\log\frac{d(x,z)}{t}\right)\right]\theta^\omega(z)\\&\le\sum_{k=K}^{\infty}\sum_{z\in B^\omega(x,r\zeta^{k+1})\setminus B^\omega(x,r\zeta^k)}c_{1.3}\exp[-c_{1.4}d(x,z)]\theta^\omega(z)\\&\qquad\left(\text{since }1\vee\log\frac{d(x,z)}{t}\ge1\right)\\&\le\sum_{k=K}^{\infty}c_9\exp[-c_{1.4}(r\zeta^k)](r\zeta^{k+1})^\alpha\quad\text{(use (1.6))}\end{aligned}$$

$$
\begin{aligned}
&\le c_{10}(\zeta,\delta)\sum_{k=K}^{\infty}\exp[-(c_{1.4}-\delta)r\zeta^k] \quad (\text{use } (2.6))\\
&= c_{10}(\zeta,\delta)\sum_{k=K}^{\infty}h_2(c_{1.4}-\delta, r\zeta^k)\\
&\le c_{11}(\zeta,\delta)h_2(c_{1.4}-\delta, r\zeta^K)\sum_{k=0}^{\infty}h_2(c_{1.4}-\delta,1)^k \quad (\text{use } (2.3))\\
&\le c_{12}(\zeta,\delta)\exp[-c_{13}(\zeta,\delta)t], \quad (\text{since } r\zeta^K \le t < r\zeta^{K+1}).
\end{aligned}
$$

Therefore, by (2.8), (2.10), (2.11) and adjusting the constants, we obtain (2.1). We thus complete the proof. □

Again recall the notations $c_{1.2}$ and $c_{1.4}$ in (1.4).

LEMMA 2.2. *Suppose Assumption* 1.1 (1) (3). *Then for all* $\delta \in (0, c_{1.2}\wedge c_{1.4})$ *there exist positive constants* $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, $c_3 = c_3(\delta)$ *such that*

$$
\tag{2.12} P_x^{\omega}\left(\sup_{0\le s\le t} d(x, Y_s^{\omega}) \ge 2r\right) \le c_1 \exp\left[-(c_{1.2}-\delta)\left(\frac{r}{(2t)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2\exp[-c_3 t]
$$

$$
\tag{2.13} P_x^{\omega}\left(\sup_{0\le s\le t} d(y, Y_s^{\omega}) \ge 4r\right) \le c_1 \exp\left[-(c_{1.2}-\delta)\left(\frac{r}{(2t)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2\exp[-c_3 t]
$$

for almost all $\omega\in\Omega$, *all* $x, y \in V(G^{\omega})$, $t\ge 1$ *and* $r \ge 1$ *with* $d(x,y)\le 2r$, $t\ge \max_{u\in B(x,2r)} N_u(\omega)$ *and* $r \ge \max_{u\in B(x,2r)} N_u(\omega)$.

Proof. This is standard (see the proof of [4, Lemma 3.9 (c)]), so we omit the proof. □

LEMMA 2.3. *Suppose Assumption* 1.1 (1) (2) (3). *Then there exist positive constants* $\eta \ge 1$, $c_1, c_2 > 0$ *such that*

$$
\tag{2.14} P_x^{\omega}\left(\sup_{0\le s\le t} d(x, Y_s^{\omega}) \le 3\eta r\right) \ge c_1\exp\left[-c_2\frac{t}{r^{\beta}}\right]
$$

for almost all $\omega\in\Omega$, *all* $x\in V(G^{\omega})$, $t\ge r\ge 1$ *with* $r^{1/\beta} \ge \max_{z\in B(y,3\eta r)} N_z(\omega)$.

Proof. The proof is quite similar to that of [30, Proposition 3.3], so we omit the proof. □

Let c_1, c_2 be as in Lemma 2.3. Note that we can assume that $c_1 < 1$ (and therefore $c_1 \exp[-c_2] \in (0,1)$). We define ρ_1, a_k, b_k, λ_k, u_k, σ_k as

$$\begin{aligned} &\rho_1 = c_1 \exp[-c_2], \quad a_k^\beta = e^{k^2}, \quad b_k^\beta = e^k, \\ &\lambda_k = \frac{2}{3|\log \rho_1|} \log(1+k), \quad u_k = \lambda_k a_k^\beta, \quad \sigma_k = \sum_{i=1}^{k-1} u_i. \end{aligned} \tag{2.15}$$

Corollary 2.4 (Corollary of Lemma 2.3). *Let $\eta \geq 1$ be as in Lemma 2.3. Then under Assumption 1.1 (1) (2) (3) we have*

$$\inf_{z \in B^\omega(x, a_k)} P_z^\omega\left(\sup_{0 \leq s \leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k\right) \geq \rho_1^{\lambda_k} \tag{2.16}$$

for almost all $\omega \in \Omega$, all k with $\max_{z \in B(x, 4\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$.

Proof. We can see from Lemma 2.3 that

$$P_z^\omega\left(\sup_{0 \leq s \leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k\right) \geq c_1 \exp\left[-c_2 \frac{u_k}{a_k^\beta}\right] \geq \rho_1^{\lambda_k}$$

for all $k \geq 1$ with $\max_{v \in B(z, 3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$. Hence (2.16) holds for k with $\max_{z \in B(x, a_k)} \max_{v \in B(z, 3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$. □

Lemma 2.5. *Suppose Assumption 1.1 (1) (3). Then there exist positive constants c_1, c_2 such that*

$$P_x^\omega\left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \leq r\right) \leq c_1 \exp\left(-c_2 \frac{t}{r^\beta}\right)$$

for almost all environment $\omega \in \Omega$, all $x \in V(G^\omega)$, $t \geq 1$ and $r \geq 1$ with $\max_{y \in B(x,r)} N_y(\omega) \leq 2r$.

Proof. The proof is quite similar to that of [30, Lemma 3.2], so we omit it. □

We will need the following version of 0–1 law.

Theorem 2.6 (0–1 law for tail events). *For almost all environment $\omega \in \Omega$, the following holds; Let A^ω be a tail event, i.e. $A^\omega \in \bigcap_{t=0}^\infty \sigma\{Y_s^\omega : s \geq t\}$. Then either $P_x^\omega(A^\omega) = 0$ for all x or $P_x^\omega(A^\omega) = 1$ for all x.*

The proof of the above theorem is quite similar to that of [7, Proposition 2.3] (see also [3, Theorem 4]), so we omit the proof here.

2.2. Green function

In this subsection, we deduce the Green function estimates. We define the Green function as

$$g^{\omega}(x,y)=\int_0^{\infty} q_t^{\omega}(x,y)\,dt. \tag{2.17}$$

Recall that $\theta^{\omega}(x)=\pi^{\omega}(x)$ in the case of CSRW and $\theta^{\omega}(x)=1$ in the case of VSRW.

PROPOSITION 2.7. *Let $\alpha>\beta$ and suppose Assumption* 1.1 (1) (2) (4). *In addition we assume there exists a positive constant $c>0$ such that $\theta^{\omega}(x)\geq c$ for all $x\in V(G^{\omega})$ in the case of CSRW. Then there exist positive constants c_1, c_2 such that*

$$\frac{c_1}{d(x,y)^{\alpha-\beta}}\leq g^{\omega}(x,y)\leq\frac{c_2}{d(x,y)^{\alpha-\beta}} \tag{2.18}$$

for almost all $\omega\in\Omega$, all $x,y\in V(G^{\omega})$ with $d(x,y)\geq N_x(\omega)\wedge N_y(\omega)$.

Proof. This proof is similar to [6, Proposition 6.2]. We first prove the upper bound of (2.18).

$$\begin{aligned}g^{\omega}(x,y)&=\int_0^{(c_{4.3}d(x,y))\wedge N_x(\omega)} q_t^{\omega}(x,y)\,dt+\int_{(c_{4.3}d(x,y))\wedge N_x(\omega)}^{N_x(\omega)} q_t^{\omega}(x,y)\,dt\\&\quad+\int_{N_x(\omega)}^{d(x,y)} q_t^{\omega}(x,y)\,dt+\int_{d(x,y)}^{\infty} q_t^{\omega}(x,y)\,dt\\&:=J_1+J_2+J_3+J_4.\end{aligned} \tag{2.19}$$

We estimate J_1, J_2, J_3, J_4 as follows.

$$\begin{aligned}J_1&\leq\int_0^{(c_{4.3}d(x,y))\wedge N_x(\omega)}\frac{c_{4.4}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}}\exp[-c_{4.5}d(x,y)]\,dt\quad(\text{use }(1.7))\\&\leq c_1d(x,y)\exp[-c_2d(x,y)],\\J_2&\leq\int_{(c_{4.3}d(x,y))\wedge N_x(\omega)}^{N_x(\omega)}\frac{c_{4.1}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}}\exp\left[-c_{4.2}\frac{d(x,y)^2}{t}\right]dt\quad(\text{use }(1.7))\\&\leq c_3N_x(\omega)\exp\left[-c_4\frac{d(x,y)^2}{N_x(\omega)}\right]\leq c_3d(x,y)\exp[-c_4d(x,y)]\\&\quad(\text{use }d(x,y)\geq N_x(\omega)),\\J_3&\leq\int_{N_x(\omega)}^{d(x,y)}c_{1.3}\exp[-c_{1.4}d(x,y)]\,dt\quad(\text{use }(1.4))\end{aligned} \tag{2.20}$$

$$\leq c_{1.3} d(x,y) \exp[-c_{1.4} d(x,y)],$$

$$J_4 \leq \int_{d(x,y)}^{\infty} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt \leq \frac{c_5}{d(x,y)^{\alpha-\beta}}.$$

By (2.19) and (2.20) we have $g^{\omega}(x,y) \leq \dfrac{c_6}{d(x,y)^{\alpha-\beta}}$ for $d(x,y) \geq N_x(\omega)$. Note that $g^{\omega}(x,y) = g^{\omega}(y,x)$. Thus we complete the upper bound of (2.18).

Next we prove the lower bound of (2.18). We can obtain the lower bound in the following way.

$$g^{\omega}(x,y) \geq \int_{d(x,y)^{\beta}}^{\infty} q_t^{\omega}(x,y)\, dt \geq \int_{d(x,y)^{\beta}}^{\infty} \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt$$

$$\geq \frac{c_7}{d(x,y)^{\alpha-\beta}}.$$

We thus complete the proof. □

2.3. Consequences of the Green function and Assumption 1.1

In this subsection we give some preliminary results of Assumption 1.1 (1) (2) (3) (4) in the case of $\alpha > \beta$. This subsection is based on [35, Section 4.1]. In this subsection we assume the following conditions.

Assumption 2.8. (1) $\alpha > \beta$,
(2) (*CSRW case*) *There exists a positive constant* c *such that* $\theta^{\omega}(x) \geq c$ *for almost all* $\omega \in \Omega$ *and all* $x \in V(G^{\omega})$.

Recall that Proposition 2.7 holds under Assumptions 1.1 (1) (2) (4) and 2.8.

We write $e_F^{\omega}(x) = P_x^{\omega}(\sigma_F^{+\omega} = \infty)1_F(x)$ as the equilibrium measure of $F \subset V(G^{\omega})$, and define $\mathrm{Cap}^{\omega}(F) = \sum_{x\in F} e_F^{\omega}(x)\theta^{\omega}(x)$ as the capacity of $F \subset V(G^{\omega})$. Then we have

$$P_x^{\omega}(\sigma_F^{+\omega} < \infty) = \sum_{y\in F} g^{\omega}(x,y) e_F^{\omega}(y)\theta^{\omega}(y) \tag{2.21}$$

for any finite set F and for any $x \in V(G^{\omega})$ since

$$\begin{aligned} &P_x^{\omega}(\sigma_F^{+\omega} < \infty) \\ &\quad = \int_0^{\infty} \sum_{y\in F} P_x^{\omega}(Y_t^{\omega} = y,\, Y_s^{\omega} \notin F \text{ for any } s > t)\, dt \quad \text{(last exit decomposition)} \\ &\quad = \int_0^{\infty} \sum_{y\in F} q_t^{\omega}(x,y)\theta^{\omega}(y) P_y^{\omega}(\sigma_F^{+\omega} = \infty)\, dt \quad \text{(by the Markov property)} \\ &\quad = \sum_{y\in F} g^{\omega}(x,y) e_F^{\omega}(y)\theta^{\omega}(y). \end{aligned}$$

LEMMA 2.9. *Under Assumptions* 1.1 (1) (2) (3) (4) *and* 2.8, *there exists a positive constant c such that*

$$\mathrm{Cap}^{\omega}(B^{\omega}(x,2r)) \geq cr^{\alpha-\beta}$$

for almost all $\omega \in \Omega$, *all* $x \in V(G^{\omega})$ *and* $r \geq 1$ *with* $r \geq \max_{v \in B(x,r)} N_v(\omega)$.

Proof. Recall the notations in (1.3).

$$\begin{aligned}
1 &= \frac{1}{\theta^{\omega}(B(x,r))} \sum_{y \in B^{\omega}(x,r)} P_y^{\omega}(\sigma_{B(x,2r)}^{+\omega} < \infty)\theta^{\omega}(y) \\
&= \frac{1}{\theta^{\omega}(B(x,r))} \sum_{y \in B^{\omega}(x,r)} \sum_{\substack{z \in B^{\omega}(x,2r) \\ d(x,z)=2r}} g^{\omega}(y,z) e^{\omega}_{B^{\omega}(x,2r)}(z)\theta^{\omega}(z)\theta^{\omega}(y) \quad \text{(we use (2.21))} \\
&\leq \frac{c_1}{\theta^{\omega}(B(x,r))} \frac{1}{r^{\alpha-\beta}} \sum_{\substack{z \in B^{\omega}(x,2r) \\ d(x,z)=2r}} \sum_{y \in B^{\omega}(x,r)} e^{\omega}_{B^{\omega}(x,2r)}(z)\theta^{\omega}(z)\theta^{\omega}(y) \\
&\quad \text{(since } d(y,z) \geq r \geq N_y(\omega) \text{ and Proposition 2.7)} \\
&= \frac{c_1}{\theta^{\omega}(B(x,r))} \frac{\theta^{\omega}(B(x,r))}{r^{\alpha-\beta}} \sum_{\substack{z \in B^{\omega}(x,2r) \\ d(x,z)=2r}} e^{\omega}_{B^{\omega}(x,2r)}(z)\theta^{\omega}(z) \\
&= \frac{c_1}{r^{\alpha-\beta}} \mathrm{Cap}^{\omega}(B^{\omega}(x,2r)).
\end{aligned}$$

We thus complete the proof. □

Recall the notations in (1.3) and set

$$\begin{aligned}
\gamma^{\omega}_{x,F}(K_1) &= P_x^{\omega}(Y^{\omega}_{\sigma_F^+} \in K_1), \\
\pi^{\omega}_{x,F}(dt, K_2) &= P_x^{\omega}(Y^{\omega}_{\sigma_F^+} \in K_2, \sigma_F^+ \in dt)
\end{aligned}$$

for $F, K_1, K_2 \subset V(G^{\omega})$. Note that $\int_0^{\infty} \pi^{\omega}_{x,F}(dt, K) = \gamma^{\omega}_{x,F}(K)$ and $\gamma^{\omega}_{x,F}(F) = P_x^{\omega}(\sigma_F^{\omega+} < \infty)$.

LEMMA 2.10. *For almost all* $\omega \in \Omega$,

$$g^{\omega}(x,y) = \sum_{v \in F^{\omega}} g^{\omega}(v,y)\gamma^{\omega}_{x,F^{\omega}}(v) \tag{2.22}$$

for any finite set $F^{\omega} \subset V(G^{\omega})$, $x \notin F^{\omega}$ *and* $y \in F^{\omega}$. *In particular we have*

$$P_x^{\omega}(Y_t^{\omega} \in F^{\omega} \text{ for some } t > 0) \leq \inf_{y \in F^{\omega}} \left(\frac{g^{\omega}(x,y)}{\inf_{z \in F^{\omega}} g^{\omega}(z,y)} \right). \tag{2.23}$$

Proof. We write $F = F^\omega$ and $\sigma = \sigma_{F^\omega}^{\omega+} = \inf\{t > 0 \mid Y_t^\omega \in F\}$ for notational simplification. Then for any $x \notin F$, $y \in F$ we have

$$P_x^\omega(Y_t^\omega = y) = E_x^\omega[1_{\{\sigma \le t\}} P_{Y_\sigma^\omega}^\omega(Y_{t-\sigma}^\omega = y)] = \sum_{v \in F} E_x^\omega[1_{\{\sigma \le t\}} 1_{\{Y_\sigma^\omega = v\}} P_{Y_\sigma}^\omega(Y_{t-\sigma}^\omega = y)]$$

$$= \sum_{v \in F} \int_0^t P_v^\omega[Y_{t-s}^\omega = y] \pi_{x,F}^\omega(ds, v).$$

Hence we have

$$g^\omega(x, y) = \int_0^\infty \sum_{v \in F} \int_0^t q_{t-s}^\omega(v, y) \pi_{x,F}^\omega(ds, v)\, dt = \int_0^\infty \sum_{v \in F} \int_s^\infty q_{t-s}^\omega(v, y)\, dt \pi_{x,F}^\omega(ds, v)$$

$$= \int_0^\infty \sum_{v \in F} g^\omega(v, y) \pi_{x,F}^\omega(ds, v) = \sum_{v \in F} g^\omega(v, y) \gamma_{x,F}^\omega(v).$$

We thus complete the proof of (2.22). (2.23) is immediate from (2.22). □

LEMMA 2.11. *Under Assumptions* 1.1 (1) (2) (3) (4) *and* 2.8 *there exist positive constants* c_1, c_2 *such that for almost all* $\omega \in \Omega$ *the following hold.*

(1) $P_x^\omega(\sigma_{B(x_0,2r)}^{+\omega} < \infty) \le c_1 \dfrac{r^{\alpha-\beta}}{(d(x, x_0) - r)^{\alpha-\beta}}$ *for all* $x, x_0 \in V(G^\omega)$, $r \ge 1$ *with* $d(x, x_0) \ge 2r + 1$ *and* $r \ge \max_{v \in B(x_0,r)} N_v(\omega)$.

(2) $P_x^\omega(\sigma_{B(x_0,2r)}^{+\omega} < \infty) \ge c_2 \dfrac{r^{\alpha-\beta}}{(d(x, x_0) + 2r)^{\alpha-\beta}}$ *for all* $x, x_0 \in V(G^\omega)$, $r \ge 1$ *with* $d(x, x_0) \ge 2r$, $r \ge N_x(\omega)$ *and* $r \ge \max_{v \in B(x_0,r)} N_v(\omega)$.

Proof. We first prove (1) by using (2.23). Let $x, x_0 \in V(G^\omega)$ satisfy $d(x, x_0) \ge 2r + 1$. For any $y \in B(x_0, r)$ we have

$$d(x, y) \ge d(x, x_0) - d(x_0, y) \ge d(x, x_0) - r \ge 2r - r = r.$$

By Proposition 2.7, for any $y \in B^\omega(x_0, r)$ and for any r with $r \ge \max_{y \in B(x_0,r)} N_y(\omega)$ we have

$$g^\omega(x, y) \le \frac{c_1}{d(x, y)^{\alpha-\beta}} \le \frac{c_1}{(d(x, x_0) - r)^{\alpha-\beta}}. \tag{2.24}$$

Next note that $B(x_0, 2r) \subset B(y, 3r)$ for any $y \in B(x_0, r)$. Since $g^\omega(\cdot, y)$ is a superharmonic function, using the minimum principle and Proposition 2.7 we have

$$\inf_{z \in B^\omega(x_0,2r)} g^\omega(z, y) \ge \inf_{z \in B^\omega(y,3r)} g^\omega(z, y) \ge \inf_{\substack{z \in B^\omega(y,3r+1) \\ d(y,z)=3r+1}} g^\omega(z, y) \ge \frac{c_2}{r^{\alpha-\beta}} \tag{2.25}$$

for all $r \geq 1$ and $y \in B^{\omega}(x_0, r)$ with $3r + 1 \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Hence by (2.23), (2.24) and (2.25) we have

$$P_x^{\omega}(\sigma^{+}_{B(x_0,2r)} < \infty) \leq \inf_{y \in B^{\omega}(x_0,r)} \left(\frac{g^{\omega}(x,y)}{\inf_{z \in B^{\omega}(x_0,2r)} g(z,y)} \right) \leq c_3 \frac{r^{\alpha-\beta}}{(d(x,x_0)-r)^{\alpha-\beta}}$$

for all r with $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Thus we complete the proof of (1).

Next we prove (2). Note that

$$\begin{aligned} P_x^{\omega}(\sigma^{+\omega}_{B(x_0,2r)} < \infty) &= \sum_{y \in B^{\omega}(x_0,2r)} g^{\omega}(x,y) e^{\omega}_{B^{\omega}(x_0,2r)}(y)\theta^{\omega}(y) \quad \text{(use (2.21))} \\ &\geq \left(\inf_{y \in B^{\omega}(x_0,2r)} g^{\omega}(x,y) \right) \sum_{y \in B^{\omega}(x_0,2r)} e^{\omega}_{B(x_0,2r)}(y)\theta^{\omega}(y) \\ &= \left(\inf_{y \in B^{\omega}(x_0,2r)} g^{\omega}(x,y) \right) \mathrm{Cap}^{\omega}(B(x_0,2r)). \end{aligned}$$

By $B(x_0,2r) \subset B(x, d(x,x_0)+2r)$, the minimum principle for superharmonic functions and our assumptions we have

$$\begin{aligned} \inf_{y \in B^{\omega}(x_0,2r)} g^{\omega}(x,y) &\geq \inf_{y \in B^{\omega}(x,d(x,x_0)+2r)} g^{\omega}(x,y) \geq \inf_{\substack{y \in B^{\omega}(x,d(x,x_0)+2r+1) \\ d(y,x)=d(x,x_0)+2r+1}} g^{\omega}(x,y) \\ &\geq \frac{c_4}{(d(x,x_0)+2r)^{\alpha-\beta}} \end{aligned}$$

for $r \geq N_x(\omega)$. By Lemma 2.9 $\mathrm{Cap}^{\omega}(B(x_0,r)) \geq c_5 r^{\alpha-\beta}$ for $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Hence

$$P_x^{\omega}(\sigma^{+\omega}_{B(x_0,2r)} < \infty) \geq \frac{c_6 r^{\alpha-\beta}}{(d(x,x_0)+2r)^{\alpha-\beta}}$$

for $r \geq N_x(\omega)$ and $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. We thus complete the proof. □

LEMMA 2.12. *Under Assumptions* 1.1 (1) (2) (3) (4) *and* 2.8 *there exist positive constants* c_1 *and* T_0 *such that*

$$P_x^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > t) \leq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, *all* $t \geq T_0$, $r \geq 1$ *and* $x, x_0 \in V(G^{\omega})$ *with* $t^{1/\beta} \geq r$, $d(x,x_0) \leq r$ *and* $r \geq \max_{z \in B(x_0,r)} N_z(\omega)$.

Proof. First note that

$$\begin{aligned} &P_x^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > t) \\ &\quad = \sum_{y \in V(G^{\omega})} P_x^{\omega}(Y_t^{\omega} = y) P_y^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > 0) \end{aligned}$$

$$= \sum_{y;\, t^{1/\beta}<d(x_0,y)-r} P_x^\omega(Y_t^\omega = y) P_y^\omega(d(x_0, Y_s^\omega) \le 2r \text{ for some } s>0)$$

$$+ \sum_{y;\, r<d(x_0,y)-r\le t^{1/\beta}} P_x^\omega(Y_t^\omega = y) P_y^\omega(d(x_0, Y_s^\omega) \le 2r \text{ for some } s>0)$$

$$+ \sum_{y;\, d(x_0,y)\le 2r} P_x^\omega(Y_t^\omega = y) P_y^\omega(d(x_0, Y_s^\omega) \le 2r \text{ for some } s>0)$$

$$:= J_1 + J_2 + J_3.$$

We estimate J_1, J_2 and J_3 in the following way.

For $t, r \ge 1$ with $t \ge N_x(\omega)$ and $r \ge \max_{z\in B(x_0,r)} N_z$ (note that $t \ge N_x(\omega)$ follows from our assumptions), using (1.4), Lemma 2.11, (1.6) we have

$$J_1 \le \sum_{y;\, t^{1/\beta}<d(x_0,y)-r} \frac{c_1 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}}$$

$$\cdot \left\{ \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_{1.3}\exp[-c_{1.4}d(x,y)] \right\} \theta^\omega(y)$$

(use (1.4) and Lemma 2.11)

$$\le \sum_{\ell=1}^\infty \sum_{y;\, d(x_0,y)\in[\ell t^{1/\beta}+r,\,(\ell+1)t^{1/\beta}+r]} \frac{c_2 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}}$$

$$\times \exp\left[-c_{1.2}\left(\frac{d(y,x_0)-r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \theta^\omega(y)$$

$$+ \sum_{\ell=1}^\infty \sum_{y;\, d(x_0,y)\in[\ell t^{1/\beta}+r,\,(\ell+1)t^{1/\beta}+r]} \frac{c_3 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \exp[-c_{1.4}(d(y,x_0)-r)]\theta^\omega(y)$$

(since $d(x,y) \ge d(y,x_0) - d(x_0,x)$ and $d(x_0,x) \le r$)

$$\le \sum_{\ell=1}^\infty \frac{c_2 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}} \exp[-c_{1.2}\ell^{\beta/(\beta-1)}]\theta^\omega(B(x_0,(\ell+1)t^{1/\beta}+r))$$

$$+ \sum_{\ell=1}^\infty \frac{c_3 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \exp[-c_{1.4}\ell t^{1/\beta}]\theta^\omega(B(x_0,(\ell+1)t^{1/\beta}+r))$$

$$\le \frac{c_4 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} \sum_{\ell=1}^\infty \ell^\beta \exp[-c_{1.2}\ell^{\beta/\beta-1}] + \frac{c_5 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} t^{\alpha/\beta} \sum_{\ell=1}^\infty \ell^\beta \exp[-c_{1.4}\ell t^{1/\beta}]$$

(use $\theta^\omega(B(x_0,(\ell+1)t^{1/\beta}+r)) \le c(\ell t^{1/\beta})^\alpha$ since $t^{1/\beta} \ge r$)

$$\le \frac{c_6 r^{\alpha-\beta}}{t^{\alpha/\beta-1}}, \quad (\text{since } t \mapsto t^{\alpha/\beta}\sum_{\ell=1}^\infty \ell^\beta \exp[-c_{1.4}\ell t^{1/\beta}] \text{ is bounded}).$$

Next we see J_2. First note that for $r \geq N_{x_0}(\omega)$ we have

$$
\begin{aligned}
(2.26)\qquad & \sum_{y; r \leq d(x_0,y)-r \leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y,x_0)-r)^{\alpha-\beta}} \\
&= \sum_{k \in [2r, r+t^{1/\beta}]} \frac{\theta^\omega(B(x_0,k)\backslash B(x_0,k-1))}{(k-r)^{\alpha-\beta}} \\
&\leq \sum_{\ell \in [0, \log_2(t^{1/\beta}/r)]} \sum_{k \in [r+2^\ell r, r+2^{(\ell+1)} r]} \frac{\theta^\omega(B(x_0,k)\backslash B(x_0,k-1))}{(k-r)^{\alpha-\beta}} \\
&\leq c_7 \sum_{\ell \in [0, \log_2(t^{1/\beta}/r)]} \frac{(r+2^{\ell+1} r)^\alpha}{(2^\ell r)^{\alpha-\beta}} \leq c_8 r^\beta \sum_{\ell \in [0, \log_2(t^{1/\beta}/r)]} 2^{\ell\beta} \leq c_9 t.
\end{aligned}
$$

We go back to estimate J_2. Note that for y with $r \leq d(x_0,y) - r \leq t^{1/\beta}$ we see $d(x,y) \leq d(x,x_0) + d(x_0,y) \leq 3t^{1/\beta}$. For $r \geq 1$, $t \geq 1$ with $t \geq T_0 := 3^{\beta/(\beta-1)}$ (so that $3t^{1/\beta} \leq t$ for $t \geq T_0$) and $r \geq \max_{z \in B(x_0,r)} N_z(\omega)$ (in particular $t \geq N_x(\omega)$), using Lemma 2.11, (1.4) and (2.26) we have

$$
\begin{aligned}
J_2 &\leq \sum_{y; r<d(x_0,y)-r \leq t^{1/\beta}} \frac{c_{10} r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \frac{\theta^\omega(y)}{t^{\alpha/\beta}} \\
&= \frac{c_{10} r^{\alpha-\beta}}{t^{\alpha/\beta}} \sum_{y; r \leq d(x_0,y)-r \leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y,x_0)-r)^{\alpha-\beta}} \\
&\leq \frac{c_{11} r^{\alpha-\beta} t}{t^{\alpha/\beta}}, \quad \text{(use (2.26))}.
\end{aligned}
$$

Finally we see J_3. For $t \geq T_0 := 3^{\beta/(\beta-1)}$, $N_x(\omega) \leq t$ and $N_x(\omega) \leq r$, using (1.4) we have

$$
\begin{aligned}
J_3 &\leq \sum_{y; d(y,x_0) \leq 2r} P_x^\omega(Y_t^\omega = y) = \sum_{y; d(y,x_0) \leq 2r} q_t^\omega(x,y)\theta^\omega(y) \\
&\leq \sum_{y; d(x,y) \leq 3r} q_t^\omega(x,y)\theta^\omega(y) \leq \frac{c_{12} r^\alpha}{t^{\alpha/\beta}} \leq \frac{c_{12} r^{\alpha-\beta} t}{t^{\alpha/\beta}}.
\end{aligned}
$$

We thus complete the proof. □

LEMMA 2.13. *Under Assumptions* 1.1 (1) (2) (3) (4) *and* 2.8 *there exist constants* $c_1 > 0, c_2, T_0 \geq 1$ *such that*

$$
P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) \geq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}
$$

for almost all $\omega \in \Omega$, *all* $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ *with* $d(x,x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. Take a constant c_2 such that $c_{3.1}c_2^{\alpha} - c_{3.2}2^{\alpha} > 0$. Note that by (1.6) we have $\theta^{\omega}(\{y \in V(G) \mid d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]\}) \geq (c_{3.1}c_2^{\alpha} - c_{3.2}2^{\alpha})t^{\alpha/\beta}$, and for y and sufficiently large t (say $t \geq T_0$) with $d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]$ we have $d(x, y)^{1+\varepsilon} \leq (d(x, x_0) + d(x_0, y))^{1+\varepsilon} \leq \{(c_2 + 1)t^{1/\beta}\}^{1+\varepsilon} \leq t$ since $1 + \varepsilon < \beta$ (see Assumption 1.1). Then by Lemma 2.11 (2), (1.5), (1.6), for t, r as in the statement above we have

$$
\begin{aligned}
&P_x^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > t) \\
&\quad = \sum_{y \in V(G^{\omega})} q_t^{\omega}(x, y)\theta^{\omega}(y)P_y^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > 0) \\
&\quad \geq \sum_{y : d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} q_t^{\omega}(x, y)\theta^{\omega}(y)P_y^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > 0) \\
&\quad \geq \sum_{y : d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right]\theta^{\omega}(y)\frac{c_3 r^{\alpha-\beta}}{(d(x_0, y) + 2r)^{\alpha-\beta}} \\
&\qquad \text{(use (1.5), Lemma 2.11 and } d(x, y)^{1+\varepsilon} \leq t\text{, note that } t \geq N_x(\omega) \\
&\qquad \text{follows from our assumptions)} \\
&\quad \geq \sum_{y : d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_4}{t^{\alpha/\beta}}\theta^{\omega}(y)\frac{r^{\alpha-\beta}}{(t^{1/\beta})^{\alpha-\beta}} \\
&\qquad \text{(use } d(x, y) \leq d(x, x_0) + d(x_0, y) \leq (c_2 + 1)t^{1/\beta} \text{ for } y \in B(x_0, c_2 t^{1/\beta})\text{)} \\
&\quad \geq \frac{c_5(c_{3.1}c_1^{\alpha} - c_{3.2}2^{\alpha})r^{\alpha-\beta}t}{t^{\alpha/\beta}}.
\end{aligned}
$$

We thus complete the proof by taking $c_1 = c_5(c_{3.1}c_2^{\alpha} - c_{3.2}2^{\alpha})$. □

LEMMA 2.14. *Under Assumptions* 1.1 (1) (2) (3) (4) *and* 2.8 *there exist positive constants* c_1, c_2, η_0, T_0 *such that for any* $\eta \geq \eta_0$ *the following holds*;

$$P_x^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s \in (t, \eta t]) \geq \frac{c_1 r^{\alpha-\beta}t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, *all* $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^{\omega})$ *with* $d(x, x_0) \leq r$, $t \geq r^{\beta}$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. By Lemmas 2.12 and 2.13 there exist positive constants c_1, c_2, c_3, T_0 such that for almost all $\omega \in \Omega$

$$\frac{c_1 r^{\alpha-\beta}t}{t^{\alpha/\beta}} \leq P_x^{\omega}(d(x_0, Y_s^{\omega}) \leq 2r \text{ for some } s > t) \leq \frac{c_2 r^{\alpha-\beta}t}{t^{\alpha/\beta}}$$

for $r \geq 1,\ t \geq T_0,\ x, x_0 \in V(G^\omega)$ with $d(x, x_0) \leq r,\ t \geq r^\beta,\ r \geq \max_{z \in B(x_0, c_3 t^{1/\beta})} N_z(\omega)$. Take η_0 such that $c_2 - \dfrac{c_1}{\eta^{\alpha/\beta - 1}} > \dfrac{c_2}{2}$ for all $\eta \geq \eta_0$. Then we have

$$\begin{aligned} &P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s \in (t, \eta t]) \\ &\quad \geq P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) - P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > \eta t) \\ &\quad \geq c_2 \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} - c_1 \frac{r^{\alpha-\beta}(\eta t)}{(\eta t)^{\alpha/\beta}} = \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} \left(c_2 - \frac{c_1}{\eta^{\alpha/\beta-1}} \right). \end{aligned}$$

We complete the proof by adjusting the constants. □

2.4. Consequences of Assumption 1.3

In this subsection, we give easy consequences of Assumption 1.3. We use $\varphi(q) = \varphi_C(q) = Cq^{1/\beta}(\log\log q)^{1-1/\beta}$ in this subsection.

LEMMA 2.15. (1) *Under Assumption* 1.3 (1), *for all* $\gamma_1, \gamma_2 > 0,\ q > 1$ *and for almost all* $\omega \in \Omega$ *there exists a positive number* $L^{(1)}(\omega) = L^{(1)}_{x,\varepsilon,\gamma_1,\gamma_2,q}(\omega)$ *such that*

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega), \quad \gamma_1 \varphi(q^n) \geq \max_{y \in B(x, \gamma_2 \varphi(q^n))} N_y(\omega),$$

for all $n \geq L^{(1)}(\omega)$.

(2) *Under Assumption* 1.3 (2), *for all* $\gamma_1, \gamma_2 > 0,\ q > 1$ *and for almost all* $\omega \in \Omega$ *there exists a positive number* $L^{(2)}(\omega) = L^{(2)}_{x,\varepsilon,\gamma_1,\gamma_2,q}(\omega)$ *such that*

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^n)} N_y(\omega)$$

for all $n \geq L^{(2)}(\omega)$.

(3) *Set* $\psi(t) := t^{1/\beta} h(t)$, *where* $h(t)$ *is non-increasing and* $\psi(t)$ *is increasing function. Under Assumption* 1.3 (3), *for all* $\gamma_1, \gamma_2 > 0,\ q > 1$ *and for almost all* $\omega \in \Omega$ *there exists a positive number* $L^{(3)}(\omega) = L^{(3)}_{x,\varepsilon,\gamma_1,\gamma_2,q}(\omega)$ *such that*

$$\gamma_1 \psi(q^n) \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega)$$

for all $n \geq L^{(3)}(\omega)$.

Proof. We can prove (1) (2) (3) similarly, so we prove only the first inequality in (1). Since

$$\begin{aligned} \mathbf{P}\left(\gamma_1 q^{n/\beta} < \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y \right) &\leq \sum_{y \in B(x, \gamma_2 q^{n/\beta})} \mathbf{P}(\gamma_1 q^{n/\beta} < N_y) \\ &\leq c(\gamma_2 q^{n/\beta})^\alpha f(\gamma_1 q^{n/\beta}), \end{aligned}$$

where we use union bound in the first inequality and use (1.2) in the second inequality. The conclusion follows by the Borel-Cantelli lemma. □

3. Proof of Theorem 1.4

In this section we give the proof of Theorem 1.4.

3.1. Proof of the LIL

We follow the strategy as in [16].

THEOREM 3.1. *Let* $\varphi(t) = \varphi_C(t) = Ct^{1/\beta}(\log\log t)^{1-1/\beta}$, *where* $C > 2^{1+1/\beta}c_{1.2}^{-(\beta-1)/\beta}$. *Then under Assumptions* 1.1 (1) (2) (3) *and* 1.3 (1) *the following hold for almost all* $\omega \in \Omega$;

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{\varphi(t)} \le 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \tag{3.1}$$

$$P_x^\omega\left(\sup_{0\le s\le t} d(x, Y_s^\omega) \le \varphi(t) \text{ for all sufficiently large } t\right) = 1, \tag{3.2}$$

$$\text{for all } x \in V(G^\omega).$$

In particular, we have

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \le 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega),$$

$$P_x^\omega(d(x, Y_t^\omega) \le \varphi(t) \text{ for all sufficient large } t) = 1, \quad \text{for all } x \in V(G^\omega).$$

Proof. Take $\eta > 0$ and $\delta \in (0, c_{1.2} \wedge c_{1.4})$ sufficiently small constants which satisfy $C > 2^{1/\beta}(1+\eta)^{1/\beta}\left(\frac{1}{c_{1.2}-\delta}\right)^{(\beta-1)/\beta}$. Set $t_n = (1+\eta)^n$.

First we estimate $P_x^\omega(\sup_{0\le s\le t_{n+1}} d(x, Y_s^\omega) \ge 2\varphi_C(t_n))$. For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$, using Lemma 2.2 we have

$$\begin{aligned} P_x^\omega&\left(\sup_{0\le s\le t_{n+1}} d(x, Y_s^\omega) \ge 2\varphi(t_n)\right) \\ &\le c_1 \exp\left[-(c_{1.2}-\delta)\left(\frac{\varphi(t_n)}{(2t_{n+1})^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2 \exp[-c_3 t_{n+1}] \\ &\le c_1 \exp\left[-(c_{1.2}-\delta)\left(\frac{\varphi(t_n)}{(2(1+\eta)t_n)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2 \exp[-c_3 t_{n+1}] \end{aligned} \tag{3.3}$$

for $\sup_{z \in B(x, 2\varphi(t_n))} N_z(\omega) \le \varphi(t_n) \wedge t_{n+1}$. Note that $\sup_{z \in B(x, 2\varphi(t_n))} N_z(\omega) \le \varphi(t_n) \wedge t_{n+1}$ for all n larger than a certain constant $L = L(\omega)$ by Lemma 2.15 (1). (3.1) is immediate from (3.3) and the Borel-Cantelli Lemma.

We prove (3.2). Let $C > 2^{1/\beta}(1+\eta)^{1/\beta}\left(\dfrac{1}{c_{1.2} - \delta}\right)^{(\beta-1)/\beta}$ be as above. Since

$$P_x^\omega\left(\sup_{0 \le s \le t_n} d(x, Y_s^\omega) \ge 2\varphi(t)\right) \le P_x^\omega\left(\sup_{0 \le s \le t_{n+1}} d(x, Y_s^\omega) \ge 2\varphi(t_n)\right)$$

for $t \in [t_n, t_{n+1}]$ and the last term of (3.3) is summable by the definition of η and δ. By the Borel-Cantelli lemma we have

$$P_x^\omega\left(\sup_{0 \le s \le t} d(Y_0^\omega, Y_s^\omega) \le 2\varphi(t) \text{ for all sufficiently large } t\right) = 1, \tag{3.4}$$

for all $x \in V(G^\omega)$.

We thus complete the (3.2) by adjusting the constants. □

Theorem 3.2. *Let* $\varphi(t) = \varphi_C(t) = Ct^{1/\beta}(\log\log t)^{1-1/\beta}$, *where* $0 < C < \dfrac{1}{2^{1+1/\beta}}\left(\dfrac{c_{3.1}}{c_{3.2}}\right)^{1/\alpha}\left(\dfrac{1}{c_{2.2}}\right)^{(\beta-1)/\beta}$. *Then under Assumptions* 1.1 (1) (2) (3) *and* 1.3 (1) *the following holds*;

$$\limsup_{t \to \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \ge 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

In particular, *we have*

$$P_x^\omega(d(Y_0^\omega, Y_t^\omega) \ge \varphi(t) \text{ for sufficiently large } t) = 1, \quad \text{for all } x \in V(G^\omega),$$

$$\limsup_{t \to \infty} \frac{\sup_{0 \le s \le t} d(Y_0^\omega, Y_s^\omega)}{\varphi(t)} \ge 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

Proof. Define $\Phi(q) = q^{1/\beta}(\log\log q)^{1-1/\beta}$ and let C be as above. Take $\eta > 0$ as a sufficiently small constant such that

$$C < \frac{1}{2^{1/\beta}}\left\{\frac{1}{2}\left(\frac{c_{3.1}}{c_{3.2}}\right)^{1/\alpha} - \eta\right\}\left(\frac{1}{c_{2.2}}\right)^{(\beta-1)/\beta}.$$

Set $\dfrac{1}{\lambda} = \dfrac{1}{2}\left(\dfrac{c_{3.1}}{c_{3.2}}\right)^{1/\alpha} - \eta$. Note that $c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha > 0$ and $c_{2.2}(2^{1/\beta}C\lambda)^{\beta/(\beta-1)} < 1$.

We prove that

$$\sum_n P_x^\omega(A_n^\omega \mid \mathcal{F}_{2^n}^\omega) = \infty, \tag{3.5}$$

where $A_n^\omega = \{d(Y_{2^n}^\omega, Y_{2^{n+1}}^\omega) \ge 2\varphi(2^{n+1})\}$ and $\mathcal{F}_t^\omega = \sigma(Y_s^\omega \mid s \le t)$. To prove (3.5), first note that by Theorem 3.1 there exists a sufficiently large constant C_1 such that for almost all $\omega \in \Omega$

$$d(x, Y_{2^n}^{\omega}) \le C_1 \Phi(2^n) \quad \text{for sufficiently large } n \text{ (say } n \ge \tilde{N}_1), \quad P_x^{\omega}\text{-a.s.}$$

Set $B_n^{\omega} = A_n^{\omega} \cap \{d(Y_0^{\omega}, Y_{2^n}^{\omega}) \le C_1 \Phi(2^n)\}$. Then we have

$$
\begin{aligned}
(3.6) \qquad P_x^{\omega}(A_n^{\omega} \mid \mathcal{F}_{2^n}^{\omega}) &\ge P_x^{\omega}(B_n^{\omega} \mid \mathcal{F}_{2^n}^{\omega}) \\
&= 1_{\{d(Y_0^{\omega}, Y_{2^n}^{\omega}) \le C_1\Phi(2^n)\}} P_{Y_{2^n}^{\omega}}^{\omega}(d(Y_0^{\omega}, Y_{2^{n+1}-2^n}^{\omega}) \ge 2\varphi(2^{n+1})) \\
&\ge \left(\inf_{u \in B^{\omega}(x, C_1\Phi(2^n))} P_u^{\omega}(d(Y_0^{\omega}, Y_{2^n}^{\omega}) \ge 2\varphi(2^{n+1})) \right) \\
&\quad \cdot 1_{\{d(Y_0^{\omega}, Y_{2^n}^{\omega}) \le C_1\Phi(2^n)\}}, \quad P_x^{\omega}\text{-a.s.}
\end{aligned}
$$

We consider the first term of (3.6). Take $u \in B^{\omega}(x, C_1\Phi(2^n))$. Since $1 + \varepsilon < \beta$, there exists a positive integer $\tilde{N}_2 = \tilde{N}_2(\lambda)$ (which does not depend on u, ω) such that $d(u, v)^{1+\varepsilon} \le 2^n$ for all $n \ge \tilde{N}_2$ and $v \in B^{\omega}(u, \lambda\varphi(2^{n+1}))$. So for all $n \ge \tilde{N}_2$ with $2^n \wedge 2\varphi(2^{n+1}) \ge N_u(\omega)$, using (1.5) and (1.6) we have

$$
\begin{aligned}
&P_u^{\omega}(d(Y_0^{\omega}, Y_{2^n}^{\omega}) \ge 2\varphi(2^{n+1})) \\
&\quad \ge P_u^{\omega}(2\varphi(2^{n+1}) \le d(Y_0^{\omega}, Y_{2^n}^{\omega}) \le \lambda\varphi(2^{n+1})) \\
&\quad = \sum_{\substack{v \in V(G^{\omega}) \\ 2\varphi(2^{n+1}) \le d(u,v) \le \lambda\varphi(2^{n+1})}} q_{2^n}^{\omega}(u, v)\theta^{\omega}(v) \\
&\quad \ge \sum_{\substack{v \in V(G^{\omega}) \\ 2\varphi(2^{n+1}) \le d(u,v) \le \lambda\varphi(2^{n+1})}} \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp\left[-c_{2.2} \left(\frac{d(u,v)}{(2^n)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \theta^{\omega}(v) \\
&\quad \ge \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp\left[-c_{2.2} \left(\frac{\lambda\varphi(2^{n+1})}{(2^n)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \\
&\qquad \times \theta^{\omega}(\{v \in V(G^{\omega}) \mid 2\varphi(2^{n+1}) \le d(u,v) \le \lambda\varphi(2^{n+1})\}) \\
&\quad \ge c_{2.1}(c_{3.1}\lambda^{\alpha} - c_{3.2}2^{\alpha})C^{\alpha} \left(\frac{1}{(n+1)\log 2} \right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} (\log\log 2^{n+1})^{(\beta-1)\alpha/\beta}.
\end{aligned}
$$

By the above estimate we have

$$
\begin{aligned}
(3.7) \qquad &\inf_{u \in B^{\omega}(x, C_1\Phi(2^n))} P_u^{\omega}(d(Y_0^{\omega}, Y_{2^n}^{\omega}) \ge 2\varphi(2^{n+1})) \\
&\quad \ge c_{2.1}(c_{3.1}\lambda^{\alpha} - c_{3.2}2^{\alpha})C^{\alpha} \left(\frac{1}{(n+1)\log 2} \right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} \\
&\quad \times (\log\log 2^{n+1})^{(\beta-1)\alpha/\beta}
\end{aligned}
$$

for $n \geq \tilde{N}_2$ with $\max_{u \in B(x, C_1\Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$. By Lemma 2.15 (1), $\max_{u \in B(x, C_1\Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$ holds for sufficiently large n (say $n \geq \tilde{N}_3 = \tilde{N}_3(\omega)$). Hence by (3.6) and (3.7) we have

$$(3.8) \qquad P_x^\omega(A_n^\omega \mid \mathcal{F}_{2^n}^\omega) \geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha \left(\frac{1}{(n+1)\log 2}\right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} \times (\log\log 2^{n+1})^{(\beta-1)\alpha/\beta}$$

for $n \geq \tilde{N}_1 \vee \tilde{N}_2 \vee \tilde{N}_3$. We thus complete to show (3.5).

By (3.5) and the second Borel-Cantelli lemma, we have $d(Y_{2^n}^\omega, Y_{2^{n+1}}^\omega) \geq 2\varphi(2^{n+1})$ for infinitely many n. This implies $d(x, Y_{2^n}^\omega) \geq \varphi(2^n)$ or $d(x, Y_{2^{n+1}}^\omega) \geq \varphi(2^{n+1})$ for infinitely many n. Hence

$$\limsup_{t \to \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \geq 1.$$

We thus complete the proof. □

By Theorems 3.1, 3.2 and 2.6 we obtain (1.9).

3.2. Another law of the iterated logarithm

The proof of Theorem 1.4 (2) is quite similar to that of [31, Theorem 4.1] by using Lemmas 2.2, 2.5, 2.15 (2), Corollary 2.4 and Theorem 2.6. So we omit the proof.

4. Lower rate function

In this section we give the proof of Theorem 1.5. We follow the strategy as in [35, Section 4.1].

THEOREM 4.1. *Suppose Assumption* 1.1 (1) (2) (3) (4). *In addition suppose that there exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$ in the case of CSRW. Let $\alpha/\beta > 1$, $h : [0, \infty) \to (0, \infty)$ be a function such that $h(t) \searrow 0$ as $t \to \infty$, $\varphi(t) := t^{1/\beta}h(t)$ be increasing for all sufficiently large t and satisfy Assumption* 1.3 (3). *If the function $h(t)$ satisfies*

$$(4.1) \qquad \int_1^\infty \frac{1}{t} h(t)^{\alpha-\beta}\, dt < \infty$$

then for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$ we have

$$P_x^\omega(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t) = 1.$$

Proof. Set $\varphi(t) := t^{1/\beta}h(t)$, $t_n := 2^n$ and $A_n^\omega := \{d(x, Y_s^\omega) \leq \varphi(s)$ for some $s \in (t_n, t_n + 1]\}$. Note that there exists a constant c_1 such that $\varphi(s) \leq 2c_1\varphi(t_n)$ for all sufficiently large n (say $n \geq N_1$) and for all $s \in (t_n, t_{n+1}]$. Then by Lemma 2.12 we have

$$P_x^\omega(A_n^\omega) \leq P_x^\omega(d(x, Y_s^\omega) \leq 2c_1\varphi(t_n) \text{ for some } s > t_n) \leq \frac{c_2\varphi(t_n)^{\alpha-\beta}t_n}{t_n^{\alpha/\beta}}$$

for n with

(4.2)
$$n \geq N_1, \quad 2^n \geq T_0, \text{ where } T_0 \text{ is as in Lemma 2.12}, \quad t_n^{1/\beta} \geq c_1\varphi(t_n),$$
$$c_1\varphi(t_n) \geq \max_{z \in B(x, c_1\varphi(t_n))} N_z(\omega).$$

Note that (4.2) is satisfied for sufficiently large n (say $n \geq N_2 = N_2(\omega)$) by Assumption 1.3 (3) and Lemma 2.15 (3). Thus

$$\begin{aligned}\sum_{n \geq N_2(\omega)} P_x^\omega(A_n^\omega) &\leq \sum_{n \geq N_2(\omega)} \frac{c_2\varphi(t_n)^{\alpha-\beta}t_n}{t_n^{\alpha/\beta}} = \sum_{n \geq N_2(\omega)} \frac{c_2 h(t_n)^{\alpha-\beta}t_n}{t_n} \\ &\leq \sum_{n \geq N_2(\omega)} \frac{c_3 h(t_n)^{\alpha-\beta}(t_n - t_{n-1})}{t_n} \leq c_4 \int_{t_{N_2-1}}^\infty \frac{h(s)^{\alpha-\beta}}{s}\, ds.\end{aligned}$$

Since the last expression above is integrable by (4.1), by the Borel-Cantelli lemma we have

$$P_x^\omega(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t) = 1.$$

We thus complete the proof. □

THEOREM 4.2. *Under the same setting as in Theorem* 4.1, *if the function* $h(t)$ *satisfies*

(4.3)
$$\int_1^\infty \frac{1}{t}h(t)^{\alpha-\beta}\, dt = \infty$$

then for almost all $\omega \in \Omega$ *and all* $x \in V(G^\omega)$

(4.4)
$$P_x^\omega(d(x, Y_t^\omega) \geq \varphi(t) \text{ for all sufficiently large } t) = 0.$$

We cite the following form of the Borel-Cantelli Lemma (see [35, Lemma 4.15], [40, Lemma B], [12, Theorem 1]).

LEMMA 4.3. *Let* $\{A_k\}_{k \geq 1}$ *be a family of event which satisfies the following conditions;*

(1) $\sum_k P(A_k) = \infty$,
(2) $P(\limsup A_k) = 0$ *or* 1,
(3) *There exist two constants* c_1, c_2 *such that for each* A_j *there exist* $A_{j_1}, \dots, A_{j_s} \in \{A_k\}_{k \geq 1}$ *such that*

(a) $\sum_{i=1}^{s} P(A_j \cap A_{j_i}) \le c_1 P(A_j)$,
(b) *for any* $k \in \{j+1, j+2, \ldots\} \setminus \{j_1, j_2, \ldots, j_s\}$ *we have* $P(A_j \cap A_k) \le c_2 P(A_j) P(A_k)$.

Then infinitely many events $\{A_k\}_{k \ge 1}$ *occur with probability* 1.

Proof of Theorem 4.2. First we prepare preliminary facts. Since $h(t) \searrow 0$ as $t \to \infty$, there exists a positive constant T_1 such that $h(t) < 1$ for all $t \ge T_1$. So there exists a constant $\kappa \in (0,1)$ such that $\varphi(t) \le (\kappa t)^{1/\beta}$ for $t \ge T_1$. Take $\eta > 1 \vee \eta_0$ (where η_0 is as in Lemma 2.14) with $1 - \dfrac{1}{\eta} \ge \kappa$ and $c_1 = c_1(\eta) \in (0,1)$ such that $2c_1(\eta^{n+1})^{1/\beta} \le (\eta^n)^{1/\beta}$ for all n. Note that for all s with $\eta^{n+1} \le s \le \eta^{n+2}$ we have

$$\varphi(\eta^{n+1}) = (\eta^{n+1})^{1/\beta} h(\eta^{n+1}) \ge 2c_1(\eta^{n+2})^{1/\beta} h(s) \ge 2c_1 \varphi(s), \tag{4.5}$$

and for all sufficiently large i, j with $i \ge j+2$ and $\eta^j \ge T_1$ (say $j \ge N_1$) we have

$$(2c_1\varphi(\eta^{i+1}))^\beta \overset{(4.5)}{\le} \varphi(\eta^i)^\beta \le \kappa \eta^i \overset{1-1/\eta \ge \kappa}{\le} \eta^i - \eta^{i-1} \le \eta^i - \eta^{j+1}. \tag{4.6}$$

Now we prove (4.4). Set $A_n^\omega := \{d(Y_0^\omega, Y_s^\omega) \le 2c_1\varphi(\eta^{n+1})$ for some $s \in (\eta^n, \eta^{n+1}]\}$. We use Lemma 4.3 to show that infinitely many A_n^ω occur with probability 1.

Note that $\eta^n \ge (c_1\varphi(\eta^{n+1}))^\beta$ for sufficiently large n (say $n \ge N_2 = N_2(\eta)$) by (4.6). By Lemma 2.14 we have

$$P_x^\omega(A_n^\omega) \ge c_2 \frac{(c_1\varphi(\eta^{n+1}))^{\alpha-\beta}\eta^n}{\eta^{n\alpha/\beta}}$$

for $\eta \ge \eta_0$ (where η_0 is as in Lemma 2.14) and $n \ge N_2$ with

$$\eta^n \ge T_0, \text{ where } T_0 \text{ is as in Lemma 2.14}, \quad c_1\varphi(\eta^{n+1}) \ge \max_{z \in B(x, c_2\eta^{n/\beta})} N_z(\omega). \tag{4.7}$$

Note that (4.7) holds for sufficiently large n (say $n \ge N_3(\omega)$) by Assumption 1.3 (3) and Lemma 2.15 (3). Hence

$$\begin{aligned}
\sum_{n \ge N_3} P_x^\omega(A_n^\omega) &\ge \sum_{n \ge N_3} \frac{c_2(c_1\varphi(\eta^{n+1}))^{\alpha-\beta}\eta^n}{\eta^{n\alpha/\beta}} = \sum_{n \ge N_3} c_2 c_1^{\alpha-\beta}\eta^{\alpha/\beta} \frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta \cdot \eta^{n+1}} \eta^{n+1} \\
&= \sum_{n \ge N_3} \frac{c_2 c_1^{\alpha-\beta}\eta^{\alpha/\beta}}{\eta \cdot (\eta-1)} \frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta^{n+1}} (\eta^{n+2} - \eta^{n+1}) \\
&\ge \frac{c_2 c_1^{\alpha-\beta}\eta^{\alpha/\beta}}{\eta(\eta-1)} \int_{\eta^{N_3+1}}^\infty \frac{h(s)^{\alpha-\beta}}{s} \, ds.
\end{aligned}$$

Thus we have $\sum_n P_x^\omega(A_n^\omega) = \infty$ by (4.5).

The condition (2) in Lemma 4.3 is immediate from Theorem 2.6, since $\limsup_k A_k^\omega$ is a tail event.

Next we show the condition (3) in Lemma 4.3. Set $\sigma_n^\omega := \inf\{t \in (\eta^n, \eta^{n+1}] \mid d(Y_0^\omega, Y_t^\omega) \le 2c_1\varphi(\eta^{n+1})\}$. Then for $i \ge j+2$ we have

$$
\begin{aligned}
(4.8) \qquad P_x^\omega(A_i^\omega \cap A_j^\omega) &= P_x^\omega(\sigma_j \le \eta^{j+1}, \sigma_i \le \eta^{i+1}) \\
&= E_x^\omega[1_{\{\sigma_j \le \eta^{j+1}\}} P_{Y_{\sigma_j}}^\omega(d(x, Y_t^\omega) \le 2c_1\varphi(\eta^{i+1}) \\
&\qquad \text{for some } t \in (\eta^i - \sigma_j, \eta^{i+1} - \sigma_j])] \\
&\le E_x^\omega[1_{\{\sigma_j \le \eta^{j+1}\}} P_{Y_{\sigma_j}}^\omega(d(x, Y_t^\omega) \le 2c_1\varphi(\eta^{i+1}) \\
&\qquad \text{for some } t > \eta^i - \eta^{j+1})] \\
&\le \Bigg(\sup_{z: d(x,z) \le 2c_1\varphi(\eta^{j+1})} P_z^\omega(d(x, Y_t^\omega) \le 2c_1\varphi(\eta^{i+1}) \\
&\qquad \text{for some } t > \eta^i - \eta^{j+1}) \Bigg) \cdot P_x^\omega(\sigma_j \le \eta^{j+1}).
\end{aligned}
$$

By Lemma 2.12, for any $i \ge j+2$ with

$$
\begin{aligned}
(4.9) \qquad & \eta^i - \eta^{j+1} \ge (c_1\varphi(\eta^{i+1}))^\beta, \quad 2c_1\varphi(\eta^{j+1}) \le c_1\varphi(\eta^{i+1}), \\
& \varphi(\eta^{i+1}) \ge \max_{z \in B(x, \varphi(\eta^{i+1}))} N_z(\omega)
\end{aligned}
$$

we have

$$
\begin{aligned}
(4.10) \qquad & \sup_{z: d(x,z) \le 2c_1\varphi(\eta^{j+1})} P_z^\omega(d(x, Y_t^\omega) \le 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}) \\
& \le \frac{c_3(c_1\varphi(\eta^{i+1}))^{\alpha-\beta}(\eta^i - \eta^{j+1})}{(\eta^i - \eta^{j+1})^{\alpha/\beta}} \le \frac{c_4(c_1\varphi(\eta^{i+1}))^{\alpha-\beta}\eta^i}{(\eta^i)^{\alpha/\beta}}.
\end{aligned}
$$

(4.9) holds for sufficiently large i, j with $i \ge j+2$ (say $j \ge N_4 = N_4(\omega)$) by (4.5), (4.6), Assumption 1.3 (3) and Lemma 2.15 (3). By Lemma 2.14, for any i with

$$
\begin{aligned}
(4.11) \qquad & \eta^i \ge T_0, \text{ where } T_0 \text{ is as in Lemma 2.14}, \quad \eta^i \ge (c_1\varphi(\eta^{i+1}))^\beta, \\
& c_1\varphi(\eta^{i+1}) \ge \max_{v \in B(x, c_5\eta^{i/\beta})} N_v(\omega)
\end{aligned}
$$

we have

$$
\begin{aligned}
(4.12) \qquad \frac{(c_1\varphi(\eta^{i+1}))^{\alpha-\beta}\eta^i}{(\eta^i)^{\alpha/\beta}} &\le c_6 P_x^\omega(d(x, Y_t^\omega) \le 2c_1\varphi(\eta^{i+1}) \text{ for some } t \in (\eta^i, \eta^{i+1}]) \\
&= c_6 P_x^\omega(A_i^\omega).
\end{aligned}
$$

(4.11) holds for sufficiently large j (say $j \geq N_5 = N_5(\omega)$) by (4.5), Assumption 1.3 (3) and Lemma 2.15 (3). Hence by (4.8), (4.10) and (4.12) we have $P_x^{\omega}(A_i^{\omega} \cap A_j^{\omega}) \leq cP_x^{\omega}(A_i^{\omega})P_x^{\omega}(A_j^{\omega})$ for sufficiently large j ($j \geq N_6 := N_4 \vee N_5$) and $i \geq j+2$. In the case of $i = j+1$ we have $P_x^{\omega}(A_{j+1}^{\omega} \cap A_j^{\omega}) \leq P_x^{\omega}(A_j^{\omega})$. Thus we obtain the condition (3) of Lemma 4.3 for $\{A_i^{\omega}\}_{i \geq N_6}$.

By Lemma 4.3, we thus complete the proof. □

By Theorems 4.1 and 4.2 we complete the proof of Theorem 1.5.

5. Ergodic media

In this section, we consider the case $G = (V, E) = \mathbf{Z}^d$ and obtain Theorem 1.7 under Assumption 1.6. We follow the strategy as in [16].

5.1. Ergodicity of the shift operator on $\Omega^{\mathbf{Z}}$

We consider Markov chains on the random environment, which is called the environment seen from the particle, according to Kipnis and Varadhan [28].

Let $\Omega = [0, \infty)^E$ and define $\mathscr{B}$ as the natural σ-algebra (generated by coordinate maps). We write $\mathcal{Y} = \Omega^{\mathbf{Z}}$, $\mathscr{Y} = \mathscr{B}^{\otimes \mathbf{Z}}$. If each conductance may take the value 0, we regard 0 as the base point and define $\mathcal{C}_0(\omega) = \{x \in \mathbf{Z}^d \mid 0 \overset{\omega}{\leftrightarrow} x\} = V(G^{\omega})$, where $0 \overset{\omega}{\leftrightarrow} x$ means that there exists a path $\gamma = e_1 e_2 \cdots e_k$ from 0 to x such that $\omega(e_i) > 0$ for all $i = 1, 2, \ldots, k$. Define $\Omega_0 = \{\omega \in \Omega \mid \#\mathcal{C}_0(\omega) = \infty\}$ and $\mathbf{P}_0 = \mathbf{P}(\cdot \mid \Omega_0)$.

Next we consider the Markov chains seen from the particle. Recall that $\{X_n^{\omega}\}_{n \geq 0}$ is the discrete time random walk which is introduced in Section 1.1. Let $\omega_n(\cdot) = \omega(\cdot + X_n^{\omega}) = \tau_{X_n^{\omega}}\omega(\cdot) \in \Omega$. We can regard this Markov chain $\{\omega_n\}_{n \geq 0}$ as being defined on $\mathcal{Y} = \Omega^{\mathbf{Z}}$. We define a probability kernel $Q : \Omega_0 \times \mathscr{B} \to [0, 1]$ as

$$Q(\omega, A) = \frac{1}{\sum_{e':|e'|=1} \omega_{e'}} \sum_{v:|v|=1} \omega_{0v} 1_{\{\tau_v \omega \in A\}}.$$

This is nothing but the transition probability of the Markov chain $\{\omega_n\}_{n \geq 0}$.

Next we define the probability measure on $(\mathcal{Y}, \mathscr{Y})$ as

$$\mu((\omega_{-n}, \ldots, \omega_n) \in B) = \int_B \mathbf{P}_0(d\omega_{-n}) Q(\omega_{-n}, d\omega_{-n+1}) \cdots Q(\omega_{n-1}, d\omega_n).$$

By the above definition, $\{\tau_{X_k^{\omega}}\omega\}_{k \geq 0}$ has the same law in $\mathbf{E}_0(P_0^{\omega}(\cdot))$ as $(\omega_0, \omega_1, \ldots)$ has in μ, that is,

$$\mathbf{E}_0[P_0^{\omega}(\{\tau_{X_k^{\omega}}\omega\}_{k \geq 0} \in B)] = \mu((\omega_0, \omega_1, \ldots) \in B) \tag{5.1}$$

for any $B \in \mathscr{Y}$.

We need the following Theorem. Let $T : \mathcal{Y} \to \mathcal{Y}$ be a shift operator of $\mathcal{Y}$, that is,

$$(T\omega)_n = \omega_{n+1}.$$

Theorem 5.1. *Under Assumption* 1.6, T *is ergodic with respect to* μ.

The proof is similar to [8, Proposition 3.5], so we omit it.

We also need the following Zero-One law (see Proposition 5.2). Let $a \geq 0$ and $A_1^\omega(a)$, $A_2^\omega(a)$, $A_3^\omega(a)$ be the events

$$A_1^\omega(a) = \left\{ \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} > a \right\},$$

$$A_2^\omega(a) = \left\{ \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} > a \right\},$$

$$A_3^\omega(a) = \left\{ \liminf_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{-1/\beta}} > a \right\}.$$

Define

$$\tilde{A}_i(a) = \{\omega \in \Omega \mid A_i^\omega(a) \text{ holds for } P_x^\omega\text{-a.s. and for all } x \in \mathcal{C}_0(\omega)\}.$$

Proposition 5.2. $\mathbf{P}_0(\tilde{A}_i(a))$ *is either* 0 *or* 1.

Proof. See [31, Proposition 5.2]. □

5.2. Proof of Theorem 1.7

In this subsection we discuss the proof of Theorem 1.7. Recall $T_0^\omega = 0$, $T_{n+1}^\omega = \inf\{t > T_n^\omega \mid Y_t^\omega \neq Y_{T_n^\omega}^\omega\}$ and $X_n^\omega = Y_{T_n^\omega}^\omega$.

First we consider the CSRW. $\{T_{n+1}^\omega - T_n^\omega\}_{n\geq 0}$ is a family of i.i.d. random variables whose distributions are exponential with mean 1, so the law of large number gives us

$$\frac{T_n^\omega}{n} \to 1 \quad P_0^\omega\text{-a.s.}$$

Thus

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = \liminf_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{-1/\beta}}.$$

By Assumption 1.6, Proposition 5.2 and Theorem 1.4 we obtain Theorem 1.7.

Next we consider the VSRW. $\{T^\omega_{n+1} - T^\omega_n\}_{n\geq 0}$ are non-i.i.d., and the distribution of $T^\omega_{n+1} - T^\omega_n$ is exponential with mean $\dfrac{1}{\pi^\omega(X^\omega_n)}$. Write S^ω_x be a exponential random variable with parameter $\pi^\omega(x)$ and $\overline{S}_x(\bar\omega) := S^{\bar\omega_0}_x$, $(\bar\omega \in \mathcal{Y})$. Then by (5.1) and the ergodicity we have

$$\begin{aligned}\frac{1}{n}T^\omega_n &= \frac{1}{n}\sum_{k=0}^{n-1} S^\omega_{X^\omega_k} \overset{d}{=} \frac{1}{n}\sum_{k=0}^{n-1} \overline{S}_0(T^k\bar\omega) \to \mathbf{E}^\mu[\overline{S}_0] \\ &= \mathbf{E}[E^\omega_0[S^\omega_0]] = \int_\Omega \int_0^\infty x\pi^\omega(0)\exp(-\pi^\omega(0)x)\,dxd\mathbf{P} = \mathbf{E}\left[\frac{1}{\pi^\omega(0)}\right].\end{aligned}$$

Thus

$$\limsup_{t\to\infty} \frac{d(Y^\omega_0, Y^\omega_t)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \left(\frac{1}{\mathbf{E}\left[\dfrac{1}{\pi^\omega(0)}\right]}\right)^{1/\beta} \limsup_{n\to\infty} \frac{d(X^\omega_0, X^\omega_n)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y^\omega_0, Y^\omega_s)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \left(\frac{1}{\mathbf{E}\left[\dfrac{1}{\pi^\omega(0)}\right]}\right)^{1/\beta} \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X^\omega_0, X^\omega_k)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y^\omega_0, Y^\omega_s)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = \left(\frac{1}{\mathbf{E}\left[\dfrac{1}{\pi^\omega(0)}\right]}\right)^{1/\beta} \liminf_{t\to\infty} \frac{\sup_{0\leq k\leq n} d(X^\omega_0, X^\omega_k)}{n^{1/\beta}(\log\log n)^{-1/\beta}}.$$

By Assumption 1.6, Proposition 5.2 and Theorem 1.4 we obtain Theorem 1.7.

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