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A NOTE ON "ON THE APPEARANCE OF EISENSTEIN SERIES THROUGH DEGENERATION"

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Abstract

Let $M = \Gamma\backslash H$ be a geometrically finite hyperbolic surface, realized as the quotient of the hyperbolic upper half plane H by a geometrically finite discrete group of isometries acting on H. To a parabolic element of the uniformizing group Γ, there is an associated 1-form parabolic Eisenstein series. To a primitive hyperbolic element, then, following ideas due to Kudla–Millson, there is a corresponding 1-form hyperbolic Eisenstein series. In this article, we study the limiting behavior of these hyperbolic Eisenstein series on a degenerating family of hyperbolic Riemann surfaces of finite type, using basically the limiting behavior of counting functions associated to degenerating hyperbolic Riemann surfaces. In this sense, we generalize the results obtained in Garbin, Jorgenson and Munn (Comment Math Helv 83:701–721, 2008) to the case of geometrically finite hyperbolic surfaces of infinite volume and form-valued parabolic and hyperbolic Eisenstein series.

1. Introduction

There is a vast literature addressing problems in the study of spectral theory degenerating hyperbolic Riemann surfaces and within, on degeneration of Poincaré series and Eisenstein series, see [3], [7], [8], [15], [18], [19], [20] to cite some examples.

Our context and our aim are the following. Let Γ contained in $PSL(2,\mathbf{R})$ be a Fuchsian group finitely generated of the first or second kind acting on the upper half plane H without elliptic elements. The quotient $\Gamma\backslash H$ is a hyperbolic geometrically finite surface. This means that Γ admits a finite sided polygonal fundamental domain in H. Throughout this article we refer to parabolic Eisenstein series $\hat{p}^s$ associated to a parabolic element of the uniformizing group Γ or equivalently to a cusp p and hyperbolic Eisenstein series $\hat{c}^s$ associated to a primitive hyperbolic element or equivalently to a simple closed oriented geodesic c.

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Precise definitions and references to all concepts will be given in Section 2 below. However, with these comments made, we are able to state the main result of the paper.

Main Theorem

Let M_l be a degenerating family of geometrically finite hyperbolic surfaces with limit surface M_0.

(1) *Let $\hat{c}_l^s$ be the hyperbolic Eisenstein series on M_l associated to a non-separating simple closed geodesic of length l, then*

$$\lim_{l\to 0} \frac{1}{l^s}\hat{c}_l^s = \hat{p}^s - \hat{q}^s,$$

where p and q are the cusps arising from the pinching geodesic c_l.

(2) *Let $\hat{c}_l^s$ be the hyperbolic Eisenstein series on M_l associated to the boundary of a funnel then*

$$\lim_{l\to 0} \frac{1}{l^s}\hat{c}_l^s = \hat{p}^s.$$

In all instances, the convergence is uniform on compact subsets of M_0 bounded away from the developing cusps, and in half-planes of the form $\mathrm{Re}(s) \geq 1+\delta$ for any $\delta > 0$.

Remark 1.1. The main tool of the demonstration is the study of the limiting behavior of counting functions as in [7].

In the cited article the authors are working with scalar-values hyperbolic Eisenstein series. Casually we point out that there is a difference between scalar-values Eisenstein series and form-valued Eisenstein series studied in [17] an in [3]: if the degenerating Riemann surface has a single pinching geodesic which is non-separating, then the associated hyperbolic Eisenstein series does not converge to the sum of two parabolic Eisenstein series corresponding to the two newly formed cusps but to the difference.

At the end of this paper, we make the remark that a same result occurs in the general infinite volume case.

2. Background material

2.1. Geometrically finite hyperbolic surface. Let us recall the standard geometric notations which will be used.

A topologically finite (i.e. finite Euler characteristic) surface is a surface homeomorphic to a compact surface with finitely many points excised and a geometrically finite hyperbolic surface M is a topologically finite, complete Riemann surface of constant curvature -1. It can be decomposed into a compact core K plus cusps C_i and funnels F_j ([1]):

$$M = K \cup (C_1 \cup \cdots \cup C_{n_c}) \cup (F_1 \cup \cdots \cup F_{n_f}).$$

The boundary of K consists of n_f closed geodesics (uniquely determined) and n_c horocycles (the choice of which is not unique) along which K is glued to the funnel and cusp ends, respectively.

A hyperbolic transformation $T \in PSL(2, \mathbf{R})$ generates a cyclic hyperbolic group $\langle T \rangle$. The quotient $C_l = \langle T \rangle \backslash H$ is a hyperbolic cylinder of diameter $l = l(T)$. By conjugation we can identify the generator T with the map $\sigma_l : z \mapsto e^l z$, and we define Γ_{σ_l} to be the corresponding cyclic group. A natural fundamental domain for Γ_{σ_l} would be the region $\mathscr{F}_l = \{z \in H, 1 \leq |z| \leq e^l\}$. The y-axis is the lift of the only simple closed geodesic on C_l, whose length is l. The standard funnel of diameter $l > 0$, F_l, is the half hyperbolic cylinder $\Gamma_{\sigma_l} \backslash H$, $F_l = (\mathbf{R}^+)_r \times (\mathbf{R} \backslash \mathbf{Z})_x$ with the metric $ds^2 = dr^2 + l^2 \cosh^2(r)\, dx^2$.

We can always conjugate a parabolic cyclic group $\langle T \rangle$ to the group Γ_∞ generated by $z \mapsto z + 1$, so the parabolic cylinder is unique up to isometry. A natural fundamental domain for Γ_∞ is $\mathscr{F}_\infty = \{0 \leq \operatorname{Re} z \leq 1\} \subset H$. The standard cusp C_∞ is the half parabolic cylinder $\Gamma_\infty \backslash H$, $C_\infty = ([0, \infty[)_r \times (\mathbf{R} \backslash \mathbf{Z})_x$ with the metric $ds^2 = dr^2 + e^{-2r}\, dx^2$. The funnels F_j and the cusps C_i are isometric to the preceding standard models.

2.2. Hodge operator. We define the Hodge operator (or conjugation operator) on smooth differential forms on a Riemann surface M as follows: for a 1-form w given in local coordinate $z = x + iy$ on M by $\omega = f\, dx + g\, dy$, we associate $*\omega = -g\, dx + f\, dy$. To define the operator $*$ on functions and 2-forms, we denote by $v_H = y^{-2}\, dx \wedge dy$ the volume form. If f is a function, we set $*f = f(z) v_H$. For a 2-form Ω, we set $*\Omega = \Omega / v_H$.

We are interested primarily in 1-forms. If ω is given in complex notation by $u(z)\, dz + v(z)\, \overline{dz}$, then $*\omega = -iu(z)\, dz + iv(z)\, \overline{dz}$. We define a pointwise scalar product at z of two 1-forms φ and ψ by $\varphi \wedge *\bar{\psi} = \langle \varphi, \psi \rangle v_H$ and the pointwise norm of a 1-form ω is defined by $\omega \wedge *\bar{\omega} = \|\omega\|^2 v_H$.

2.3. Hyperbolic and parabolic Eisenstein series. The study of parabolic Eisenstein series is a classical part of mathematical literature (see [16] just to cite one reference) and more precisely in the case of infinite area hyperbolic Riemann surfaces the study of such series can be found also in [1], p. 102.

As underlined by Gérardin in [9], an explicit construction of hyperbolic Eisenstein series can be found in [6] and the convergence of these Eisenstein series can be found in [6], p. 184. Kudla and Millson give an invariant construction of hyperbolic Eisenstein series that we follow here (for more details see [9] and [4]). Let us recall the definitions of hyperbolic and parabolic Eisenstein series.

If X is an horocycle of H with the direct orientation, we denote by $\mathrm{d}_X(z)$ the oriented distance between X and $z \in H$, $(z : X) = e^{\mathrm{d}_X(z)}$, v_X the volume form on X invariant under Γ_X, the stabilizer of X in Γ, p_X the orthogonal projection from H to X. Then define a 1-form on H, $w_X = *p_X^* v_X$ such that $\|w_X\| = (z : X)$.

If Y is an oriented geodesic of H, we denote by $\mathrm{d}_Y(z)$ the oriented distance between Y and $z \in H$, $(z : Y) = 1/\cosh \mathrm{d}_Y(z)$, v_Y the volume form on Y invariant under Γ_Y, the stabilizer of Y in Γ, p_Y the orthogonal projection from H to Y. Then define a 1-form on H, $w_Y = *p_Y^* v_Y$ such that $\|w_Y\| = (z : Y)$.

Let ξ an oriented horocycle on M associated to a point p and $H(\xi)$ the set of horocycles on H that project under the canonical projection $H \to M$ on ξ. The Eisenstein series associated to ξ is the 1-form

$$\hat{\xi}^s = \sum_{X \in H(\xi)} \|w_X\|^{s-1} w_X,$$

defined for $\mathrm{Re}\, s > 1$ and called horocyclic Eisenstein series.

If we denote by $|\xi|$ the width of the horocycle ξ then the form $|\xi|^{-s}\hat{\xi}^s$ is independent of the choice of the horocycle ξ associated to the point p. We denoted this series by $\hat{p}^s$ and we will call it a parabolic Eisenstein series.

In the same way let η be a closed oriented geodesic on M and $H(\eta)$ the set of oriented geodesics on H that project to η. The Eisenstein series associated to η is, up to some normalization, the 1-form

$$\hat{\eta}^s = \sum_{Y \in H(\eta)} \|w_Y\|^{s-1} w_Y,$$

defined for $\mathrm{Re}\, s > 1$ and called hyperbolic Eisenstein series.

In each case, for $s \in \mathbf{C}$, $\mathrm{Re}\, s > 1$, we define the 1-form on M with $Z = X$ (respectively, $Z = Y$) and the notation $\|w_Z\|^{s-1} w_Z = w_Z^s$:

$$\sum w_Z^s$$

called an horocyclic Eisenstein series (respectively, an hyperbolic Eisenstein series).

Fix Y_0 in $H(\eta)$ and denote by Γ_{Y_0} its stabilizer in Γ, then $H(\eta) = \Gamma Y_0 = (\Gamma \backslash \Gamma_{Y_0}) Y_0$

$$\hat{\eta}^s = \sum_{\gamma \in \Gamma \backslash \Gamma_{Y_0}} w_{\gamma Y_0}^s.$$

Choose and fix any point $z \in M$, which we lift to a point $z \in H$. As $\mathrm{d}_{\gamma Y_0}(z) = \mathrm{d}_{Y_0}(\gamma^{-1} z)$, we have also

$$\hat{\eta}^s(z) = \sum_{\delta \in \Gamma_{Y_0} \backslash \Gamma} \frac{1}{\cosh \mathrm{d}_{Y_0}(\delta z)^{s-1}} \frac{d\mathrm{d}_{Y_0}(\delta z)}{\cosh \mathrm{d}_{Y_0}(\delta z)}.$$

Remark 2.1. d_{Y_0} is the Fermi-coordinate x_2 in [17].

In the same way, fix X_0 in $H(\xi)$ and denote by Γ_{X_0} its stabilizer in Γ, then $H(\xi) = \Gamma X_0 = (\Gamma \backslash \Gamma_{X_0}) X_0$ then

$$\hat{\xi}^s(z) = \sum_{\gamma \in \Gamma \backslash \Gamma_{X_0}} w_{\gamma X_0}^s(z).$$

Choose and fix any point $z \in M$, which we lift to a point $z \in H$. As $d_{\gamma X_0}(z) = d_{X_0}(\gamma^{-1}z)$, we have also

$$\hat{\xi}^s(z) = \sum_{\delta \in \Gamma_{X_0} \backslash \Gamma} e^{s d_{X_0}(\delta z)} \, d d_{X_0}(\delta z).$$

2.4. Stieltjes integrals. In order to be consistent with the notations of [7] we will fix Z_0 in the set of oriented geodesics of H that project to η (respectively, in the set of oriented horocycles of H that project to ξ) and we will write $d_{\mathrm{hyp}}(z, Z_0)$ the geodesic distance and as before $d_{Z_0}(z)$, the *oriented* geodesic distance from z to Z_0. With all this, we will re-write the counting functions in [7], p. 705, in the following way: the hyperbolic counting function (respectively, parabolic counting function associated to X_0) is define as

$$N_{\mathrm{hyp}, M, \eta}(T; z) = \mathrm{card}\{\delta \in \Gamma_{Y_0} \backslash \Gamma, -T < d_{Y_0}(\delta z) < T\}$$

$$(\text{respectively, } N_{\mathrm{par}, M_0, p}(T; z, \xi) = \mathrm{card}\{\delta \in \Gamma_{X_0} \backslash \Gamma, -T < d_{X_0}(\delta z) < T\}).$$

As η is non-separating one needs to take into account that geodesic lengths from z to η enter the cylinder about the pinching geodesic from the two different sides.

$$\hat{\eta}^s(z) = \sum_{\delta \in \Gamma_{Y_0} \backslash \Gamma} \frac{1}{\cosh d_{Y_0}(\delta z)^{s-1}} \frac{d d_{Y_0}(\delta z)}{\cosh d_{Y_0}(\delta z)} \tag{1}$$

$$= \sum_{\substack{\delta \in \Gamma_{Y_0} \backslash \Gamma \\ d_{Y_0}(\delta z) \geq 0}} \frac{1}{\cosh d_{Y_0}(\delta z)^{s-1}} \frac{d d_{Y_0}(\delta z)}{\cosh d_{Y_0}(\delta z)} + \sum_{\substack{\delta \in \Gamma_{Y_0} \backslash \Gamma \\ d_{Y_0}(\delta z) < 0}} \frac{1}{\cosh d_{Y_0}(\delta z)^{s-1}} \frac{d d_{Y_0}(\delta z)}{\cosh d_{Y_0}(\delta z)}. \tag{2}$$

Let then write

$$N_{\mathrm{hyp}, M, \eta}(x; z) = N^L_{\mathrm{hyp}, M, \eta}(x; z) + N^R_{\mathrm{hyp}, M, \eta}(x; z);$$

where

$$N^L_{\mathrm{hyp}, M, \eta}(x; z) = \mathrm{card}\{\delta \in \Gamma_{Y_0} \backslash \Gamma, 0 \leq d_{Y_0}(\delta z) < x\}$$

and

$$N^R_{\mathrm{hyp}, M, \eta}(x; z) = \mathrm{card}\{\delta \in \Gamma_{Y_0} \backslash \Gamma, -x < d_{Y_0}(\delta z) \leq 0\}.$$

They are increasing step-functions and give rise to a Stieltjes measure $dN_{\mathrm{hyp}, M, \eta}$ (respectively, $dN_{\mathrm{par}, M_0, p}$, $dN^L_{\mathrm{hyp}, M, \eta}$, $dN^R_{\mathrm{hyp}, M, \eta}$).

If we denote $w_{Y_0}(x) = \dfrac{dx}{\cosh x}$, we can express the hyperbolic Eisenstein series as a Stieltjes integral, namely

$$\begin{aligned}\hat{\eta}^s(z) &= \int_0^\infty \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN_{\mathrm{hyp},\, M,\eta}(x; z) \\ &= \int_0^\infty \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},\, M,\eta}(x; z) \\ &\quad - \int_0^\infty \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^R_{\mathrm{hyp},\, M,\eta}(x; z).\end{aligned}$$

We have the following inequality

$$\begin{aligned}\|\hat{\eta}^s(z)\| &\le \int_0^\infty \left\| \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x) \right\| dN_{\mathrm{hyp},\, M,\eta}(x; z) \\ &= \int_0^\infty \left(\frac{1}{\cosh x}\right)^{(\mathrm{Re}\, s)} dN_{\mathrm{hyp},\, M,\eta}(x; z).\end{aligned}$$

We can choose X_0 such that $N_{\mathrm{par},\, M_0,p}(T; z, \xi) = \mathrm{card}\{\delta \in \Gamma_{X_0}\backslash\Gamma, -T < \mathrm{d}_{X_0}(\delta z) \le 0\}$. If we denote $w_{X_0}(x) = e^x\, dx$, we can express the parabolic Eisenstein series as a Stieltjes integral, namely

$$\hat{\xi}^s(z) = \int_0^\infty (e^{-x})^{(s-1)} w_{X_0}(-x)\, dN_{\mathrm{par},\, M_0,p}(x; z);$$

and we have the same preceding remark.

3. Convergence

A family of degenerating geometrically finite hyperbolic surfaces consists of a surface M and a smooth family $(g_l)_{l>0}$ of Riemannian metrics that meet the following assumptions:

(1) The Riemannian manifold $M_l = (M, g_l)$ is a geometrically finite hyperbolic surface for each l.
(2) There are finitely many disjoint open subsets $\mathscr{C}_{l,i} \subset M$ that are diffeomorphic to cylinders $\mathbf{R}\backslash\mathbf{Z} \times J_i$ where $J_i \subset \mathbf{R}$ is a connected neighborhood of 0 with the metric $(x, a) \mapsto (l_i(l)^2 + a^2)\, dx^2 + ((l_i(l)^2 + a^2)^{-1}\, da^2$ and $l_i(l) \to 0$ as $l \to 0$. The curve $c_i = \mathbf{R}\backslash\mathbf{Z} \times \{0\}$ is a closed geodesic of length $l_i(l)$.
(3) The complement of $(C_1 \cup \cdots \cup C_{n_c}) \cup (F_1 \cup \cdots \cup F_{n_f}) \cup_i \mathscr{C}_{l,i}$ where we may have some $F_j \subset \mathscr{C}_{l,i}$ is relatively compact.
(4) On $M_0 := M \backslash \bigcup_i c_i$, the metrics g_l converge smoothly to a hyperbolic metric g_0 as $l \to 0$. M_0 is a possibly non connected hyperbolic surface that contains a pair of cusps for each i.

In the following, we will assume that M_l has a single family of degenerating geodesics; the more general situation is easily obtained with only a slight modification of notation. More precisely we contemplate two cases: the case the degenerating geodesic is non-separating and the case the degenerating geodesic is the boundary of a funnel. In the first case we have for any $0 < \varepsilon < 1/2$, $\mathscr{C}_{l,\varepsilon} = \mathbf{R}\backslash\mathbf{Z} \times]-\varepsilon/2, +\varepsilon/2[$ with total volume equal to ε. In the second case $\mathscr{C}_{l,\varepsilon} = \mathbf{R}\backslash\mathbf{Z} \times]-\varepsilon/2, +\infty[$ contains the funnel F_l.

In both cases we consider a degenerating family of groups $\{\Gamma_l\}$ with $M_l = H\backslash\Gamma_l$ degenerating to the surface M_0, Γ_l containing the transformation $\sigma_l(z) = e^l z$ and its stabilizer Γ_{σ_l}. We also write σ_l for the associated closed geodesic. Then the geodesic in H fixed by σ_l is the line $Y_0 = \{\mathrm{Re}(z) = 0\} \cap H$. For any point $z \in M_l$, which we lift to a point $z \in H$, let $\mathrm{d}_l(z)$ denote the geodesic distance from z to Y_0. We denote by p and q the two cusps of M_0 arising from pinching σ_l, the limit of respectively the right side and the left side of the σ_l-collar $\mathscr{C}_{l,\varepsilon}$.

To prevent burdensome notation, we write

$$N_{\mathrm{hyp},l} := N_{\mathrm{hyp},M_l,c_l}$$
$$N^{L(R)}_{\mathrm{hyp},l} := N^{L(R)}_{\mathrm{hyp},M_l,c_l}$$

In the case the degenerating geodesic c_l is non-separating, we denote by $\partial\mathscr{C}^L_{l,\varepsilon}$ (respectively, $\partial\mathscr{C}^R_{l,\varepsilon}$) the left (respectively, right) boundary of the collar $\mathscr{C}_{l,\varepsilon}$ and the corresponding counting functions $N_{\mathrm{hyp},\partial\mathscr{C}^L_{l,\varepsilon}}(x;z) = \mathrm{card}\{\delta \in \Gamma_{\sigma_l}\backslash\Gamma_l, 0 \le \mathrm{d}_{\partial\mathscr{C}^L_{l,\varepsilon}}(\delta z) < x\}$ (respectively, $N_{\mathrm{hyp},\partial\mathscr{C}^R_{l,\varepsilon}}(x;z) = \mathrm{card}\{\delta \in \Gamma_{\sigma_l}\backslash\Gamma_l, -x < \mathrm{d}_{\partial\mathscr{C}^R_{l,\varepsilon}}(\delta z) \le 0\}$).

In the case the degenerating geodesic c_l is the boundary of the funnel F_l we are only interested in the right side of the collar and the corresponding definitions.

3.1. Convergence of counting functions. We can rewrite Lemma 3.3 of [7] in the following way

LEMMA 3.1. *Assume $\varepsilon > 0$ is sufficiently small so that $\mathscr{C}_{l,\varepsilon}$ is embedded in M_l. Let $\tau(\varepsilon, l)$ being the half width of the collar $\mathscr{C}_{l,\varepsilon}$, then for any $x > 0$ we have*:

(1) *In the case the degenerating geodesic c_l is non-separating*

$$N_{\mathrm{hyp},\partial\mathscr{C}^L_{l,\varepsilon}}(x;z) = N^L_{\mathrm{hyp},l}(x + \tau(\varepsilon,l);z);$$
$$\lim_{l\to 0} N^L_{\mathrm{hyp},l}(x + \tau(\varepsilon,l);z) = N_{\mathrm{par},M_0,q}(x;z,\xi_{q,\varepsilon})$$

with $|\xi_{q,\varepsilon}| = \varepsilon/2$.

In the same way

$$N_{\mathrm{hyp},\partial\mathscr{C}^R_{l,\varepsilon}}(x;z) = N^R_{\mathrm{hyp},l}(x + \tau(\varepsilon,l);z);$$
$$\lim_{l\to 0} N^R_{\mathrm{hyp},l}(x + \tau(\varepsilon,l);z) = N_{\mathrm{par},M_0,p}(x;z,\xi_{p,\varepsilon})$$

with $|\xi_{p,\varepsilon}| = \varepsilon/2$.

(2) *In the case the degenerating geodesic* c_l *is the boundary of a funnel,* $N_{\mathrm{hyp},l}(T;z)$ *is equal to* $\operatorname{card}\{\delta \in \Gamma_{\sigma_l}\backslash\Gamma_l, -T < d_l(\delta z) \le 0\}$ *and we have*

$$\lim_{l\to 0} N_{\mathrm{hyp},l}(x+\tau(\varepsilon,l);z) = N_{\mathrm{par},M_0,p}(x;z,\xi_{p,\varepsilon}).$$

In all instances, the convergence is uniform on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$.

We will denote by X_q (respectively, X_p) a horocycle in H corresponding to $\xi_{q,\varepsilon}$ (respectively, $\xi_{p,\varepsilon}$).

Let us illustrate this result by a change of variables. To study the left side of the collar use the change of variables $l\zeta = -\log(-z)$, with the principal branch: then $\left(\frac{1}{l}\frac{dz}{z}\right)^2 = (d\zeta)^2$ and $\left(\frac{|dz|}{\operatorname{Im} z}\right)^2 = \left(\frac{l|d\zeta|}{\sin lb}\right)^2$ for $\zeta = a+ib$.

We consider q the cusp of M_0 limit of the left side of the σ_l-collar. Now, as above, let $l\zeta = -\log(-z)$, $z\in H$, and conjugate Γ_l by the map $\zeta(z)$ to obtain $\tilde{\Gamma}_l$ acting on $\mathscr{S}_l = \{\zeta \mid 0 < \operatorname{Im}\zeta < \pi/l\}$. $\tilde{\Gamma}_l$ is a (non-Möbius) group of desk transformations acting on $\mathscr{S}_l$; the quotient $\mathscr{S}_l\backslash\tilde{\Gamma}_l$ is M_l.

There exist homeomorphisms f_l from $M_l - \{c_l\}$ to M_0, with f_l tending to isometries C^2-uniformly on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$; f_l has a lift $\tilde{f}_l$, a homeomorphism from a sub domain of $\mathscr{S}_l$ (containing the left half-collar $\{-1 < \operatorname{Re}\zeta \le 0, c < \operatorname{Im}\zeta < \pi/2l\}$) to H; f_l induces a group homomorphism $\rho_l : \Gamma_0 \to \tilde{\Gamma}_l$ by the rule $A \to \tilde{f}_l^{-1}A\tilde{f}_l$, $A\in\Gamma_0$. We call $\rho_l(A)\in\tilde{\Gamma}_l$ the element corresponding to $A\in\Gamma_0$. Now by our normalizations for $\tilde{\Gamma}_l$ and Γ_0, the translation $\zeta\mapsto\zeta-1$ corresponds to itself. If we specify the further normalization $\tilde{f}_l(i) = i$, then the lifts $\tilde{f}_l$ are uniquely determined and tend uniformly on compact subsets to the identity, and thus for $A\in\Gamma_0$, the corresponding elements $\rho_l(A)$ tend uniformly on compact subsets to A.

3.2. Convergence of Eisenstein series. In this section, we prove the Main Theorem.

First assume c_l is non-separating. Let then write

$$\hat{c}_l^s(z) = \hat{c}_{lL}^s(z) - \hat{c}_{lR}^s(z)$$

with

$$\hat{c}_{lL}^s(z) = \int_0^\infty \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN_{\mathrm{hyp},l}^L(x;z)$$

and corresponding definition for $\hat{c}_{lR}^s(z)$. To begin, we write

$$\hat{c}_{lL}^s(z) = \int_0^{T_0+\tau(\varepsilon,l)} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN_{\mathrm{hyp},l}^L(x;z) + \int_{T_0+\tau(\varepsilon,l)}^\infty \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN_{\mathrm{hyp},l}^L(x;z), \tag{3}$$

where $\tau(\varepsilon,l)$ is given in Lemma 3.1.

For the integral over $[T_0 + \tau(\varepsilon, l), \infty)$, we have

$$\left\| \int_{T_0+\tau(\varepsilon,l)}^{\infty} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},l}(x; z) \right\| \le \int_{T_0+\tau(\varepsilon,l)}^{\infty} \left(\frac{1}{\cosh x}\right)^{\mathrm{Re}\, s} dN^L_{\mathrm{hyp},l}(x; z).$$

Now, we recall the fundamental geometric lemma which applies in our context (see Lemma 1.4 of [14]):

Lemma 3.2. *Let $M = \Gamma\backslash H$ be a hyperbolic Riemann surface of finite type. For any point $z \in M$ with injectivity radius r and any $x > T_0 > r$, we have*

$$(4) \qquad N_{\mathrm{hyp},M,\eta}(x; z) \le N_{\mathrm{hyp},M,\eta}(T_0; z) + \frac{\sinh^2\left(\frac{x+r}{2}\right) - \sinh^2\left(\frac{T_0 - r}{2}\right)}{\sinh^2\left(\frac{r}{2}\right)}.$$

From this lemma we deduce the following inequality, as in [7] p. 718, with $\sigma = \mathrm{Re}\, s$ and r the injectivity radius of M_l at z:

$$(5) \quad 2^{-\sigma} e^{\sigma\tau(\varepsilon,l)} \int_{T_0+\tau(\varepsilon,l)}^{\infty} \left(\frac{1}{\cosh x}\right)^{\mathrm{Re}\, s} dN^L_{\mathrm{hyp},l}(x; z) \le e^{(-\sigma+1)T_0} \frac{e^r}{\sinh^2(r/2)} \left(\frac{\sigma}{\sigma - 1}\right).$$

By choosing

$$T_0 \ge \frac{1}{\sigma - 1}\left(-\ln \mu + \ln\left(\frac{e^r}{\sinh^2(r/2)}\left(\frac{\sigma}{\sigma-1}\right)\right)\right),$$

we have that the upper bound in (5) can be made smaller than any $\mu > 0$.

In the same way,

$$\hat{\xi}^s_{q,\varepsilon}(z) = \int_0^{\infty} (e^{-x})^{(s-1)} w_{X_q}(-x)\, dN_{\mathrm{par},M_0,q}(x; z)$$

and

$$(6) \quad \left\| \int_{T_0}^{\infty} (e^{-x})^{(s-1)} w_{X_q}(-x)\, dN_{\mathrm{par},M_0,q}(x; z) \right\| \le \frac{e^{-T_0(\sigma-1)}}{4\sinh^2(r/2)} \left(1 + \frac{2\sinh(r)}{\sinh^2(r/2)}\right),$$

can be made, for T_0 sufficiently big, as small as we want uniformly on compact subsets of M_0 bounded away from the developing cusps and in half-planes of the form $\mathrm{Re}(s) \ge 1 + \delta$ for any $\delta > 0$.

For the first integral in (3), with an adequate T_0 chosen, we begin by writing

$$\int_0^{T_0+\tau(\varepsilon,l)} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},l}(x; z)$$
$$= \int_0^{T_0} \left(\frac{1}{\cosh(x + \tau(\varepsilon, l))}\right)^{(s-1)} w_{Y_0}(x + \tau(\varepsilon, l))\, dN_{\mathrm{hyp},\partial\mathscr{C}^L_{l,\varepsilon}}(x; z).$$

Let us assume, for convenience, that T_0 is a point of continuity of $N_{\mathrm{par},M_0,q}(x; z, \xi_{q,\varepsilon})$, meaning there is no geodesic path from z to $\xi_{q,\varepsilon}$ on M_0 with length equal to T_0. Then, as $\lim_{l\to 0} N^L_{\mathrm{hyp},l}(T_0+\tau(\varepsilon,l);z) = N_{\mathrm{par},M_0,q}(T_0;z,\xi_{q,\varepsilon})$, there exists $l_0 = l_0(T_0,\varepsilon)$ such that, for $l < l_0$, $N = N^L_{\mathrm{hyp},l}(T_0+\tau(\varepsilon,l);z) = N_{\mathrm{par},M_0,q}(T_0;z,\xi_{q,\varepsilon})$. Let $\{t_{k,l}, 1 \le k \le n_l\} \subset [0,T_0]$ (respectively, $\{t_k, 1 \le k \le n\} \subset [0,T_0]$) be the set of lengths on M_l (respectively, M_0) such that for any $\eta > 0$ we have

$$N^L_{\mathrm{hyp},l}(t_{k,l}+\tau(\varepsilon,l)-\eta;z) < N^L_{\mathrm{hyp},l}(t_{k,l}+\tau(\varepsilon,l)+\eta;z).$$

(respectively, $N_{\mathrm{par},M_0,q}(t_k-\eta;z,\xi_{q,\varepsilon}) < N_{\mathrm{par},M_0,q}(t_k+\eta;z,\xi_{q,\varepsilon})$).

We denote by $\{m_{k,l}, 1 \le k \le n_l\}$ (respectively, $\{m_k, 1 \le k \le n\}$) the multiplicities of $\{t_{k,l}\}$ (respectively, $\{t_k, 1 \le k \le n\}$).

Then we have

$$\int_0^{T_0+\tau(\varepsilon,l)} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},l}(x;z) = \sum_{k=1}^{n_l} \cosh(t_{k,l}+\tau(\varepsilon,l))^{-s} m_{k,l} d(t_{k,l}+\tau(\varepsilon,l)).$$

In the same way,

$$\hat{\xi}^s_{q,\varepsilon}(z) = \int_0^\infty (e^{-x})^{(s-1)} w_{X_q}(-x)\, dN_{\mathrm{par},M_0,q}(x;z)$$

and

$$\int_0^{T_0} (e^{-x})^{(s-1)} w_{X_q}(-x)\, dN_{\mathrm{par},M_0,q}(x;z) = -\sum_{k=1}^{n} e^{-t_k s} m_k\, dt_k.$$

In the following take $l < l_0$. As $\lim_{l\to 0} N^L_{\mathrm{hyp},l}(t_1+\tau(\varepsilon,l);z) = 0$, $\exists l_1 = l_1(t_1)$, $l < l_1$, $N^L_{\mathrm{hyp},l}(t_1+\tau(\varepsilon,l);z) = 0$, so $t_1 \le t_{1,l}$.

In a similar way, $\lim_{l\to 0} N^L_{\mathrm{hyp},l}(t_2+\tau(\varepsilon,l);z) = m_{1,0}$, for l sufficiently small, $N^L_{\mathrm{hyp},l}(t_2+\tau(\varepsilon,l);z) = m_{1,0} > 0$, so $t_2 > t_{1,l}$.

In conclusion there exists $l_2 = l_2(T_2) < l_1$ and sufficiently small so that for $l < l_2$, $t_1 \le t_{1,l} < t_2 \le t_{2,l}$ and $m_{1,l} = m_{1,0}$. Repeating this argument there exists l_i sufficiently small so that for $l < l_i$, $\forall j$, $1 \le j \le i$, $t_j \le t_{j,l} < t_{j+1}$ and $m_{j,l} = m_{j,0}$. As $\sum_{k=1}^n m_k = \sum_{k=1}^{n_l} m_{k,l}$, there exists l_n sufficiently small so that for $l < l_n$, $n_l = n$ and $\forall j$, $1 \le j \le n$, $t_j \le t_{j,l} < t_{j+1}$ and $m_{j,l} = m_{j,0}$.

Moreover as for all T, $t_1 < T \le t_2$, we have $\lim_{l\to 0} N^L_{\mathrm{hyp},l}(T+\tau(\varepsilon,l);z) = m_{1,0}$, we deduce that $\lim_{l\to 0} t_{1,l} = t_1$ and the same for all others $t_{j,l}$, $1 \le j \le n$.

Then for $l < l_n$ we can write,

$$\int_0^{T_0+\tau(\varepsilon,l)} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},l}(x;z) = \sum_{k=1}^{n} \cosh(t_{k,l}+\tau(\varepsilon,l))^{-s} m_k d(t_{k,l}+\tau(\varepsilon,l)).$$

Now we use the preceding change of variables $l\zeta = -\log(-z)$ to see that $\lim_{l\to 0}\|d(t_{k,l} + \tau(\varepsilon, l)) - dt_k\| = 0$.

The hyperbolic metric on the collar is given in polar coordinates by $ds^2 = \dfrac{dr^2 + r^2\, d\theta^2}{r^2 \sin^2\theta}$. Then, with the substitution $\ln r = -la$, $\theta = \pi - lb$ where $\zeta = a + ib$, $ds^2 = \dfrac{l^2(da^2 + db^2)}{\sin^2 lb}$, which tends to $\dfrac{da^2 + db^2}{b^2}$ as l tends to zero. The convergence is uniform for y bounded, and for instance is not uniform for $ly \le \pi/2$.

In Fermi coordinates $\sin\theta = \dfrac{1}{\operatorname{ch} x_2}$ and $dx_2 = -\dfrac{l}{\sin lb}\, db$ tends to $-\dfrac{db}{b}$ as l tends to zero. Now, remember that, with simplified notations: $t_{k,l} + \tau(\varepsilon, l) = x_2(Z) = \mathrm{d}_l(Z)$, $t_k = -\mathrm{d}_{X_0}(Z)$ for some $Z = a + ib$ and $\|d(t_{k,l} + \tau(\varepsilon, l)) - dt_k\| = (lb/\sin(lb) - 1)$. It follows that $\|d(t_{k,l} + \tau(\varepsilon, l)) - dt_k\|$ tends to zero as l tends to zero.

Moreover for fixed $x > 0$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$, we have

$$\lim_{r\to\infty} 2^{-s} e^{rs}((\cosh(x + r))^{-s} = e^{-sx}$$

and the limit is uniform for all $x > 0$ and $\mathrm{Re}(s) \ge 1 + \delta$. Then $\lim_{l\to 0} 2^{-s} e^{\tau(\varepsilon,l)s} \cosh(t_{k,l} + \tau(\varepsilon, l))^{-s} = e^{-st_k}$ and $\lim_{l\to 0}\|d(t_{k,l}) - d(t_k)\| = 0$ give

$$\lim_{l\to 0}\left(2^{-s} e^{\tau(\varepsilon,l)s} \sum_{k=1}^{n} \cosh(t_{k,l} + \tau(\varepsilon, l))^{-s} d(t_{k,l} + \tau(\varepsilon, l)) m_k\right) = \sum_{k=1}^{n} e^{-st_k}\, dt_k m_k.$$

In other words we have

$$\lim_{l\to 0}\left(2^{-s} e^{\tau(\varepsilon,l)s} \int_0^{T_0+\tau(\varepsilon,l)} \left(\frac{1}{\cosh x}\right)^{(s-1)} w_{Y_0}(x)\, dN^L_{\mathrm{hyp},l}(x; z)\right)$$
$$= -\int_0^{T_0} (e^{-x})^{(s-1)} w_{X_q}(-x)\, dN_{\mathrm{par},M_0,q}(x; z)$$

and the convergence is uniform on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$ and in half-planes of the form $\mathrm{Re}(s) \ge 1 + \delta$ for any $\delta > 0$.

Then we write

$$\frac{1}{l^s} \hat{c}^s_{lL}(z) = \frac{1}{l^s} \frac{2^s}{e^{s\tau(\varepsilon,l)}} 2^{-s} e^{s\tau(\varepsilon,l)} \hat{c}^s_{lL}(z).$$

We have

$$\tau(\varepsilon, l) = \int_{\cot^{-1}(\varepsilon/2l)}^{\pi/2} \frac{d\theta}{\sin\theta} = \log\left(\frac{\varepsilon}{2l} + \sqrt{\left(\frac{\varepsilon}{2l}\right)^2 + 1}\right),$$

such that

$$\frac{1}{l^s} \frac{2^s}{e^{s\tau(\varepsilon,l)}} = \frac{2^s}{\varepsilon^s}(1 - sO(l^2))$$

converges uniformly on compact subsets of $\mathrm{Re}(s) > 1$ to $\left(\dfrac{\varepsilon}{2}\right)^{-s}$.

Now

$$\hat{q}^s = |\xi_{q,\varepsilon}|^{-s}\hat{\xi}^s_{q,\varepsilon}$$
$$= \left(\frac{\varepsilon}{2}\right)^{-s}\left[\int_0^{T_0} (e^{-x})^{(s-1)} w_{X_0}(-x)\, dN_{\text{par},\,M_0,\,q}(x;z) + \int_{T_0}^{\infty} (e^{-x})^{(s-1)} w_{X_0}(-x)\, dN_{\text{par},\,M_0,\,q}(x;z)\right].$$

We now use (5), (6), the preceding limit and the triangle inequality in order to prove that

$$\lim_{l\to 0} \frac{1}{l^s}\hat{c}^s_{lL} = -\hat{q}^s,$$

uniformly on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$ and on compact subsets of $\mathrm{Re}(s) > 1$.

To study the right side of the collar, use the change of variables $l\omega = \log(z)$, with the principal branch: then $\left(\frac{1}{l}\frac{dz}{z}\right)^2 = (d\omega)^2$ and $\left(\frac{|dz|}{\mathrm{Im}\, z}\right)^2 = \left(\frac{l|d\omega|}{\sin lv}\right)^2$ for $\omega = u + iv$.

The hyperbolic metric on the collar is given in polar coordinates by $ds^2 = \dfrac{dr^2 + r^2\, d\theta^2}{r^2 \sin^2\theta}$. Then, with the substitution $\ln r = lu$, $\theta = lv$ and $ds^2 = \dfrac{l^2(da^2 + db^2)}{\sin^2 lb}$, which tends to $\dfrac{da^2 + db^2}{b^2}$ as l tends to zero. The convergence is uniform for y bounded. In the same way we show that

$$\lim_{l\to 0} \frac{1}{l^s}\hat{c}^s_{lR} = -\hat{p}^s.$$

In the case c_l is the boundary of the funnel F_l, for z away from the developing cusps, we have only to consider the right side of the σ_l-collar, $N_{\text{hyp},\,l}(T;z) = N^R_{\text{hyp},\,l}(T;z)$, and from the preceding study

$$\lim_{l\to 0} \frac{1}{l^s}\hat{c}^s_l = \hat{p}^s.$$

3.3. Final remarks. For geometrically infinite surfaces, that is to say a surface of infinite genus or homeomorphic to a compact surface with infinitely many points removed, the notion of geometry ‘at infinity’ is ill-defined, and there is virtually nothing we can say about the spectral theory of the Laplacian. However we can make the following remarks.

First note that it has already been pointed out (see [14]) that one can find results for spectral counting functions on degenerating hyperbolic surfaces of infinite volume analogues to those obtained for finite volume surfaces and with the same techniques.

The parabolic and hyperbolic Eisenstein series $\hat{p}^s$ and $\hat{c}^s$, we work with are well defined. For $\operatorname{Re} s > 1$, il follows from the fundamental lemma (see [12], p. 178, [10], p. 27):

Proposition 3.1. *For any Fuchsian group* Γ, *there exists a* $\mathscr{C}(q, \Gamma)$ *such that for all* $z \in H$,

$$\sum_{\gamma \in \Gamma} \frac{y(\gamma z)^q}{[1 + |\gamma z|]^{2q}} \leq \mathscr{C}(q, \Gamma).$$

The constant $\mathscr{C}(q, \Gamma)$ *depends only on* q *and* Γ.

In fact these series converge for $\operatorname{Re} s > \delta$ where δ is the exponent of convergence of the (relative) Poincaré series

$$\delta = \inf\left\{ s > 0, \sum_{T \in \Gamma} e^{-sd(z, Tw)} < \infty \right\}$$

for some $z, w \in H$, where $d(z, w)$ again denotes the hyperbolic distance from $z \in H$ to $w \in H$. We have $0 < \delta < 1$ for a geometrically infinite surface.

There is no decomposition in a finite number of trousers and funnels as in the geometrically finite case, we have the following result though. First we recall the definition (see [13], p. 84)

Definition 3.1. A family Y of simple closed curves on a surface S is called a multicurve if the elements of Y are disjoint, no two are homotopic to each other, and none is homotopic to a point.

And then give the theorem (see [13], p. 84)

Theorem 3.1. *Let* X *be a connected hyperbolic Riemann surface that is not simply connected, with its hyperbolic metric. Then there exists a multicurve* Y *on* X *such that if* $\bar{Z}$ *denotes the closure of* $Z = \{x \in \gamma, \gamma \in Y\}$, *then the closure of* $X - \bar{Z}$ *is isometric to either*

(1) *a trouser, with anywhere from zero to three cusps,*

(2) *a half-annulus* $|z| \geq 1$ *in* $\{1/R < |z| < R\}$ *for some* $0 < R < \infty$, *with its hyperbolic metric, or*

(3) *a half plane* $\operatorname{Re} z \leq 0$ *in* H, *with its hyperbolic metric.*

Moreover, each component of $\bar{Z} - Z$ *is a simple infinite geodesic bounding a half plane (i.e., case* 3 *above).*

A geometrically infinite hyperbolic surface contains an infinite multicurve or case 3 is checked, or both. This decomposition allows us to construct a degenerating family of geometrically infinite surfaces $(M_l)_{l>0}$, $M_l = \Gamma_l \backslash H$, by letting the lengths of a finite number of geodesics approaching zero as l tends to zero. These pinching geodesics can be taken as boundary components of a finite number of trousers appearing in Theorem 3.1. Denote by P_l the union of such trousers, in the general case the injectivity radius of M_l may not be always strictly positive outside the collars of the small geodesics and a thick-thin decomposition is no more possible, however, using the same methods, we obtain the previous results of degeneration on every compact of $M_l \backslash P_l$.

In the following we give a more precise description of this claim. We may suppose without loss of generality that there is only one pinching geodesic, c_l, of length l, which is the boundary of a trouser of the previous decomposition. The existence for any $0 < \varepsilon < 1/2$, of the collar $\mathscr{C}_{l,\varepsilon}$ (see Section 3 (2)) is found for example in [13], p. 90. With this collar, one can construct homeomorphisms f_l (see end of Section 3.1) from $M_l - \{c_l\}$ to M_0, with f_l quasi-isometries outside a tiny neighborhood of c_l, tending to isometries C^2-uniformly on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$ in analogous manner to the geometrically finite case (see for example [2], Proposition 3.1 p. 359, [5], [11], Theorem 1.18 p. 50). The proof of Lemma 3.1 Section 3.1, in the case of geometrically infinite hyperbolic surfaces, follows. Proof of Lemma 3.2 Section 3.2, which essentially uses the universal covering H and the fact that Γ_l is a discrete subgroup of $PSL(2, \mathbf{R})$ is also adapted to this case. The following theorem ensues

THEOREM 3.2. *Let* $(M_l)_{l>0}$ *be a degenerating family of geometrically infinite hyperbolic surfaces with limit surface* M_0, *as described above. Let* $\hat{c}_l^s$ *be the hyperbolic Eisenstein series on* M_l *associated to a simple closed geodesic of length* l, *with* $\mathscr{C}_{l,\varepsilon}$ *the associated collar.*

(1) *If* c_l *is non-separating, then*

$$\lim_{l \to 0} \frac{1}{l^s} \hat{c}_l^s = \hat{p}^s - \hat{q}^s,$$

where p *and* q *are the cusps arising from the pinching geodesic* c_l.

(2) *If* c_l *is the boundary of a funnel, then*

$$\lim_{l \to 0} \frac{1}{l^s} \hat{c}_l^s = \hat{p}^s;$$

and the convergence is uniform on compact subsets of the complement of $\mathscr{C}_{l,\varepsilon}$ *and in half-planes of the form* $\mathrm{Re}(s) \geq 1 + \delta$ *for any* $\delta > 0$.

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