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COMPLETE FLAT RESOLUTIONS, TATE HOMOLOGY AND THE DEPTH FORMULA

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Abstract

We introduce Tate homology of complexes of finite Gorenstein flat dimension based on complete flat resolutions and give a new method of computing Tate homology in Christensen and Jorgensen's sense. We also investigate the relationship between Tate homology and Tate cohomology. As an application, a more brief proof of the main result on derived depth formula of [Vanishing of Tate homology and depth formula over local rings, J. Pure Appl. Algebra 219 (2015) 464–481] is given.

1. Introduction

Throughout this paper, R denotes a ring, R-Mod the category of left R-modules and $\mathscr{C}(R)$ the category of complexes of left R-modules. R° denotes the opposite ring.

The notion of Tate cohomology originated from the study of representations of finite groups. Avramov and Martsinkovsky [4] extended the definition, based on complete projective resolutions (or Tate projective resolutions), so that it can work well for finitely generated modules of finite G-dimension over a noetherian ring. Later, Veliche [20] and Christensen and Jorgensen [6] studied a Tate cohomology theory for complexes. The parallel theory of Tate homology has been treated by Iacob [16] and Christensen and Jorgensen [6, 7]. They defined Tate homology of complexes based on complete projective resolutions. That is, given an R°-complex M with a complete projective resolution $U \xrightarrow{\tau} P \xrightarrow{\pi} M$ (that is, a diagram of morphisms of complexes, where U is a totally acyclic complexes of projective R°-modules, π is a dg-projective resolution of M, and τ_n is an isomorphism for $n \gg 0$), Tate homology of the R°-complex M with coefficients in an arbitrary R-complex N is defined as

$$\widehat{\operatorname{Tor}}_i^R(M, N) = \mathrm{H}_i(U \otimes_R N).$$

However we note that there is no relationship between Tate homology and Tate cohomology when R is a coherent ring. The motivation of current paper is to

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establish a Tate homology theory based on complete flat resolutions, which can form a correspondence with the classical Tor homology. More precisely, given an R°-complex M admitting a complete flat resolution $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$ (see Definition 3.1 for details). For an R-complex N, the Tate homology with respect to the complete flat resolution of M with coefficients in N is defined as

$$\overline{\mathrm{Tor}}_i^R(M,N) = \mathrm{H}_i(T \otimes_R N).$$

We find a relationship between Tate homology and Tate cohomology and provide a new method of computing Tate homology in Christensen and Jorgensen's sense.

Let M and N be finitely generated modules over a commutative noetherian local ring R. We say that (M,N) satisfies the depth formula if

$$\mathrm{depth}_R(M \otimes_R N) = \mathrm{depth}_R M + \mathrm{depth}_R N - \mathrm{depth}\, R$$

and satisfies the derived depth formula if

$$\mathrm{depth}_R(M \otimes_R^{\mathrm{L}} N) = \mathrm{depth}_R M + \mathrm{depth}_R N - \mathrm{depth}\, R.$$

For the width, under suitable conditions on M and N, there is a width formula

$$\mathrm{width}_R(\mathbf{R}\,\mathrm{Hom}_R(M,N)) = \mathrm{depth}_R M + \mathrm{width}_R N - \mathrm{depth}\, R.$$

Several authors have proved that the two formulas hold under certain conditions in different ways, such as [1, 2, 11, 15, 18, 22]. Especially, Christensen and Jorgensen [7] proved the depth formula holds for every pair of Tate Tor-independent complexes, one of which is of finite Gorenstein projective dimension and the other is bounded above. It subsumes previous generalizations of Auslander's depth formula obtained over the half-century that has passed since [2] appeared. Also the width formula holds for every pair of Tate Ext-independent complexes, one of which is of finite Gorenstein injective dimension and the other is bounded above. However, in the previous passages, the study of depth formula and width formula is always independent. As applications of the Tate homology we have established in Section 3, we simplify the study of depth formula and width formula and provide a more brief proof to Christensen and Jorgensen's main result.

The layout of this paper is as follows: In section 2, we recall the definitions and basic notations of complexes and depth. Section 3 is devoted to discussing complete flat resolutions of complexes and Tate homology. We prove that when R is a left coherent ring over which each flat right R-module has finite projective dimension, the Tate homology defined here coincides with Tate homology defined by complete projective resolutions treated by Christensen and Jorgensen [6] and establish the relationship between Tate homology and Tate cohomology. In section 4, we give some applications.

2. Preliminaries and basic facts

We first review some basic facts on complexes. For terminologies we follow [3, 12, 10].

COMPLEX. Given a complex X

$$\cdots \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \xrightarrow{\delta_0} X_{-1} \xrightarrow{\delta_{-1}} \cdots,$$

the n-th homology module of X is $\mathrm{H}_n(X) = \mathrm{Z}_n(X)/\mathrm{B}_n(X)$, where $\mathrm{Z}_n(X) = \mathrm{Ker}(\delta_n^X)$, $\mathrm{B}_n(X) = \mathrm{Im}(\delta_{n+1}^X)$; we set $\mathrm{H}^n(X) = \mathrm{H}_{-n}(X)$, $\mathrm{C}_n(X) = \mathrm{Coker}(\delta_{n+1}^X)$. A complex X is called acyclic or exact if the homology complex $\mathrm{H}(X)$ is the zero-complex.

Given a complex X and an integer i, $\Sigma^i X$ denotes the complex such that $(\Sigma^i X)_n = X_{n-i}$ and whose boundary operators are $(-1)^i \delta_{n-i}^X$.

For a complex X we associate the numbers

$$\sup X = \sup\{i \mid X_i \neq 0\} \quad \text{and} \quad \inf X = \inf\{i \mid X_i \neq 0\}.$$

The complex X is called bounded above when $\sup X < \infty$, bounded below when $\inf X > -\infty$ and bounded when it is bounded below and above.

A homomorphism $\varphi : X \to Y$ of degree n is a family $(\varphi_i)_{i\in\mathbf{Z}}$ of homomorphisms of R-modules $\varphi_i : X_i \to Y_{n+i}$. All such homomorphisms form an abelian group, denoted by $\mathrm{Hom}_R(X, Y)_n$; it is clearly isomorphic to $\prod_{i\in\mathbf{Z}} \mathrm{Hom}_R(X_i, Y_{n+i})$. We let $\mathrm{Hom}_R(X, Y)$ denote the complex of abelian groups with n-th component $\mathrm{Hom}_R(X, Y)_n$ and boundary operator

$$\delta_n((\varphi_i)_{i\in\mathbf{Z}}) = (\delta_{n+i}^Y \varphi_i - (-1)^n \varphi_{i-1} \delta_i^X)_{i\in\mathbf{Z}}.$$

A homomorphism $\varphi \in \mathrm{Hom}_R(X, Y)_n$ is called a chain map if $\delta(\varphi) = 0$, that is, if $\delta_{n+i}^Y \varphi_i = (-1)^n \varphi_{i-1} \delta_i^X$ for all $i \in \mathbf{Z}$. A chain map of degree 0 is called a morphism. A morphism $\varphi : X \to Y$ is called a quasi-isomorphism if the induced morphisms $\mathrm{H}_n(\varphi) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ are isomorphisms for all $n \in \mathbf{Z}$. Complexes X and Y are quasi-isomorphic (denoted as $X \simeq Y$) if they are linked by a chain of quasi-isomorphisms.

If X is a complex of right R-modules and Y is a complex of left R-modules, the tensor product of X and Y is the complex of abelian groups $X \otimes_R Y$ with $(X \otimes_R Y)_n = \bigoplus_{t\in\mathbf{Z}} (X_t \otimes_R Y_{n-t})$ and $\delta(x \otimes_R y) = \delta_t^X(x) \otimes y + (-1)^t x \otimes \delta_{n-t}^Y(y)$ for all $x \in X_t$, $y \in Y_{n-t}$.

Let X be an R-complex and u, v integers. The hard above-truncation, $X_{\leq u}$, of X at u and the hard below-truncation, $X_{\geq v}$, of X at v are given by:

$$X_{\leq u} = 0 \to X_u \to X_{u-1} \to X_{u-2} \to \cdots,$$

$$X_{\geq v} = \cdots \to X_{v+2} \to X_{v+1} \to X_v \to 0.$$

The soft above-truncation, $X_{\subset_u}$, of X at u and the soft below-truncation, $X_{\supset_v}$, of X at v are given by:

$$X_{\subset_u} = 0 \to C_u(X) \to X_{u-1} \to X_{u-2} \to \cdots,$$

$$X_{\supset_v} = \cdots \to X_{v+2} \to X_{v+1} \to Z_v(X) \to 0.$$

Cotorsion pair. A pair $(\mathscr{A}, \mathscr{B})$ of subcategories in R-Mod is called a cotorsion pair provided that $\mathscr{A}^{\perp} = \mathscr{B}$ and ${}^{\perp}\mathscr{B} = \mathscr{A}$, where ${}^{\perp}\mathscr{B} = \{A \in R\text{-Mod} \mid \operatorname{Ext}_R^1(A, B) = 0, \forall B \in \mathscr{B}\}$ and $\mathscr{A}^{\perp} = \{B \in R\text{-Mod} \mid \operatorname{Ext}_R^1(A, B) = 0, \forall A \in \mathscr{A}\}$. We say that a homomorphism $\phi : M \to B$ is a $\mathscr{B}$-preenvelope if $B \in \mathscr{B}$ and the abelian group homomorphism $\operatorname{Hom}_R(\phi, B') : \operatorname{Hom}_R(B, B') \to \operatorname{Hom}_R(M, B')$ is surjective for each $B' \in \mathscr{B}$. Moreover, if $\operatorname{Coker} \phi \in {}^{\perp}\mathscr{B}$, then such a preenvelope is called a special $\mathscr{B}$-preenvelope of M. Dually we have the definitions of $\mathscr{A}$-precover and special $\mathscr{A}$-precover. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is said to be hereditary if $\operatorname{Ext}_R^i(A, B) = 0$ for all $i \geq 1$ and $A \in \mathscr{A}$ and $B \in \mathscr{B}$. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called complete if every R-module M has a special $\mathscr{B}$-preenvelope and a special $\mathscr{A}$-precover.

Definition 2.1 ([13]). Let $(\mathscr{A}, \mathscr{B})$ be a cotorsion pair in R-Mod. Let X be a complex.

(1) X is called an $\mathscr{A}$ complex if it is exact and $Z_n(X) \in \mathscr{A}$ for all n.

(2) X is called a $\mathscr{B}$ complex if it is exact and $Z_n(X) \in \mathscr{B}$ for all n.

(3) X is called a dg-$\mathscr{A}$ complex if $X_n \in \mathscr{A}$ for each n, and $\operatorname{Hom}(X, B)$ is exact whenever B is a $\mathscr{B}$ complex.

(4) X is called a dg-$\mathscr{B}$ complex if $X_n \in \mathscr{B}$ for each n, and $\operatorname{Hom}(A, X)$ is exact whenever A is an $\mathscr{A}$ complex.

We denote the class of $\mathscr{A}$ complexes by $\tilde{\mathscr{A}}$ and the class of dg-$\mathscr{A}$ complexes by dg $\tilde{\mathscr{A}}$. Similarly, the class of $\mathscr{B}$ complexes are denoted by $\tilde{\mathscr{B}}$ and the class of dg-$\mathscr{B}$ complexes by dg $\tilde{\mathscr{B}}$.

In the following, we denote $\mathscr{P}$, $\mathscr{I}$, $\mathscr{F}$, $\mathscr{C}$ the class of projective, injective, flat and cotorsion modules respectively. Let $(\mathscr{A}, \mathscr{B}) = (\mathscr{F}, \mathscr{C})$ be the flat cotorsion pair in R-Mod. Then $\mathscr{A}$ (resp. $\mathscr{B}$) complex and dg-$\mathscr{A}$ (resp. dg-$\mathscr{B}$) complex are flat (resp. cotorsion) complex and dg-flat (resp. dg-cotorsion) complex. Also the induced cotorsion pair $(\tilde{\mathscr{F}}, dg\,\tilde{\mathscr{C}})$ and $(dg\,\tilde{\mathscr{F}}, \tilde{\mathscr{C}})$ are both complete and hereditary in $\mathscr{C}(R)$ by [13, Corollary 3.13, 4.18]. Similarly, let $(\mathscr{A}, \mathscr{B}) = (\mathscr{P}, R\text{-}Mod)$, then dg-$\mathscr{A}$ complex is dg-projective complex and let $(\mathscr{A}, \mathscr{B}) = (R\text{-}Mod, \mathscr{I})$, then dg-$\mathscr{B}$ complex is dg-injective.

The projective and injective dimension of an R-complex M are defined as

$$\operatorname{pd}_R M = \inf\{\sup P \mid P \xrightarrow{\simeq} M \text{ is a dg-projective resolution}\},$$
$$\operatorname{id}_R M = \inf\{-\inf I \mid M \xrightarrow{\simeq} I \text{ is a dg-injective resolution}\}.$$

Depth and width. Let $(R, \mathfrak{m}, k)$ be a local ring. Let M be an R-complex.
(1) The depth of M is defined as

$$\operatorname{depth}_R M = -\sup \mathrm{H}(\mathbf{R}\operatorname{Hom}_R(k, M)).$$

For every R-complex M one has

$$\operatorname{depth}_R M \geq -\sup \mathrm{H}(M).$$

If $\sup \mathrm{H}(M) = s < \infty$, then the equality holds if and only if $\mathfrak{m}$ is an associated prime ideal of the homology module $\mathrm{H}_s(M)$ (that is, $\mathfrak{m} = \mathrm{Ann}(x)$ for some $x \in \mathrm{H}_s(M)$).

(2) The width of M is defined as

$$\mathrm{width}_R\, M = \inf \mathrm{H}(k \otimes^{\mathrm{L}}_R M).$$

There is an obvious inequality

$$\mathrm{width}_R\, M \leq \inf \mathrm{H}(M),$$

and the equality holds if $\mathrm{H}(M)$ is bounded below with each item being a finitely generated module.

Definition 2.2 ([7]). Let M and N be R-complexes. We say that the derived depth formula holds for M and N if there is an equality

$$\mathrm{depth}_R(M \otimes^{\mathrm{L}}_R N) = \mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R.$$

By [11, Lemma 2.1] the derived depth formula holds for complexes M and N, if M has finite projective dimension and $\mathrm{H}(N)$ is bounded above.

3. Complete flat resolution and tate homology

In this section, R is an associative ring (not necessarily commutative noetherian). Right modules over R are treated as (left) modules over the opposite ring R°. We establish Tate homology with respect to complete flat resolutions.

Recall that a totally acyclic complex of flat R-modules is an exact complex of flat R-modules which remains exact after applying $I \otimes_R -$ for any injective R°-module I. An R-module G is called Gorenstein flat if there exists a totally acyclic complex T of flat R-modules such that $G \cong \mathrm{Ker}(T_{-1} \to T_{-2})$. By [14, Lemma 2.3.2], if R is a right coherent ring, an R-module M is Gorenstein flat if and only if $\mathrm{Ext}^i_R(M, B) = 0$ for any $i \geq 1$ and $B \in \mathscr{F} \cap \mathscr{C}$, and there exists a $\mathrm{Hom}_R(-, \mathscr{F} \cap \mathscr{C})$-exact exact sequence

$$0 \to M \to B^0 \to B^1 \to \cdots$$

with each $B^i \in \mathscr{F} \cap \mathscr{C}$.

Definition 3.1 ([14]). For an R-complex M, a complete flat resolution of M is a diagram

$$T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$$

of morphisms of complexes satisfying:

(1) $F \xrightarrow{\pi} C \xleftarrow{q} M$ is a flat-cotorsion resolution of M, that is, q is a special dg-cotorsion preenvelope and π is a special dg-flat precover;

(2) T is an exact complex with each entry in $\mathscr{F} \cap \mathscr{C}$ and $\mathrm{Hom}_R(-, \mathscr{F} \cap \mathscr{C})$-exact;

(3) $\tau : T \to F$ is a morphism such that $\tau_i = \mathrm{id}_{T_i}$ for all $i \gg 0$.

The Gorenstein flat dimension $\mathrm{Gfd}_R\, M$ is defined by:

$$\mathrm{Gfd}_R\, M = \inf\left\{ g \in \mathbf{Z} \;\middle|\; \begin{array}{c} T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M \text{ is a complete flat resolution} \\ \text{with } \tau_i : T_i \to F_i \text{ bijective for each } i \geq g \end{array} \right\}.$$

Lemma 3.2 ([14]). *Let R be a right coherent ring. For an R-complex M, the following are equivalent for each integer $n \in \mathbf{Z}$.*

(1) $\mathrm{Gfd}_R\, M \leq n$.

(2) *There exists a quasi-isomorphism $F \to M$ with F dg-flat such that $\sup \mathrm{H}(F) \leq n$ and $C_j(F)$ is Gorenstein flat for any integer $j \geq n$.*

(3) *For each flat-cotorsion resolution $F \xrightarrow{\pi} C \xleftarrow{q} M$ of M, there exists a complete flat resolution $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$ of M such that $\tau_i = \mathrm{id}_{T_i}$ for all $i \geq n$.*

Remark 3.3. By Lemma 3.2, the Gorenstein flat dimension defined here coincides with Iacob's definition in [17] over a right coherent ring.

Lemma 3.4. *Let $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$ and $T' \xrightarrow{\tau'} F' \xrightarrow{\pi'} C' \xleftarrow{q'} M'$ be complete flat resolutions. For every morphism $\alpha : M \to M'$ there exists a morphism $\overline{\alpha}$ such that the right hand square of the diagram*

$$\begin{array}{ccccccc} T & \xrightarrow{\tau} & F & \xrightarrow{\pi} & C & \xleftarrow{q} & M \\ \downarrow{\scriptstyle\hat{\alpha}} & & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\overline{\alpha}} & & \downarrow{\scriptstyle\alpha} \\ T' & \xrightarrow{\tau'} & F' & \xrightarrow{\pi'} & C' & \xleftarrow{q'} & M' \end{array}$$

commutes; for each choice of $\overline{\alpha}$, there exists a unique up to homotopy morphism $\tilde{\alpha}$, making the middle square commute up to homotopy; for each choice of $\tilde{\alpha}$ there exists a unique up to homotopy morphism $\hat{\alpha}$, making the left hand square commute up to homotopy. If one has $M = M'$ and α is the identity map, then $\overline{\alpha}$, $\tilde{\alpha}$ and $\hat{\alpha}$ are homotopy equivalences.

Proof. By [14, Lemma 2.2.4], for each morphism $\alpha : M \to M'$ there exists a morphism $\overline{\alpha}$ such that the right hand square of the diagram

$$\begin{array}{ccccc} F & \xrightarrow{\pi} & C & \xleftarrow{q} & M \\ \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\overline{\alpha}} & & \downarrow{\scriptstyle\alpha} \\ F' & \xrightarrow{\pi'} & C' & \xleftarrow{q'} & M' \end{array}$$

commutes; for each choice of $\overline{\alpha}$ there exists a unique up to homotopy morphism $\tilde{\alpha}$, making the left-hand square commute. If $\alpha = \mathrm{id}_M$, then $\overline{\alpha}$ and $\tilde{\alpha}$ are homotopy equivalences. We only prove the remaining conclusions.

Note that τ' can be factored as $T' \xrightarrow{\iota} T'' \xrightarrow{\beta} F'$, where ι is a homotopy equivalence and β is a morphism with each β_i a split epimorphism. Thus without loss of generality, we assume that each τ'_i is a split epimorphism.

Thus there exists an exact sequence $0 \to \mathrm{Ker}\,\tau' \to T' \xrightarrow{\tau'} F' \to 0$ with $\mathrm{Ker}\,\tau'$ bounded above and each item in $\mathscr{F} \cap \mathscr{C}$. Applying $\mathrm{Hom}_R(T,-)$ to this sequence, we have an exact sequence of complexes $0 \to \mathrm{Hom}_R(T, \mathrm{Ker}\,\tau') \to \mathrm{Hom}_R(T,T') \xrightarrow{\mathrm{Hom}_R(T,\tau')} \mathrm{Hom}_R(T,F') \to 0$. Note that $\mathrm{Hom}_R(T, \mathrm{Ker}\,\tau')$ is exact by [5, Lemma 2.5]. Then $\mathrm{Hom}_R(T,T') \xrightarrow{\mathrm{Hom}_R(T,\tau')} \mathrm{Hom}_R(T,F') \to 0$ is a surjective quasi-isomorphism. Hence there exists a unique up to homotopy morphism $\hat{\alpha}$ such that $\tilde{\alpha}\tau = \tau'\hat{\alpha}$ by [4, (1.1.1)]. The homotopy equivalence is easy to get. □

Definition 3.5. Let R be a left coherent ring and M an R°-complex with a complete flat resolution $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$. For an R-complex N, the Tate homology with respect to complete flat resolutions of M with coefficients in N is defined as

$$\overline{\mathrm{Tor}}_i^R(M,N) = \mathrm{H}_i(T \otimes_R N).$$

Remark 3.6. (1) It follows from Lemma 3.4 that the definition is independent of the choice of complete flat resolutions.

(2) Let R be a left coherent ring and M an R°-complex of finite Gorenstein flat dimension. If $\mathrm{fd}_{R^\circ} M < \infty$, then $\overline{\mathrm{Tor}}_i^R(M,N) = 0$ for all $i \in \mathbf{Z}$.

In fact, $0 \to F \xrightarrow{\pi} C \xleftarrow{q} M$ is a complete flat resolution of M, so one has $\overline{\mathrm{Tor}}_i^R(M,N) = 0$ for all $i \in \mathbf{Z}$.

For an R-complex M of finite Gorenstein projective dimension with a chosen complete projective resolution $U \xrightarrow{\tau} P \xrightarrow{\pi} M$ and an arbitrary R-complex N, Veliche defined in [20] the Tate cohomology group by

$$\widehat{\mathrm{Ext}}_R^i(M,N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(U,N)).$$

Christensen and Jorgensen [6] defined the Tate homology of an R°-complex M with coefficients in an arbitrary R-complex N as

$$\widehat{\mathrm{Tor}}_i^R(M,N) = \mathrm{H}_i(U \otimes_R N).$$

In the following, we compare these two different definitions of Tate homology.

Lemma 3.7. *Let T be a $\mathrm{Hom}_R(-,\mathscr{P})$-exact exact R-complex of projective modules. Then T is $\mathrm{Hom}_R(-,Q)$-exact for any R-module Q with finite projective dimension.*

Proof. This follows by induction on the projective dimension of Q. □

Theorem 3.8. *Let R be a left coherent ring over which each flat R°-module has finite projective dimension and let M be an R°-complex of finite Gorenstein projective dimension. Then for any bounded above R-complex N, one has*

$$\overline{\mathrm{Tor}}_i^R(M,N) \cong \widehat{\mathrm{Tor}}_i^R(M,N).$$

Proof. Since each flat R°-module has finite projective dimension, $\mathrm{Gfd}_{R^\circ} M \leq \mathrm{Gpd}_{R^\circ} M < \infty$ by Remark 3.3 and [17, Proposition 3.6]. Choose a complete projective resolution

$$T \xrightarrow{\tau} P \xrightarrow{\pi} M$$

and a complete flat resolution

$$T' \xrightarrow{\tau'} F' \xrightarrow{\pi'} C' \xleftarrow{q'} M$$

of M. To prove the desired isomorphism we firstly show that there exists the following commutative diagram

$$\begin{array}{ccccccc} T & \xrightarrow{\tau} & P & \xrightarrow{\pi} & M & \xleftarrow{=} & M \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle q'}\downarrow & & {\scriptstyle =}\downarrow \\ T' & \xrightarrow{\tau'} & F' & \xrightarrow{\pi'} & C' & \xleftarrow{q'} & M'. \end{array}$$

Since P is dg-projective and π' a quasi-isomorphism, there is a morphism of complexes $\alpha : P \to F'$ such that $\pi'\alpha \simeq q'\pi$ by [19, Proposition 1.4]. Note that α is also a quasi-isomorphism, because $q'\pi$ is so. Next we prove the existence of β. Assume that τ_i and τ_j' are bijective for $i \geq s$ and $j \geq t$ respectively. Set $n = \max\{s, t\}$. If $i \geq n$, then we set $\beta_i = \alpha_i$. For $i < n$, consider the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & T_{n+1} & \longrightarrow & T_n & \longrightarrow & T_{n-1} & \longrightarrow & \cdots \\ & & {\scriptstyle\alpha_{n+1}}\downarrow & & {\scriptstyle\alpha_n}\downarrow & & {\scriptstyle\beta_{n-1}}\downarrow & & \\ \cdots & \longrightarrow & T'_{n+1} & \longrightarrow & T'_n & \longrightarrow & T'_{n-1} & \longrightarrow & \cdots. \end{array}$$

The existence of β_i for $i < n$ is by Lemma 3.7 because that the upper row is $\mathrm{Hom}_R(-, \mathscr{P})$ exact and $\mathrm{pd}_{R^\circ} T_i' < \infty$ for all $i \in \mathbf{Z}$. Hence $\beta = \{\beta_i\}_{i \in \mathbf{Z}}$ is the desired morphism.

In the following we prove $T \otimes_R N \to T' \otimes_R N$ is a quasi-isomorphism. Consider the short exact sequences $0 \to T_{\leq n-1} \to T \to T_{\geq n} \to 0$ and $0 \to T'_{\leq n-1} \to T' \to T'_{\geq n} \to 0$. By above $T_{\geq n} = T'_{\geq n}$. So $T_{\geq n} \otimes_R N = T'_{\geq n} \otimes_R N$. We denote $\tilde{\beta} : T_{\leq n-1} \to T'_{\leq n-1}$. Then $C(\tilde{\beta})$ is bounded above and degreewise flat. We show $\tilde{\beta} \otimes_R N : T_{\leq n-1} \otimes_R N \to T'_{\leq n-1} \otimes_R N$ is a quasi-isomorphism, that is, $C(\tilde{\beta}) \otimes_R N$ is exact. Assume that $\gamma : N \xrightarrow{\simeq} I$ is a dg-injective resolution of N. Then $C(\gamma)$ is exact. $C(\tilde{\beta}) \otimes_R C(\gamma)$ is exact by [5, Lemma 2.13]. Hence $C(\tilde{\beta}) \otimes_R N \simeq C(\tilde{\beta}) \otimes_R I$. We only need to show $C(\tilde{\beta}) \otimes_R I \cong C(\tilde{\beta} \otimes_R I)$ is exact. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\leq n-1} & \longrightarrow & T & \longrightarrow & T_{\geq n} & \longrightarrow & 0 \\ & & {\scriptstyle\tilde{\beta}}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle \mathrm{id}_{T_{\geq n}}}\downarrow & & \\ 0 & \longrightarrow & T'_{\leq n-1} & \longrightarrow & T' & \longrightarrow & T'_{\geq n} & \longrightarrow & 0. \end{array}$$

We have an exact sequence

$$0 \to C(\tilde{\beta}) \to C(\beta) \to C(\mathrm{id}_{T_{\geq n}}) \to 0.$$

Since $C(\mathrm{id}_{T_{\geq n}}) \otimes_R I_i$ and $C(\beta) \otimes_R I_i$ are exact by assumption, $C(\tilde{\beta}) \otimes_R I_i$ is exact. So $C(\tilde{\beta}) \otimes_R I$ is exact by [5, Lemma 2.13]. Therefore $T \otimes_R N \to T' \otimes_R N$ is a quasi-isomorphism by five lemma. □

In classical homology algebra, for an R°-module M and R-module N,

$$(\mathrm{Tor}_i^R(M, N))^+ \cong \mathrm{Ext}_R^i(N, M^+)$$

for all $i \geq 0$. But we find that Tate cohomology and Tate homology in Christensen and Jorgensen [6]'s sense don't have such isomorphisms when the ring is coherent. In the following, we discuss the relationship of Tate cohomology and Tate homology based on complete flat resolutions.

Lemma 3.9. *Let R be a left coherent ring and M an R°-complex with a complete flat resolution $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$. Then $M^+ \to F^+ \to T^+$ is a complete injective resolution of $M^+ = \mathrm{Hom}_{\mathbf{Z}}(M, Q/\mathbf{Z})$.*

Proof. Let $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$ be a complete flat resolution of M. Applying the functor $\mathrm{Hom}_{\mathbf{Z}}(-, Q/\mathbf{Z})$ to $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$, we have $M^+ \xleftarrow{q^+} C^+ \xrightarrow{\pi^+} F^+ \xrightarrow{\tau^+} T^+$, where F^+ is dg-injective and π^+, q^+ are quasi-isomorphisms. Hence there exists a quasi-isomorphism $\alpha : M^+ \to F^+$ such that $\pi^+ \simeq \alpha q^+$ by [19, Proposition 1.5]. We show $M^+ \to F^+ \to T^+$ is a complete injective resolution of M^+. Assume that $T_i = F_i$ for all $i \gg 0$. Then $T_i^+ = F_i^+$ for all $i \ll 0$. Note that T^+ is degreewise injective. It remains to show that $\mathrm{Hom}_R(E, T^+)$ is exact for arbitrary injective R-module E. Indeed, we have isomorphism $(T \otimes_R E)^+ \cong \mathrm{Hom}_R(T, E^+)$. Since E is an injective R-module, $E^+ \in \mathscr{F} \cap \mathscr{C}$. Thus $\mathrm{Hom}_R(T, E^+)$ is exact by assumption, and hence $\mathrm{Hom}_R(E, T^+) \cong (T \otimes_R E)^+$ is exact. □

Theorem 3.10. *Let R be a left coherent ring and M a bounded above R°-complex of finite Gorenstein flat dimension. Then $(\overline{\mathrm{Tor}}_i^R(M, N))^+ \cong \widehat{\mathrm{Ext}}_R^i(N, M^+)$ for all $i \in \mathbf{Z}$ and any bounded above R-complex N.*

Proof. By Lemma 3.2, we have a complete flat resolution $T \xrightarrow{\tau} F \xrightarrow{\pi} C \xleftarrow{q} M$. Thus Lemma 3.9 implies that $M^+ \to F^+ \to T^+$ is a complete injective resolution of M^+. Then we have the following sequence of equalities

$$\begin{aligned}(\overline{\mathrm{Tor}}_i^R(M, N))^+ &= (\mathrm{H}_i(T \otimes_R N))^+ \\ &\cong \mathrm{H}_{-i}((T \otimes_R N)^+) \\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(N, T^+)) \\ &\cong \widehat{\mathrm{Ext}}_R^i(N, M^+),\end{aligned}$$

where the last isomorphism is by [6, Definition 5.5]. □

In the rest part of this section we study some properties of Tate homology.

PROPOSITION 3.11. *Let R be a left coherent ring and M an R°-complex of finite Gorenstein flat dimension. For every bounded above R-complex N of finite flat dimension, one has $\overline{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\in\mathbf{Z}$.*

Proof. Let $T\xrightarrow{\tau}F\xrightarrow{\pi}C\xleftarrow{q}M$ be a complete flat resolution of M over R. For every dg-flat resolution $\pi':F'\to N$ over R, applying the functor $T\otimes_R-$ to the exact sequence $0\to N\to\mathrm{Cone}(\pi')\to\Sigma F'\to 0$, we get a short exact sequence in homology and further yields an isomorphism $\mathrm{H}(T\otimes_R N)\cong\mathrm{H}(T\otimes_R\mathrm{Cone}(\pi'))$, as one has $\mathrm{H}(T\otimes_R F')=0$ because F' is dg-flat. If N is bounded above and of finite flat dimension, then we can assume that F' and therefore, $\mathrm{Cone}(\pi')$ is bounded above, and then $\mathrm{H}(T\otimes_R\mathrm{Cone}(\pi'))=0$ by [5, Lemma 2.13]. Thus $\overline{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\in\mathbf{Z}$. □

PROPOSITION 3.12. *Let R be a left coherent ring and M an R°-complex of finite Gorenstein flat dimension. For every exact sequence $0\to N'\to N\to N''\to 0$ of R-complexes, there is an exact sequence of $\mathbf{Z}$-modules*

$$\cdots\to\overline{\mathrm{Tor}}_{i+1}^R(M,N'')\to\overline{\mathrm{Tor}}_i^R(M,N')\to\overline{\mathrm{Tor}}_i^R(M,N)\to\overline{\mathrm{Tor}}_i^R(M,N'')\to\cdots.$$

Proof. Let $T\xrightarrow{\tau}F\xrightarrow{\pi}C\xleftarrow{q}M$ be a complete flat resolution of M. The sequence

$$0\to T\otimes_R N'\to T\otimes_R N\to T\otimes_R N''\to 0$$

is exact because T is a complex of flat R°-modules. The associated exact sequence in homology is the desired one. □

The next result establishes the dimension shifting for Tate homology.

LEMMA 3.13. *Let R be a left coherent ring and M an R°-complex of finite Gorenstein flat dimension and let N be an R-complex.*

(1) *For every complete flat resolution $T\to F\to C\leftarrow M$ and for every $m\in\mathbf{Z}$, there are isomorphisms*

$$\overline{\mathrm{Tor}}_i^R(M,N)\cong\overline{\mathrm{Tor}}_{i-m}^R(C_m(T),N)\quad\text{for all } i\in\mathbf{Z}.$$

(2) *For every dg-flat resolution $F\xrightarrow{\simeq}N$ and every integer $n\geq\sup N$, one has isomorphisms*

$$\overline{\mathrm{Tor}}_i^R(M,N)\cong\overline{\mathrm{Tor}}_{i-n}^R(M,C_n(F))\quad\text{for all } i\in\mathbf{Z}.$$

Proof. (1) For every $m\in\mathbf{Z}$, $\Sigma^{-m}T\to\Sigma^{-m}F_{\geq m}\xrightarrow{\beta}\Sigma^{-m+1}F_{\leq m-1}\xleftarrow{\alpha}C_m(T)$ is a complete flat resolution of $C_m(T)$. Indeed, since $\Sigma^{-m}F_{\leq m-1}$ is a bounded above complex of cotorsion modules, it is dg-cotorsion by [13, Lemma 3.4 (2)], $\Sigma^{-m}F_{\geq m}$ is a dg-flat complex by [13, Lemma 3.4 (1)] since it is a bounded below complex of flat modules. Also it is easy to check that α and β are quasi-isomorphisms. Hence one has

$$\begin{aligned}\overline{\mathrm{Tor}}^R_{i-m}(C_m(T),N) &= \mathrm{H}_{i-m}((\Sigma^{-m}T)\otimes_R N)\\ &= \mathrm{H}_{i-m}(\Sigma^{-m}(T\otimes_R N))\\ &\cong \mathrm{H}_i(T\otimes_R N)\\ &= \overline{\mathrm{Tor}}^R_i(M,N).\end{aligned}$$

(2) We may assume that N is bounded above. For every dg-flat resolution $F \xrightarrow{\simeq} N$ and $n \geq \sup N$ there is a quasi-isomorphism $\alpha : F_{\subset n} \to N$. Since the exact complex $\mathrm{Cone}(\alpha)$ is bounded above, $T \otimes_R \mathrm{Cone}(\alpha)$ is exact by [5, Lemma 2.13]. By Proposition 3.12, the exact sequence $0 \to N \to \mathrm{Cone}(\alpha) \to \Sigma F_{\subset n} \to 0$ yields isomorphisms

$$\overline{\mathrm{Tor}}^R_i(M,N) \cong \overline{\mathrm{Tor}}^R_i(M,F_{\subset n}) \quad \text{for all } i \in \mathbf{Z}.$$

Consider the exact sequence of R-complexes $0 \to F_{\leq n-1} \to F_{\subset n} \to \Sigma^n C_n(F) \to 0$. Then $F_{\leq n-1}$ has finite flat dimension. Indeed, since $0 \to F_{\leq n-1} \to F \to F_{\geq n} \to 0$ is exact, the complexes F and $F_{\geq n}$ are dg-flat, so $F_{\leq n-1}$ is dg-flat. Note that $F_{\leq n-1}$ is bounded above. Then Proposition 3.11 and 3.12 imply that

$$\overline{\mathrm{Tor}}^R_i(M,F_{\subset n}) \cong \overline{\mathrm{Tor}}^R_i(M,\Sigma^n C_n(F)) \quad \text{for all } i \in \mathbf{Z}.$$

The desired isomorphisms follows from these two assertions. □

Next we use pinched tensor product complexes introduced in [6] to investigate the balancedness for Tate homology based on complete flat resolutions.

(Construction) Let R be a left coherent ring and T an R°-complex and A an R-complex. The pinched tensor product of T and A is defined as follows:

$$(T \otimes_R^{\bowtie} A)_n = \begin{cases} (T_{\geq 0} \otimes_R A_{\geq 0})_n, & n \geq 0,\\ (T_{\leq -1} \otimes_R \Sigma(A_{\leq -1}))_n, & n \leq -1.\end{cases}$$

The differential on $T \otimes_R^{\bowtie} A$ is defined by

$$\partial_n^{T\otimes_R^{\bowtie}A} = \begin{cases} \partial_n^{T_{\geq 0}\otimes_R A_{\geq 0}}, & n \geq 1,\\ \partial_0^T \otimes_R (\sigma\partial_0^A), & n = 0,\\ \partial_n^{T_{\leq -1}\otimes_R \Sigma(A_{\leq -1})}, & n \leq -1.\end{cases}$$

where σ denotes the canonical map $A \to \Sigma A$.

There are also equalities

$$\begin{aligned}(T \otimes_R^{\bowtie} A)_{\geq 0} &= T_{\geq 0} \otimes_R A_{\geq 0},\\ (T \otimes_R^{\bowtie} A)_{\leq -1} &= T_{\leq -1} \otimes_R \Sigma(A_{\leq -1}).\end{aligned}$$

LEMMA 3.14. *Let R be a left coherent ring and M an R°-complex with a complete flat resolution $T \to F \to C \leftarrow M$ and let A be an exact R-complex and set $N = C_0(A)$. For every $i \in \mathbf{Z}$ there is an isomorphism of $\mathbf{Z}$-modules*

$$\mathrm{H}_i(T \otimes_R^{\bowtie} A) \cong \overline{\mathrm{Tor}}_i^R(M, N).$$

Proof. Note that $\overline{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N)$ by definition, so we must show $\mathrm{H}(T \otimes_R N) \cong \mathrm{H}(T \otimes_R^{\bowtie} A)$. The quasi-isomorphisms

$$\pi : A_{\geq 0} \xrightarrow{\simeq} N \quad \text{and} \quad \gamma : N \xrightarrow{\simeq} \Sigma(A_{\leq -1}) \quad \text{with } \gamma_0\pi_0 = \sigma\partial_0^A,$$

induce quasi-isomorphisms

$$(T \otimes_R^{\bowtie} A)_{\geq 0} \xrightarrow{\simeq} T_{\geq 0} \otimes_R N \quad \text{and} \quad T_{\leq -1} \otimes_R N \xrightarrow{\simeq} (T \otimes_R^{\bowtie} A)_{\leq -1}.$$

The first quasi-isomorphism is because $T_{\geq 0}$ is *dg*-flat and the second is by [5, Proposition 2.14], because N and $\Sigma(A_{\leq -1})$ is bounded above and T is degreewise flat. It follows that there are isomorphisms $\mathrm{H}_i(T \otimes_R N) \cong \mathrm{H}_i(T \otimes_R^{\bowtie} A)$ for all $i \in \mathbf{Z} \backslash \{0, 1\}$. The isomorphisms in the remaining two degrees are by the proof of [6, Theorem 3.5]. □

PROPOSITION 3.15. *Let R be a left and right coherent ring and M an R°-complex and N an R-complex, both of which are bounded above and of finite Gorenstein flat dimension. For every $i \in \mathbf{Z}$ there is an isomorphism of $\mathbf{Z}$-modules*

$$\overline{\mathrm{Tor}}_i^R(M, N) \cong \overline{\mathrm{Tor}}_i^{R^\circ}(N, M).$$

Proof. Choose complete flat resolutions $T \to F \to C \leftarrow M$ and $T' \to F' \to C' \leftarrow N$. Set $m = \max\{\sup M, \mathrm{Gfd}_{R^\circ} M\}$ and $n = \max\{\sup N, \mathrm{Gfd}_R N\}$. Then the modules $C_m(F) \cong C_m(T)$ and $C_n(F') \cong C_n(T')$ are Gorenstein flat modules with complete flat resolutions

$$\Sigma^{-m} T \to \Sigma^{-m} F_{\geq m} \to \Sigma^{-m+1} F_{\leq m-1} \leftarrow C_m(T),$$
$$\Sigma^{-n} T' \to \Sigma^{-n} F'_{\geq n} \to \Sigma^{-n+1} F'_{\leq n-1} \leftarrow C_n(T').$$

Hence we have the following isomorphisms by Lemma 3.13, 3.14 and [6, Proposition 3.6],

$$\begin{aligned}
\overline{\mathrm{Tor}}_i^R(M, N) &\cong \overline{\mathrm{Tor}}_{i-m-n}^R(C_m(F), C_n(F')) \\
&\cong \mathrm{H}_{i-m-n}((\Sigma^{-m} T) \otimes_R^{\bowtie} (\Sigma^{-n} T')) \\
&\cong \mathrm{H}_{i-n-m}((\Sigma^{-n} T') \otimes_{R^\circ}^{\bowtie} (\Sigma^{-m} T)) \\
&\cong \overline{\mathrm{Tor}}_{i-n-m}^{R^\circ}(C_n(F'), C_m(F)) \\
&\cong \overline{\mathrm{Tor}}_i^{R^\circ}(N, M).
\end{aligned}$$

This completes the proof. □

4. Applications

In this section, R always denotes a commutative noetherian local ring.

The depth formula and width formula are always studied independently. Using the relationship of Tate cohomology and Tate homology we have established in Section 3, we can simplify this study and also provide a brief proof to a special case of the main result of [7].

LEMMA 4.1 ([7, Proposition 6.4]). *Let N be an R-complex of finite Gorenstein injective dimension and M a bounded above R-complex. If one has $\widehat{\mathrm{Ext}}^i_R(M,N)=0$ for all $i\in\mathbf{Z}$, then the next equality holds*

$$\mathrm{width}_R(\mathbf{R}\,\mathrm{Hom}_R(M,N)) = \mathrm{depth}_R\, M + \mathrm{width}_R\, N - \mathrm{depth}\, R.$$

PROPOSITION 4.2 ([7, Theorem 2.3]). *Let M be an R-complex of finite Gorenstein projective dimension and N a bounded above R-complex. If one has $\widehat{\mathrm{Tor}}_i^R(M,N)=0$ for all $i\in\mathbf{Z}$, then the derived depth formula holds for M and N. That is, one has*

$$\mathrm{depth}_R(M\otimes^{\mathrm{L}}_R N) = \mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R.$$

Proof. By [9, Corollary 3.5], $\mathrm{Gfd}_R\, M<\infty$. Let $E(k)$ be the injective hull of the residue field. Since M is an R-complex of finite Gorenstein flat dimension, $M^\vee=\mathrm{Hom}_R(M,E(k))$ has finite Gorenstein injective dimension by [21, Corollary 3.2]. Note that for an injective module E, we have $\mathrm{Ext}^1_R(F,E^\vee)\cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(F,E),E(k))=0$ for any flat R-module F by [8, Theorem 3.2.1]. So $E^\vee$ is cotorsion. Thus $E^\vee\in\mathscr{F}\cap\mathscr{C}$. So for a complete flat resolution $T\to F\to C\leftarrow M$ of M (the existence of complete flat resolutions is ensured by Lemma 3.2), $M^\vee\to F^\vee\to T^\vee$ is a complete injective resolution of $M^\vee$ by analogy with the proof of Lemma 3.9. Then by Theorem 3.8 we have

$$\begin{aligned}(\widehat{\mathrm{Tor}}_i^R(M,N))^\vee &= (\mathrm{H}_i(T\otimes_R N))^\vee\\ &\cong \mathrm{H}_{-i}((T\otimes_R N)^\vee)\\ &\cong \mathrm{H}_{-i}(\mathrm{Hom}_R(N,T^\vee))\\ &\cong \widehat{\mathrm{Ext}}^i_R(N,M^\vee).\end{aligned}$$

Hence Lemma 4.1 implies that

$$\begin{aligned}\mathrm{depth}_R(M\otimes^{\mathrm{L}}_R N) &= \mathrm{width}_R(M\otimes^{\mathrm{L}}_R N)^\vee\\ &= \mathrm{width}_R(\mathbf{R}\,\mathrm{Hom}_R(N,M^\vee))\\ &= \mathrm{depth}_R\, N + \mathrm{width}_R\, M^\vee - \mathrm{depth}\, R\\ &= \mathrm{depth}_R\, M + \mathrm{depth}_R\, N - \mathrm{depth}\, R.\end{aligned}$$

The first equality follows from the similar proof of [22, Lemma 2.2] and the second is by adjoint isomorphism. □

Remark 4.3. Let M and N be R-complexes. Set $(-)^{\vee} = \mathrm{Hom}_R(-, E(k))$. Then

$$\mathrm{depth}_R(M \otimes^{\mathrm{L}}_R N) = \mathrm{width}_R(\mathbf{R}\,\mathrm{Hom}_R(N, M^{\vee})) = \mathrm{width}_R(\mathbf{R}\,\mathrm{Hom}_R(M, N^{\vee})).$$

Combined with Lemma 3.9, the depth formula and the width formula of [7] can only be discussed one of them.

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