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HERZ-TYPE BESOV SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

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Abstract

In this paper, Herz-type Besov spaces with variable smoothness and integrability are introduced. Our scale contains variable Besov spaces as special cases. We prove several basic properties, especially the Sobolev-type embeddings.

1. Introduction

The Herz spaces initially appeared in the paper of Herz [15] to study the absolute convergence of Fourier transforms. The theory of these spaces had a remarkable development in part due to its usefulness in applications to other fields of applied mathematics. For instance, they appear in the characterization of multipliers on Hardy spaces [3], in the summability of Fourier transforms [11] and in regularity theory for elliptic equations in divergence form [26] and [27]. We refer the monograph [25] for further details and references on recent developments on Herz spaces.

Function spaces with variable exponents have been intensively studied in the recent years by a significant number of authors. The motivation to study such function spaces comes from applications to other fields of applied mathematics, such that fluid dynamics and image processing, see [4] and [28].

Herz spaces $K_{q(\cdot)}^{\alpha,p}$ and $\dot{K}_{q(\cdot)}^{\alpha,p}$ with variable exponent q but fixed $\alpha \in \mathbf{R}$ and $p \in (0, \infty]$ were recently studied by Izuki [16, 17]. These spaces with variable exponents $\alpha(\cdot)$ and $q(\cdot)$ were studied in [2], where they gave the boundedness results for a wide class of classical operators on these function spaces. The spaces $K_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbf{R}^n)$ and $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbf{R}^n)$, were first introduced by Izuki and Noi in [18]. In [7] the authors gave a new equivalent norms of these function spaces. See [29] where new variable Herz spaces are given.

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C. Shi and J. Xu [31] and [30] studied Herz-type Besov spaces $\dot{K}^{\alpha,p}_{q(\cdot)}B^s_\beta$ with variable q, but fixed α, p, s and β, where the characterization of these function spaces by so-called Peetre maximal functions are obtained. B. Dong and J. Xu [10] also considered $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}B^s_\beta$ with variables q and α. These function spaces (with fixed exponents) are introduced earlier in the papers of J. Xu and D. Yang [36], [37] and [38]. The interest in these spaces comes not only from the theoretical reasons but also from their applications to several classical problems in analysis. In [24], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. We refer the reader to the recent paper [5] for further results for these function spaces.

Based on Besov spaces with variable smoothness and integrability $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, we introduce Herz-type Besov spaces with variable smoothness and integrability, which covers Herz-type Besov spaces with fixed exponents. We will give several properties of these new family of function spaces.

Let us now present the contents of this paper. Section 2 collects fundamental notation and concepts. Some necessary tools are given in Section 3. In particular we generalize the classical Plancherel-Polya-Nikolskij inequality on $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}$ spaces instead of $L^{p(\cdot)}$ spaces. For making the presentation clearer, we give their proofs later in Section 6. In Section 4 we define Herz-type Besov spaces with variable smoothness and integrability and present a few aspects of their properties. Finally, in Section 5 we present some embeddings properties. In particular we will prove the Sobolev embedding theorem for these function spaces and present some consequences.

2. Preliminaries

As usual, we denote by $\mathbf{R}^n$ the n-dimensional real Euclidean space, and by $\mathbf{N}$ the collection of all natural numbers. We write $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. The symbol $\mathbf{Z}$ stands for the set of all integer numbers. For any $u > 0$, $k \in \mathbf{Z}$ we set $C(u) := \{x \in \mathbf{R}^n : u/2 \leqslant |x| < u\}$ and $C_k := C(2^k)$. We use c for various positive constant, i.e. a constant whose value may change from appearance to appearance. The expression $f \lesssim g$ means that $f \leqslant cg$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

The notation $X \hookrightarrow Y$ stands for continuous embeddings from X to Y, where X and Y are quasi-normed spaces. If $E \subset \mathbf{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. By $\operatorname{supp} f$ we denote the support of the function f.

By ℓ^q, $q \in (0, \infty]$, we denote the discrete Lebesgue space equipped with the usual quasi-norm. Mostly we will deal with sequences defined either on $\mathbf{N}$ or $\mathbf{Z}$.

The variable exponents that we consider are always measurable functions on $\mathbf{R}^n$ with range in (c, ∞) for some $c > 0$. We denote the set of such functions by $\mathscr{P}_0(\mathbf{R}^n)$. The subset of variable exponents with range $[1, \infty)$ is denoted by $\mathscr{P}(\mathbf{R}^n)$. For $p \in \mathscr{P}_0(\mathbf{R}^n)$, we use the notation $p^+ = \operatorname{ess\,sup}_{x\in\mathbf{R}^n} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\mathbf{R}^n} p(x)$.

By $\mathscr{S}(\mathbf{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbf{R}^n$. The topology in the complete locally convex space $\mathscr{S}(\mathbf{R}^n)$ is generated by

$$p_N(\varphi) := \sup_{x\in\mathbf{R}^n}(1+|x|)^N \sum_{|\alpha|\leqslant N}|D^\alpha\varphi(x)|, \quad N=1,2,3,\ldots.$$

By $\mathscr{S}'(\mathbf{R}^n)$ the dual space of all tempered distributions on $\mathbf{R}^n$. We define the Fourier transform of a function $f\in\mathscr{S}(\mathbf{R}^n)$ by $\mathscr{F}(f)(\xi)=(2\pi)^{-n/2}\int_{\mathbf{R}^n}e^{-ix\cdot\xi}f(x)\,dx$. Its inverse is denoted by $\mathscr{F}^{-1}f$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are extended to the dual Schwartz space $\mathscr{S}'(\mathbf{R}^n)$ in the usual way.

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ is the class of all measurable functions f on $\mathbf{R}^n$ such that the modular $\varrho_{p(\cdot)}(f) := \int_{\mathbf{R}^n}|f(x)|^{p(x)}\,dx$, is finite. If $p(\cdot) := p$ is constant, then $L^{p(\cdot)}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ is the classical Lebesgue space. A useful property is that $\varrho_{p(\cdot)}(f)\leqslant 1$ if and only if $\|f\|\leqslant 1$ (unit ball property). This property is clear for constant exponents due to the obvious relation between the norm and the modular in that case. For variable exponents, Hölder's inequality takes the form

$$\|fg\|_{s(\cdot)} \leqslant 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}$$

where s is defined pointwise by $\dfrac{1}{s(x)} := \dfrac{1}{p(x)}+\dfrac{1}{q(x)}$. Often we use the particular case $s(x):=1$ corresponding to the situation when $q=p'$ is the conjugate exponent of p.

We say that a function $g:\mathbf{R}^n\to\mathbf{R}$ is *locally* log-*Hölder continuous*, abbreviated to $g\in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g)>0$ such that

$$|g(x)-g(y)| \leqslant \frac{c_{\log}}{\log(e+1/|x-y|)} \tag{2.1}$$

for all $x,y\in\mathbf{R}^n$. If

$$|g(x)-g(0)| \leqslant \frac{c_{\log}}{\log(e+1/|x|)}$$

for all $x\in\mathbf{R}^n$, then we say that g is log-*Hölder continuous at the origin* (or has a log *decay at the origin*). If, for some $g_\infty\in\mathbf{R}$ and $c_{\log}>0$,

$$|g(x)-g_\infty| \leqslant \frac{c_{\log}}{\log(e+|x|)}$$

for all $x\in\mathbf{R}^n$, then we say that g is log-*Hölder continuous at infinity* (or has a log *decay at infinity*). We note that all functions g are log-*Hölder continuous at infinity* always belong to L^∞.

The notation $\mathscr{P}^{\log}(\mathbf{R}^n)$ is used for all those exponents $p\in\mathscr{P}(\mathbf{R}^n)$ which are locally log-Hölder continuous and have a log decay at infinity, with $p_\infty := \lim_{|x|\to\infty}p(x)$. The class $\mathscr{P}^{\log}_0(\mathbf{R}^n)$ is defined analogously.

Let $p, q \in \mathscr{P}_0(\mathbf{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\left\{\lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \leqslant 1\right\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \leqslant 1\right\}. \tag{2.2}$$

If $q^+ < \infty$, then we can replace (2.2) by the simpler expression $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \| |f_v|^{q(\cdot)}\|_{p(\cdot)/q(\cdot)}$. Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. It is known, cf. [1] and [20], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geqslant 1$ is constant almost everywhere (a.e.) on $\mathbf{R}^n$ and $p(\cdot) \geqslant 1$, or if $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} \leqslant 1$ a.e. on $\mathbf{R}^n$, or if $1 \leqslant q(x) \leqslant p(x) < \infty$ a.e. on $\mathbf{R}^n$. If $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$, then convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \leqslant c\|\varphi\|_1 \|f\|_{p(\cdot)}. \tag{2.3}$$

Recall that $\eta_{v,m}(x) := 2^{nv}(1 + 2^v|x|)^{-m}$, for any $x \in \mathbf{R}^n$, $v \in \mathbf{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v.

3. Basic tools

In this section we present some results which are useful for us. The following lemma is from [8, Lemma 6.1], see also [19, Lemma 19].

LEMMA 3.1. *Let $\alpha \in C^{\log}_{\mathrm{loc}}$ and let $M \geqslant c_{\log}(\alpha)$, where $c_{\log}(\alpha)$ is the constant from* (2.1) *for α. Then*

$$2^{v\alpha(x)}\eta_{v,m+M}(x-y) \leqslant c2^{v\alpha(y)}\eta_{v,m}(x-y)$$

with $c > 0$ independent of $x, y \in \mathbf{R}^n$ and $v, m \in \mathbf{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)}\eta_{v,m+M} * f(x) \leqslant c\eta_{v,m} * (2^{v\alpha(\cdot)}f)(x).$$

The next Lemma tells us that in most circumstances two convolutions are as good as one, is from [8, Lemma A.3].

LEMMA 3.2. *For j_0 and $j_1 \geqslant 0$ and $m > n$, we have*

$$\eta_{j_0,m} * \eta_{j_1,m} \approx \eta_{\min(j_0,j_1),m}$$

with the constant depending only on m and n.

The next lemma often allows us to deal with exponents which are smaller than 1, see [8, Lemma A.7].

LEMMA 3.3. *Let $r > 0$, $j \in \mathbf{N}_0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbf{R}^n)$ with* $\operatorname{supp} \mathcal{F} g \subset \{\xi \in \mathbf{R}^n : |\xi| \leqslant 2^{j+1}\}$, *we have*

$$|g(x)| \leqslant c(\eta_{j,m} * |g|^r(x))^{1/r}, \quad x \in \mathbf{R}^n.$$

Very often we have to deal with the norm of the characteristic functions on balls (or cubes) when studying the behavior of various operators in Harmonic Analysis. In classical L^p spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. Nevertheless, it is known that for $p \in \mathcal{P}(\mathbf{R}^n)$ be *locally log-Hölder continuous and has a log decay at infinity* there holds $\|\chi_B\|_{p(\cdot)} \approx |B|^{1/p(x)}$, $x \in B$ for small balls $B \subset \mathbf{R}^n$, and $\|\chi_B\|_{p(\cdot)} \approx |B|^{1/p_\infty}$ for large balls, with constants only depending on the log-Hölder constant of p, see, for example, [9, Section 4.5].

For characteristic functions defined on (dyadic) annulus we have similar norm estimates, without requiring the log-Hölder continuity at every point.

The following Lemma plays an important role in the proof of the main results of this work, see [2].

LEMMA 3.4. *Let $p \in \mathcal{P}(\mathbf{R}^n)$ be log-Hölder continuous at infinity, and $R = B(0, r) \backslash B(0, \frac{r}{2})$. If $|R| > 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)} \approx |R|^{1/p(x)} \approx |R|^{1/p_\infty}$$

with the implicit constants independent of r and $x \in \mathbf{R}^n$.

The left-hand side equivalence remains true for every $|R| > 0$ if we assume, additionally, p is log-Hölder continuous, both at the origin and at infinity.

For convenience, we set

$$B_k := B(0, 2^k), \quad R_k := B_k \backslash B_{k-1} \quad \text{and} \quad \chi_k := \chi_{R_k}, \quad k \in \mathbf{Z}.$$

DEFINITION 3.5. Let $p, q \in \mathcal{P}_0(\mathbf{R}^n)$ and $\alpha : \mathbf{R}^n \to \mathbf{R}$. The *homogeneous Herz space* $\dot{K}_{q(\cdot)}^{\alpha(\cdot), p(\cdot)}(\mathbf{R}^n)$ is defined as the set of all $f \in L_{\mathrm{loc}}^{q(\cdot)}(\mathbf{R}^n \backslash \{0\})$ such that

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p(\cdot)}} := \|(2^{k\alpha(\cdot)} f \chi_k)_{k \in \mathbf{Z}}\|_{\ell^{p(\cdot)}(L^{q(\cdot)})} < \infty. \tag{3.6}$$

If α and p, q are constant, then $\dot{K}_{q(\cdot)}^{\alpha(\cdot), p(\cdot)}(\mathbf{R}^n)$ is the classical Herz space $\dot{K}_q^{\alpha, p}(\mathbf{R}^n)$. A detailed discussion of the properties of these spaces my be found in [13], [14], [22], [23], [25], and references therein. Let us denote

$$\|(g_k)\|_{\ell_>^p(L^{q(\cdot)})} := \left(\sum_{k=0}^{\infty} \|g_k\|_{q(\cdot)}^p\right)^{1/p} \quad \text{and} \quad \|(g)\|_{\ell_<^p(L^{q(\cdot)})} := \left(\sum_{k=-\infty}^{-1} \|g_k\|_{q(\cdot)}^p\right)^{1/p}$$

for sequences $\{g_k\}_{k\in\mathbf{Z}}$ of measurable functions (with the usual modification when $q=\infty$). Let $\alpha\in L^\infty(\mathbf{R}^n)$, $p,q\in\mathscr{P}_0(\mathbf{R}^n)$. Recently, the authors in [7] proved that if α and q are log-Hölder continuous, both at the origin and at infinity, then

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \approx \|(2^{k\alpha(0)}f\chi_k)\|_{\ell_<^{p(0)}(L^{q(\cdot)})} + \|(2^{k\alpha_\infty}f\chi_k)\|_{\ell_>^{p_\infty}(L^{q(\cdot)})}.$$

Let $p,q,\theta\in\mathscr{P}_0(\mathbf{R}^n)$ and $\alpha:\mathbf{R}^n\to\mathbf{R}$ be log-Hölder continuous, both at the origin and at infinity. If $(p-\theta)^-\geqslant 0$, then

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}, \tag{3.7}$$

is a simple consequence of the embedding $\ell^{\theta(\cdot)}(L^{q(\cdot)})\hookrightarrow\ell^{p(\cdot)}(L^{q(\cdot)})$.

Let T be a sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x\notin \operatorname{supp} f \tag{3.8}$$

for integrable and compactly supported functions f. Condition (3.8) is satisfied by several classical operators in Harmonic Analysis, such as Calderón-Zygmund operators, the Carleson maximal operator and the Hardy-Littlewood maximal operator (see [21] and [32]). Various important results have been proved in the space $\dot{K}_q^{\alpha,p}$ under some assumptions on α, p and q. The conditions $-\frac{n}{q}<\alpha<n\left(1-\frac{1}{q}\right)$, $1<q<\infty$ and $0<p\leqslant\infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_q^{\alpha,p}$ spaces. This fact was first realized by Li and Yang [21] with the proof of the boundedness of the maximal function. The proof of the main result of this section is based on the following result, see [7].

Theorem 3.9. *Let $p\in\mathscr{P}_0(\mathbf{R}^n)$, $q\in\mathscr{P}(\mathbf{R}^n)$ with $1<q^-\leqslant q^+<\infty$, and let α, p and q be* log-*Hölder continuous, both at the origin and at infinity such that*

$$-\frac{n}{q^+}<\alpha^-\leqslant\alpha^+<n\left(1-\frac{1}{q^-}\right).$$

Then every sublinear operator T satisfying (3.8) *which is bounded on $L^{q(\cdot)}(\mathbf{R}^n)$ can also be extended to a linear operator bounded on $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbf{R}^n)$.*

Let $p,q,\beta\in\mathscr{P}_0(\mathbf{R}^n)$ and $\alpha:\mathbf{R}^n\to\mathbf{R}$ with $\beta^+<\infty$. The space $\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})$ is defined to be the set of all sequences of $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbf{R}^n)$-functions by the modular

$$\varrho_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})}((f_j)_j) := \sum_{j=0}^{\infty} \|\,|f_j|^{\beta(\cdot)}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}}.$$

The quasi-norm is defined from this as usual:

$$\|(f_j)_j\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} := \inf\left\{\mu>0:\varrho_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})}\left(\frac{1}{\mu}(f_j)_j\right)\leqslant 1\right\}.$$

Lemma 3.10. *Let $q, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$. Let $p \in \mathscr{P}_0(\mathbf{R}^n)$ and $\alpha : \mathbf{R}^n \to \mathbf{R}$ be log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + \dfrac{n}{q^+} > 0$. Let $M \geqslant c_{\log}(1/\beta)$, where $c_{\log}(1/\beta)$ is the constant from (2.1) for $\frac{1}{\beta}$. For $m > n$, there exists $c > 0$ such that*

$$\|((\eta_{j,m+M} * |f_j|^r)^{1/r})_j\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \leqslant c\|(f_j)_j\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \tag{3.11}$$

for any $0 < r < \min\left(q^-, \dfrac{n}{\alpha^+ + n/q^-}\right)$ whenever the quasi-norm on the right hand side is finite.

Proof. By the scaling argument, it suffices to consider the case $\|(f_j)_j\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} = 1$ and show that the modular of $(\eta_{j,m+M} * |f_j|^r)^{1/r}$ on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \| \, |c(\eta_{j,m+M} * |f_j|^r)^{\beta(\cdot)/r} \|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} \leqslant 1 \tag{3.12}$$

for some constant $c > 0$. Our estimate (3.12), clearly follows from the inequality

$$\| \, |c(\eta_{j,m+M} * |f_j|^r)^{\beta(\cdot)/r} \|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} \leqslant \| \, |f_j|^{\beta(\cdot)} \|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} + 2^{-j} =: \delta. \tag{3.13}$$

This claim can be reformulated as showing that

$$\|c\delta^{-1}(\eta_{j,m+M} * |f_j|^r)^{\beta(\cdot)/r}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} \leqslant 1.$$

Since β is log-Hölder continuous and $\delta \in [2^{-j}, 1 + 2^{-j}]$, we can move $\delta^{-1/\beta(\cdot)}$ inside the convolution by Lemma 3.1:

$$\|c\delta^{-1}(\eta_{j,m+M} * |f_j|^r)^{\beta(\cdot)/r}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} \lesssim \|(\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r)^{\beta(\cdot)/r}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}}.$$

Let us prove that

$$\|(\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r)^{\beta(\cdot)/r}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha(\cdot)\beta(\cdot),p(\cdot)/\beta(\cdot)}} \lesssim 1,$$

which is equivalent to

$$\sum_{k=-\infty}^{\infty} \|(2^{k\alpha(\cdot)r}\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r)^{p(\cdot)/r}\chi_k\|_{q(\cdot)/p(\cdot)} \lesssim 1.$$

Again this is equivalent to $\|(\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r)^{1/r}\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \lesssim 1$. Observe that

$$\|(\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r)^{1/r}\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} = \|\eta_{j,m} * \delta^{-r/\beta(\cdot)}|f_j|^r\|_{\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r}}^{1/r},$$

thanks to Theorem 3.9, under the assumption $0 < r < \min\left(q^-, \dfrac{n}{\alpha^+ + n/q^-}\right)$, the right-hand side is bounded by $c\|\delta^{-1/\beta(\cdot)}f_j\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \lesssim 1$, which follows immediately from the definition of δ. This finishes the proof. □

Remark 3.14. Let $q \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$. Let $p \in \mathscr{P}_0(\mathbf{R}^n)$ and $\alpha : \mathbf{R}^n \to \mathbf{R}$ be log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + \frac{n}{q^+} > 0$. Observe that $\eta_{j,m} * |f_j|^r$, $j \in \mathbf{N}_0$, satisfying the size condition (3.8). Using Theorem 3.9, one can find a constant $c > 0$ such that

$$\sup_{j \geqslant 0} \|(\eta_{j,m} * |f_j|^r)^{1/r}\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \leqslant c \sup_{j \geqslant 0} \|f_j\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}$$

for any $m > n$ and any $0 < r < \min\left(q^-, \frac{n}{\alpha^+ + n/q^-}\right)$ whenever the quasi-norm on the right hand side is finite.

The classical Plancherel-Polya-Nikolskij inequality (cf. [33, 1.3.2/5, Rem. 1.4.1/4]), says that $\|f\|_q$ can be estimated by

$$cR^{n(1/p-1/q)}\|f\|_p$$

for any $0 < p \leqslant q \leqslant \infty$, $R > 0$ and any $f \in L^p(\mathbf{R}^n) \cap \mathscr{S}'(\mathbf{R}^n)$ with $\operatorname{supp} \mathscr{F}f \subset \bar{B}(0,R)$. The constant $c > 0$ is independent of R. This inequality plays an important role in theory of function spaces and PDE's. Our aim is to extend this result to the variable Herz spaces. Let us start with the following lemma.

Lemma 3.15. *Let $\sigma, \beta \in C_{\mathrm{loc}}^{\log}$, $p \in \mathscr{P}_0(\mathbf{R}^n)$, $\alpha : \mathbf{R}^n \to \mathbf{R}$ and $R \geqslant \max(1,H)$. Let q be log-Hölder continuous, both at the origin and at infinity. Then there exists a constant $c > 0$ independent of R and H such that*

$$\sup_{x \in B(0,1/H)} \lambda^{-1/\beta(x)} R^{\sigma(x)}|f(x)| \leqslant c\left(\frac{R}{H}\right)^{n/d} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \tag{3.16}$$

for all $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} \cap \mathscr{S}'(\mathbf{R}^n)$ with $\operatorname{supp} \mathscr{F}f \subset \bar{B}(0,R)$, any $0 < d < \min(q^-, n/(\alpha + n/q)^+)$ and any $\lambda \in [R^{-1}, 1 + R^{-1}]$.

If α, p, σ, β and q are constants, this result is [5, Lemma 3.3].

The following Lemma is the $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbf{R}^n)$-version of the Plancherel-Polya-Nikolskij inequality.

Lemma 3.17. *Let $R \geqslant 1$, $p, r \in \mathscr{P}_0(\mathbf{R}^n)$ with $p^+, r^+ < \infty$, $q, t, s, r \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$, $\beta \in C_{\mathrm{loc}}^{\log}$ and $\alpha_1, \alpha_2 \in C_{\mathrm{loc}}^{\log}$ such that $(\alpha_2 - \alpha_1)^- > 0$ or $\alpha_1(\cdot) = \alpha_2(\cdot)$. We suppose that $q(\cdot) \leqslant t(\cdot)$ and $(\alpha_1 + n/t)^- > 0$. Then there exist a positive constant $c > 0$ independent of R such that*

$$\| |R^{s(\cdot)} f|^{\beta(\cdot)}\|_{\dot{K}_{t(\cdot)/\beta(\cdot)}^{\alpha_1(\cdot)\beta(\cdot),r(\cdot)/\beta(\cdot)}} \leqslant c \| |R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{\beta(\cdot)}\|_{\dot{K}_{q(\cdot)/\beta(\cdot)}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}} + \frac{1}{R} \tag{3.18}$$

for all $f \in \dot{K}_{q(\cdot)}^{\alpha_2(\cdot),p(\cdot)} \cap \mathscr{S}'(\mathbf{R}^n)$ with $\operatorname{supp} \mathscr{F}f \subset \bar{B}(0,R)$ such that the norm on the right hand side is at most one, where

$$\theta(\cdot) = \begin{cases} r(\cdot) & \textit{if } \alpha_1(\cdot) = \alpha_2(\cdot) \\ p(\cdot) & \textit{if } (\alpha_2 - \alpha_1)^- > 0. \end{cases}$$

We would like to mention that this lemma improves the Plancherel-Polya-Nikolskij inequality of [1, Lemma 6.3] by taking $\alpha_2(\cdot) = \alpha_1(\cdot) = 0$, $r(\cdot) = t(\cdot)$, $\dot{K}^{0,t(\cdot)/\beta(\cdot)}_{t(\cdot)/\beta(\cdot)} = L^{t(\cdot)/\beta(\cdot)}$ and by using the embedding $L^{q(\cdot)/\beta(\cdot)} = \dot{K}^{0,q(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)} \hookrightarrow \dot{K}^{0,t(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}$.

Lemma 3.19. *Let* $R \geqslant 1$, $p, r \in \mathscr{P}_0(\mathbf{R}^n)$ *with* $p^+, r^+ < \infty$, $q, t, s, r \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$, $\beta \in C^{\log}_{\mathrm{loc}}$ *and* $\alpha_1, \alpha_2 \in C^{\log}_{\mathrm{loc}}$ *such that* $\alpha_2(\cdot) + n/q(\cdot) = \alpha_1(\cdot) + n/t(\cdot)$ *or* $(\alpha_2 + n/q - \alpha_1 - n/t)^- > 0$. *We suppose that* $t(\cdot) \leqslant q(\cdot)$ *and* $(\alpha_1 + n/t)^- > 0$. *Then there exist a positive constant* $c > 0$ *independent of* R *such that for all* $f \in \dot{K}^{\alpha_2(\cdot),p(\cdot)}_{q(\cdot)} \cap \mathscr{S}'(\mathbf{R}^n)$ *with* $\operatorname{supp} \mathscr{F}f \subset \bar{B}(0,R)$, *we have* (3.18), *such that the norm on the right hand side is at most one, where*

$$\theta(\cdot) = \begin{cases} r(\cdot) & \text{if } \ \alpha_2(\cdot) + n/q(\cdot) = \alpha_1(\cdot) + n/t(\cdot) \\ p(\cdot) & \text{if } \ (\alpha_2 + n/q - \alpha_1 - n/t)^- > 0. \end{cases}$$

The proof of Lemmas 3.15, 3.17 and 3.19 is postponed to the Appendix.

4. Variable Herz-type Besov spaces

In this section we present the Fourier analytical definition of Herz type Besov spaces of variable smoothness and integrability and we prove the basic properties in analogy to the Herz type Besov spaces with constant exponents. We first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathscr{S}(\mathbf{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leqslant 1$ and $\Psi(x) = 0$ for $|x| \geqslant 2$. We define φ_0 and φ_1 by $\mathscr{F}\varphi_0(x) = \Psi(x)$, $\mathscr{F}\varphi_1(x) = \Psi(x) - \Psi(2x)$ and

$$\mathscr{F}\varphi_j(x) = \mathscr{F}\varphi_1(2^{-j}x) \quad \text{for } \ j = 2, 3, \ldots.$$

Then $\{\mathscr{F}\varphi_j\}_{j \in \mathbf{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{j=0}^{\infty} \mathscr{F}\varphi_j(x) = 1$ for all $x \in \mathbf{R}^n$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \varphi_j * f$ of all $f \in \mathscr{S}'(\mathbf{R}^n)$ (convergence in $\mathscr{S}'(\mathbf{R}^n)$).

Now, we introduce the Herz-type Besov spaces of variable smoothness and integrability.

Definition 4.1. Let $\{\mathscr{F}\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity, $\alpha, s : \mathbf{R}^n \to \mathbf{R}$ and $p, q, \beta \in \mathscr{P}_0(\mathbf{R}^n)$. The Herz-type Besov space $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}$ is the collection of all distributions $f \in \mathscr{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}} := \|(2^{js(\cdot)}\varphi_j * f)_j\|_{\ell^{\beta(\cdot)}(\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)})} < \infty. \tag{4.2}$$

Herz-type Besov spaces $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}$ with variable exponents p and α but fixed s, q and β were recently studied in [31], [30] and [10]. While the first time we introduce the spaces $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} B^{s(\cdot)}_{\beta(\cdot)}$ with the quasi-norm (4.2). When, $\beta := \infty$

the Herz-type Besov space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}$ consist of all distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\sup_{v \geqslant 0} \|2^{vs(\cdot)}\varphi_v * f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} < \infty.$$

One recognizes immediately that if α, s, p, q and β are constants, then the spaces $\dot{K}_q^{\alpha,p}B_\beta^s$ are just the usual Herz-type Besov spaces were first introduced by J. Xu and D. Yang [36] and [37]. See [5] and [40] for further results.

We state the definition of the spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$, which introduced and investigated in [1].

Definition 4.3. Let $\{\mathcal{F}\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity, $s : \mathbf{R}^n \to \mathbf{R}$ and $p, q \in \mathcal{P}_0(\mathbf{R}^n)$. The Besov space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot)}} := \|(2^{js(\cdot)}\varphi_j * f)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Taking $s \in \mathbf{R}$ and $q \in (0, \infty]$ as constants we derive the spaces $B_{p(\cdot),q}^{s}$ studied by Xu in [39]. We refer the reader to the recent papers [20], [19] and [6] for further details, historical remarks and more references on these function spaces. For any $p, q \in \mathcal{P}_0^{\log}(\mathbf{R}^n)$ and $s \in C_{\text{loc}}^{\log}$, the space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_j\}_{j \in \mathbf{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathcal{S}(\mathbf{R}^n) \hookrightarrow B_{p(\cdot),q(\cdot)}^{s(\cdot)} \hookrightarrow \mathcal{S}'(\mathbf{R}^n).$$

Moreover, if p, q, s are constants, we re-obtain the usual Besov spaces $B_{p,q}^s$, studied in detail by H. Triebel in [33], [34] and [35]. Clearly, $\dot{K}_{p(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)} = B_{p(\cdot),\beta(\cdot)}^{s(\cdot)}$.

Now, we are ready to show that the definition of the spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}$ is independent of the chosen resolution of unity $\{\mathcal{F}\varphi_j\}_{j \in \mathbf{N}_0}$. This justifies our omission of the subscript φ in the sequel.

Theorem 4.4. *Let* $\{\mathcal{F}\varphi_j\}_{j=0}^{\infty}$, $\{\mathcal{F}\psi_j\}_{j=0}^{\infty}$ *be two resolutions of unity,* $p, q, \beta \in \mathcal{P}_0^{\log}(\mathbf{R}^n)$ *and* $s \in C_{\text{loc}}^{\log}$. *Let* $\alpha : \mathbf{R}^n \to \mathbf{R}$ *be log-Hölder continuous, both at the origin and at infinity such that* $\alpha^- + \dfrac{n}{q^+} > 0$. *Then* $\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\varphi} \approx \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}$.

Proof. It is sufficient to show that for all $f \in \mathcal{S}'(\mathbf{R}^n)$ we have $\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\varphi} \leqslant c\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}^{\psi}$ with $c > 0$. Interchanging the roles of ψ and φ we obtain the desired result. Putting $\psi_{-1} = 0$ we see $\mathcal{F}\varphi_v = \mathcal{F}\varphi_v \sum_{k=-1}^{k=1} \mathcal{F}\psi_{v+k}$ for all $v \in \mathbf{N}_0$. By the properties of the Fourier transform

$$\varphi_v * f = \sum_{k=-1}^{k=1} \varphi_v * \psi_{v+k} * f.$$

Fix $0 < r < \min\left(1, \frac{n}{\alpha^+ + n/q^-}\right)$ and $m > n + 2c_{\log}(s) + c_{\log}(1/\beta)$ large. Since $|\varphi_v| \leqslant c\eta_{v,2m/r}$, with $c > 0$ independent of v, we obtain

$$|\varphi_v * \psi_{v+k} * f| \lesssim \eta_{v,m/r} * |\psi_{v+k} * f| \lesssim \eta_{v,m/r} * (\eta_{v+k,m} * |\psi_{v+k} * f|^r)^{1/r},$$

where in the second inequality we used Lemma 3.3. By Minkowski's integral inequality the left-hand side is bounded by

$$\begin{aligned} &c((\eta_{v,m/r} * \eta_{v+k,m}^{1/r})^r * |\psi_{v+k} * f|^r)^{1/r} \\ &\quad = c2^{n(v+k)(1/r-1)}((\eta_{v,m/r} * \eta_{v+k,m/r})^r * |\psi_{v+k} * f|^r)^{1/r}. \end{aligned}$$

By Lemma 3.2 we have $\eta_{v,m/r} * \eta_{v+k,m/r} \approx \eta_{v+k,m/r}$. Then the last expression is bounded by $c(\eta_{v+k,m} * |\psi_{v+k} * f|^r)^{1/r}$. This, together with Lemma 3.1, gives

$$\begin{aligned} &\|(2^{vs(\cdot)}\varphi_v * f)_{v\geqslant 0}\|_{\ell^{\beta(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} \\ &= \|(2^{vs(\cdot)r}|\varphi_v * f|^r)_{v\geqslant 0}\|^{1/r}_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})} \\ &\lesssim \sum_{k=-1}^{k=1} \|(2^{vs(\cdot)r}\eta_{v+k,m} * |\psi_{v+k} * f|^r)_{v\geqslant 0}\|^{1/r}_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})} \\ &\lesssim \sum_{k=-1}^{k=1} \|(\eta_{v+k,m-c_{\log}(s)} * 2^{(v+k)s(\cdot)r}|\psi_{v+k} * f|^r)_{v\geqslant 0}\|^{1/r}_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})}. \end{aligned}$$

By the change of variable $v + k = i$, this expression is bounded by

$$\begin{aligned} &\sum_{k=-1}^{k=1} \|(\eta_{i,m-c_{\log}(s)} * 2^{is(\cdot)r}|\psi_i * f|^r)_{i\geqslant k}\|^{1/r}_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})} \\ &\qquad \lesssim 3\|(2^{is(\cdot)r}|\psi_i * f|^r)_{i\geqslant 0}\|^{1/r}_{\ell^{\beta(\cdot)/r}(\dot{K}_{q(\cdot)/r}^{\alpha(\cdot)r,p(\cdot)/r})} \leqslant 3\|f\|^{\psi}_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}}, \end{aligned}$$

where in the first inequality we have used Lemma 3.10. □

Remark 4.5. Let $p, q \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ and $s \in C_{\text{loc}}^{\log}$. Let $\alpha : \mathbf{R}^n \to \mathbf{R}$ be log-Hölder continuous, both at the origin and at infinity such that $\alpha^- + n/q^+ > 0$. As in the last proof and by Remark 3.14 we can prove that the definition of the spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}$ is independent of the chosen resolution of unity $\{\mathscr{F}\varphi_j\}_{j\in\mathbf{N}_0}$.

5. Embeddings

The following Theorem gives basic embeddings of the spaces $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\beta(\cdot)}^{s(\cdot)}$.

Theorem 5.1. *Let* $p, q, \beta_1, \beta_2, p_1, p_2 \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ *and* $s, s_1, s_2 \in C_{\mathrm{loc}}^{\log}$. *Let* α *be log-Hölder continuous, both at the origin and at infinity such that* $\alpha^- + \frac{n}{q^+} > 0$.

(i) *If* $\beta_1(\cdot) \leqslant \beta_2(\cdot)$, *then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1(\cdot)}^{s(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2(\cdot)}^{s(\cdot)}.$$

(ii) *If* $p_1(\cdot) \leqslant p_2(\cdot)$, *then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p_1(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p_2(\cdot)} B_{\beta(\cdot)}^{s(\cdot)}.$$

(iii) *If* $(s_1 - s_2)^- > 0$, *then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1(\cdot)}^{s_1(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2(\cdot)}^{s_2(\cdot)}.$$

Proof. (i) is a simple consequence of the embedding $\ell^{\beta_1(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}) \hookrightarrow \ell^{\beta_2(\cdot)}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})$. (ii) can be deduced from the embeddings properties of the Herz-type spaces, see (3.7). Notice that $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1(\cdot)}^{s_1(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_1^+}^{s_1(\cdot)}$ and $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2^-}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta_2(\cdot)}^{s_2(\cdot)}$. Therefore, it suffices to prove (iii) for constant exponents β_1^+ and β_2^-. We have

$$\begin{aligned}
\|(2^{vs_2(\cdot)}\varphi_v * f)_{v\geqslant 0}\|_{\ell^{\beta_2^-}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})} &\leqslant c \sup_{v\geqslant 0} \|2^{vs_1(\cdot)}\varphi_v * f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \\
&\leqslant c\|(2^{vs_1(\cdot)}\varphi_v * f)_{v\geqslant 0}\|_{\ell^{\beta_2^+}(\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)})},
\end{aligned}$$

with $c = (\sum_{v\geqslant 0} 2^{(s_1-s_2)^- v\beta_2^-})^{1/\beta_2^-}$. $\square$

Theorem 5.2. *Let* $p, q, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ *and* $s \in C_{\mathrm{loc}}^{\log}$. *Let* $\alpha : \mathbf{R}^n \to \mathbf{R}$ *be log-Hölder continuous, both at the origin and at infinity such that* $\alpha^- + \frac{n}{q^+} > 0$. *Then*

$$\mathscr{S}(\mathbf{R}^n) \hookrightarrow \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow \mathscr{S}'(\mathbf{R}^n). \tag{5.3}$$

Proof. Our proof use partially some decomposition techniques already used in [5] where the constant exponent case was studied. Also, in more general spaces given by abstract definitions, see Hedberg and Netrusov [12]. By Theorem 5.1 we need only to prove (5.3) with $\beta := \infty$.

Step 1. Let $f \in \mathscr{S}(\mathbf{R}^n)$ and $\{\mathscr{F}\varphi_j\}_{j=0}^{\infty}$ is a resolution of unity. If L, M and N are sufficiently large natural numbers, then

$$\begin{aligned}
\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\infty}^{s(\cdot)}} &= \sup_{j\geqslant 0} \|2^{js(\cdot)}\varphi_j * f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}} \\
&\leqslant \sup_{j\geqslant 0} \|2^{js(\cdot)}(1+|x|^2)^{2L}\varphi_j * f\|_{\infty} \left\| \frac{1}{(1+|x|^2)^{2L}} \right\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}.
\end{aligned}$$

Take L sufficiently large such that $(\alpha - 4L + n/q)^+ < 0$, we then have

$$\varrho_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}\left(\frac{1}{(1+|\cdot|^2)^{2L}}\right) = \sum_{k=-\infty}^{+\infty}\left\| \left|\frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}}\right|^{p(\cdot)} \chi_k \right\|_{q(\cdot)/p(\cdot)}.$$

First we remark that

$$\begin{aligned}\left\| \left|\frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}}\right|^{p(\cdot)} \chi_k \right\|_{q(\cdot)/p(\cdot)} &\leqslant c2^{k(\alpha-4L+n/q)^+p^-} \| |2^{-kn/q(\cdot)}|^{p(\cdot)} \chi_k\|_{q(\cdot)/p(\cdot)} \\ &\lesssim 2^{k(\alpha-4L+n/q)^+p^-}\end{aligned}$$

for any $k \geqslant 0$ and

$$\left\| \left|\frac{2^{k\alpha(\cdot)}}{(1+|\cdot|^2)^{2L}}\right|^{p(\cdot)} \chi_k \right\|_{q(\cdot)/p(\cdot)} \lesssim 2^{k(\alpha+n/q)^-p^-}$$

for any $k < 0$. Therefore,

$$\sum_{k=-\infty}^{+\infty}\left\| \left|\frac{2^{k\alpha(\cdot)}}{(1+|x|^2)^{2L}}\right|^{p(\cdot)} \chi_k \right\|_{q(\cdot)/p(\cdot)} \lesssim \sum_{k=-\infty}^{0} 2^{k(\alpha+n/q)^-p^-} + \sum_{k=1}^{+\infty} 2^{k(\alpha-4L+n/q)^+p^-} \lesssim 1.$$

Hence

$$\begin{aligned}\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}} &\lesssim \sup_{j\geqslant 0} 2^{js^+} \|\mathscr{F}^{-1}[1+(-\triangle)^L]\mathscr{F}\varphi_j \mathscr{F}f\|_\infty \\ &\lesssim \sup_{j\geqslant 0} 2^{js^+} \|[1+(-\triangle)^L]\mathscr{F}\varphi_j \mathscr{F}f\|_1 \\ &\lesssim \|(1+|x|)^M[1+(-\triangle)^L]\mathscr{F}f\|_\infty \\ &\lesssim p_N(\mathscr{F}f).\end{aligned}$$

Step 2. We prove the right-hand side of (5.3). Let $\{\mathscr{F}\varphi_j\}_{j\in\mathbf{N}_0}$ be the smooth dyadic resolution of unity. We put $\omega_j = \sum_{i=j-1}^{i=j+1} \mathscr{F}\varphi_i$ if $j = 1, 2, \ldots$ (with $\mathscr{F}\varphi_{-1} = 0$). If $f \in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}$ and $\psi \in \mathscr{S}(\mathbf{R}^n)$, then $f(\psi)$ denotes the value of the functional f of $\mathscr{S}'(\mathbf{R}^n)$ for the test function ψ. We obtain

$$\begin{aligned}|f(\psi)| &\leqslant \sum_{j=0}^{\infty} |\varphi_j * f(\mathscr{F}^{-1}\omega_j * \psi)| \\ &= \sum_{j=0}^{\infty} \|\varphi_j * f \cdot (\mathscr{F}^{-1}\omega_j * \psi)\|_1 \\ &= \sum_{j=0}^{\infty} \|\varphi_j * f \cdot (\mathscr{F}^{-1}\omega_j * \psi)\|_{\dot{K}_1^{0,1}}.\end{aligned}$$

Recalling the definition of $\dot{K}_1^{0,1}$ spaces, the last sum can be rewritten as

$$\sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}\|\varphi_j * f\cdot(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\leqslant \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}\sup_{x\in B(0,2^k)}|\varphi_j * f(x)|\,\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1.$$

We divide the last sum into two parts $\sum_{j=0}^{\infty}\sum_{k=-\infty}^{-j-1}\cdots+\sum_{j=0}^{\infty}\sum_{k=-j}^{\infty}\cdots$. Lemma 3.15, gives for any $0<d<\min(q^-,n/(\alpha+n/q)^+)$

$$\sum_{j=0}^{\infty}\sum_{k=-\infty}^{-j-1}\sup_{x\in B(0,2^{-j})}|\varphi_j * f(x)|\,\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\lesssim \sum_{j=0}^{\infty}\sum_{k=-\infty}^{-j-1}\|2^{(n/q(\cdot)+\alpha(\cdot))j}\varphi_j * f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}}\sum_{j=0}^{\infty}\sum_{k=-\infty}^{-j-1}2^{(n/q+\alpha-s)^+j}\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}}\|\psi\|_{B_{1,1}^{(n/q+\alpha-s)^+}}.$$

Using again Lemma 3.15, we have for any $0<d<\min(q^-,n/(\alpha+n/q)^+)$

$$\sum_{j=0}^{\infty}\sum_{k=-j}^{\infty}\sup_{x\in B(0,2^k)}|\varphi_j * f(x)|\,\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\leqslant \sum_{j=0}^{\infty}\sum_{k=-j}^{\infty}\|2^{jn/d}2^{k(n/d-n/q(\cdot)-\alpha(\cdot))}\varphi_j * f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}}\|(\mathscr{F}^{-1}\omega_j*\psi)\chi_k\|_1$$
$$\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}}\big(\|\psi\|_{\dot{K}_1^{(n/d-n/q-\alpha)^+,1}B_1^{(n/d-s)^+}}+\|\psi\|_{\dot{K}_1^{(n/d-n/q-\alpha)^-,1}B_1^{(n/d-s)^+}}\big).$$

Consequently

$$|f(\psi)|\leqslant c\mu\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}B_{\infty}^{s(\cdot)}},$$

where

$$\mu=\max(\|\psi\|_{B_{1,1}^{(n/q+\alpha-s)^+}},\|\psi\|_{\dot{K}_1^{n/d-(n/q+\alpha)^+,1}B_1^{(n/d-s)^+}},\|\psi\|_{\dot{K}_1^{n/d-(n/q+\alpha)^-,1}B_1^{(n/d-s)^+}}).$$

By our assumption on d we have

$$\mathscr{S}(\mathbf{R}^n)\hookrightarrow \dot{K}_1^{n/d-(n/q+\alpha)^+,1}B_1^{(n/d-s)^+}$$

and

$$\mathscr{S}(\mathbf{R}^n)\hookrightarrow \dot{K}_1^{n/d-(n/q+\alpha)^-,1}B_1^{(n/d-s)^+}.$$

From this and the embedding $\mathscr{S}(\mathbf{R}^n) \hookrightarrow B_{1,1}^{(n/d-n/q-\alpha)^+}$, we obtain

$$|f(\psi)| \leqslant c p_N(\psi) \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}}.$$

This proves that $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_\infty^{s(\cdot)}$ is continuously embedded in $\mathscr{S}'(\mathbf{R}^n)$. This completes the proof. $\square$

Applying Lemmas 3.17 and 3.19, we obtain the following Sobolev type embeddings.

THEOREM 5.4. *Let $p, r \in \mathscr{P}_0(\mathbf{R}^n)$ with $p^+, r^+ < \infty$, $q_1, q_2, \beta, r \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ and $s_1, s_2 \in C_{\mathrm{loc}}^{\log}$. Let $\alpha_1, \alpha_2 \in C_{\mathrm{loc}}^{\log}$ be such that $(\alpha_1 + n/q_1)^- > 0$ and $(\alpha_2 + n/q_2)^- > 0$. Assume that*

$$s_1(\cdot) - n/q_1(\cdot) - \alpha_1(\cdot) \leqslant s_2(\cdot) - n/q_2(\cdot) - \alpha_2(\cdot). \tag{5.5}$$

Let $\varrho(\cdot) = \alpha_2(\cdot) + n/q_2(\cdot) - \alpha_1(\cdot) - n/q_1(\cdot)$. The embedding

$$\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_1(\cdot)}^{\alpha_1(\cdot),r(\cdot)} B_{\beta(\cdot)}^{s_1(\cdot)}, \tag{5.6}$$

holds if $q_2(\cdot) \leqslant q_1(\cdot)$ with $\alpha_2(\cdot) = \alpha_1(\cdot)$ or $(\alpha_2 - \alpha_1)^- > 0$ or $q_1(\cdot) \leqslant q_2(\cdot)$ with

$$\varrho(\cdot) = 0 \quad \text{or} \quad \varrho^- > 0, \tag{5.7}$$

where

$$\theta(\cdot) = r(\cdot) \quad \text{if} \quad \varrho(\cdot) = 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \text{or} \quad \alpha_1(\cdot) = \alpha_2(\cdot), \quad q_2(\cdot) \leqslant q_1(\cdot)$$

and

$$\theta(\cdot) = p(\cdot) \quad \text{if} \quad \varrho^- > 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \text{or} \quad (\alpha_2 - \alpha_1)^- > 0, \quad q_2(\cdot) \leqslant q_1(\cdot).$$

THEOREM 5.8. *Let $p \in \mathscr{P}_0(\mathbf{R}^n)$ with $p^+ < \infty$, $q_1, q_2, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ and $s_1, s_2 \in C_{\mathrm{loc}}^{\log}$. Let $\alpha_2 \in C_{\mathrm{loc}}^{\log}$ be such that $(\alpha_2 + n/q_2)^- > 0$. Assume that*

$$s_1(\cdot) - n/q_1(\cdot) \leqslant s_2(\cdot) - n/q_2(\cdot) - \alpha_2(\cdot).$$

Let $\varrho(\cdot) = \alpha_2(\cdot) + n/q_2(\cdot) - n/q_1(\cdot)$. The embedding

$$\dot{K}_{q_2(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)},$$

holds if $q_2(\cdot) \leqslant q_1(\cdot)$ with $\alpha_2(\cdot) = 0$ or $\alpha_2^- > 0$ or $q_1(\cdot) \leqslant q_2(\cdot)$ with

$$\varrho(\cdot) = 0 \quad \text{or} \quad \varrho^- > 0,$$

where

$$\theta(\cdot) = q_1(\cdot) \quad \text{if} \quad \varrho(\cdot) = 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \text{or} \quad \alpha_2(\cdot) = 0, \quad q_2(\cdot) \leqslant q_1(\cdot)$$

and

$$\theta(\cdot) = p(\cdot) \quad \text{if} \quad \varrho^- > 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \text{or} \quad \alpha_2^- > 0, \quad q_2(\cdot) \leqslant q_1(\cdot).$$

To prove this embeddings it suffices to take $r(\cdot) = q_1(\cdot)$ and $\alpha_1(\cdot) = 0$ in Theorem 5.4.

Using this result, we have the following useful consequence.

COROLLARY 5.9. *Let $q_1, q_2, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ and $s_1, s_2 \in C_{\mathrm{loc}}^{\log}$, such that $s_1(\cdot) - n/q_1(\cdot) \leqslant s_2(\cdot) - n/q_2(\cdot)$ and $q_2(\cdot) \leqslant q_1(\cdot)$. Then*

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_2(\cdot)}^{0,q_1(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow B_{q_1(\cdot),\beta(\cdot)}^{s_1(\cdot)}. \tag{5.10}$$

To prove (5.10) it suffices to take in Theorem 5.8, $\theta(\cdot) = q_1(\cdot)$ and $\alpha_2(\cdot) = 0$. However the desired embeddings follow immediately from the fact that

$$B_{q_2(\cdot),\beta(\cdot)}^{s_2(\cdot)} = \dot{K}_{q_2(\cdot)}^{0,q_2(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)} \hookrightarrow \dot{K}_{q_2(\cdot)}^{0,q_1(\cdot)} B_{\beta(\cdot)}^{s_2(\cdot)}.$$

Let us define $\sigma_{q(\cdot)} := n\left(\dfrac{1}{\min(1, q(\cdot))} - 1\right)$ and $\bar{q}(\cdot) := \max(1, q(\cdot))$.

PROPOSITION 5.11. *Let $p, q, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ and $s \in C_{\mathrm{loc}}^{\log}$. Let $\alpha \in C_{\mathrm{loc}}^{\log}$ such that $\alpha^- > 0$. If $(s - \sigma_q - \alpha)^- > 0$, then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow L^{\bar{q}(\cdot)}.$$

To prove this proposition it suffices to use the embedding

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)} B_{\beta(\cdot)}^{s(\cdot)} \hookrightarrow B_{q(\cdot),\beta(\cdot)}^{s(\cdot)-\alpha(\cdot)} \hookrightarrow L^{\bar{q}(\cdot)},$$

where the first embedding is follows from Theorem 5.8, and the second embedding is given in [1, Proposition 6.9].

Let C_u be the space of all bounded uniformly continuous functions on $\mathbf{R}^n$ equipped with the sup norm. Concerning embeddings into C_u, we have the following result.

COROLLARY 5.12. *Let $p, q \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ with $p^+ < \infty$ and $\alpha \in C_{\mathrm{loc}}^{\log}$ such that $\alpha^- > 0$ or $\alpha(\cdot) = 0$. Then*

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} B_1^{\alpha(\cdot)+n/q(\cdot)} \hookrightarrow C_u, \tag{5.13}$$

where

$$\theta(\cdot) = \begin{cases} \infty & \text{if } \alpha(\cdot) = 0 \\ p(\cdot) & \text{if } \alpha^- > 0. \end{cases}$$

Proof. It follows from Theorem 5.8 that

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot),\theta(\cdot)} B_1^{\alpha(\cdot)+n/q(\cdot)} \hookrightarrow B_{q(\cdot),1}^{n/q(\cdot)} \hookrightarrow B_{\infty,1}^0.$$

Hence the result follows by the embedding $B_{\infty,1}^0 \hookrightarrow C_u$, see [33]. $\square$

The following statement holds by Theorem 5.4 and the fact that $\dot{K}^{0,q(\cdot)}_{q(\cdot)} B^{s_2(\cdot)}_{\beta(\cdot)} = B^{s_2(\cdot)}_{q(\cdot),\beta(\cdot)}$.

THEOREM 5.14. *Let* $p \in \mathscr{P}_0(\mathbf{R}^n)$ *with* $p^+ < \infty, q_1, q_2, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ *and* $s_1, s_2 \in C^{\log}_{\mathrm{loc}}$. *Let* $\alpha_1 \in C^{\log}_{\mathrm{loc}}$ *be such that* $(\alpha_1 + n/q_1)^- > 0$. *Assume that*

$$s_1(\cdot) - n/q_1(\cdot) - \alpha_1(\cdot) \leqslant s_2(\cdot) - n/q_2(\cdot).$$

Let $\varrho(\cdot) = n/q_2(\cdot) - \alpha_1(\cdot) - n/q_1(\cdot)$. *The embedding*

$$B^{s_2(\cdot)}_{q_2(\cdot),\beta(\cdot)} \hookrightarrow \dot{K}^{\alpha_1(\cdot),\theta(\cdot)}_{q_1(\cdot)} B^{s_1(\cdot)}_{\beta(\cdot)}, \tag{5.15}$$

holds if $q_2(\cdot) \leqslant q_1(\cdot)$ *with* $\alpha_1(\cdot) = 0$ *or* $\alpha_1^+ < 0$ *or* $q_1(\cdot) \leqslant q_2(\cdot)$ *with* $\varrho(\cdot) = 0$ *or* $\varrho^- > 0$, *where*

$$\theta(\cdot) = q_2(\cdot) \quad \textit{if} \quad \varrho(\cdot) = 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \textit{or} \quad \alpha_1(\cdot) = 0, \quad q_2(\cdot) \leqslant q_1(\cdot)$$

and

$$\theta(\cdot) = p(\cdot) \quad \textit{if} \quad \varrho^- > 0, \quad q_1(\cdot) \leqslant q_2(\cdot) \quad \textit{or} \quad \alpha_1^+ < 0, \quad q_2(\cdot) \leqslant q_1(\cdot).$$

Using this result, we obtain:

COROLLARY 5.16. *Let* $q_1, q_2, \beta \in \mathscr{P}_0^{\log}(\mathbf{R}^n)$ *and* $s_1, s_2 \in C^{\log}_{\mathrm{loc}}$, *such that* $s_1(\cdot) - n/q_1(\cdot) \leqslant s_2(\cdot) - n/q_2(\cdot)$ *and* $q_2(\cdot) \leqslant q_1(\cdot)$. *Then*

$$B^{s_2(\cdot)}_{q_2(\cdot),\beta(\cdot)} \hookrightarrow \dot{K}^{0,q_2(\cdot)}_{q_1(\cdot)} B^{s_1(\cdot)}_{\beta(\cdot)} \hookrightarrow B^{s_1(\cdot)}_{q_1(\cdot),\beta(\cdot)}. \tag{5.17}$$

To prove this it suffices to take in Theorem 5.14, $\theta(\cdot) = q_2(\cdot)$ and $\alpha_1(\cdot) = 0$. Then the desired embedding is an immediate consequence of the fact that $\dot{K}^{0,q_2(\cdot)}_{q_1(\cdot)} B^{s_1(\cdot)}_{\beta(\cdot)} \hookrightarrow \dot{K}^{0,q_1(\cdot)}_{q_1(\cdot)} B^{s_1(\cdot)}_{\beta(\cdot)} = B^{s_1(\cdot)}_{q_1(\cdot),\beta(\cdot)}$.

6. Appendix

Here we present the more technical proofs of the Lemmas 3.15, 3.17 and 3.19. Our proofs use partially some decomposition techniques already used in [5] where the constant exponent case was studied.

Proof of Lemma 3.15. By Lemma 3.3 we have for $d, R > 0$, $N > n$ and any $x \in B(0, 1/H)$

$$\begin{aligned} |f(x)|^{\varrho} &\leqslant c\left(\int_{\mathbf{R}^n} |f(y)|^d \eta_{R,N+M}(x-y)\, dy\right)^{\varrho/d} \\ &\leqslant c\left(\int_{B(0,2^2/H)} (\cdots)\, dy\right)^{\varrho/d} + c\left(\int_{\mathbf{R}^n \setminus B(0,2^2/H)} (\cdots)\, dy\right)^{\varrho/d}, \end{aligned}$$

where $\varrho := \min(1, d)$ and $M \geqslant c_{\log}(\sigma) + c_{\log}(1/\beta)$ with $c_{\log}(\sigma)$, $c_{\log}(1/\beta)$ are the constants from (2.1) for σ and $\frac{1}{\beta}$, respectively. Here

$$\eta_{R,N+M}(\cdot) := R^n(1 + R|\cdot|)^{-N-M}.$$

Using the following decomposition

$$\int_{B(0,2^2/H)} (\cdots)\, dy = \sum_{j=0}^{\infty} \int_{C(2^{2-j}/H)} (\cdots)\, dy,$$

$$\int_{\mathbf{R}^n \setminus B(0,2^2/H)} (\cdots)\, dy = \sum_{j=0}^{\infty} \int_{C(2^{j+3}/H)} (\cdots)\, dy$$

and the well-known inequality

$$\left(\sum_{j=0}^{\infty} |a_j|\right)^{\tau} \leqslant \sum_{j=0}^{\infty} |a_j|^{\tau}, \quad \{a_j\}_j \subset \mathbf{C},\ \tau \in [0,1] \tag{6.1}$$

we obtain that $|f(x)|^{\varrho}$ can be estimated by

$$c \sum_{j=0}^{\infty} (V_{j,R,H}^1(x) + V_{j,R,H}^2(x)), \tag{6.2}$$

where

$$V_{j,R,H}^1(x) := (\eta_{R,N+M} * |f\chi_{C(2^{2-j}/H)}|^d(x))^{\varrho/d}, \quad V_{j,R,H}^2(x) := V_{-j-1,R,H}^1(x). \tag{6.3}$$

Here N is chosen large enough such that $N > \max(n, n/d - (n/q + \alpha)^-)$. Let us give the estimation of the first term in (6.2). Lemma 3.1, a simple change of variables, the Hölder inequality (with $\frac{1}{d} = \frac{1}{q(\cdot)} + \frac{1}{d} - \frac{1}{q(\cdot)}$) and Lemma 3.4, yield for any $d, R > 0$ and any $x \in B(0, 1/H)$

$$\begin{aligned}\sum_{j=0}^{\infty} R^{\sigma(x)\varrho} \lambda^{-\varrho/\beta(x)} V_{j,R,H}^1(x) &\lesssim R^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} \|\lambda^{-1/\beta(\cdot)} R^{\sigma(\cdot)} f\chi_{\tilde{C}_k}\|\chi_{\tilde{C}_k}\|_{t(\cdot)}\|_{q(\cdot)}^{\varrho} \\ &\lesssim R^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} \|\lambda^{-1/\beta(\cdot)} R^{\sigma(\cdot)} 2^{kn(1/d-1/q(\cdot))} f\chi_{\tilde{C}_k}\|_{q(\cdot)}^{\varrho},\end{aligned}$$

where $\frac{1}{t(\cdot)} := \frac{1}{d} - \frac{1}{q(\cdot)}$, $\tilde{C}_k := \{x \in \mathbf{R}^n : 2^{k-2} \leqslant |x| < 2^k\}$ and $[a]$ is the integer part of the real number a. This term is bounded by

$$c\left(\frac{R}{H}\right)^{n\varrho/d} \sum_{k=-\infty}^{2-[\log_2 H]} (2^k H)^{\varrho(n/d-(n/q+\alpha)^+)} \|\lambda^{-1/\beta(\cdot)} 2^{k\alpha(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\chi_{\tilde{C}_k}\|_{q(\cdot)}^{\varrho}.$$

Using the fact that $n/d > (n/q+\alpha)^+$, $2^{k-3}H < 1$ and the embedding $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} \hookrightarrow \dot{K}^{\alpha(\cdot),\infty}_{q(\cdot)}$ we obtain that the right-hand side of the last expression is bounded by

$$c\Big(\frac{R}{H}\Big)^{n\varrho/d} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\|^{\varrho}_{\dot{K}^{\alpha(\cdot),\infty}_{q(\cdot)}} \sum_{k=-\infty}^{2-[\log_2 H]} (2^k H)^{\varrho(n/d-(n/q+\alpha)^+)} \tag{6.4}$$

$$\lesssim \Big(\frac{R}{H}\Big)^{n\varrho/d} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\|^{\varrho}_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}},$$

where the implicit positive constant not depending on R and H.

Now we estimate the second term in (6.2). Notice that for any $y \in C(2^{3+j}/H)$ and any $x \in B(0,1/H)$, we have $|x-y| > 2^j/H$, so for any $d > 0$, $N \in \mathbf{N}$ and $j \in \mathbf{N}_0$,

$$\eta_{R,N}(x-y) \leqslant R^n \Big(\frac{2^j R}{H}\Big)^{-N} \leqslant 2^{-jN} R^n.$$

Hence by Lemma 3.1, a simple change of variables, the Hölder inequality (with $\frac{1}{d} = \frac{1}{q(\cdot)} + \frac{1}{d} - \frac{1}{q(\cdot)}$) and Lemma 3.4, we obtain

$$\sum_{j=0}^{\infty} R^{\sigma(x)\varrho} \lambda^{-\varrho/\beta(x)} V^2_{j,R,H}(x)$$

$$\lesssim R^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} \|\lambda^{-1/\beta(\cdot)} (2^k H)^{-N} R^{\sigma(\cdot)} f \chi_{\tilde{C}_k}\|^{\varrho}_d$$

$$\lesssim R^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} \|\lambda^{-1/\beta(\cdot)} (2^k H)^{-N} R^{\sigma(\cdot)} f \chi_{\tilde{C}_k} \|\chi_{\tilde{C}_k}\|_{t(\cdot)}\|^{\varrho}_{q(\cdot)}$$

$$\lesssim \Big(\frac{R}{H}\Big)^{n\varrho/d} \sum_{k=2-[\log_2 H]}^{\infty} (2^k H)^{(n/d-(n/q+\alpha)^- - N)\varrho} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} 2^{k\alpha(\cdot)} R^{\sigma(\cdot)} f \chi_{\tilde{C}_k}\|^{\varrho}_{q(\cdot)},$$

where the implicit positive constant does not depend on R. Using again the embedding $\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)} \hookrightarrow \dot{K}^{\alpha(\cdot),\infty}_{q(\cdot)}$, and since $N > n/d - (n/q+\alpha)^-$ and $2^k H > 1$, for any $k \geqslant 2 - [\log_2 H]$, the right-hand side of the last expression is bounded by

$$c\Big(\frac{R}{H}\Big)^{n\varrho/d} \Big(\sup_{k \geqslant 2-[\log_2 H]} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} 2^{k\alpha(\cdot)} R^{\sigma(\cdot)} f \chi_{\tilde{C}_k}\|_{q(\cdot)}\Big)^{\varrho} \tag{6.5}$$

$$\leqslant c\Big(\frac{R}{H}\Big)^{n\varrho/d} \|\lambda^{-1/\beta(\cdot)} H^{n/q(\cdot)+\alpha(\cdot)} R^{\sigma(\cdot)} f\|^{\varrho}_{\dot{K}^{\alpha(\cdot),p(\cdot)}_{q(\cdot)}}.$$

Finally, we obtain the desired estimate from (6.4) and (6.5) taking into account the decomposition (6.2). This finishes the proof.

Proof of Lemma 3.17. Our estimate (3.18), clearly follows from the inequality

$$\sum_{k=-\infty}^{\infty} \| |2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \lesssim 1,$$

where

$$\lambda := \| |R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{\beta(\cdot)} \|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} + \frac{1}{R}.$$

Note that the assumption on the norm implies that $\lambda \in [R^{-1}, R^{-1}+1]$. We divide the sum $\sum_{k=-\infty}^{\infty} \cdots$ into two parts,

$$\sum_{2^k<2/R} (\cdots) + \sum_{2^k \geqslant 2/R} (\cdots) =: I_R + II_R. \tag{6.6}$$

Estimate of I_R. By Lemma 3.15 we get

$$\begin{aligned} &\sup_{x \in B(0,2/R)} (R^{s(x)-\alpha_1(x)-n/t(x)} \lambda^{-1/\beta(x)} |f(x)|) \\ &\qquad \lesssim \| R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f \|_{\dot{K}^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}}, \end{aligned} \tag{6.7}$$

where the implicit positive constant not depending on R. The norm on the right hand side is bounded by 1. To show this, we investigate the corresponding modular:

$$\begin{aligned} &\varrho_{\ell^{\theta(\cdot)}(L^{q(\cdot)})}((2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f \chi_k)_k) \\ &= \sum_{k=-\infty}^{\infty} \| |2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f|^{\theta(\cdot)} \chi_k \|_{q(\cdot)/\theta(\cdot)} \\ &= \sum_{k=-\infty}^{\infty} \| |\lambda^{-1} |2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{\beta(\cdot)}|^{\theta(\cdot)/\beta(\cdot)} \chi_k \|_{q(\cdot)/\theta(\cdot)}. \end{aligned}$$

This term is bounded by 1 if and only if

$$\| \lambda^{-1} |R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{\beta(\cdot)} \|_{\dot{K}^{\alpha_2(\cdot)\beta(\cdot),\theta(\cdot)/\beta(\cdot)}_{q(\cdot)/\beta(\cdot)}} \leqslant 1,$$

which follows immediately from the definition of λ. Therefore,

$$I_R \lesssim \sum_{2^k<2/R} (2^k R)^{r^-(\alpha_1+n/t)^-} \| |2^{-nk/t(\cdot)}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \lesssim \sum_{2^k<2/R} (2^k R)^{r^-(\alpha_1+n/t)^-} \lesssim 1,$$

where the second inequality follows by the fact that $\| |2^{-nk/t(\cdot)}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \lesssim 1$. To show this, we investigate the corresponding modular: $\varrho_{t(\cdot)/r(\cdot)}(2^{-nkr(\cdot)/t(\cdot)}) = \int_{C_k} 2^{-nk}\, dx = c < \infty$. In the last inequality we use the fact that $(\alpha_1 + n/t)^- > 0$.

Estimation of II_R. By Lemma 3.3 we have for $R > 0$, $N > n/\tau$ and any $x \in C_k$

$$\begin{aligned}(6.8)\qquad & \lambda^{-1/\beta(x)} R^{s(x)-\alpha_1(x)-n/t(x)}|f(x)| \\ & \lesssim \left(\int_{\mathbf{R}^n} |\lambda^{-1/\beta(y)} R^{s(y)-\alpha_1(y)-n/t(y)} f(y)|^\tau \eta_{R,N\tau+\delta}(x-y)\, dy \right)^{1/\tau} \\ & \lesssim \left(\int_{B(0,2^{k-2})} (\cdots)\, dy \right)^{1/\tau} + c\left(\int_{\tilde{C}_k} (\cdots)\, dy \right)^{1/\tau} \\ & \quad + c\left(\int_{\mathbf{R}^n \setminus B(0,2^{k+2})} (\cdots)\, dy \right)^{1/\tau} \\ & = V^1_{R,k}(x) + V^2_{R,k}(x) + V^3_{R,k}(x).\end{aligned}$$

Here $\tilde{C}_k := \{x \in \mathbf{R}^n : 2^{k-2} \leqslant |x| \leqslant 2^{k+2}\}$, $0 < \tau < q^-$ and $\delta := c_{\log}(\alpha_2) + c_{\log}\left(\frac{1}{r}\right) + c_{\log}\left(\frac{1}{q}\right)$. We choose N such that

$$\begin{aligned}(6.9)\qquad N > \max(& n/d + n/\tau + (\alpha_1 + n/t)^+, \\ & n/d + n/\tau - (\alpha_2 + n/q)^- + (\alpha_1 + n/t)^+),\end{aligned}$$

where d is chosen as in Lemma 3.15. It is easy to verify that if $x \in C_k$ and $y \in B(0, 2^{k-2})$, then $|x - y| > 2^{k-2}$. This estimate and Lemma 3.15 yield for any $x \in C_k$ and any $k \in \mathbf{Z}$ such that $2^k R \geqslant 2$

$$\begin{aligned}(6.10)\qquad & 2^{k\alpha_1(x)} V^1_{R,k}(x) \\ & \lesssim 2^{k\alpha_1(x)} \sup_{y \in B(0,2^{k-2})} |\lambda^{-1/\beta(y)} R^{s(y)-\alpha_1(y)-n/t(y)} f(y)| \\ & \quad \times \left(\int_{2^{k-2}<|t|<2^{k+1}} \eta_{R,N\tau+\delta}(t)\, dt \right)^{1/\tau} \\ & \lesssim (2^k R)^{n/d+n/\tau-N} 2^{k\alpha_1(x)} \|2^{-k(\alpha_2(\cdot)+n/q(\cdot))} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\|_{K^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}}.\end{aligned}$$

Observe that $2^k R \geqslant 2$. Hence

$$\begin{aligned}& 2^{k\alpha_1(x)} V^1_{R,k}(x) \\ & \quad \lesssim (2^k R)^{n/d+n/\tau-(\alpha_2+n/q)^- - N} 2^{k\alpha_1(x)} \|\lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f\|_{K^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}} \\ & \quad \lesssim (2^k R)^{n/d+n/\tau-(\alpha_2+n/q)^- - N} 2^{k\alpha_1(x)}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{2^k \geqslant 2/R} \| \, |2^{k\alpha_1(\cdot)} R^{\alpha_1(\cdot)+n/t(\cdot)} V^1_{R,k}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \\
&\quad \lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{(n/d+n/\tau-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+ - N)r^-} \| \, |2^{-kn/t(\cdot)}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \\
&\quad \lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{(n/d+n/\tau-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+ - N)r^-} \lesssim 1,
\end{aligned}$$

by our assumption on N. Now we observe that

$$V^2_{R,k} := (\eta_{R,N\tau} + \delta * |\lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\tilde{C}_k}|^\tau)^{1/\tau}.$$

Hölder's inequality gives

$$\begin{aligned}
&|2^{k\alpha_2(x)} R^{\alpha_2(x)} V^2_{R,k}(x) \chi_k(x)|^\tau \\
&\quad \lesssim \eta_{R,N\tau} * (|2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\tilde{C}_k}|^\tau)(x) \\
&\quad \lesssim \|\eta_{R,N\tau}(x-\cdot) R^{-n\tau/q(\cdot)}\|_{(q(\cdot)/\tau)'} \\
&\qquad \times \| \, |2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\tilde{C}_k}|^\tau \|_{q(\cdot)/\tau} \\
&\quad \leqslant \|R^{-n\tau/q(\cdot)} \eta_{R,N\tau}(x-\cdot)\|_{(q(\cdot)/\tau)'} \|R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f\|^\tau_{\dot{K}^{\alpha_2(\cdot),\theta(\cdot)}_{q(\cdot)}},
\end{aligned}$$

where $k\bar{\alpha}_2 := \begin{cases} k(\alpha_2)_\infty & \text{if } k \geqslant 0 \\ k\alpha_2(0) & \text{if } k < 0 \end{cases}$. The second norm on the right hand side is bounded by 1 due to the choice of λ, see the estimation of I_R. To show that the first norm is also bounded, we investigate the corresponding modular:

$$\varrho_{(q(\cdot)/\tau)'}(\eta_{R,N\tau}(x-\cdot) R^{-n\tau/q(\cdot)}) = R^n \int_{\mathbf{R}^n} (1 + R|x-y|)^{-N(q(\cdot)/\tau)'(y)\tau} \, dy < \infty.$$

Let us prove that

$$\| \, |(2^k R)^{\alpha_2(\cdot)} R^{n/t(\cdot)} V^2_{R,k}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \lesssim 1 \tag{6.11}$$

for any $k \in \mathbf{Z}$. We have

$$\begin{aligned}
&|(2^k R)^{\alpha_2(x)} R^{n/t(x)} V^2_{R,k}(x)|^{t(x)} \\
&\quad = |(2^k R)^{\alpha_2(x)} V^2_{R,k}(x)|^{t(x)-q(x)} |(2^k R)^{\alpha_2(x)} R^{n/q(x)} V^2_{R,k}(x)|^{q(x)} \\
&\quad \lesssim |(2^k R)^{\alpha_2(x)} R^{n/q(x)} V^2_{R,k}(x)|^{q(x)},
\end{aligned}$$

for any $x \in C_k$, where the implicit positive constant does not depend on R and k. Therefore,

$$\begin{aligned}\int_{R_k} &|(2^k R)^{\alpha_2(x)} R^{n/t(x)} V_{R,k}^2(x)|^{t(x)}\, dx \\ &\lesssim \int_{\mathbf{R}^n} |(2^k R)^{\alpha_2(x)} R^{n/q(x)} V_{R,k}^2(x)|^{q(x)}\, dx \\ &\lesssim \int_{\mathbf{R}^n} (\eta_{R,N\tau} * (|2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\chi_{\tilde{C}_k}|^\tau)(x))^{q(x)/\tau}\, dx.\end{aligned}$$

This term is bounded by 1 if and only if

$$\|\eta_{R,N\tau} * (|2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\chi_{\tilde{C}_k}|^\tau)\|_{q(\cdot)/\tau} \leqslant 1.$$

Since convolution is bounded in $L^{p(\cdot)}$ when $p \in \mathscr{P}^{\log}(\mathbf{R}^n)$, see (2.3), we obtain that the left-hand side is bounded by

$$\begin{aligned}c&\|2^{k\bar{\alpha}_2} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\chi_{\tilde{C}_k}\|_{q(\cdot)}^\tau \\ &\lesssim \|R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} \lambda^{-1/\beta(\cdot)} f\|_{\dot{K}_{q(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)}}^\tau \lesssim 1.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{2^k \geqslant 2/R} &\| |2^{k\alpha_1(\cdot)} R^{\alpha_1(\cdot)+n/t(\cdot)} V_{R,k}^2|^{r(\cdot)} \chi_k\|_{t(\cdot)/r(\cdot)} \\ &\lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{-(\alpha_2-\alpha_1)^- r^-} \| |(2^k R)^{\alpha_2(\cdot)} R^{n/t(\cdot)} V_{R,k}^2|^{r(\cdot)} \chi_k\|_{t(\cdot)/r(\cdot)} \\ &\lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{-(\alpha_2-\alpha_1)^- r^-} \lesssim 1,\end{aligned}$$

if $(\alpha_2 - \alpha_1)^- > 0$. Let us treat the case $\alpha_1(\cdot) = \alpha_2(\cdot)$ more carefully. Let recall that for $\alpha_1(\cdot) = \alpha_2(\cdot)$ we have $\theta(\cdot) = r(\cdot)$. Hence it suffices to prove that

$$\begin{aligned}\| &|(2^k R)^{\alpha_1(\cdot)} R^{n/t(\cdot)} V_{R,k}^2|^{r(\cdot)} \chi_k\|_{t(\cdot)/r(\cdot)} \\ &\lesssim \| |2^{k\alpha_1} \lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)-n/t(\cdot)} f\chi_{\tilde{C}_k}|^{r(\cdot)} \chi_k\|_{q(\cdot)/r(\cdot)} + \frac{1}{2^k R} =: \eta,\end{aligned}$$

which is equivalent to

$$\|\eta^{-1/r(\cdot)} (2^k R)^{\alpha_1(\cdot)} R^{n/t(\cdot)} V_{R,k}^2 \chi_k\|_{t(\cdot)} \lesssim 1. \tag{6.12}$$

We have

$$\begin{aligned}\eta^{-1/r(x)} &= \eta^{1/r(y)-1/r(x)} \eta^{-1/r(y)} \\ &\lesssim (2^k R)^{|1/r(x)-1/r(y)|} \eta^{-1/r(y)}, \quad x \in C_k,\ y \in \tilde{C}_k.\end{aligned}$$

We use the log-Hölder continuity of r to get the equivalence

$$2^{k(1/r(x)-1/r(y))} \approx 1$$

and

$$R^{1/r(x)-1/r(y)} \lesssim (1+R|x-y|)^{c_{\log}(r)}$$

for any $x \in C_k$ and any $y \in \tilde{C}_k$. Therefore,

$$\eta^{-1/r(x)} V_{R,k}^2(x) \lesssim (\eta_{R,N\tau} * |\eta^{-1/r(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{\tilde{C}_k}|^{\tau})^{1/\tau}.$$

Now (6.12) can be obtained by repeating the same arguments used in the proof of (6.11).

We see that $\int_{\mathbf{R}^n \setminus B(0,2^{k+2})} (\cdots)\, dy$ can be rewritten as $\sum_{i=0}^{\infty} \int_{C_{k+i+3}} (\cdots)\, dy$. Then, using (6.2), we get for any $x \in C_k$

$$(V_{R,k}^3(x))^{\varrho} \leqslant c \sum_{i=0}^{\infty} (\eta_{R,N\tau} + \delta * |\lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{C_{k+i+3}}|^{\tau}(x))^{\varrho/\tau}, \tag{6.13}$$

with $\varrho := \min(1,\tau)$. Since $|x-y| > 3 \cdot 2^{k+i}$ for any $x \in C_k$ and any $y \in C_{k+i+3}$, the right-hand side of (6.13) is bounded by

$$\begin{aligned} & cR^{\varrho(n/\tau-N)} \sum_{i=0}^{\infty} 2^{-\varrho(k+i)N} \|\lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{C_{k+i+3}}\|_{\tau}^{\varrho} \\ & \leqslant cR^{\varrho(n/\tau-N)} \sum_{j=k+3}^{\infty} 2^{-\varrho j N} \|\lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f \chi_{C_j}\|_{\tau}^{\varrho}. \end{aligned}$$

By Lemma 3.15 we get

$$\begin{aligned} & \sup_{x \in B(0,2^j)} |\lambda^{-1/\beta(x)} R^{s(x)-\alpha_1(x)-n/t(x)} f(x)| \\ & \lesssim (2^j R)^{n/d} \|2^{-j(n/q(\cdot)+\alpha_2(\cdot))} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\|_{\dot{K}_{q(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)}} \\ & \leqslant (2^j R)^{n/d} (2^j R)^{-(\alpha_2+n/q)^-} \|\lambda^{-1/\beta(\cdot)} R^{n/q(\cdot)+s(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)-n/t(\cdot)} f\|_{\dot{K}_{q(\cdot)}^{\alpha_2(\cdot),\theta(\cdot)}} \\ & \lesssim (2^j R)^{n/d-(\alpha_2+n/q)^-}, \end{aligned}$$

since $2^k R \geqslant 2$ we estimate $(V_{R,k}^3)^{\varrho}$ by

$$cR^{\varrho(n/\tau+n/d-N-(\alpha_2+n/q)^-)} \sum_{j=k+3}^{\infty} 2^{\varrho j(n/d+n/\tau-N-(\alpha_2+n/q)^-)} \lesssim (2^k R)^{\varrho(n/d+n/\tau-N-(\alpha_2+n/q)^-)}.$$

Hence

$$
\begin{aligned}
&\sum_{2^k \geqslant 2/R} \| \, |2^{k\alpha_1(\cdot)} R^{\alpha_1(\cdot)+n/t(\cdot)} V^3_{R,k}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \\
&\quad \lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{(n/d+n/\tau-N-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+)r^-} \| \, |2^{-kn/t(\cdot)}|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \\
&\quad \lesssim \sum_{2^k \geqslant 2/R} (2^k R)^{(n/d+n/\tau-N-(\alpha_2+n/q)^- + (\alpha_1+n/t)^+)r^-} \lesssim 1.
\end{aligned}
$$

This finishes the proof.

Proof of Lemma 3.19. We employ the notations I_R and II_R from (6.6). The estimate of I_R follows easily from the previous proof. We only need to estimate the part II_R with $(\alpha_2 - \alpha_1 + n/q - n/t)^- > 0$. Hölder's inequality gives

$$
\| \, |2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f|^{r(\cdot)} \chi_k \|_{t(\cdot)/r(\cdot)} \lesssim \| \, |2^{k\alpha_1(\cdot)} \lambda^{-1/\beta(\cdot)} R^{s(\cdot)} f|^{r(\cdot)} \chi_k \| \chi_k \|_{h(\cdot)/r(\cdot)} \|_{q(\cdot)/r(\cdot)},
$$

where $\dfrac{1}{t(\cdot)} := \dfrac{1}{q(\cdot)} + \dfrac{1}{h(\cdot)}$. By Lemma 3.4 we get $\|\chi_k\|_{h(\cdot)/r(\cdot)} \approx 2^{kn(1/t(x)-1/q(x))r(x)}$ for any $x \in R_k$. Therefore,

$$
\begin{aligned}
II_R &\lesssim \sup_{k \in \mathbf{Z}} \| \, |\lambda^{-1/\beta(\cdot)} 2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{r(\cdot)} \chi_k \|_{q(\cdot)/r(\cdot)} \\
&\qquad \times \sum_{2^k \geqslant 2/R} (2^k R)^{-(\alpha_2-\alpha_1+n/q-n/t)^- r^-} \\
&\lesssim \sup_{k \in \mathbf{Z}} \| \, |\lambda^{-1/\beta(\cdot)} 2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f|^{r(\cdot)} \chi_k \|_{q(\cdot)/r(\cdot)}.
\end{aligned}
$$

The last norm is bounded by 1 if and only if

$$
\| \lambda^{-1/\beta(\cdot)} |2^{k\alpha_2(\cdot)} R^{n/q(\cdot)-n/t(\cdot)+\alpha_2(\cdot)-\alpha_1(\cdot)+s(\cdot)} f| \chi_k \|_{q(\cdot)} \lesssim 1, \quad k \in \mathbf{Z}
$$

which follows immediately from (6.7). The proof is completed.

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