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NILPOTENT ADMISSIBLE INDIGENOUS BUNDLES VIA CARTIER OPERATORS IN CHARACTERISTIC THREE

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Abstract

In the present paper, we study the *p-adic Teichmüller theory* in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*. By means of this characterization, we prove that, for every nilpotent *ordinary* indigenous bundle over a projective smooth curve in characteristic three, there exists a connected finite étale covering of the curve on which the indigenous bundle is *not ordinary*. We also prove that every projective smooth curve *of genus two* in characteristic three is *hyperbolically ordinary*. These two applications yield *negative*, *partial positive* answers to *basic questions* in the *p-adic Teichmüller theory*, respectively.

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Introduction

In the present paper, we study the *p-adic Teichmüller theory* established by S. Mochizuki [cf. [6], [7]] in the case where $p = 3$. In particular, we discuss *nilpotent admissible/ordinary* indigenous bundles over a projective smooth curve in characteristic three. In the Introduction, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbf{F}_p$], and $f : X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F : X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S and $\Phi : X \to X^F$ for the relative Frobenius morphism over S. We use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

First, let us recall the notion of an *indigenous bundle* and some properties on an indigenous bundle. We shall say that a pair

$$(\pi : P \to X, \nabla_P)$$

consisting of a $\mathbf{P}^1$-bundle $\pi : P \to X$ over X and a connection ∇_P on P relative to X/S is an *indigenous bundle* over X/S if there exists a [uniquely determined—cf. [6], Chapter I, Proposition 2.4, p. 1004] section [i.e., the *Hodge section*] $\sigma : X \to P$ of $\pi : P \to X$ such that the Kodaira-Spencer homomorphism $\sigma^*\omega_{P/X} \to \omega_{X/S}$ at σ relative to ∇_P [i.e., the homomorphism obtained by differentiating σ by means of ∇_P] is an isomorphism [cf. [6], Chapter I, Definition 2.2, pp. 1002–1003]. The notion of an indigenous bundle was introduced and studied by R. C. Gunning [cf. [2], §2] and enables one to understand the theory of uniformization of [algebraic] Riemann surfaces in a somewhat more algebraic setting.

Let $(\pi : P \to X, \nabla_P)$ be an indigenous bundle over X/S. Then the connection ∇_P on P determines a *horizontal* homomorphism [i.e., the *p-curvature*]

$$\mathscr{P} : \Phi^*\tau_{X^F/S} \to \mathscr{A}d(P) \stackrel{\text{def}}{=} \pi_*\tau_{P/X}.$$

We shall say that the indigenous bundle $(\pi : P \to X, \nabla_P)$ is *nilpotent* (respectively, *admissible*; *dormant*) if the square of $\mathscr{P}$ is zero (respectively, the zero locus of $\mathscr{P}$ is empty; $\mathscr{P} = 0$) [cf. [6], Chapter II, Definition 2.4, p. 1030 (respectively, [6], Chapter II, Definition 2.4, p. 1030; [7], Chapter II, Definition 1.1, p. 127)]. Moreover, we shall refer to the composite of the p-curvature $\mathscr{P}$ and the surjection $\mathscr{A}d(P) \twoheadrightarrow \tau_{X/S}$ determined by the Hodge section of $(\pi : P \to X, \nabla_P)$ as the *square Hasse invariant* of $(\pi : P \to X, \nabla_P)$ [cf. [6], Chapter II, Proposition 2.6, (1), p. 1032]. Then, by means of this *square Hasse invariant*, one may define the *Frobenius on* $\mathbf{R}^1 f_*\tau_{X/S}$ *induced by* $(\pi : P \to X, \nabla_P)$ [cf. the discussion following [6], Chapter II, Lemma 2.11, pp. 1036–1037]. We shall say that the indigenous bundle $(\pi : P \to X, \nabla_P)$ is *ordinary* if the Frobenius on $\mathbf{R}^1 f_*\tau_{X/S}$ induced by $(\pi : P \to X, \nabla_P)$ is an isomorphism [cf. [6], Chapter II, Definition 3.1, p. 1044]. A *nilpotent admissible/ordinary* indigenous bundle plays a central role in the "*classical*" *p-adic Teichmüller theory*, i.e., the *p-adic Teichmüller theory* discussed in [not [7] but] [6].

First, we verify the following *uniqueness* of a *dormant* indigenous bundle in characteristic three [cf. Theorem 2.1, Corollary 2.4]:

Theorem A. *In the notation introduced at the beginning of the Introduction, suppose that* $p = 3$. *Then there exists a* **unique dormant** *indigenous bundle over* X/S. *In particular, there exists a natural* **bijection** *between*

- $H^0(S, f_*\omega_{X/S}^{\otimes 2}) = H^0(X, \omega_{X/S}^{\otimes 2})$ *and*
- *the set of isomorphism classes of indigenous bundles over* X/S

such that, for $\theta \in H^0(S, f_*\omega_{X/S}^{\otimes 2})$, *the* **dormant locus** *in* S *of the indigenous bundle over* X/S *corresponding to* θ **coincides** *with the zero locus in* S *of* θ.

If an indigenous bundle $(\pi : P \to X, \nabla_P)$ over X/S is *nilpotent admissible*, then there exist an invertible sheaf $\mathscr{H}$ on X and a global section χ of $\mathscr{H}$ such that $\mathscr{H}^{\otimes 2} \cong \mathscr{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})$, and, moreover, the square of χ coincides with the square Hasse invariant of $(\pi : P \to X, \nabla_P)$ [cf. [6], Chapter II, Proposition 2.6, (3), p. 1032]. We shall refer to χ as the *Hasse invariant* of $(\pi : P \to X, \nabla_P)$ [cf. [6], Chapter II, Proposition 2.6, (3), p. 1032] and to the zero locus of the Hasse invariant as the *supersingular divisor* of $(\pi : P \to X, \nabla_P)$ [cf. [6], Chapter II, Proposition 2.6, (3), p. 1032]. The *supersingular divisor* is an important invariant of a nilpotent admissible indigenous bundle; for instance, if S is *reduced*, then the isomorphism class of a nilpotent admissible indigenous bundle over X/S is *completely determined* by the *supersingular divisor* [cf. [6], Chapter II, Proposition 2.6, (4), p. 1032]. The main result of the present paper is a characterization of the *supersingular divisors* of nilpotent admissible/ordinary indigenous bundles in characteristic three by means of various *Cartier operators*.

In order to present the main result of the present paper, let us recall some notions related to the *Cartier operator*. Let $(\mathscr{L}, \Theta)$ be a square-trivialized invertible sheaf on X, i.e., a pair consisting of an invertible sheaf $\mathscr{L}$ on X and a trivialization Θ of the square of $\mathscr{L}$ [cf. Definition A.3]. Then the [usual] Cartier operator $\Phi_*\omega_{X/S} \to \omega_{X^F/S}$, together with the trivialization Θ, determines a homomorphism of $\mathcal{O}_S$-modules

$$C_{(\mathscr{L},\Theta)} : f_*(\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

—where we write $\mathscr{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathscr{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S. We shall refer to this homomorphism as the *Cartier operator associated to* $(\mathscr{L}, \Theta)$ [cf. Definition A.4]. On the other hand, the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism. We shall say that a global section u of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $(\mathscr{L}, \Theta)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{(\mathscr{L},\Theta)}(u) = -u^F$ [cf. Definition A.8, (i)].

A part of the main result of the present paper is as follows [cf. Theorem 5.2, (ii)]:

THEOREM B. *In the notation introduced at the beginning of the Introduction, suppose that* $p = 3$. *Let* D *be a relative effective Cartier divisor of* X/S. *Then it holds that* D *is the* **supersingular divisor** *of a* **nilpotent admissible** (*respectively*, **nilpotent ordinary**) *indigenous bundle over* X/S *if and only if* D *is* **of CE-type** (*respectively*, **of CEO-type**) [*cf. Definition* 5.1, (iii)], *i.e.*, *there exist an invertible sheaf* $\mathscr{L}$ *on* X, *a trivialization* Θ *of the square of* $\mathscr{L}$, *and a global section* χ *of* $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ *such that the following two* (*respectively*, *three*) *conditions* (1), (2) (*respectively*, (1), (2), (3)) *are satisfied*:

(1) *The divisor* D *is* **étale** *over* S *and coincides with the zero locus of* $\chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$.

(2) *The global section* $\chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is a* **normalized Cartier eigenform** *associated to* $(\mathscr{L}, \Theta)$.

(3) *The invertible sheaf* $\mathscr{L}$ *is* **parabolically ordinary** [*cf. Definition* A.7], *i.e.*, *the Cartier operator associated to* $(\mathscr{L}, \Theta)$ *is* **injective** *at every point of* S, *or*, *equivalently* [*cf. Proposition* A.6], *one of the following two conditions is satisfied*:

- $\mathscr{L}$ *is* **of relative order one** [*cf. Definition* A.2], *and*, *moreover*, X *is* **parabolically ordinary** [*cf. Definition* A.5, (i)].
- $\mathscr{L}$ *is* **of relative order two** [*cf. Definition* A.2], *and*, *moreover*, *the connected finite étale double covering of* X *which trivializes* $\mathscr{L}$ [*determined by* Θ] *is* **parabolically new-ordinary** [*cf. Definition* A.5, (ii)].

Here, let us recall the following two *basic questions* in the *p-adic Teichmüller theory* discussed in [7], Introduction, §2.1 [cf. [7], Introduction, §2.1, (1), (2), p. 72]:

(1) Is every pointed stable curve [of type (g, r), where $2g - 2 + r > 0$] *hyperbolically ordinary*? That is to say, does every pointed stable curve [of type (g, r), where $2g - 2 + r > 0$] over S admit, étale locally on S, a *nilpotent ordinary* indigenous bundle?

(2) Let P be a *nilpotent ordinary* indigenous bundle over a pointed stable curve X [of type (g, r), where $2g - 2 + r > 0$] and $Y \to X$ a connected finite [log] étale covering of X. Then is the pull-back of P to Y still *ordinary*?

As a corollary of Theorem B, we obtain the following theorem, which yields a *negative answer* to the above basic question (2) [cf. Corollary 5.4]:

THEOREM C. *Let* X *be a projective smooth curve of genus* ≥ 2 *over an algebraically closed field* k *of characteristic* 3. *Then*, *for every* **nilpotent ordinary** *indigenous bundle* P *over* X/k, *there exists a connected finite étale covering* $Y \to X$ *of* X *such that the* [*necessarily* **nilpotent admissible**] *indigenous bundle* $(Y \to X)^* P$ *over* Y/k *is* **not ordinary**.

In §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1]. Moreover, we prove the following theorem, which yields a *partial positive answer* to the above basic question (1) [cf. Corollary 6.6, Remark 6.6.1]:

THEOREM D. *Every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is* **hyperbolically ordinary** [*cf.* [6], *Chapter* II, *Definition* 3.3, *p.* 1044].

1. Construction of a dormant indigenous bundle

In the present §1, we construct a *dormant indigenous bundle* over a projective smooth curve of genus ≥ 2 of characteristic 3 [cf. Proposition 1.1 below]. In the present §1, let $g \geq 2$ be an integer, S a connected noetherian scheme *of characteristic* 3 [i.e., over $\mathbf{F}_3$], and $f : X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g over S. Write $f^F : X^F \to S$ for the projective smooth curve over S obtained by base-changing $X \to S$ via the absolute Frobenius morphism of S, $\Phi : X \to X^F$ for the relative Frobenius morphism over S, $\mathscr{I} \subseteq \mathcal{O}_{X \times_S X}$ for the ideal of $\mathcal{O}_{X \times_S X}$ which defines the diagonal morphism with respect to X/S, and $X_{(n)} \subseteq X \times_S X$ for the closed subscheme of $X \times_S X$ defined by the ideal $\mathscr{I}^{n+1} \subseteq \mathcal{O}_{X \times_S X}$ [where n is a nonnegative integer]. In particular, it follows that $\mathscr{I}/\mathscr{I}^2 = \omega_{X/S}$ (respectively, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_X) = \tau_{X/S}$), where we use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

We shall write

$$\mathscr{B}_\circ \stackrel{\text{def}}{=} \operatorname{Coker}(\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X)$$

for the $\mathcal{O}_{X^F}$-module obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$ and

$$\mathscr{E}_\circ \stackrel{\text{def}}{=} \Phi^*\mathscr{B}_\circ.$$

Since the homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$ admits a natural splitting after pulling back via Φ, which thus determines a natural isomorphism of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \oplus \mathscr{E}_\circ,$$

and Φ is *finite flat of degree* 3, it follows that $\mathscr{B}_\circ$, hence also $\mathscr{E}_\circ$, is *locally free of rank* 2. We shall write

$$\pi_\circ : P_\circ \stackrel{\text{def}}{=} \mathbf{P}(\mathscr{E}_\circ) \to X$$

for the $\mathbf{P}^1$*-bundle over* X *associated to* $\mathscr{E}_\circ$.

Next, let us observe that one verifies immediately that the natural morphism

$$X \times_{X^F} X \to X \times_S X$$

determines an isomorphism

$$X \times_{X^F} X \xrightarrow{\sim} X_{(2)}.$$

In particular, the closed immersion $X_{(1)} \hookrightarrow X \times_S X$ determines a closed immersion $X_{(1)} \hookrightarrow X \times_{X^F} X$. Thus, it follows that the $\mathcal{O}_X$-module $\mathcal{E}_\circ$, hence also the $\mathbf{P}^1$-bundle $P_\circ$, on X admits a *natural connection* relative to X/S. We shall write

$$\nabla_{\mathcal{E}_\circ}, \quad \nabla_{P_\circ}$$

for the respective natural connections on $\mathcal{E}_\circ$, $P_\circ$. [So one verifies immediately that the connection $\nabla_{\mathcal{E}_\circ}$ *coincides* with the connection on $\mathcal{E}_\circ = \Phi^*\mathcal{B}_\circ$ determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$.] Moreover, the above isomorphism $X \times_{X^F} X \xrightarrow{\sim} X_{(2)}$, together with the cartesian diagram

$$\begin{array}{ccc} X \times_{X^F} X & \xrightarrow{\mathrm{pr}_2} & X \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \Phi} \\ X & \xrightarrow[\Phi]{} & X^F, \end{array}$$

determines *isomorphisms* of $\mathcal{O}_X$-modules

$$\Phi^*\Phi_*\mathcal{O}_X \xrightarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^F} X} \xleftarrow{\sim} \mathrm{pr}_{1*}\mathcal{O}_{X_{(2)}},$$

which are *compatible* with the respective natural surjections onto $\mathcal{O}_X$ [arising from the diagonal morphism with respect to X/X^F] from each of these three modules. In particular, by forming the kernels of the respective natural surjections onto $\mathcal{O}_X$, we obtain *isomorphisms* of $\mathcal{O}_X$-modules

$$\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{Ker}(\mathrm{pr}_{1*}\mathcal{O}_{X \times_{X^F} X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3).$$

We shall write

$$\sigma_\circ : X \to P_\circ$$

for the section of $\pi_\circ : P_\circ \to X$ determined by the composite $\mathcal{E}_\circ \twoheadrightarrow \omega_{X/S}$ of the above isomorphism $\mathcal{E}_\circ \xrightarrow{\sim} \mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3)$ and the natural surjection $\mathrm{pr}_{1*}(\mathcal{I}/\mathcal{I}^3) \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 = \omega_{X/S}$. Then one verifies easily that the *Kodaira-Spencer homomorphism* $\sigma_\circ^*\omega_{P_\circ/X} \to \omega_{X/S}$ at $\sigma_\circ$ relative to $\nabla_{P_\circ}$ [i.e., the homomorphism obtained by differentiating $\sigma_\circ$ by means of $\nabla_{P_\circ}$] is an *isomorphism*. Thus, it follows immediately from our construction that the following proposition holds:

Proposition 1.1. *The pair* $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ *is an* **indigenous bundle** [*cf. Introduction*] *over* X/S *whose* **Hodge section** [*cf.* [6], *Chapter* I, *Proposition* 2.4, *p.* 1004] *is given by* $\sigma_\circ$. *Moreover, the indigenous bundle* $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ *is* **dormant** [*cf. Introduction*].

Proof. The fact that the pair $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ is an *indigenous bundle* over X/S has already been verified. The fact that the indigenous bundle $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ is *dormant* follows immediately from the definition of the connection $\nabla_{P_\circ}$ [i.e., the construction of $\nabla_{P_\circ}$ via the relative Frobenius morphism Φ]. This completes the proof of Proposition 1.1. $\square$

In the remainder of the present §1, let us consider the invertible sheaves

$$\det(\mathscr{E}_\circ), \quad \det(\mathscr{B}_\circ), \quad \det(\Phi_* \omega_{X/S}).$$

Write $\mathscr{M} \stackrel{\text{def}}{=} \mathscr{H}om_{\mathcal{O}_{X^F}}(\det(\mathscr{B}_\circ), \omega_{X^F/S})$. First, let us observe that since the $\mathcal{O}_X$-module $\mathrm{pr}_{1*}(\mathscr{I}/\mathscr{I}^3) \cong \mathscr{E}_\circ = \Phi^* \mathscr{B}_\circ$ fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \omega_{X/S}^{\otimes 2} \to \mathrm{pr}_{1*}(\mathscr{I}/\mathscr{I}^3) \to \omega_{X/S} \to 0,$$

it follows that

$$\det(\mathscr{E}_\circ) \cong \omega_{X/S}^{\otimes 3},$$

hence also

$$\Phi^* \mathscr{M} \cong \mathcal{O}_X.$$

Next, let us recall from the discussion preceding [9], Théorème 4.1.1, that the map

$$\begin{aligned} \Phi_* \mathcal{O}_X \times \Phi_* \mathcal{O}_X &\to \omega_{X^F/S} \\ (f, g) &\mapsto c(f \cdot \Phi_* d(g)) \end{aligned}$$

—where we write $d : \mathcal{O}_X \to \omega_{X/S}$ for the exterior differentiation operator and $c : \Phi_* \omega_{X/S} \to \omega_{X^F/S}$ for the *Cartier operator*—determines an *isomorphism* of $\mathcal{O}_{X^F}$-modules

$$\mathscr{B}_\circ \xrightarrow{\sim} \mathscr{H}om_{\mathcal{O}_{X^F}}(\mathscr{B}_\circ, \omega_{X^F/S}),$$

which thus implies that

$$\mathscr{M}^{\otimes 2} \cong \mathcal{O}_{X^F}.$$

Thus, we obtain:

LEMMA 1.2. *It holds that*

$$\det(\mathscr{E}_\circ) \cong \omega_{X/S}^{\otimes 3}, \quad \det(\mathscr{B}_\circ) \cong \omega_{X^F/S}, \quad \det(\Phi_* \omega_{X/S}) \cong \omega_{X^F/S}^{\otimes 2}.$$

Proof. The first "$\cong$" has already been verified. Since the homomorphism between the relative Jacobian varieties of X^F/S, X/S induced by Φ is *finite flat of degree* 3^g, it follows from the fact that $\Phi^* \mathscr{M} \cong \mathcal{O}_X$, $\mathscr{M}^{\otimes 2} \cong \mathcal{O}_{X^F}$ verified above that $\mathscr{M}$ lies in $(f^F)^* \operatorname{Pic}(S)$. Thus, again by the fact that $\Phi^* \mathscr{M} \cong \mathcal{O}_X$, the second "$\cong$" follows. The third "$\cong$" follows from the second "$\cong$", together with the *well-known* exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_* \mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_* \omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

[cf., e.g., [4], Theorem 7.2]. $\square$

2. The dormant trivialization of the Schwarz torsor

In the present §2, we maintain the notation of the preceding §1. In particular, we have a projective smooth curve $f : X \to S$ and a *dormant indigenous bundle* $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ over X/S [cf. Proposition 1.1]. We shall write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 and

$$\mathcal{N}_g[\infty]$$

for the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *dormant indigenous bundles*. The starting point of the present §2 is the following theorem:

THEOREM 2.1. *Every* **dormant** *indigenous bundle over X/S is* **isomorphic** *to the dormant indigenous bundle $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ of Proposition* 1.1.

Proof. To verify Theorem 2.1, let us first recall some facts on the *p-adic Teichmüller theory* [cf. [6], [7]]. The natural (1-)morphism

$$\mathcal{N}_g[\infty] \to \mathcal{M}_g$$

is *finite* and *faithfully flat*; moreover, there exists a *dense open* substack of $\mathcal{M}_g$ on which this (1-)morphism is *étale* [cf. the final portion of [7], Chapter II, Theorem 2.8, p. 153]. Thus, to complete the verification of Theorem 2.1, it suffices to verify Theorem 2.1 for a "*sufficiently general*" [i.e., in $\mathcal{M}_g$] projective smooth curve of genus g over an algebraically closed field of characteristic three.

Next, let us observe that it follows from [11], Corollary 5.4, together with [5], Theorem 2.1, that, for every *odd* prime number p and an integer $g \geq 2$, the number of isomorphism classes of *dormant indigenous bundles* over a "*sufficiently general*" projective smooth curve of genus g over an algebraically closed field of characteristic p is equal to

$$\frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{i=1}^{p-1} \frac{1}{\sin^{2g-2}\left(\dfrac{\pi \cdot i}{p}\right)} = \frac{(-p)^{g-1}}{2} \cdot \sum_{\zeta^p = 1, \zeta \neq 1} \frac{\zeta^{g-1}}{(\zeta - 1)^{2g-2}}.$$

On the other hand, one verifies easily that the above quantity in the case where $p = 3$ is always *equal to* 1. This completes the proof of Theorem 2.1. $\square$

Remark 2.1.1. Let us observe that Theorem 2.1 also follows from the theory of *molecules* given in [7] [or the theory of *Ehrhart quasi-polynomials* discussed in [5]—cf. [5], Theorem 3.9] as follows: By considering dormant indigenous bundles over not only smooth curves but also *stable curves*, we have a natural extension of the (1-)morphism $\mathcal{N}_g[\infty] \to \mathcal{M}_g$ whose codomain is the moduli stack of *stable curves* of genus g of characteristic 3 [i.e., "$\bar{\mathscr{M}}_g$"]. Then it follows from [7], Chapter II, Theorem 2.8, p. 153, together with a similar argument to the argument applied in the first paragraph of the proof of Theorem 2.1, that, to complete the verification of Theorem 2.1, it suffices to verify that

> a structure of *dormant molecule* [cf. [7], Chapter V, §0, p. 229] on a *fixed* [nonpointed] totally degenerate stable curve of characteristic 3 is *unique*.

On the other hand, this follows immediately from [7], Introduction, Theorem 1.3, pp. 41–42, together with the fact that $\#((\mathbf{F}_3/\{\pm 1\})\backslash\{0\}) = 1$.

Remark 2.1.2. One may also replace the second paragraph of the proof of Theorem 2.1 by the *local computation of the p-curvature* given in the discussion preceding Proposition 3.1 below [cf. Remark 3.1.1 below].

It follows from Theorem 2.1 [together with the discussion given in the first paragraph of proof of Theorem 2.1] that the natural (1-)morphism

$$\mathscr{N}_g[\infty] \to \mathscr{M}_g$$

is an *isomorphism*, hence also *étale*. Thus, by the final portion of [11], Theorem 3.3, we obtain:

COROLLARY 2.2. *Every dormant indigenous bundle over X/S is* **dormant ordinary** [*cf.* [11], *Definition* 3.2].

We shall write

$$\mathscr{C}_g \to \mathscr{M}_g$$

for the universal curve over $\mathscr{M}_g$ and

$$\mathscr{S}_g \to \mathscr{M}_g$$

for the *Schwarz torsor* over $\mathscr{M}_g$ [cf. [7], Introduction, §0.4, pp. 7–9], i.e., the torsor over the locally free coherent $\mathcal{O}_{\mathscr{M}_g}$-module of rank $3g-3$

$$(\mathscr{C}_g \to \mathscr{M}_g)_* \omega_{\mathscr{C}_g/\mathscr{M}_g}^{\otimes 2}$$

obtained by forming the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *indigenous bundles* [cf. also [6], Chapter I, Corollary 2.9, p. 1007]. By considering the composite of the above natural isomorphism $\mathscr{M}_g \xleftarrow{\sim} \mathscr{N}_g[\infty]$ and the natural closed immersion $\mathscr{N}_g[\infty] \hookrightarrow \mathscr{S}_g$ of stacks, we obtain a trivialization

$$\mathscr{M}_g \to \mathscr{S}_g$$

of the Schwarz torsor.

DEFINITION 2.3. We shall refer to this trivialization $\mathscr{M}_g \to \mathscr{S}_g$ of the Schwarz torsor as the *dormant trivialization*.

By the dormant trivialization of Definition 2.3, we obtain an isomorphism of $\mathscr{S}_g$ with the geometric vector bundle over $\mathscr{M}_g$ associated to $(\mathscr{C}_g \to \mathscr{M}_g)_* \omega_{\mathscr{C}_g/\mathscr{M}_g}^{\otimes 2}$. Thus:

COROLLARY 2.4. *There exists a natural* **bijection** *between the following two sets*:

- $\Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$.
- *The set of isomorphism classes of indigenous bundles over X/S.*

For $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, the indigenous bundle over X/S corresponding to θ is given as follows: *Let us recall the pair $(\mathscr{E}_\circ, \nabla_{\mathscr{E}_\circ})$ and the exact sequence of $\mathscr{O}_X$-modules*

$$0 \to \omega_{X/S}^{\otimes 2} \to \mathscr{E}_\circ \to \omega_{X/S} \to 0$$

discussed in §1. *Write $\phi^\theta : \mathscr{E}_\circ \to \mathscr{E}_\circ \otimes_{\mathscr{O}_X} \omega_{X/S}$ for the homomorphism of $\mathscr{O}_X$-modules obtained by forming the composite*

$$\mathscr{E}_\circ \twoheadrightarrow \omega_{X/S} \xrightarrow{\theta} \omega_{X/S}^{\otimes 3} = \omega_{X/S}^{\otimes 2} \otimes_{\mathscr{O}_X} \omega_{X/S} \hookrightarrow \mathscr{E}_\circ \otimes_{\mathscr{O}_X} \omega_{X/S}.$$

We shall write

$$\nabla_{P_\circ}^\theta$$

for the connection on $P_\circ$ determined by the connection

$$\nabla_{\mathscr{E}_\circ}^\theta \stackrel{\text{def}}{=} \nabla_{\mathscr{E}_\circ} + \phi^\theta$$

on $\mathscr{E}_\circ$. Then the indigenous bundle over X/S corresponding to θ is given by

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ : P_\circ \to X, \nabla_{P_\circ}^\theta).$$

Moreover, for $\theta \in \Gamma(S, f_*\omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2})$, the **dormant locus** *in S of P_θ* [*i.e., the maximal closed subscheme $F \subseteq S$ of S such that the restriction of P_θ to $X \times_S F$ is* **dormant**] **coincides** *with the zero locus in S of θ* [*i.e., the maximal closed subscheme $F \subseteq S$ of S such that the restriction of θ to $X \times_S F$ is identically zero*].

Remark 2.4.1. We note that since $\det(\mathscr{E}_\circ) \cong \omega_{X/S}^{\otimes 3} \not\cong \mathscr{O}_X$ [cf. Lemma 1.2], the pair $(\mathscr{E}_\circ, \nabla_{\mathscr{E}_\circ})$, as well as the pair $(\mathscr{E}_\circ, \nabla_{\mathscr{E}_\circ}^\theta)$ [cf. Corollary 2.4], is *not an indigenous vector bundle* [cf. [6], Chapter I, Definition 2.2, pp. 1002–1003; also the discussion preceding [6], Chapter I, Definition 2.2, p. 1002]. One verifies easily from the fact that $\det(\mathscr{B}_\circ) \cong \omega_{X^F/S}$ [cf. Lemma 1.2] that if $\mathscr{L}$ is an invertible sheaf on X^F such that $\mathscr{L}^{\otimes 2} \cong \tau_{X^F/S}$ [note that since 2 is *invertible* on S, such an invertible sheaf always exists after étale localizing S], then an *indigenous vector bundle* whose projectivization is isomorphic to $(\pi_\circ : P_\circ \to X, \nabla_{P_\circ})$ is given by tensoring $(\mathscr{E}_\circ, \nabla_{\mathscr{E}_\circ})$ with the invertible sheaf $\Phi^*\mathscr{L}$ equipped with the connection determined by the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$. On the other hand, one also verifies easily that the operation of taking tensor product with a *dormant invertible sheaf* [i.e., an invertible sheaf equipped with a connection whose p-curvature is identically zero] does *not affect* the *local computation of the p-curvature* as given in the discussion preceding Proposition 3.1 below.

3. Local criteria

In the present §3, we prove *local criteria* for some properties on indigenous bundles [cf. Proposition 3.1; Proposition 3.8, (ii), below]. We maintain the notation introduced at the beginning of §1.

Let

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2})$$

be a global section of $\omega_{X/S}^{\otimes 2}$. Thus, it follows from Corollary 2.4 that we obtain a connection

$$\nabla_{P_\circ}^{\theta}$$

on the $\mathbf{P}^1$-bundle $P_\circ$ such that the pair

$$P_\theta \stackrel{\text{def}}{=} (\pi_\circ : P_\circ \to X, \nabla_{P_\circ}^{\theta})$$

forms an *indigenous bundle* over X/S.

Let $x \in X$ be a point of X and $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x. Write $\phi_x = \phi \in \mathcal{O}_X$ for the local function on X at x which fits into the equality

$$\theta = \phi \cdot dt \otimes dt.$$

Then one verifies immediately that the local sections

$$e_1 \stackrel{\text{def}}{=} 1 \otimes t - t \otimes 1, \quad e_2 \stackrel{\text{def}}{=} e_1^2 \in \mathrm{pr}_{1*}\, \mathcal{O}_{X \times_{X^F} X} \xleftarrow{\sim} \Phi^*\Phi_*\mathcal{O}_X$$

[cf. the discussion preceding Proposition 1.1] are contained in the submodules

$$\mathrm{Ker}(\mathrm{pr}_{1*}\, \mathcal{O}_{X \times_{X^F} X} \twoheadrightarrow \mathcal{O}_X) \xleftarrow{\sim} \mathscr{E}_\circ,$$

and that, in the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \to \omega_{X/S}^{\otimes 2} \to \mathscr{E}_\circ \to \omega_{X/S} \to 0,$$

the local section e_2 determines a *local trivialization* of the invertible sheaf $\omega_{X/S}^{\otimes 2}$, and the local section e_1 determines a *local splitting* of the surjection $\mathscr{E}_\circ \twoheadrightarrow \omega_{X/S}$; in particular, $\{e_1, e_2\}$ forms a *local basis* of $\mathscr{E}_\circ$.

Next, let us observe that it follows immediately from the definition of $\nabla_{\mathscr{E}_\circ}$ that

$$\nabla_{\mathscr{E}_\circ}(e_1, e_2) = (e_1, e_2) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes dt.$$

Thus, one verifies immediately from the definition of $\nabla^{\theta}_{\mathscr{E}_\circ}$ [cf. Corollary 2.4] that

$$\nabla^{\theta}_{\mathscr{E}_\circ}(e_1, e_2) = (e_1, e_2) \cdot \begin{pmatrix} 0 & 1 \\ \phi & 0 \end{pmatrix} \otimes dt.$$

In particular, it follows that the *p-curvature* $\mathscr{P}^{\theta}$ of the connection $\nabla^{\theta}_{\mathscr{E}_\circ}$ [cf., e.g., the discussion preceding [4], Theorem 5.1] is given by

$$\begin{aligned} \mathscr{P}^{\theta} : \Phi^* \tau_{X^F/S} &\to \mathscr{A}d_{\mathcal{O}_X}(\mathscr{E}_\circ) \\ \Phi^{-1}\delta_{t^F} &\mapsto \left((e_1, e_2) \mapsto (e_1, e_2) \cdot \begin{pmatrix} -\phi' & \phi \\ \phi^2 + \phi'' & \phi' \end{pmatrix} \right) \end{aligned}$$

—where we write $t^F \in \mathcal{O}_{X^F}$ for the local parameter of X^F/S determined by the local parameter $t \in \mathcal{O}_X$, δ_{t^F} (respectively, δ_t) for the local trivialization of $\tau_{X^F/S}$ (respectively, $\tau_{X/S}$) which maps dt^F (respectively, dt) to 1, ∂_t for the local derivation corresponding to δ_t, "$(-)'$" for "$\partial_t(-)$" [i.e., "$(-)'$" is the "derivative of $(-)$ with respect to t"], and

$$\mathscr{A}d_{\mathcal{O}_X}(\mathscr{E}_\circ) \subseteq \mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}_\circ)$$

for the submodule of $\mathscr{E}nd_{\mathcal{O}_X}(\mathscr{E}_\circ)$ consisting of *trace zero* endomorphisms of locally free coherent $\mathcal{O}_X$-module $\mathscr{E}_\circ$. This local computation [cf. Remark 2.4.1] leads us to the following *local criteria* for some properties on indigenous bundles:

PROPOSITION 3.1. *The following hold*:

(i) *The indigenous bundle* P_θ *is* **nilpotent** [*cf. Introduction*] *if and only if*, *for every point* $x \in X$, *the equality*

$$(\phi_x')^2 + \phi_x \cdot \phi_x'' + \phi_x^3 = 0$$

holds.

(ii) *Suppose that* S *is the spectrum of an algebraically closed field* [*of characteristic* 3]. *Then the indigenous bundle* P_θ *is* **admissible** [*cf. Introduction*] *if and only if*, *for every closed point* $x \in X$, *it holds that*

$$\mathrm{ord}_x(\phi_x) \leq 2.$$

Proof. Assertion (i) follows from the definition, together with the above local computation. To verify assertion (ii), let us observe that

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

forms a local basis of the locally free coherent $\mathcal{O}_X$-module $\mathscr{A}d_{\mathcal{O}_X}(\mathscr{E}_\circ)$. Thus, assertion (ii) follows immediately from the definition, together with the above local computation. □

Remark 3.1.1. If $\mathscr{P}^{\theta} = 0$, then it follows from the above local computation that $\phi = 0$, hence also $\theta = 0$. By means of this observation, one can give an *alternative proof* of Theorem 2.1 [cf. Remark 2.1.2].

Next, let us observe that the natural exact sequence of $\mathcal{O}_X$-modules

$$0 \to \omega_{X/S}^{\otimes 2} \to \mathcal{E}_\circ \to \omega_{X/S} \to 0$$

determines a homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{A}d_{\mathcal{O}_X}(\mathcal{E}_\circ) \hookrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}_\circ) \to \mathcal{H}om_{\mathcal{O}_X}(\omega_{X/S}^{\otimes 2}, \omega_{X/S}) \cong \tau_{X^F/S};$$

moreover, the *square Hasse invariant* [cf. Introduction] of the indigenous bundle P_θ is defined as the composite of the p-curvature $\mathcal{P}^\theta$ and this homomorphism. Thus, by the above local computation, we obtain:

Proposition 3.2. *The* **square Hasse invariant** *of the indigenous bundle P_θ is, up to multiplication by a global section of $\mathcal{O}_S^\times$, given by*

$$\theta \in \Gamma(X, \omega_{X/S}^{\otimes 2}) \cong \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\Phi^*\tau_{X^F/S}, \tau_{X/S})).$$

In particular, if, moreover, the indigenous bundle P_θ is **admissible**, *then the* **double supersingular divisor** [*cf. Introduction*] *of P_θ* **coincides** *with the zero locus of θ.*

In particular, we obtain the following two corollaries:

Corollary 3.3. *Suppose that the indigenous bundle P_θ is* **nilpotent** *and* **admissible**. *Then the* **supersingular divisor** [*cf. Introduction*] *of P_θ is* **finite étale** *over S.*

Proof. Since [it follows from the definition that] the supersingular divisor of P_θ is *finite flat* over S [cf. also [6], Chapter II, Proposition 2.6, (2), p. 1032], to complete the verification of Corollary 3.3, it suffices to verify the *unramifiedness*. Thus, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the *unramifiedness* follows from Proposition 3.1, (ii); Proposition 3.2, together with the definition of the *supersingular divisor*. $\square$

Corollary 3.4. *Suppose that S is* **reduced**. *Then the isomorphism class of* **nilpotent** *indigenous bundle over X/S is* **completely determined** *by the zero locus of the square Hasse invariant.*

Proof. First, let us observe that since S is *reduced*, it follows from [7], Chapter I, Proposition 1.5, p. 91, that, to verify Corollary 3.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. Next, let us observe that one verifies easily that if ϕ is *nonzero* and *satisfies* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i), then, for every $c \in k \backslash \{0, 1\}$, $c \cdot \phi$ does *not satisfy* the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i). Thus, Corollary 3.4 follows from Proposition 3.1, (i); Proposition 3.2, together with Corollary 2.4. $\square$

Remark 3.4.1. Observe that Corollary 3.4 is a *generalization* of [6], Chapter II, Proposition 2.6, (4), p. 1032, in the case where $p = 3$.

Next, let us observe that it follows from the equality of Proposition 3.1, (i), that the following lemma holds:

Lemma 3.5. *Suppose that S is the spectrum of an algebraically closed field* [*of characteristic* 3], *and that the indigenous bundle P_θ is* **nilpotent**. *Then, for every closed point $x \in X$, it holds that* $\mathrm{ord}_x(\phi_x) \notin 3\mathbf{Z} + 1$.

Proof. Assume that $n \stackrel{\mathrm{def}}{=} \mathrm{ord}_x(\phi_x) \in 3\mathbf{Z} + 1$ for some closed point $x \in X$. Write

$$\phi_x = \sum_{i=0}^{\infty} a_i t_x^i$$

by regarding ϕ_x as an element of the completion $\mathcal{O}_{X,x}^{\wedge}$. Then, by considering the coefficient of the "t_x^{2n-2}" of the left-hand side of the equality "$(\phi')^2 + \phi \cdot \phi'' + \phi^3 = 0$" of Proposition 3.1, (i), we obtain that $a_n = 0$. Thus, we obtain a *contradiction*. □

By Lemma 3.5, we obtain:

Corollary 3.6. *Suppose that $g = 2$. If a* **nilpotent** *indigenous bundle over X/S is* **active** [*cf.* [7], *Chapter* II, *Definition* 1.1, *p.* 127], *then it is* **admissible**.

Proof. Let us first observe that it follows from the definition of *admissibility* that, to verify Corollary 3.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field k [of characteristic 3]. On the other hand, in this case, since $\deg(\omega_{X/S}^{\otimes 2}) = 4$, it follows immediately from Proposition 3.1, (ii), together with Lemma 3.5, that every *nilpotent* and *active* indigenous bundle over X/S is *admissible*. □

We shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* [cf. the discussion preceding [7], Introduction, Theorem 0.1, p. 24] of genus g of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with *nilpotent indigenous bundles*. Note that it follows from [6], Chapter II, Theorem 2.3, p. 1029 [cf. also the discussion following [6], Chapter II, Definition 2.4, p. 1030], that the natural (1-)morphism

$$\mathcal{N}_g \to \mathcal{M}_g$$

is *finite flat of degree* 3^{3g-3}.

Corollary 3.7. *Suppose that* $g = 2$. *Then the open substack of* $\mathcal{N}_2$

$$\mathcal{N}_2 \backslash \mathcal{N}_2[\infty]$$

is **smooth** *over* $\mathbf{F}_3$.

Proof. This follows from Corollary 3.6, together with [6], Chapter II, Corollary 2.16, p. 1043. □

Proposition 3.8. *Suppose that* S *is the spectrum of an algebraically closed field* k [*of characteristic* 3], *and that the indigenous bundle* P_θ *is* **nilpotent**. *Then the following hold*:

(i) *We shall write*

$$T_\theta$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ *at the* k*-valued point of* $\mathcal{N}_g$ *corresponding to* P_θ. *Then* T_θ *is naturally isomorphic to the subspace of* $\Gamma(X, \omega_{X/S}^{\otimes 2})$ *consisting of global sections* η *of* $\omega_{X/S}^{\otimes 2}$ *such that if*, *for some closed point* $x \in X$, *we write*

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' = 0.$$

(ii) *It holds that the indigenous bundle* P_θ *is* **ordinary** [*cf. Introduction*] *if and only if the following condition is satisfied*: *For every nonzero global section* η *of* $\omega_{X/S}^{\otimes 2}$, *if*, *for some closed point* $x \in X$, *we write*

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then it holds that

$$(\phi_x \cdot \psi_x)'' \neq 0.$$

Proof. Assertion (ii) follows immediately from assertion (i). Thus, to complete the verification of Proposition 3.8, it suffices to verify assertion (i). Write $A \stackrel{\text{def}}{=} k[\varepsilon]/(\varepsilon^2)$, where ε is an indeterminate. Then it follows from Proposition 3.1, (i), that T_θ is naturally isomorphic to the subspace of $\Gamma(X, \omega_{X/S}^{\otimes 2})$ consisting of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi_x \cdot dt_x \otimes dt_x,$$

then the equality

$$((\phi + \varepsilon\psi)')^2 + (\phi + \varepsilon\psi) \cdot (\phi + \varepsilon\psi)'' + (\phi + \varepsilon\psi)^3 = 0$$

—where write $\psi \stackrel{\text{def}}{=} \psi_x$—in $A \otimes_k \Gamma(X, \omega_{X/S}^{\otimes 2}) = \Gamma(X, \omega_{X/S}^{\otimes 2}) \oplus \varepsilon \cdot \Gamma(X, \omega_{X/S}^{\otimes 2})$ holds. On the other hand, again by Proposition 3.1, (i), one verifies easily that it holds that this equality holds if and only if the equality

$$\phi'' \cdot \psi + \phi \cdot \psi'' - \phi' \cdot \psi' (= (\phi \cdot \psi)'') = 0$$

holds. This completes the proof of assertion (i). □

Remark 3.8.1. Proposition 3.8, (ii), also follows immediately from Proposition 3.2; Lemma A.9, (i) [in the case where we take the pair "$(\mathcal{L}, \Theta)$" of Lemma A.9, (i), to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$—cf. Remark A.4.1], together with [6], Chapter II, Proposition 2.12, p. 1037.

Thus, we obtain:

COROLLARY 3.9. *Suppose that S is the spectrum of an algebraically closed field k* [*of characteristic* 3], *and that the indigenous bundle P_θ is* **nilpotent**. *Then the following conditions are equivalent*:
(1) *The indigenous bundle P_θ is* **dormant**.
(2) *The vector space T_θ over k of Proposition* 3.8, (i), *is* **of dimension $3g - 3$**.

Proof. If P_θ is *dormant*, then $\theta = 0$ [cf. Corollary 2.4]. Thus, the implication $(1) \Rightarrow (2)$ follows from Proposition 3.8, (i). On the other hand, if condition (2) is satisfied, then it follows from Proposition 3.8, (i) [in the case where we take the "η" of Proposition 3.8, (i), to be θ], that $(\phi^2)'' = 0$. Thus, since $0 = (\phi^2)'' = -(\phi')^2 - \phi \cdot \phi'' = \phi^3$ [cf. Proposition 3.1, (i)], we conclude that $\phi = 0$, hence also $\theta = 0$, i.e., that condition (1) is satisfied [cf. Corollary 2.4]. This completes the proof of Corollary 3.9. $\square$

4. Indigenous bundles arising from squares

In the present §4, we discuss some properties on an indigenous bundle which arises from the square of a "*twisted*" differential form, i.e., the square of a global section of a "*square root*" of the square of the relative cotangent sheaf [cf. Proposition 4.1, Proposition 4.2, Proposition 4.4 below]. In the present §4, we maintain the notation introduced at the beginning of §1.

Let

$$\mathscr{L} = (\mathcal{L}, \Theta : \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X [cf. Definition A.3] and

$$\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Let us recall [cf. the discussion following Definition A.3] that we have isomorphisms of invertible sheaves

$$\mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\otimes 3} \xrightarrow{\sim} \Phi^* \mathcal{L}^F$$

$$\Theta(l \otimes l) \cdot l \mapsto l \otimes l \otimes l \mapsto \Phi^{-1} l^F$$

—where we write $\mathcal{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathcal{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathcal{L}$, and l^F is the local section of $\mathcal{L}^F$ determined by l.

Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, and $l_x = l \in \mathscr{L}$ a local trivialization of $\mathscr{L}$ at x. Then the global trivialization Θ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x = \delta \stackrel{\text{def}}{=} \Theta(l \otimes l) \in \mathcal{O}_X^\times$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi = \phi \cdot l \otimes dt$$

at x.

Next, let us observe that the trivialization Θ determines an isomorphism

$$\Theta : \Gamma(X, (\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Thus, by considering the image via this isomorphism of the *square*

$$\theta \stackrel{\text{def}}{=} \chi \otimes \chi \in \Gamma(X, (\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$$

of χ, we obtain a global section

$$\Theta(\theta) \in \Gamma(X, \omega_{X/S}^{\otimes 2})$$

of $\omega_{X/S}^{\otimes 2}$. On the other hand, it follows from Corollary 2.4 that this global section $\Theta(\theta)$ gives rise to an *indigenous bundle* over X/S

$$P_{\Theta(\theta)} \stackrel{\text{def}}{=} (\pi_\circ : P_\circ \to X, \nabla_{P_\circ}^{\Theta(\theta)}).$$

Proposition 4.1. *Suppose that χ defines a relative effective Cartier divisor of X/S. Then the following conditions are equivalent*:

(1) *The indigenous bundle $P_{\Theta(\theta)}$ is* **nilpotent** *and* **active**.

(2) *The global section $\chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a* **normalized Cartier eigenform** *associated to $\mathscr{L} = (\mathscr{L}, \Theta)$ [cf. Definition* A.8, (i)].

Proof. Let us first observe that it follows from the definitions of $\Theta(\theta)$ that $\Theta(\theta)$ fits into the equality

$$\Theta(\theta) = \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.1, (i), that it holds that $P_{\Theta(\theta)}$ is *nilpotent* if and only if, for every point $x \in X$,

$$\begin{aligned}
&((\phi^2 \cdot \delta)')^2 + (\phi^2 \cdot \delta) \cdot (\phi^2 \cdot \delta)'' + (\phi^2 \cdot \delta)^3 \\
&\quad = \phi^4 \cdot (\delta')^2 - \phi^3 \cdot \phi' \cdot \delta \cdot \delta' - \phi^3 \cdot \phi'' \cdot \delta^2 + \phi^4 \cdot \delta \cdot \delta'' + \phi^6 \cdot \delta^3 \\
&\quad = \phi^3 \cdot \delta^3 \cdot (-(\phi \cdot \delta^{-1})'' + \phi^3)
\end{aligned}$$

is equal to zero. In particular, Proposition 4.1 follows from Lemma A.9, (ii), together with Corollary 2.4. $\square$

PROPOSITION 4.2. *Suppose that the indigenous bundle* $P_{\Theta(\theta)}$ *is* **nilpotent** *and* **active**. *Then the following conditions are equivalent*:

(1) *The indigenous bundle* $P_{\Theta(\theta)}$ *is* **nilpotent** *and* **admissible**.

(2) *The zero locus of the global section* $\chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *is* **finite étale** *over* S.

Proof. Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.2, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.1, (ii), together with the definition of $\Theta(\theta)$. □

Remark 4.2.1.

(i) Note that condition (2) of Proposition 4.1 does *not imply* condition (2) of Proposition 4.2. Such a counter-example is as follows: Let k be an algebraically closed field of characteristic 3. Let us consider the following polynomial:

$$f(t) = t^{12} + t^{10} + 1 \in k[t].$$

Then one verifies easily that $f(t)$ does *not have* any multiple root, which thus implies that the equation

$$s^2 = f(t)$$

determines a *hyperelliptic* projective smooth curve C of genus five over k.

Write $\omega \in \Gamma(C, \omega_{C/k})$ for the global section of $\omega_{C/k}$ whose restriction to the open subscheme of X on which f is invertible is of the form

$$\frac{\alpha \cdot t^4}{s}\, dt$$

—where $\alpha \in k$ satisfies that $\alpha^2 = 2$. Then one verifies easily from Lemma A.9, (ii), that ω is a *normalized Cartier eigenform* associated to $\mathcal{O}_C$ [equipped with the natural identification $\mathcal{O}_C \otimes_{\mathcal{O}_C} \mathcal{O}_C = \mathcal{O}_C$]. On the other hand, it is immediate that if we write $c \in C$ for the closed point corresponding to $(t, s) = (0, 1)$, then $\mathrm{ord}_c(\omega) = 4$.

(ii) It follows from Corollary 3.6 that a *nilpotent active* indigenous bundle over a projective smooth curve *of genus two* in characteristic three is *admissible*. On the other hand, it follows from the discussion of (i), together with Proposition 4.1 and Proposition 4.2, that there exists a *nilpotent active* indigenous bundle over a projective smooth curve in characteristic three which is *not admissible*.

PROPOSITION 4.3. *Suppose that S is the spectrum of an algebraically closed field k* [*of characteristic* 3], *and that the indigenous bundle* $P_{\Theta(\theta)}$ *is* **nilpotent** *and* **admissible**. *Write*

$$T_{\Theta(\theta)}$$

for the relative tangent space of $\mathcal{N}_g/\mathcal{M}_g$ *at the* k*-valued point of* $\mathcal{N}_g$ *corresponding to* $P_{\Theta(\theta)}$. *Thus, it follows from Proposition* 3.8, (i), *that* $T_{\Theta(\theta)}$ *may be regarded as a subspace of* $\Gamma(X, \omega_{X/S}^{\otimes 2})$:

$$T_{\Theta(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2}).$$

Then the map

$$\Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \otimes_k \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to \Gamma(X, \omega_{X/S}^{\otimes 2})$$
$$\alpha \otimes \beta \mapsto \Theta(\alpha \otimes \beta)$$

induces an **isomorphism** *of vector spaces over* k

$$\mathrm{Ker}(C_{\mathscr{L}}) \xrightarrow{\sim} T_{\Theta(\theta)}$$
$$\sigma \mapsto \Theta(\sigma \otimes \chi)$$

—where we write $C_{\mathscr{L}}$ *for the Cartier operator associated to* $\mathscr{L} = (\mathscr{L}, \Theta)$ [*cf. Definition* A.4].

Proof. Let us first observe that [one verifies easily that] the homomorphism of vector spaces over k

$$\Xi : \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to \Gamma(X, \omega_{X/S}^{\otimes 2})$$
$$\alpha \mapsto \Theta(\alpha \otimes \chi)$$

is *injective*. Thus, to verify Proposition 4.3, it suffices to verify the following two assertions:

(a) $\Xi(\mathrm{Ker}(C_{\mathscr{L}})) \subseteq T_{\Theta(\theta)}$.

(b) The resulting [cf. (a)] homomorphism $\Xi : \mathrm{Ker}(C_{\mathscr{L}}) \to T_{\Theta(\theta)}$ is *surjective*.

Next, let us recall from the proof of Proposition 4.1 that $\Theta(\theta)$ fits into the equality

$$\Theta(\theta) = \phi^2 \cdot \delta \cdot dt \otimes dt$$

at x. Thus, it follows from Proposition 3.8, (i), that the subspace $T_{\Theta(\theta)} \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$ consists of global sections η of $\omega_{X/S}^{\otimes 2}$ such that if, for some closed point $x \in X$, we write

$$\eta = \psi \cdot dt \otimes dt,$$

then it holds that

$$(\phi^2 \cdot \delta \cdot \psi)'' = 0.$$

Now we verify the assertion (a). Let $\sigma \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ be such that $C_{\mathscr{L}}(\sigma) = 0$. Write

$$\sigma = \mu \cdot l \otimes dt$$

at x. Since $\sigma \in \mathrm{Ker}(C_{\mathscr{L}})$, it holds that $(\mu \cdot \delta^{-1})'' = 0$ [cf. Lemma A.9, (i)]. Thus, since

$$\Theta(\sigma \otimes \chi) = \phi \cdot \mu \cdot \delta \cdot dt \otimes dt$$

at x, and

$$(\phi^3 \cdot \mu \cdot \delta^2)'' = (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) - (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})'' = 0,$$

we conclude that $\Xi(\sigma) \in T_{\Theta(\theta)}$. This completes the proof of the assertion (a).

Next, we verify the assertion (b). Let η be a global section of $\omega_{X/S}^{\otimes 2}$ which belongs to $T_{\Theta(\theta)}$. Write

$$\eta = \psi \cdot dt \otimes dt$$

at x. Then since

$$\begin{aligned} 0 &= (\phi^2 \cdot \delta \cdot \psi)'' \\ &= (-(\phi')^2 - \phi \cdot \phi'') \cdot \delta \cdot \psi + \phi^2 \cdot \delta'' \cdot \psi + \phi^2 \cdot \delta \cdot \psi'' \\ &\quad + \phi \cdot \phi' \cdot \delta' \cdot \psi - \phi^2 \cdot \delta' \cdot \psi' + \phi \cdot \phi' \cdot \delta \cdot \psi', \end{aligned}$$

and ϕ is *of order* ≤ 1 [at x] by Proposition 4.2, it holds that $\mathrm{ord}_x(\phi) \geq 1$ implies $\mathrm{ord}_x(\psi) \geq 1$. Thus, it follows that $V(\chi) = V(\chi)_{\mathrm{red}} \subseteq V(\eta)_{\mathrm{red}} \subseteq V(\eta)$, where we write "$V(-)$" for the zero locus of "$(-)$", i.e., that $\eta \in \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \subseteq \Gamma(X, \omega_{X/S}^{\otimes 2})$. Now let us observe that since $(\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, which thus implies that $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \omega_{X/S}^{\otimes 2}(-V(\chi))$, we have an isomorphism

$$\begin{aligned} \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) &\xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2}(-V(\chi))) \\ \sigma &\mapsto \Theta(\sigma \otimes \chi). \end{aligned}$$

Thus, we conclude that there exists a global section σ of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ such that $\eta = \Theta(\sigma \otimes \chi)$. Write

$$\sigma = \mu \cdot l \otimes dt$$

at x, which thus implies that

$$\psi = \mu \cdot \phi \cdot \delta$$

at x. Then since

$$\begin{aligned} 0 = (\phi^2 \cdot \delta \cdot \psi)'' &= (\mu \cdot \phi^3 \cdot \delta^2)'' \\ &= (\phi^3 \cdot \delta^3)'' \cdot (\mu \cdot \delta^{-1}) - (\phi^3 \cdot \delta^3)' \cdot (\mu \cdot \delta^{-1})' + (\phi^3 \cdot \delta^3) \cdot (\mu \cdot \delta^{-1})'' \\ &= \phi^3 \cdot \delta^3 \cdot (\mu \cdot \delta^{-1})'', \end{aligned}$$

it holds that $(\mu \cdot \delta^{-1})'' = 0$, i.e., that $\sigma \in \mathrm{Ker}(C_{\mathscr{L}})$ [cf. Lemma A.9, (i)]. This completes the proof of the assertion (b), hence also of Proposition 4.3. $\square$

Proposition 4.4. *Suppose that the indigenous bundle* $P_{\Theta(\theta)}$ *is* **nilpotent** *and* **admissible**. *Then the following conditions are equivalent*:

(1) *The indigenous bundle* $P_{\Theta(\theta)}$ *is* **nilpotent** *and* **ordinary**.

(2) *The invertible sheaf* $\mathscr{L}$ *is* **parabolically ordinary** [*cf. Definition* A.7].

Proof. Since [one verifies immediately that] the locus [in S] on which condition (1) (respectively, (2)) is satisfied is *open*, to complete the verification of Proposition 4.4, we may assume without loss of generality that S is the spectrum of an algebraically closed field [of characteristic 3]. Then Proposition 4.4 follows from Proposition 4.3. □

5. Nilpotent admissible indigenous bundles via Cartier operators

In the present §5, we prove the main result of the present paper [cf. Theorem 5.2 below], as well as some corollaries to the main result. In the present §5, we maintain the notation introduced at the beginning of §1.

Definition 5.1.

(i) We shall say that a pair

$$(\mathscr{L}, \chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}))$$

consisting of an invertible sheaf $\mathscr{L}$ on X and a global section χ of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is *of CE-type* [where "CE" stands for "Cartier Eigenform"] if

- $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$,
- $\chi \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to $\mathscr{L}$ [cf. Definition A.8, (ii)], and
- the zero locus of χ is *étale* over S.

(ii) We shall say that a pair $(\mathscr{L}, \chi)$ of CE-type is *of CEO-type* [where "CEO" stands for "Cartier Eigenform and Ordinary"] if $\mathscr{L}$ is parabolically ordinary.

(iii) We shall say that a relative effective Cartier divisor D of X/S is *of CE-type* (respectively, *of CEO-type*) if there exists a pair $(\mathscr{L}, \chi)$ of CE-type (respectively, of CEO-type) such that D coincides with the zero locus of χ.

The main result of the present paper is as follows:

Theorem 5.2. *Let* $g \geq 2$ *be an integer*, S *a connected noetherian scheme of characteristic* 3 [*i.e., over* $\mathbf{F}_3$], *and* $f : X \to S$ *a projective smooth curve of genus* g. *Write* $\omega_{X/S}$ *for the relative cotangent bundle of* X/S. *Then the following hold*:

(i) *Let* P *be a* **nilpotent admissible** *indigenous bundle over* X/S. *Write* $\mathscr{L}_P$ *for the* **Hasse defect** *of* P [*cf. Definition* B.2] *and* $\chi_P \in \Gamma(X, \mathscr{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})$ *for the* **Hasse invariant** *of* P [*cf. also the final portion of Proposition* B.4]. *Then the pair*

$$(\mathscr{L}_P, \chi_P)$$

is **of CE-type** [*cf. Definition* 5.1, (i)]. *Moreover, it holds that* P *is* **nilpotent ordinary** *if and only if the pair* $(\mathscr{L}_P, \chi_P)$ *is* **of CEO-type** [*cf. Definition* 5.1, (ii)].

(ii) *Let D be a relative effective Cartier divisor of X/S. Then it holds that D is the* **supersingular divisor** *of a* **nilpotent admissible** (*respectively*, **nilpotent ordinary**) *indigenous bundle over X/S if and only if D is* **of CE-type** (*respectively*, **of CEO-type**) [*cf. Definition* 5.1, (iii)].

(iii) *Suppose that S is* **reduced**. *Then, by considering the* **supersingular divisors**, *we have a* **bijection** *between the following two sets*:

· *The set of isomorphism classes of* **nilpotent admissible** (*respectively*, **nilpotent ordinary**) *indigenous bundles over X/S.*

· *The set of relative effective Cartier divisors of X/S* **of CE-type** (*respectively*, **of CEO-type**).

Proof. First, we verify the first assertion of assertion (i). Let us first observe that it follows from the final portion of Proposition B.3 that $\mathscr{L}_P^{\otimes 2} \cong \mathcal{O}_X$. Moreover, it follows from Corollary 3.3 that the zero locus of χ_P is *finite étale* over S. Thus, to complete the verification of the first assertion of assertion (i), it suffices to verify that there exists a trivialization $\Theta : \mathscr{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ such that χ_P is a *normalized Cartier eigenform* associated to $(\mathscr{L}_P, \Theta)$.

Let us write $\theta_P \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ for the global section of $\omega_{X/S}^{\otimes 2}$ corresponding, via the bijection of Corollary 2.4, to the indigenous bundle P. Fix a trivialization $\Theta : \mathscr{L}_P^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathscr{L}_P^{\otimes 2}$ and write θ for the image via the isomorphism $\Gamma(X, (\mathscr{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by Θ of the square $\chi_P \otimes \chi_P \in \Gamma(X, (\mathscr{L}_P \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ_P. Then it follows from Proposition 3.2 that there exists a global unit $u \in \Gamma(S, \mathcal{O}_S^{\times})$ such that $\theta_P = u \cdot \theta$. Thus, we may assume without loss of generality, by replacing Θ by $u^{-1} \cdot \Theta$, that $\theta_P = \theta$. In particular, it follows from Proposition 4.1 that χ_P is a *normalized Cartier eigenform* associated to $(\mathscr{L}_P, \Theta)$. This completes the proof of the first assertion of assertion (i). Moreover, the final assertion of assertion (i) follows from the first assertion of assertion (i), together with Proposition 4.4 [cf. also the equality "$\theta_P = \theta$" in the proof of the first assertion of assertion (i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The *necessity* follows from assertion (i). To verify the *sufficiency*, let D be a relative effective Cartier divisor of X/S *of CE-type* (respectively, *of CEO-type*). Thus, it follows from the definition that there exists a pair $(\mathscr{L}, \chi)$ *of CE-type* (respectively, *of CEO-type*) such that D is defined by χ. Now since $(\mathscr{L}, \chi)$ is *of CE-type*, the zero locus of χ is *étale* over S, and there exists a trivialization $\Theta : \mathscr{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$ of $\mathscr{L}^{\otimes 2}$ such that χ is a *normalized Cartier eigenform* associated to $(\mathscr{L}, \Theta)$. Thus, it follows from Proposition 4.1 and Proposition 4.2 that the indigenous bundle P over X/S corresponding, via the bijection of Corollary 2.4, to the image via the isomorphism $\Gamma(X, (\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2}) \xrightarrow{\sim} \Gamma(X, \omega_{X/S}^{\otimes 2})$ induced by Θ of the square $\chi \otimes \chi \in \Gamma(X, (\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})^{\otimes 2})$ of χ is *nilpotent* and *admissible*. Moreover, it follows from Proposition 4.4 that if $(\mathscr{L}, \chi)$ is *of CEO-type*, then the indigenous bundle P is *ordinary*. Write χ_P for the *Hasse invariant* of P. Then it follows from Proposition 3.2 that the zero locus of χ_P, i.e., the supersingular divisor of

P, *coincides* with the zero locus of χ, i.e., D. This completes the proof of the *sufficiency*, hence also of assertion (ii).

The *injectivity* of the map of assertion (iii) follows from Corollary 3.4 [cf. also [6], Chapter II, Proposition 2.6, (4), p. 1032]. The *surjectivity* of the map of assertion (iii) follows from assertion (ii). This completes the proof of Theorem 5.2. $\square$

Corollary 5.3. *Let X be a projective smooth curve of genus $g \geq 2$ over an algebraically closed field k of characteristic* 3 *and $\mathscr{L}$ an invertible sheaf on X such that $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$. Then the following hold*:

(i) *Suppose that $\mathscr{L} \cong \mathcal{O}_X$ (respectively, $\mathscr{L} \not\cong \mathcal{O}_X$). Then the number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k whose Hasse defects are isomorphic to $\mathscr{L}$ is*

$$\leq \#\mathbf{P}^{g-1}(\mathbf{F}_3) = \frac{3^g - 1}{3 - 1} \quad \left(\textit{respectively}, \ \leq \#\mathbf{P}^{g-2}(\mathbf{F}_3) = \frac{3^{g-1} - 1}{3 - 1}\right).$$

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/k is*

$$\begin{aligned} &\leq \#\mathbf{P}^{g-1}(\mathbf{F}_3) + (\#((\mathbf{Z}/2\mathbf{Z})^{\oplus 2g}) - 1) \cdot \#\mathbf{P}^{g-2}(\mathbf{F}_3) \\ &= \frac{1}{2}((3^g - 1) + (2^{2g} - 1)(3^{g-1} - 1)). \end{aligned}$$

Proof. First, we verify assertion (i). Fix a trivialization $\Theta : \mathscr{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$. Let us first observe that it follows from Theorem 5.2, (iii) [cf. also Remark A.8.1], that, to verify assertion (i), it suffices to verify that the number of subspaces of $V_{\mathscr{L}} \stackrel{\text{def}}{=} \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ *of dimension* 1 which are *preserved* and *not annihilated* by the Cartier operator $V_{\mathscr{L}} \to \Gamma(X^F, \mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})) \xleftarrow{\sim} V_{\mathscr{L}}$ associated to $(\mathscr{L}, \Theta)$ is

$$\leq \#\mathbf{P}^{g-1}(\mathbf{F}_3) = \frac{3^g - 1}{3 - 1} \quad \left(\text{respectively}, \ \leq \#\mathbf{P}^{g-2}(\mathbf{F}_3) = \frac{3^{g-1} - 1}{3 - 1}\right)$$

if $\mathscr{L} \cong \mathcal{O}_X$ (respectively, $\mathscr{L} \not\cong \mathcal{O}_X$). To this end, let us observe that one verifies easily from the *Riemann-Roch theorem* that the vector space $V_{\mathscr{L}}$ over k is *of dimension g* (respectively, $g - 1$) if $\mathscr{L} \cong \mathcal{O}_X$ (respectively, $\mathscr{L} \not\cong \mathcal{O}_X$). Thus, assertion (i) follows immediately from elementary linear algebra [cf. also [8], Corollary, p. 143]. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i). This completes the proof of Corollary 5.3. $\square$

Corollary 5.4. *Let X be a projective smooth curve of genus ≥ 2 over an algebraically closed field k of characteristic* 3. *Then, for every* **nilpotent ordinary** *indigenous bundle P over X/k, there exists a connected finite étale covering $Y \to X$*

of X such that the [*necessarily* **nilpotent admissible**] *indigenous bundle* $(Y \to X)^* P$ *over* Y/k *is* **not ordinary**.

Proof. Write $\mathcal{L}_P$ for the *Hasse defect* of P and $Y_1 \to X$ for the connected finite étale covering of X which trivializes $\mathcal{L}_P$. [So if $\mathcal{L}_P \cong \mathcal{O}_X$ (respectively, $\mathcal{L}_P \not\cong \mathcal{O}_X$), then $Y_1 \to X$ is *of degree* 1 (respectively, 2).] Next, let $Y_2 \to X$ be a connected finite étale covering of X such that Y_2 is *not parabolically ordinary* [cf., e.g., [10], Théorème 2] and $Y \to X$ a connected finite étale covering of X which dominates $Y_1 \to X$ and $Y_2 \to X$.

Now let us observe that since $Y \to X$ *factors* through $Y_1 \to X$, one verifies immediately that the Hasse defect of the indigenous bundle $(Y \to X)^* P$ over Y/k is *trivial*. Thus, it follows from the final portion of Theorem 5.2, (i), that the indigenous bundle $(Y \to X)^* P$ over Y/k is *ordinary* if and only if Y is *parabolically ordinary* [cf. also Proposition A.6]. On the other hand, since $Y \to X$ *factors* through $Y_2 \to X$, and Y_2 is *not parabolically ordinary*, it holds that Y is *not parabolically ordinary* [cf., e.g., the discussion entitled "The p-rank" of [8], pp. 146–147 and [8], Corollary 1, p. 174], which thus implies that the indigenous bundle $(Y \to X)^* P$ over Y/k is *not ordinary*. $\square$

Remark 5.4.1. Corollary 5.4 yields a *negative answer* to the basic question (2) of [7], Introduction, §2.1, p. 72.

Finally, we discuss the various *moduli stacks* related to the main result of the present paper. We shall apply the notational conventions for the various stacks established in the Appendix C [in the case where we take the "p" of the Appendix C to be 3]. The following corollary follows immediately from the final portion of Theorem 5.2, (i) [cf. also Proposition C.5]:

Corollary 5.5. *We have a* **cartesian** *diagram of stacks*

$$\begin{tikzcd} \mathcal{N}_g^{\mathrm{ord}} \arrow[r] \arrow[d] & \mathcal{I}_g[2]^{\mathrm{pb\text{-}ord}} \arrow[d] \\ \mathcal{N}_g^{\mathrm{adm}} \arrow[r] & \mathcal{I}_g[2] \end{tikzcd}$$

—where the vertical arrows are the natural open immersions of stacks [*cf. Definition* C.4], *and the lower horizontal arrow is the* **Hasse defect morphism** [*cf. Definition* C.1].

Next, for a nonnegative integer r, write

$$\mathcal{M}_{g,[r]}$$

for the moduli stack of *hyperbolic curves of type* (g, r) of characteristic 3, i.e., the moduli stack of projective smooth curves of genus g of characteristic 3 equipped with relative effective *étale* Cartier divisors of relative degree r.

Definition 5.6. It follows from Corollary 3.3, together with Proposition B.3, that the supersingular divisor of the universal nilpotent admissible indigenous bundle over $\mathscr{C}_g \times_{\mathscr{M}_g} \mathscr{N}_g^{\mathrm{adm}} \to \mathscr{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathscr{M}_g$

$$\mathscr{N}_g^{\mathrm{adm}} \to \mathscr{M}_{g,[2g-2]}.$$

We shall refer to this (1-)morphism as the *Hasse morphism*.

The following corollary follows immediately from Theorem 5.2, (ii):

Corollary 5.7. *Let (X, D) be a hyperbolic curve of type $(g, 2g-2)$ over a connected noetherian scheme S of characteristic* 3. *Then the following conditions are equivalent*:

(1) *The classifying* (1-)*morphism $S \to \mathscr{M}_{g,[2g-2]}$ of (X, D)* **factors** *through the* **Hasse morphism** $\mathscr{N}_g^{\mathrm{adm}} \to \mathscr{M}_{g,[2g-2]}$.

(2) *The relative effective Cartier divisor D of X/S is* **of CE-type**.

6. The case of genus two

In the present §6, we give, by applying the results obtained in the present paper, a complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over a projective smooth curve *of genus two* over an algebraically closed field of characteristic three [cf. Theorem 6.1 below]. Moreover, we also prove that every projective smooth curve of genus two over a connected noetherian scheme of characteristic three is *hyperbolically ordinary* [cf. Corollary 6.6 below]. In the present §6, we maintain the notation introduced at the beginning of §1. Suppose, moreover, that $g = 2$ [i.e., that X is *of genus* 2], and that S is the spectrum of an algebraically closed field k [of characteristic 3].

Since X is of genus 2, X admits a *uniquely determined hyperelliptic involution* ι, which determines a *double* covering

$$\xi : X \to Q$$

—where we write Q for the [scheme-theoretic] quotient of X by the action of ι. [Thus, Q is isomorphic to the projective line $\mathbf{P}_k^1$ over k.] We shall write

$$\mathrm{WP} \subseteq X$$

for the ramification locus of ξ, i.e., the zero locus of the global section of the invertible sheaf on X [of degree 6]

$$\mathscr{H}om_{\mathcal{O}_X}(\xi^*\omega_{Q/S}, \omega_{X/S})$$

determined by ξ. [Thus, $\mathrm{WP} \subseteq X$ is the set of *Weierstrass points* of X.] Then, as is well-known, the closed subscheme $\mathrm{WP} \subseteq X$ is *reduced*; moreover, we have a bijection [of finite sets of cardinality 15] between

- the set of subsets of WP *of cardinality* 2

and

- the set of isomorphism classes of invertible sheaves on X *of order* 2 given by mapping $D = \{x_1, x_2\} \subseteq \mathrm{WP}$ to $\mathscr{L}_D \stackrel{\mathrm{def}}{=} \mathcal{O}_X(x_1 - x_2)$. Finally, for a subset $D \subseteq \mathrm{WP}$ of cardinality 2, write

$$\xi_D : X_D \to X$$

for the connected finite étale double covering which trivializes $\mathscr{L}_D$ and

$$E_D \ (\to Q)$$

for the elliptic curve over k obtained by considering the double covering of Q ($\cong \mathbf{P}^1_k$) whose branch locus coincides with $\xi(\mathrm{WP} \backslash D)$. Then one verifies immediately from the definition of E_D that we obtain a *cartesian* diagram

$$\begin{array}{ccc} X_D & \xrightarrow{\xi_D} & X \\ \downarrow & & \downarrow \xi \\ E_D & \longrightarrow & Q \end{array}$$

which thus implies that the "*new part*" of ξ_D [i.e., the abelian variety obtained by forming the quotient of the Jacobian variety of X_D by the image—via the homomorphism induced by ξ_D—of the Jacobian variety of X] is *isogenous* to E_D.

By this observation, together with the results obtained in the present paper, we give the following complete list of *nilpotent/nilpotent admissible/nilpotent ordinary* indigenous bundles over X/S:

THEOREM 6.1. *The following hold*:

(i) *Every* **nilpotent nondormant** *indigenous bundle over* X/S *is* **admissible**.

(ii) *Let* $D \subseteq \mathrm{WP}$ *be a subset* **of cardinality 2** *and* $\theta_D \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *a* [*uniquely determined, up to multiplication by an element of* $k^\times$] *global section of* $\omega_{X/S}^{\otimes 2}$ *such that the zero locus of* θ_D *coincides with* $2D$ [*if we naturally regard* D *as a reduced divisor of degree* 2], *and, moreover, the elliptic curve* E_D *is* **ordinary**. *Then a* [*uniquely determined—cf. Proposition* 3.2, *Corollary* 3.4] $k^\times$*-multiple of* θ_D *corresponds, via the bijection of Corollary* 2.4, *to a* **nilpotent** [*necessarily* **admissible**—*cf.* (i)] *indigenous bundle over* X/S.

(iii) *Let* $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ *be a* **Cartier eigenform** *associated to* $\mathcal{O}_X$. *Then a* [*uniquely determined—cf. Proposition* 3.2, *Corollary* 3.4] $k^\times$*-multiple of* $\omega_{\mathrm{CE}} \otimes \omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S}^{\otimes 2})$ *corresponds, via the bijection Corollary* 2.4, *to a* **nilpotent** [*necessarily* **admissible**—*cf.* (i)] *indigenous bundle over* X/S.

(iv) *Every* **nilpotent nondormant** [*i.e.*, **admissible**—*cf.* (i)] *indigenous bundle over* X/S *is obtained as the result of either* (ii) *or* (iii).

(v) *It holds that a* **nilpotent** *indigenous bundle over* X/S *is* **ordinary** *if and only if one of the following two conditions is satisfied*:

(1) *The indigenous bundle is obtained as the result of* (ii).

(2) *The indigenous bundle is obtained as the result of* (iii), *and, moreover,* X *is* **parabolically ordinary**.

Proof. Assertion (i) follows from Corollary 3.6. Next, we verify assertion (ii). Let $D \subseteq \mathrm{WP}$ be as in assertion (ii). Then it is immediate that $\mathscr{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S} \cong \mathcal{O}_X(D)$ [if we naturally regard D as a reduced divisor of degree 2]. In particular, it follows that $\Gamma(X, \mathscr{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S})$ is *of dimension* 1, which thus implies that the zero locus of every nonzero global section of $\mathscr{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ *coincides* with [the reduced closed subscheme of X whose underlying subset is] D. Moreover, since E_D is *ordinary*, it follows from Proposition A.6 [cf. also Remark A.8.1] that every nonzero global section of $\mathscr{L}_D \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathscr{L}_D$. Thus, it follows immediately from Proposition 4.1 that assertion (ii) holds. This completes the proof of assertion (ii).

Assertion (iii) follows from Proposition 4.1. Next, we verify assertion (iv). Let P be a *nilpotent admissible* indigenous bundle over X/S. If the Hasse defect of P is *trivial*, then it follows from Theorem 5.2, (iii), that P is obtained as the result of (iii). If the Hasse defect of P is *nontrivial*, then it follows from Theorem 5.2, (iii), together with Proposition A.6, that P is obtained as the result of (ii) [cf. also the proof of assertion (ii)]. This completes the proof of assertion (iv). Assertion (v) follows from the final portion of Theorem 5.2, (i), together with assertion (iv). This completes the proof of Theorem 6.1. □

Remark 6.1.1. It follows immediately from Proposition 3.2, together with the various definitions involved, that the *Hasse invariants* and the *supersingular divisors* of *nilpotent admissible* indigenous bundles over X/S are given as follows:

- Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (ii), with respect to a subset $D \subseteq \mathrm{WP}$ as in Theorem 6.1, (ii). Then the *supersingular divisor* of P is [the reduced closed subscheme of X whose underlying subset is] D. Next, let us observe that $\mathcal{O}_X(D)^{\otimes 2} \cong \omega_{X/S}^{\otimes 2}$, and, moreover, the vector space $\Gamma(X, \mathcal{O}_X(D))$ over k is *of dimension* 1. Let s be a nonzero global section of $\mathcal{O}_X(D)$. Then the *Hasse invariant* of the indigenous bundle P is a $k^\times$-multiple of

$$s \in \Gamma(X, \mathcal{O}_X(D)).$$

- Write P for the *nilpotent admissible* indigenous bundle over X/S obtained as the result of Theorem 6.1, (iii), with respect to a global section $\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S})$ as in Theorem 6.1, (iii). Then the *Hasse invariant* of P is a $k^\times$-multiple of

$$\omega_{\mathrm{CE}} \in \Gamma(X, \omega_{X/S}).$$

The *supersingular divisor* of this indigenous bundle is the zero locus of ω_{CE}.

Remark 6.1.2. One verifies immediately that an indigenous bundle [implicitly] discussed in [1], §11, is a nilpotent admissible indigenous bundle obtained as the result of Theorem 6.1, (iii) [cf. the discussion in Remark 6.1.1 concerning *supersingular divisors*; also condition (2) of Theorem 6.1, (v), and the equivalence (a) $\Leftrightarrow$ (b) of [1], Theorem 2.8, (3)].

The following corollary follows immediately from Theorem 2.1 and Theorem 6.1 [cf. also Remark 6.1.1], together with elementary linear algebra [cf. also [8], Corollary, p. 143]:

Corollary 6.2. *Write n_{WP} for the number of subsets D of* WP *of cardinality* 2 *such that the elliptic curve E_D is* **ordinary**. *Write, moreover, γ_X ($\in \{0,1,2\}$) for the p-rank of the Jacobian variety of X. Then the following hold*:

(i) *The number of isomorphism classes of* **nilpotent** *indigenous bundles over X/S is given by*

$$1 + n_{\mathrm{WP}} + \#\mathbf{P}^{\gamma_X - 1}(\mathbf{F}_3)$$

—where we write $\#\mathbf{P}^{-1}(\mathbf{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(ii) *The number of isomorphism classes of* **nilpotent admissible** *indigenous bundles over X/S is given by*

$$n_{\mathrm{WP}} + \#\mathbf{P}^{\gamma_X - 1}(\mathbf{F}_3)$$

—where we write $\#\mathbf{P}^{-1}(\mathbf{F}_3) \stackrel{\mathrm{def}}{=} 0$.

(iii) *If X is* **parabolically ordinary** (*respectively*, **not parabolically ordinary**) [*i.e.*, $\gamma_X = 2$ (*respectively*, $\gamma_X \neq 2$)], *then the number of isomorphism classes of* **nilpotent ordinary** *indigenous bundles over X/S is given by*

$$n_{\mathrm{WP}} + 4 \quad (\textit{respectively}, \ n_{\mathrm{WP}}).$$

Next, let us recall the following *well-known* lemma on the p-rank of the Jacobian variety of a projective smooth curve of genus ≤ 2 over an algebraically closed field of characteristic three. The following well-known lemma follows immediately from, for instance, the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2], together with a *well-known* explicit description of the global differential forms on a *hyperelliptic* projective smooth curve:

Lemma 6.3. *The following hold*:

(i) *Suppose that X admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}(k[s,t]/(s^2 - f(t)))$$

—where s and t are indeterminates, and

$$f(t) = t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \in k[t].$$

Then it holds that X is **parabolically ordinary** *if and only if $a_1 \neq a_2 \cdot a_4$.*

(ii) *If an elliptic curve E over k admits a dense open subscheme which is isomorphic to the affine scheme over k*

$$\mathrm{Spec}(k[s,t]/(s^2 - f(t)))$$

—*where* s *and* t *are indeterminates*, *and*

$$f(t) = t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \in k[t]$$

$$(\textit{respectively},\ f(t) = t^3 + a_2 t^2 + a_1 t + a_0 \in k[t])$$

then it holds that E *is* **ordinary** *if and only if* $a_2 \neq 0$.

The following corollary was already proved in [7] [cf. Remark 6.4.1 below]:

COROLLARY 6.4. *There exists a* **dense open** *substack of* $\mathcal{M}_2$ *such that every projective smooth curve parametrized by a geometric point on this open substack admits* **exactly 19** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles.*

Proof. It follows from Theorem 6.1 [cf. also Corollary 6.2, (iii)] that, to verify Corollary 6.4, it suffices to verify the following assertion: There exists a *dense open* substack $U_{\mathrm{ord}} \subseteq \mathcal{M}_2$ (respectively, $U_{\mathrm{WP}} \subseteq \mathcal{M}_2$) of $\mathcal{M}_2$ such that every projective smooth curve parametrized by a geometric point on U_{ord} (respectively, U_{WP}) is *parabolically ordinary* (respectively, satisfies the condition that, for every subset $D \subseteq \mathrm{WP}$ of *cardinality* 2, the elliptic curve E_D is *ordinary*). On the other hand, the existence of "U_{ord}" (respectively, "U_{WP}") as above follows immediately from Lemma 6.3, (i) (respectively, Lemma 6.3, (ii)), together with a straightforward calculation. This completes the proof of Corollary 6.4. □

Remark 6.4.1. Let us observe that the number "19" in the statement of Corollary 6.4 *coincides* with the result of the formula of [7], Chapter V, Corollary 1.3, (3), pp. 237–238, i.e., the formula

$$n_{2,0}^{\mathrm{ord}} = \frac{p}{3}(2p^2 + 1).$$

Finally, we prove the *existence* of a *nilpotent ordinary* indigenous bundle over X/S:

PROPOSITION 6.5. *The following hold*:

(i) *There exist* **at least 12** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles whose Hasse defects are* **nontrivial** *over* X/S.

(ii) *There exist* **at least 13** *isomorphism classes of* **nilpotent ordinary** *indigenous bundles over* X/S.

Proof. Let us identify Q with $\mathbf{P}^1_k$ by an isomorphism over k. Moreover, let us naturally identify the set of closed points of $\mathbf{P}^1_k = Q$ with the set $k \cup \{\infty\}$.

First, I claim that the following assertion holds:

CLAIM 6.5.A. If the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12, then, by considering a suitable automorphism of $\mathbf{P}^1_k = Q$, one may take the subset "WP" of $\mathbf{P}^1_k = Q$ to be

$$\{0, 1, \infty, -1, \alpha, -\alpha\} \subseteq k \cup \{\infty\}$$

—where $\alpha \in k$ satisfies that $\alpha^2 = 2$—i.e., X admits a dense open subscheme which is isomorphic to the affine scheme over k

$$\mathrm{Spec}(k[s,t]/(s^2 - f(t)))$$

—where s and t are indeterminates, and

$$f(t) = x(x-1)(x+1)(x-\alpha)(x+\alpha) = x^5 + 2x \in k[t].$$

Indeed, suppose that there exist 3 distinct subsets $D_1, D_2, D_3 \subseteq \mathrm{WP}$ of *cardinality* 2 such that the elliptic curves E_{D_1}, E_{D_2}, and E_{D_3} are *not ordinary*.

First, we consider the case where $\mathrm{WP} = D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbf{P}^1_k = Q$, that $\xi(D_1) = \{0, \infty\}$ and $1 \in \xi(D_2)$. Then since E_{D_2} and E_{D_3} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \backslash \{0, 1, -1\}$ such that

$$\xi(D_2) = \{1, -1\}, \quad \xi(D_3) = \{\alpha, -\alpha\}.$$

On the other hand, since

$$\alpha \cdot (-\alpha) + \alpha \cdot 1 + \alpha \cdot (-1) + (-\alpha) \cdot 1 + (-\alpha) \cdot (-1) + 1 \cdot (-1) = -\alpha^2 - 1,$$

and E_{D_1} is *not ordinary*, it follows from Lemma 6.3, (ii), that $\alpha^2 = 2$. Thus, one may take the subset "WP" to be as in Claim 6.5.A.

Next, we consider the case where $\mathrm{WP} \neq D_1 \cup D_2 \cup D_3$. Then let us observe that we may assume without loss of generality, by considering a suitable automorphism of $\mathbf{P}^1_k = Q$, that $\infty \in \xi(\mathrm{WP} \backslash (D_1 \cup D_2 \cup D_3))$, that $0 \in \xi(\mathrm{WP} \backslash (D_1 \cup D_2))$, and that $1 \in \xi(D_1 \backslash (D_1 \cap D_2))$. Then since E_{D_1} and E_{D_2} are *not ordinary*, it follows from Lemma 6.3, (ii), that there exists an element $\alpha \in k \backslash \{0, 1, -1\}$ such that

$$\xi(\mathrm{WP} \backslash D_1) = \{0, \infty, \alpha, -\alpha\}, \quad \xi(\mathrm{WP} \backslash D_2) = \{0, \infty, 1, -1\},$$

which thus implies that

$$\xi(\mathrm{WP}) = \{0, 1, -1, \alpha, -\alpha, \infty\}.$$

Thus, since $\infty \notin \xi(D_3)$, one verifies easily from Lemma 6.3, (ii), that E_{D_3} is *ordinary*—in *contradiction* to our assumption that E_{D_3} is *not ordinary*. This completes the proof of Claim 6.5.A.

Now we verify assertion (i). Suppose that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Then it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. In particular, it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Thus, assertion (i) follows from Theorem 6.1. This completes the proof of assertion (i).

Next, we verify assertion (ii). Assume that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. Then it follows from Corollary 6.2, (iii), that the number "n_{WP}" defined in the statement of Corollary 6.2 is ≤ 12. Thus, it follows from Claim 6.5.A that one may take the subset "WP" to be as in Claim 6.5.A. Then it follows from Lemma 6.3, (ii), together with a straightforward calculation, that the number "n_{WP}" defined in the statement of Corollary 6.2 is *equal to* 12. Moreover, since [it follows from Lemma 6.3, (i), that] X is *parabolically ordinary*, it follows from Corollary 6.2, (iii), that X/S admits *exactly* 16 $(= 12 + 4) > 12$ isomorphism classes of *nilpotent ordinary* indigenous bundles—in *contradiction* to our assumption that the set of isomorphism classes of *nilpotent ordinary* indigenous bundles over X/S is *of cardinality* ≤ 12. This completes the proof of assertion (ii). $\square$

It follows from Proposition 6.5, (ii), together with [6], Chapter II, Proposition 3.4, p. 1044, that the following corollary holds:

Corollary 6.6. *Every projective smooth curve of genus* 2 *over a connected noetherian scheme of characteristic* 3 *is* **hyperbolically ordinary** [*cf. Introduction*].

Remark 6.6.1. Corollary 6.6 yields a *partial positive answer* to the basic question (1) of [7], Introduction, §2.1, p. 72. By Corollary 6.6, we conclude that the image of the natural (1-)morphism discussed in the basic question (1) of [7], Introduction, §2.1, p. 72, in the case where $(g, r, p) = (2, 0, 3)$ *contains* the open substack $(\mathcal{M}_{2,0})_{\mathbf{F}_3} \subseteq (\overline{\mathcal{M}}_{2,0})_{\mathbf{F}_3}$.

Remark 6.6.2. Corollary 6.6 also yields an example of a *projective* smooth curve of positive characteristic which is *not parabolically ordinary* but *hyperbolically ordinary* [cf. also Lemma 6.3, (i)].

Appendix A. Cartier operator associated to a square-trivialized invertible sheaf

In the Appendix A, let us recall the *Cartier operator* associated to a *square-trivialized* invertible sheaf on a projective smooth curve in positive characteristic. It seems to the author that the content of the Appendix A is *well-known*; however, since a suitable literature could not be found, the author has decided to discuss it in the Appendix A.

In the Appendix A, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbf{F}_p$], and $f : X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F : X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi : X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

Proposition A.1. *Let* $\mathcal{L}$ *be an invertible sheaf on* X *such that* $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X$. *Then the following conditions are equivalent*:

(1) *The restriction of* $\mathscr{L}$ *to every fiber of* f *is* **of order two** [*i.e.*, *is* **nontrivial**].

(2) *There exists a point* $s \in S$ *of* S *such that the restriction of* $\mathscr{L}$ *to the fiber of* f *at* s *is* **of order two** [*i.e.*, *is* **nontrivial**].

(3) *The invertible sheaf* $\mathscr{L}$ *does* **not arise** *from an invertible sheaf on* S.

(4) *The image of the classifying morphism of* $\mathscr{L}$ [*from* S *to the relative Jacobian variety of* X/S] *does* **not intersect** *the image of the identity section.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are immediate. The implication (3) $\Rightarrow$ (4) follows immediately from our assumption that S is *connected*, together with the [well-known] fact that the endomorphism of the relative Jacobian variety of X/S obtained by multiplication by 2 is *finite étale*. This completes the proof of Proposition A.1. □

Definition A.2. Let $\mathscr{L}$ be an invertible sheaf on X such that $\mathscr{L}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathscr{L}$ is *of relative order two* (respectively, *one*) if $\mathscr{L}$ satisfies (respectively, does not satisfy) the four conditions in the statement of Proposition A.1.

Definition A.3. We shall refer to a pair

$$(\mathscr{L}, \Theta : \mathscr{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

consisting of an invertible sheaf $\mathscr{L}$ on X and a global trivialization Θ of the square $\mathscr{L}^{\otimes 2}$ of $\mathscr{L}$ as a *square-trivialized invertible sheaf* on X.

Let

$$\mathscr{L} = (\mathscr{L}, \Theta : \mathscr{L}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X)$$

be a *square-trivialized invertible sheaf* on X. Thus, the trivialization Θ determines isomorphisms of invertible sheaves on X

$$\mathscr{L} \xrightarrow{\sim} \mathscr{L}^{\otimes p} \xrightarrow{\sim} \Phi^* \mathscr{L}^F$$

$$\Theta(l \otimes l)^{(p-1)/2} \cdot l \mapsto l^{\otimes p} \mapsto \Phi^{-1} l^F$$

—where we write $\mathscr{L}^F$ for the invertible sheaf on X^F obtained by pulling back $\mathscr{L}$ via the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S, l is a local section of $\mathscr{L}$, and l^F is the local section of $\mathscr{L}^F$ determined by l.

Let us recall [cf., e.g., [4], Theorem 7.2] that we have an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_* \mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_* \omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

—where we write d for the exterior differentiation operator $\mathcal{O}_X \to \omega_{X/S}$, and c for the Cartier operator. We shall write

$$\mathscr{B}_\circ \stackrel{\text{def}}{=} \operatorname{Coker}(\mathcal{O}_{X^F} \to \Phi_* \mathcal{O}_X) = \operatorname{Ker}(c : \Phi_* \omega_{X/S} \to \omega_{X^F/S})$$

for the *locally free coherent* $\mathcal{O}_{X^F}$*-module of rank* $p-1$ obtained by forming the cokernel of the natural homomorphism $\mathcal{O}_{X^F} \to \Phi_*\mathcal{O}_X$, or, alternatively, the kernel of the Cartier operator $c : \Phi_*\omega_{X/S} \to \omega_{X^F/S}$. Then, by tensoring with $\mathcal{L}^F$ and applying the above isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^*\mathcal{L}^F$ determined by Θ, we obtain an exact sequence of $\mathcal{O}_{X^F}$-modules

$$0 \to \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ \to \Phi_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to \mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S} \to 0,$$

which thus determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \to f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \mathcal{B}_\circ) \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

Definition A.4. We shall write

$$C_{\mathcal{L}}$$

for the third arrow of the above exact sequence of $\mathcal{O}_S$-modules and refer to

$$C_{\mathcal{L}} : f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

as the *Cartier operator* associated to $\mathcal{L} = (\mathcal{L}, \Theta)$.

Remark A.4.1. If we take the pair "$(\mathcal{L}, \Theta)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$, then the Cartier operator $f_*\omega_{X/S} \to f_*^F\omega_{X^F/S}$ associated to $(\mathcal{L}, \Theta)$ *coincides* with the [homomorphism induced by the] *usual* Cartier operator.

Remark A.4.2. One verifies easily that since the formation of

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0$$

commutes with arbitrary change of base "$S' \to S$", the formation of

$$C_{\mathcal{L}} : f_*(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$$

commutes with arbitrary change of base "$S' \to S$".

If [the underlying invertible sheaf of] $\mathcal{L}$ is *of relative order two*, then we shall write

$$\xi_{\mathcal{L}} : X_{\mathcal{L}} \to X$$

for the connected finite étale *double* covering of X over S which *trivializes* the invertible sheaf $\mathcal{L}$ [determined by Θ] and

$$\xi_{\mathcal{L}}^F : X_{\mathcal{L}}^F \to X^F$$

for the connected finite étale covering of X^F over S obtained by base-changing $\xi_{\mathcal{L}}$ via the absolute Frobenius morphism of S. Thus, a trivialization of $\xi_{\mathcal{L}}^*\mathcal{L}$ determines respective isomorphisms of $\mathcal{O}_X$-, $\mathcal{O}_{X^F}$-modules

$$(\xi_{\mathcal{L}})_*\mathcal{O}_{X_{\mathcal{L}}} \cong \mathcal{O}_X \oplus \mathcal{L}, \quad (\xi_{\mathcal{L}}^F)_*\mathcal{O}_{X_{\mathcal{L}}^F} \cong \mathcal{O}_{X^F} \oplus \mathcal{L}^F.$$

Moreover, one verifies immediately that the natural homomorphism of $\mathcal{O}_S$-modules

$$\mathbf{R}^1(f^F \circ \xi^F_{\mathscr{L}})_*\mathcal{O}_{X^F_{\mathscr{L}}} \to \mathbf{R}^1(f \circ \xi_{\mathscr{L}})_*\mathcal{O}_{X_{\mathscr{L}}}$$

determined by the relative Frobenius morphism $X_{\mathscr{L}} \to X^F_{\mathscr{L}}$ over S is *decomposed* into the direct sum of the natural homomorphisms of $\mathcal{O}_S$-modules

$$\mathbf{R}^1 f^F_* \mathcal{O}_{X^F} \to \mathbf{R}^1 f_* \mathcal{O}_X, \quad \mathbf{R}^1 f^F_* \mathscr{L}^F \to \mathbf{R}^1 f_* \Phi^* \mathscr{L}^F \xleftarrow{\sim} \mathbf{R}^1 f_* \mathscr{L}$$

[cf. the isomorphism given in the discussion following Definition A.3].

Definition A.5.

(i) We shall say that $f : X \to S$ is *parabolically ordinary* [cf. the discussion following [6], Chapter II, Definition 3.3, p. 1044] if the Jacobian variety of every fiber of f is ordinary.

(ii) Let $Y \to S$ be a projective smooth curve over S and $\xi : Y \to X$ a finite étale covering over S. Then we shall say that $\xi : Y \to X$ is *parabolically new-ordinary* if, for every point $s \in S$ of S, the "new part" of ξ at s [i.e., the abelian variety over s obtained by forming the quotient of the Jacobian variety of $Y \times_S s$ by the image—via the homomorphism induced by ξ—of the Jacobian variety of $X \times_S s$] is ordinary.

Thus, we obtain:

Proposition A.6. *It holds that the Cartier operator $C_{\mathscr{L}} : f_*(\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f^F_*(\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S})$ associated to $\mathscr{L} = (\mathscr{L}, \Theta)$ is* **injective** *at every point of S if and only if one of the following conditions is satisfied*:

(1) *$\mathscr{L}$ is* **of relative order one**, *and X/S is* **parabolically ordinary**.

(2) *$\mathscr{L}$ is* **of relative order two**, *and the connected finite étale double covering $\xi_{\mathscr{L}} : X_{\mathscr{L}} \to X$ is* **parabolically new-ordinary**.

Proof. Let us first observe that it follows from Remark A.4.2 that, to verify Proposition A.6, we may assume without loss of generality that S is the spectrum of an algebraically closed field. Next, let us recall from the discussion preceding Definition A.4 that we have an isomorphism of $\mathcal{O}_S$-modules

$$f^F_*(\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \mathscr{B}_\circ) \xrightarrow{\sim} \mathrm{Ker}(C_{\mathscr{L}}).$$

In particular, the exact sequence of $\mathcal{O}_{X^F}$-modules in the discussion preceding Definition A.4

$$0 \longrightarrow \mathcal{O}_{X^F} \longrightarrow \Phi_*\mathcal{O}_X \xrightarrow{\Phi_* d} \Phi_*\omega_{X/S} \xrightarrow{c} \omega_{X^F/S} \longrightarrow 0,$$

together with the isomorphism $\mathscr{L} \xrightarrow{\sim} \Phi^*\mathscr{L}^F$ given in the discussion following Definition A.3, determines an exact sequence of $\mathcal{O}_S$-modules

$$0 \to \mathrm{Ker}(C_{\mathscr{L}}) \to \mathbf{R}^1 f^F_* \mathscr{L}^F \to \mathbf{R}^1 f_* \mathscr{L}.$$

Thus, it follows from the discussion preceding Definition A.5 that Proposition A.6 holds. This completes the proof of Proposition A.6. □

Definition A.7. Let $\mathscr{M}$ be an invertible sheaf on X such that $\mathscr{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that $\mathscr{M}$ is *parabolically ordinary* if, for some [or, equivalently, every] trivialization $\Theta_{\mathscr{M}} : \mathscr{M}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$, the square-trivialized invertible sheaf $(\mathscr{M}, \Theta_{\mathscr{M}})$ satisfies either (1) or (2) in the statement of Proposition A.6 [i.e., the Cartier operator associated to $(\mathscr{M}, \Theta_{\mathscr{M}})$ is injective at every point of S—cf. Proposition A.6].

Next, let us observe that the morphism $X^F \to X$ induced by the absolute Frobenius morphism of S determines a *Frobenius-semi-linear* homomorphism

$$f_*(\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}) \to f_*^F(\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}).$$

For a global section u of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$, we shall write u^F for the global section of $\mathscr{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}$ obtained by forming the image of u via this Frobenius-semi-linear homomorphism.

Definition A.8.

(i) We shall say that a global section u of $\mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *normalized Cartier eigenform* associated to $\mathscr{L} = (\mathscr{L}, \Theta)$ if u defines a relative effective Cartier divisor of X/S, and, moreover, $C_{\mathscr{L}}(u) = -u^F$.

(ii) Let $\mathscr{M}$ be an invertible sheaf on X such that $\mathscr{M}^{\otimes 2} \cong \mathcal{O}_X$. Then we shall say that a global section of $\mathscr{M} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a *Cartier eigenform* associated to $\mathscr{M}$ if there exists a trivialization $\Theta_{\mathscr{M}}$ of the square of $\mathscr{M}$ such that the global section is a normalized Cartier eigenform associated to the square-trivialized invertible sheaf $(\mathscr{M}, \Theta_{\mathscr{M}})$.

Remark A.8.1. One verifies immediately that if S is the spectrum of an algebraically closed field k [of characteristic p], then the following two conditions are equivalent:

- A global section $u \in \Gamma(X, \mathscr{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$ is a *Cartier eigenform* associated to [the underlying invertible sheaf of] $\mathscr{L}$.
- It holds that $u \neq 0$, and, moreover, $C_{\mathscr{L}}(u)$ is a $k^\times$-multiple of u^F.

Moreover, in this case, the subset of $k^\times$ consisting of $c \in k^\times$ such that cu is a *normalized Cartier eigenform* associated to $\mathscr{L} = (\mathscr{L}, \Theta)$ forms an $\mathbf{F}_p^\times$-*torsor*, which thus implies that this subset is *of cardinality* $p - 1$.

Remark A.8.2. If we take the pair "$(\mathscr{L}, \Theta)$" to be the pair consisting of $\mathcal{O}_X$ and the natural identification $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{O}_X$ [i.e., if we are in the situation of Remark A.4.1], then the property of being a [*normalized*] *Cartier eigenform* is closely related to the property of being *locally logarithmic* [cf., e.g., [3], Théorème 2.1.17].

Finally, we consider a local criterion for a *normalized Cartier eigenform*. Let $x \in X$ be a point of X, $t_x = t \in \mathcal{O}_X$ a local parameter of X/S at x, $l_x = l \in \mathcal{L}$ a local trivialization of $\mathcal{L}$ at x, and

$$\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S})$$

a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$. Then the global trivialization Θ and the local trivialization $l_x = l$ determine a local unit

$$\delta_x = \delta \stackrel{\text{def}}{=} \Theta(l \otimes l) \in \mathcal{O}_X^{\times}$$

at x. Moreover, the global section χ determines a local function $\phi_x = \phi \in \mathcal{O}_X$ on X at x which fits into the equality

$$\chi = \phi \cdot l \otimes dt$$

at x. Then it follows immediately from the characterization of the *Cartier operator* [cf., e.g., [4], Theorem 7.2; also the discussion given in [3], §2.1—especially, the equality (2.1.13) in [3], §2.1], together with the explicit description of the isomorphism $\mathcal{L} \xrightarrow{\sim} \Phi^* \mathcal{L}^F$ given in the discussion following Definition A.3, that the following lemma holds:

LEMMA A.9. *Write ∂_{t_x} for the local derivation corresponding to the local trivialization of $\tau_{X/S}$ which maps dt_x to 1 [i.e., "$\partial_{t_x}(-)$" is the "derivative of $(-)$ with respect to t_x"]. Then the following hold*:

(i) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is* **annihilated** *by the Cartier operator $C_{\mathcal{L}}$ associated to $\mathcal{L} = (\mathcal{L}, \Theta)$ if and only if, for every point $x \in X$, the equality*

$$(\overbrace{\partial_{t_x} \circ \cdots \circ \partial_{t_x}}^{p-1})(\delta_x^{-(p-1)/2} \cdot \phi_x) = 0$$

holds.

(ii) *It holds that the global section χ of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}$ is a* **normalized Cartier eigenform** *associated to $\mathcal{L} = (\mathcal{L}, \Theta)$ if and only if, for every point $x \in X$, the local function ϕ_x is* **not a zero-divisor**, *and, moreover, the equality*

$$\phi_x^p = (\overbrace{\partial_{t_x} \circ \cdots \circ \partial_{t_x}}^{p-1})(\delta_x^{-(p-1)/2} \cdot \phi_x)$$

holds.

Appendix B. The Hasse bundle of a nilpotent admissible indigenous bundle

In the Appendix B, we discuss the *Hasse bundle* of a *nilpotent admissible* indigenous bundle. In the Appendix B, let p be an *odd* prime number, $g \geq 2$ an integer, S a connected noetherian scheme of characteristic p [i.e., over $\mathbf{F}_p$], and $f : X \to S$ a *projective smooth curve* [i.e., a morphism which is projective, smooth, geometrically connected, and of relative dimension one] of genus g. Write $f^F : X^F \to S$ for the projective smooth curve obtained by base-changing f via the absolute Frobenius morphism of S and $\Phi : X \to X^F$ for the relative Frobenius morphism over S. We shall use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf.

Let

$$P = (\pi : P \to X, \nabla_P)$$

be a *nilpotent admissible* indigenous bundle over X/S. Write

$$\sigma_{\mathrm{Hdg}} : X \to P$$

for the *Hodge section* of P and

$$\mathscr{I}_{\mathrm{Hdg}} \subseteq \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{Hdg}. Thus, it follows from the definition of an indigenous bundle that the Kodaira-Spencer homomorphism at σ_{Hdg} relative to ∇_P [i.e., the homomorphism obtained by differentiating σ_{Hdg} by means of ∇_P]

$$\sigma_{\mathrm{Hdg}}^* \omega_{P/X} \to \omega_{X/S}$$

is an *isomorphism*.

Proposition B.1. *There exists a* **unique section** $\sigma_{\mathrm{con}} : X \to P$ *of* $\pi : P \to X$ *which satisfies the following conditions*:

(1) *The section* σ_{con} *is* **horizontal** *with respect to* ∇_P. *In particular, the connection* ∇_P *induces a connection on the invertible sheaf* $\sigma_{\mathrm{con}}^* \omega_{P/X}$ *on* X.

(2) *There exists a* **horizontal** *isomorphism* $\sigma_{\mathrm{con}}^* \omega_{P/X} \cong \Phi^* \tau_{X^F/S}$, *where we regard* $\Phi^* \tau_{X^F/S}$ *as an invertible sheaf equipped with a connection by equipping* $\Phi^* \tau_{X^F/S}$ *with the connection arising from the exterior differentiation operator* $\mathcal{O}_X \to \omega_{X/S}$.

We shall refer to this section σ_{con} *as the* **conjugate section** *of the indigenous bundle* P.

Proof. This follows from the second paragraph of [the statement of] [6], Chapter II, Proposition 2.5, pp. 1030–1031. □

Write

$$\sigma_{\mathrm{con}} : X \to P$$

for the *conjugate section* of P and

$$\mathscr{I}_{\mathrm{con}} \subseteq \mathcal{O}_P$$

for the ideal of $\mathcal{O}_P$ which defines the section σ_{con}.

DEFINITION B.2. We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P) = \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^* \mathcal{I}_{\mathrm{con}}, \mathcal{O}_X)$$

obtained by pulling back $\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$ via σ_{Hdg} as the *Hasse bundle* of P. We shall refer to the invertible sheaf on X

$$\sigma_{\mathrm{Hdg}}^* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P) \otimes_{\mathcal{O}_X} \tau_{X/S}^{\otimes (p-1)/2} = \mathcal{H}om_{\mathcal{O}_X}(\sigma_{\mathrm{Hdg}}^* \mathcal{I}_{\mathrm{con}}, \tau_{X/S}^{\otimes (p-1)/2})$$

obtained by tensoring the Hasse bundle with $\tau_{X/S}^{\otimes (p-1)/2}$ as the *Hasse defect* of P.

Write

$$\mathcal{H}_P \stackrel{\mathrm{def}}{=} \sigma_{\mathrm{Hdg}}^* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{O}_P)$$

for the *Hasse bundle* of P. Then let us observe that since $\pi : P \to X$ is *of genus zero*, and the invertible sheaf on P

$$\mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}})$$

is *of relative degree* 0 over X, it follows immediately that the natural homomorphisms

$$\pi_* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \to \sigma_{\mathrm{Hdg}}^* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \cong \mathcal{H}_P \otimes_{\mathcal{O}_X} \omega_{X/S},$$
$$\pi_* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \to \sigma_{\mathrm{con}}^* \mathcal{H}om_{\mathcal{O}_P}(\mathcal{I}_{\mathrm{con}}, \mathcal{I}_{\mathrm{Hdg}}) \cong \sigma_{\mathrm{con}}^* \mathcal{I}_{\mathrm{Hdg}} \otimes_{\mathcal{O}_X} \Phi^* \omega_{X^F/S}$$

[cf. the discussion preceding Proposition B.1; Proposition B.1, (2)] are *isomorphisms*. Thus, by means of the natural identification $\sigma_{\mathrm{Hdg}}^* \mathcal{I}_{\mathrm{con}} = \sigma_{\mathrm{con}}^* \mathcal{I}_{\mathrm{Hdg}}$, we obtain:

PROPOSITION B.3. *There exist isomorphisms of invertible sheaves on* X

$$\mathcal{H}_P^{\otimes 2} \cong \mathcal{H}om_{\mathcal{O}_X}(\Phi^* \tau_{X^F/S}, \tau_{X/S}) \cong \omega_{X/S}^{\otimes p-1}.$$

In particular, the **square of the Hasse defect** *is* **trivial**.

Moreover, we obtain:

PROPOSITION B.4. *The global section of*

$$\mathcal{H}om_{\mathcal{O}_X}(\Phi^* \tau_{X^F/S}, \tau_{X/S})$$

obtained, relative to the isomorphism of Proposition B.3, *by forming the square of the global section of* $\mathcal{H}_P$ *determined by the natural inclusion* $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ *coincides, up to multiplication by a global section of* $\mathcal{O}_S^{\times}$, *with the* **square Hasse invariant** *of* P. *In particular, the global section of* $\mathcal{H}_P$ *determined by the natural inclusion* $\mathcal{I}_{\mathrm{con}} \hookrightarrow \mathcal{O}_P$ *coincides, up to multiplication by a global section of* $\mathcal{O}_S^{\times}$, *with the* **Hasse invariant** *of* P.

Proof. This follows from the discussion in the proof of [6], Chapter II, Proposition 2.6, (3), p. 1032. $\square$

Appendix C. Various moduli stacks

In the Appendix C, we consider various moduli stacks related to the notions discussed in the present paper. In the Appendix C, let p be an *odd* prime number and $g \geq 2$ an integer.

We shall write

$$\mathcal{M}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p;

$$\mathcal{C}_g \to \mathcal{M}_g$$

for the universal curve over $\mathcal{M}_g$;

$$\mathcal{J}_g \to \mathcal{M}_g$$

for the relative Jacobian variety of $\mathcal{C}_g \to \mathcal{M}_g$;

$$\mathcal{J}_g[n] \subseteq \mathcal{J}_g$$

for the kernel of the endomorphism of $\mathcal{J}_g$ over $\mathcal{M}_g$ obtained by multiplication by n [where n is a nonnegative integer]. Moreover, we shall write

$$\mathcal{N}_g$$

for the moduli stack of smooth *nilcurves* of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{adm}} \subseteq \mathcal{N}_g$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent admissible indigenous bundles*;

$$\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$$

for the moduli stack of projective smooth curves of genus g of characteristic p equipped with *nilpotent ordinary indigenous bundles*.

Definition C.1. It follows from the final portion of Proposition B.3 that the *Hasse defect* of the universal nilpotent admissible indigenous bundle over $\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{N}_g^{\mathrm{adm}} \to \mathcal{N}_g^{\mathrm{adm}}$ determines a (1-)morphism over $\mathcal{M}_g$

$$\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{J}_g[2].$$

We shall refer to this (1-)morphism as the *Hasse defect morphism*.

Proposition C.2. *The following three open substacks of* $\mathcal{N}_g^{\mathrm{adm}}$ **coincide**:
(1) *The open substack* $\mathcal{N}_g^{\mathrm{ord}} \subseteq \mathcal{N}_g^{\mathrm{adm}}$.
(2) *The* **étale locus** *of the natural* (1-)*morphism* $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$.
(3) *The* **étale locus** *of the Hasse defect morphism* $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{J}_g[2]$.

Proof. The assertion that the open substack given in (1) *coincides* with the open substack given in (2) follows from the definition [cf. also the discussion following [7], Introduction, Theorem 0.1, p. 24]. On the other hand, since the (1-)morphism $\mathcal{N}_g^{\mathrm{adm}} \to \mathcal{M}_g$ is *flat* [cf. [6], Chapter II, Theorem 2.3, p. 1029], the assertion that the open substack given in (2) *coincides* with the open substack given in (3) follows from the well-known fact that the natural (1-)morphism $\mathcal{J}_g[2] \to \mathcal{M}_g$ is a *finite étale surjection.* $\square$

Now let us observe that since, as is well-known, $\mathcal{J}_g[2]$ is *finite étale* over $\mathcal{M}_g$, the identity section of $\mathcal{J}_g \to \mathcal{M}_g$ determines an isomorphism of stacks over $\mathcal{M}_g$

$$\mathcal{M}_g \sqcup (\mathcal{J}_g[2] \backslash \mathcal{J}_g[1]) \xrightarrow{\sim} \mathcal{J}_g[2].$$

Thus, by considering the Hasse defect morphism, we obtain:

Proposition C.3. *Let $U \to \mathcal{N}_g^{\mathrm{adm}}$ be a scheme over $\mathcal{N}_g^{\mathrm{adm}}$. Suppose that there exist two geometric points s_1, s_2 of U such that the Hasse defect of the nilpotent admissible indigenous bundle corresponding to s_1 (respectively, s_2) is* **of relative order one** (*respectively,* **two**). *Then U is* **not connected**.

Next, we shall write

$$\mathcal{R}_g$$

for the moduli stack of "*nontrivial*" smooth *Prym curves* of genus g of characteristic p, i.e., the moduli stack of projective smooth curves of genus g of characteristic p equipped with square-trivialized invertible sheaves whose underlying invertible sheaves are of relative order two;

$$\underline{\mathcal{R}}_g \stackrel{\mathrm{def}}{=} \mathcal{J}_g[2] \backslash \mathcal{J}_g[1].$$

Thus, we have a natural (1-)morphism $\mathcal{R}_g \to \underline{\mathcal{R}}_g$ over $\mathcal{M}_g$. For a nonnegative integer d, write

$$\mathcal{A}_d$$

for the moduli stack of principally polarized abelian varieties of dimension d of characteristic p and

$$\mathcal{A}_d^{\mathrm{ord}} \subseteq \mathcal{A}_d$$

for the moduli stack of principally polarized *ordinary* abelian varieties of dimension d of characteristic p.

Definition C.4. Since $\mathcal{M}_g \sqcup \underline{\mathcal{R}}_g$ is naturally isomorphic to $\mathcal{J}_g[2]$ over $\mathcal{M}_g$, the inverse image of $\mathcal{A}_g^{\mathrm{ord}} \subseteq \mathcal{A}_g$ via the *Torelli morphism* $\mathcal{M}_g \to \mathcal{A}_g$ and the image in $\underline{\mathscr{R}}_g$ of the inverse image of $\mathscr{A}_{g-1}^{\mathrm{ord}} \subseteq \mathscr{A}_{g-1}$ via the *Prym morphism* $\mathscr{R}_g \to \mathscr{A}_{g-1}$ determine an open substack of $\mathscr{I}_g[2]$. We shall write

$$\mathscr{I}_g[2]^{\text{pb-ord}} \subseteq \mathscr{I}_g[2]$$

for this open substack.

Thus, it follows immediately from the various definitions involved that the following proposition holds:

PROPOSITION C.5. *In the notation introduced at the beginning of the Appendix* B, *let* P *be a* **nilpotent admissible** *indigenous bundle over* X/S. *Then the following conditions are equivalent*:

(1) *The image of the composite*

$$S \to \mathscr{N}_g^{\mathrm{adm}} \to \mathscr{I}_g[2]$$

of the classifying (1-)*morphism* $S \to \mathscr{N}_g^{\mathrm{adm}}$ *of* P *and the Hasse defect morphism is* **contained** *in the open substack* $\mathscr{I}_g[2]^{\text{pb-ord}} \subseteq \mathscr{I}_g[2]$.

(2) *The Hasse defect of* P *is* **parabolically ordinary**.

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