EQUINORMALIZABILITY AND TOPOLOGICALLY TRIVIALITY OF DEFORMATIONS OF ISOLATED CURVE SINGULARITIES OVER SMOOTH BASE SPACES

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Dedicated to Professor Gert-Martin Greuel on the occasion of his 70th birthday

Abstract

We give a δ-constant criterion for equinormalizability of deformations of isolated (not necessarily reduced) curve singularities over smooth base spaces of dimension ≥ 1. For one-parametric families of isolated curve singularities, we show that their topologically triviality is equivalent to the admission of weak simultaneous resolutions.

1. Introduction

The theory of equinormalizable deformations has been initiated by B. Teissier ([14]) in the late 1970's for deformations of reduced curve singularities over $(\mathbf{C}, 0)$. It is generalized to higher dimensional base spaces by M. Raynaud and Teissier himself ([15; some insight into the background of Raynaud's argument might be gleaned from the introduction to [10]). Recently, it is developed by Chiang-Hsieh and Lipman ([5], 2006) for projective deformations of reduced complex spaces over normal base spaces, and it is studied by Kollár ([11], 2011) for projective deformations of generically reduced algebraic schemes over semi-normal base spaces.

Each reduced curve singularity is associated with a δ number (see Definition 3.3), which is a finite number and it is a topological invariant of reduced curve singularities. Teissier-Raynaud-Chiang-Hsieh-Lipman ([14], [15], [5]) showed that a deformation of a reduced curve singularity over a normal base space is equinormalizable (see Definition 3.1) if and only if it is δ-constant, that is the δ number of all of its fibers are the same. This is so-called the *δ-constant criterion* for equinormalizability of deformations of reduced curve singularities.

For isolated curve singularities with embedded components, Brücker and Greuel ([3], 1990) gave a similar δ-constant criterion (with a new definition of the

δ number, see Definition 3.3) for equinormalizability of deformations of isolated (not necessarily reduced) curve singularities over $(\mathbf{C}, 0)$. The author considered in [12] (2012) deformations of *plane curve singularities* with embedded components over smooth base spaces of dimension ≥ 1, and gave a similar δ-constant criterion for equinormalizability of these deformations, using special techniques (e.g. a corollary of Hilbert-Burch theorem), which are effective only for plane curve singularities.

The first purpose of this paper is to generalize the δ-constant criterion given in [3] and [12] to deformations of isolated (not necessarily reduced) curve singularities over normal or smooth base spaces of dimension ≥ 1. In Proposition 3.4 we show that equinormalizability of deformations of isolated curve singularities over normal base spaces implies the constancy of the δ number of fibers of these deformations. Moreover, in Theorem 3.6 we show that if the normalization of the total space of a deformation of an isolated curve singularity over $(\mathbf{C}^k, 0)$, $k \geq 1$, is Cohen-Macaulay then the converse holds. The assumption on Cohen-Macaulayness of the normalization of the total space ensures for flatness of the composition map. Moreover, Cohen-Macaulayness of the normalization of the total space is always satified for deformations over $(\mathbf{C}, 0)$, because in this case, the total space is a normal surface singularity, which is Cohen-Macaulay.

In all of known results for the δ-constant criterion for equinormalizability of deformations of isolated curve singularities, the total spaces of these deformations are always assumed to be reduced and pure dimensional. It is necessary to weaken the hypothesis on reducedness or purity of the dimension of total spaces. In section 2 we study the relationship between reducedness of the total space and that of the *generic fibers* of a flat morphism, and show in Theorem 2.5 that if the generic fibers of a flat morphism over a reduced Cohen-Macaulay space are reduced then the total space is reduced. In particular, if there exists a representative of a deformation of an isolated singularity over a reduced Cohen-Macaulay base space such that *the total space is generically reduced over the base space* then the total space is reduced (see Corollary 2.6). This gives a way to check reducedness of the total space of a deformation, and to weaken the hypothesis on reducedness of the total space of a deformation.

For families of isolated curve singularities, one of the most important things is the admission of weak simultaneous resolutions ([15]) of these families. Buchweitz and Greuel ([2], 1980) gave a list of criteria for the admission of weak simultaneous resolutions of one-parametric families of reduced curve singularities, namely, the constancy of the Milnor number, the constancy of the δ number as well as the number of branches of all fibers, and the topologically triviality of these families (see Theorem 4.3). In the last section, we use a very new result of Bobadilla, Snoussi and Spivakovsky (2014) to show that these criteria are also true for one-parametric families of isolated (not necessarily reduced) curve singularities (see Theorem 4.5).

NOTATION: Let $f : (X, x) \to (S, 0)$ be a morphism of complex germs. Denote by (X^{red}, x) the reduction of (X, x) and $i : (X^{red}, x) \hookrightarrow (X, x)$ the inclusion.

Let $v^{red} : (\overline{X}, \bar{x}) \to (X^{red}, x)$ be the normalization of (X^{red}, x), where $\bar{x} := (v^{red})^{-1}(x)$. Then the composition $v : (\overline{X}, \bar{x}) \xrightarrow{v^{red}} (X^{red}, x) \xhookrightarrow{i} (X, x)$ is called the *normalization of* (X, x). Denote $\bar{f} := f \circ v : (\overline{X}, \bar{x}) \to (S, 0)$. For each $s \in S$, we denote

$$X_s := f^{-1}(s), \quad \overline{X}_s := \bar{f}^{-1}(s).$$

2. Generic reducedness

Let $f : (X, x) \to (S, 0)$ be a flat morphism of complex germs. In this section we study the relationship between reducedness of the total space (X, x) and that of the generic fibers of f. This gives a way to check reducedness of the total space of a flat morphism.

DEFINITION 2.1. Let $f : X \to S$ be a morphism of complex spaces. Denote by $\mathrm{Red}(X)$ the set of all reduced points of X and

$$\mathrm{Red}(f) = \{x \in X \,|\, f \text{ is flat at } x \text{ and } f^{-1}(f(x)) \text{ is reduced at } x\}$$

the *reduced locus* of f. We say
  (1) X is *generically reduced* if $\mathrm{Red}(X)$ is open and dense in X;
  (2) X is *generically reduced over* S if there is an analytically open dense set V in S such that $f^{-1}(V)$ is contained in $\mathrm{Red}(X)$;
  (3) the *generic fibers of* f *are reduced* if there is an analytically open dense set V in S such that $X_s := f^{-1}(s)$ is reduced for all s in V.

We show in the following that under properness of the restriction of a flat morphism $f : (X, x) \to (S, 0)$ to its non-reduced locus, the generically reducedness of X over S implies reducedness of the generic fibers of f.

PROPOSITION 2.2. *Let* $f : (X, x) \to (S, 0)$ *be flat with* $(S, 0)$ *reduced. Assume that there is a representative* $f : X \to S$ *such that its restriction on the non-reduced locus* $\mathrm{NRed}(f) := X \backslash \mathrm{Red}(f)$ *is proper and* X *is generically reduced over* S. *Then the generic fibers of* f *are reduced.*

Proof. $\mathrm{NRed}(f)$ is analytically closed in X (cf. [8, Corollary I.1.116]). Moreover, since X is generically reduced over S, there exists an analytically open dense set U in S such that $f^{-1}(U) \subseteq \mathrm{Red}(X)$. Then, by properness of the restriction $\mathrm{NRed}(f) \to S$, $f(\mathrm{NRed}(f))$ is analytically closed and nowhere dense in S by [1, Theorem 2.1(3), p. 56]. This implies that $V := S \backslash f(\mathrm{NRed}(f))$ is analytically open dense in S, and for all $s \in V$, $X_s := f^{-1}(s)$ is reduced. Therefore the generic fibers of f are reduced. $\qquad\square$

COROLLARY 2.3. *Let* $f : (X, x) \to (S, 0)$ *be flat with* $(S, 0)$ *reduced. Assume that* $X_0 \backslash \{x\}$ *is reduced and there exists a representative* $f : X \to S$ *such that* X *is generically reduced over* S. *Then the generic fibers of* f *are reduced.*

In particular, if $X_0 \setminus \{x\}$ and (X, x) are reduced, then the generic fibers of f are reduced.

Proof. Since f is flat, we have

$$\mathrm{NRed}(f) \cap X_0 = \mathrm{NRed}(X_0) \subseteq \{x\},$$

where $\mathrm{NRed}(X_0)$ denotes the set of non-reduced points of X_0. This implies that the restriction $f : \mathrm{NRed}(f) \to S$ is finite, hence proper. Then the first assertion follows from Proposition 2.2. Moreover, if (X, x) is reduced then there exists a representative X of (X, x) which is reduced. Then X is obviously generically reduced over some representative S of (S, s). Hence we have the latter assertion. □

Remark 2.4. The assumption on reducedness of $X_0 \setminus \{x\}$ in Corollary 2.3 is necessary for reducedness of generic fibers, even for the case $S = \mathbf{C}$. In fact, let $(X_0, 0) \subseteq (\mathbf{C}^3, 0)$ be defined by the ideal

$$I_0 = \langle x^2, y \rangle \cap \langle y^2, z \rangle \cap \langle z^2, x \rangle \subseteq \mathbf{C}\{x, y, z\}$$

and $(X, 0) \subseteq (\mathbf{C}^4, 0)$ defined by the ideal

$$I = \langle x^2 - t^2, y \rangle \cap \langle y^2 - t^2, z \rangle \cap \langle z^2, x \rangle \subseteq \mathbf{C}\{x, y, z, t\}.$$

Let $f : (X, 0) \to (\mathbf{C}, 0)$ be the restriction on $(X, 0)$ of the projection on the fourth component $\pi : (\mathbf{C}^4, 0) \to (\mathbf{C}, 0)$, $(x, y, z, t) \mapsto t$. Then f is flat, $X \setminus X_0$ is reduced, hence X is generically reduced over some representative T of $(\mathbf{C}, 0)$. However the fiber $(X_t, 0)$ is not reduced for any $t \neq 0$. Note that in this case $X_0 \setminus \{0\}$ is not reduced.

As we have seen from Corollary 2.3, if the total space of a flat morphism over a reduced base space is reduced, then the generic fibers of that morphism are reduced. In the following we shows that over a reduced Cohen-Macaulay base space, the converse is also true. This generalizes [3, Proposition 3.1.1 (3)] to deformations over higher dimensional base spaces.

THEOREM 2.5. *Let $f : (X, x) \to (S, 0)$ be flat with $(S, 0)$ reduced Cohen-Macaulay of dimension $k \geq 1$. If there exists a representative $f : X \to S$ whose generic fibers are reduced, then (X, x) is reduced.*

Proof. We divide the proof of this part into two steps.

STEP 1: $S = \mathbf{C}^k$. Then $f = (f_1, \ldots, f_k) : (X, x) \to (\mathbf{C}^k, 0)$ is flat.

For $k = 1$, assume that there exists a representative $f : X \to T$ such that $X_t := f^{-1}(t)$ is reduced for every $t \neq 0$. Then for any $y \in X \setminus X_0$ we have $(X_{f(y)}, y)$ is reduced. It follows that (X, y) is reduced (cf. [8, Theorem I. 1.101]). Thus $X \setminus X_0$ is reduced. To show that (X, x) is reduced, let g be a nilpotent element of $\mathcal{O}_{X, x}$. Then we have

$$supp(g) = V(\mathrm{Ann}(g)) \subseteq X_0 = V(f).$$

It follows from Hilbert-Rückert's Nullstellensatz (cf. [8, Theorem I.1.72]) that $f^n \in \operatorname{Ann}(g)$ for some $n \in \mathbf{Z}_+$. Hence $f^n g = 0$ in $\mathcal{O}_{X,x}$. Since f is flat, it is a non-zerodivisor in $\mathcal{O}_{X,x}$. Then f^n is also a non-zerodivizor in $\mathcal{O}_{X,x}$. It follows that $g = 0$. Thus (X, x) is reduced, and the statement is true for $k = 1$.

For $k \geq 2$, suppose there is a representative $f : X \to S$ and an analytically open dense set V in S such that X_s is reduced for all $s \in V$. Let us denote by H the line

$$H := \{(t_1, \ldots, t_k) \in \mathbf{C}^k \,|\, t_1 = \cdots = t_{k-1} = 0\}.$$

Denote by A the complement of V in S. Then A is analytically closed and nowhere dense in S. We can choose coordinates $t_1, \ldots, t_k$ and a representative of $(\mathbf{C}^k, 0)$ such that $A \cap H = \{0\}$.

Denote $f' := (f_1, \ldots, f_{k-1})$. Since f is flat, $f_1, \ldots, f_{k-1}$ is an $\mathcal{O}_{X,x}$-regular sequence, hence $f' : (X, x) \to (\mathbf{C}^{k-1}, 0)$ is flat with the special fiber $(X', x) := (f'^{-1}(0), x) = (f^{-1}(H), x)$. Since f is flat, f_k is a non-zerodivisor in $\mathcal{O}_{X,x}/f'\mathcal{O}_{X,x} = \mathcal{O}_{X',x}$, hence the morphism $f_k : (X', x) \to (\mathbf{C}, 0)$ is flat. For any $t \in \mathbf{C} \setminus \{0\}$ close to 0, we have $(0, \ldots, 0, t) \notin A$, hence $f_k^{-1}(t) = f^{-1}(0, \ldots, 0, t)$ is reduced. It follows from the case $k = 1$ that the total space (X', x) of f_k is reduced. Since $f' : (X, x) \to (\mathbf{C}^{k-1}, 0)$ is flat whose special fiber is reduced, (X, x) is reduced (cf. [8, Theorem I.1.101]), and we have the proof for this step.

STEP 2: $(S, 0)$ is Cohen-Macaulay of dimension $k \geq 1$. Since $(S, 0)$ is Cohen-Macaulay, there exists an $\mathcal{O}_{S,0}$-regular sequence $g_1, \ldots, g_k$, where $g_i \in \mathcal{O}_{S,0}$ for every $i = 1, \ldots, k$. Then the morphism

$$g = (g_1, \ldots, g_k) : (S, 0) \to (\mathbf{C}^k, 0), \quad t \mapsto (g_1(t), \ldots, g_k(t))$$

is flat. We have

$$\dim(g^{-1}(0), 0) = \dim \mathcal{O}_{S,0}/(g_1, \ldots, g_k)\mathcal{O}_{S,0} = 0$$

(cf. [8, Prop. I.1.85]). This implies that g is finite. Let $g : S \to T$ be a representative which is flat and finite, where T is an open neighborhood of $0 \in \mathbf{C}^k$. Then the composition $h = g \circ f : X \to T$ (for some representative) is flat. To apply Step 1 for h, we need to show the existence of an analytically open dense set U in T such that all fibers over U are reduced. In fact, since S is reduced, its singular locus $\operatorname{Sing}(S)$ is closed and nowhere dense in S (cf. [8, Corollary I.1.111]). It follows that $A \cup \operatorname{Sing}(S)$, A as in Step 1, is closed and nowhere dense in S. Then the set $U := T \setminus g(A \cup \operatorname{Sing}(S))$ is open and dense in T by the finiteness of g. Furthermore, for any $t \in U$, $g^{-1}(t) = \{t_1, \ldots, t_r\}$, $t_i \in V \cap (S \setminus \operatorname{Sing}(S))$. It follows that $h^{-1}(t) = f^{-1}(t_1) \cup \cdots \cup f^{-1}(t_r)$ is reduced.

Now applying Step 1 for the flat map $h : X \to T$, we have reducedness of (X, x). The proof is complete. □

The following result is a direct consequence of Corollary 2.3 and Theorem 2.5.

COROLLARY 2.6. *Let $f : (X,x) \to (S,0)$ be flat with $(S,0)$ reduced Cohen-Macaulay of dimension $k \geq 1$. Suppose $X_0 \backslash \{x\}$ is reduced and there exists a representative $f : X \to S$ such that X is generically reduced over S. Then (X,x) is reduced.*

Since normal surface singularities are reduced and Cohen-Macaulay, we have

COROLLARY 2.7. *Let $f : (X,x) \to (S,0)$ be flat with $(S,0)$ a normal surface singularity. If there exists a representative $f : X \to S$ whose generic fibers are reduced, then (X,x) is reduced.*

3. Equinormalizable deformations of isolated curve singularities over smooth base spaces

In this section we focus on equinormalizability of deformations of isolated (not necessarily reduced) curve singularities over smooth base spaces of dimension ≥ 1. Because of isolatedness of singularities in the special fibers of these deformations, by Corollary 2.6, instead of assuming reducedness of the total spaces, we need only assume the generically reducedness of the total spaces over the base spaces.

First we recall a definition of equinormalizable deformations which follows Chiang-Hsieh-Lipman ([5]) and Kollár ([11]).

DEFINITION 3.1. Let $f : X \to S$ be a morphism of complex spaces. A *simultaneous normalization* of f is a morphism $n : \tilde{X} \to X$ such that
  (1) n is finite,
  (2) $\tilde{f} := f \circ n : \tilde{X} \to S$ is *normal*, i.e., for each $z \in \tilde{X}$, $\tilde{f}$ is flat at z and the fiber $\tilde{X}_{\tilde{f}(z)} := \tilde{f}^{-1}(\tilde{f}(z))$ is normal,
  (3) the induced map $n_s : \tilde{X}_s := \tilde{f}^{-1}(s) \to X_s$ is bimeromorphic for each $s \in f(X)$.
The morphism f is called *equinormalizable* if the normalization $v : \overline{X} \to X$ is a simultaneous normalization of f. It is called *equinormalizable at $x \in X$* if the restriction of f to some neighborhood of x is equinormalizable.

If $f : (X,x) \to (S,s)$ is a morphism of germs, then a *simultaneous normalization of f* is a morphism n from a multi-germ $(\tilde{X}, n^{-1}(x))$ to (X,x) such that some representative of n is a simultaneous normalization of a representative of f. The germ f is *equinormalizable* if some representative of f is equinormalizable.

The following lemma allows us to do base change, reducing deformations over higher dimensional base spaces to those over smooth 1-dimensional base spaces with similar properties.

LEMMA 3.2. *Let $f : (X,x) \to (S,0)$ be a deformation of an isolated singularity (X_0,x) with $(S,0)$ normal. Suppose that there exists some representative*

$f : X \to S$ such that X is generically reduced over S. Then there exists an open and dense set U in S such that $X_s := f^{-1}(s)$ is reduced, $\overline{X}_s := \bar{f}^{-1}(s)$ is normal for all $s \in U$. Moreover, for each $s \in U$, the induced morphism on the fibers $v_s : \overline{X}_s \to X_s$ is the normalization of X_s.

Here we recall that $v : (\overline{X}, \bar{x}) \to (X, x)$ is the normalization of (X, x) and $\bar{f} := f \circ v : (\overline{X}, \bar{x}) \to (S, 0)$.

Proof. Since $X_0 \backslash \{x\}$ is reduced, it follows from the proof of Corollary 2.3 that the set $f(\mathrm{NRed}(f))$ is closed and nowhere dense in S. Denote by $\mathrm{NNor}(f)$ (resp. $\mathrm{NNor}(\bar{f})$) the *non-normal locus of f (resp. $\bar{f}$)*, the set of points z in X (resp. $\overline{X}$) at which either f (resp. $\bar{f}$) is not flat or $X_{f(z)}$ (resp. $\overline{X}_{\bar{f}(z)}$) is not normal. Since f is flat and S is normal, we have $v(\mathrm{NNor}(\bar{f}) \cap \overline{X}_0)$ $\subseteq \mathrm{NNor}(f) \cap X_0 = \mathrm{NNor}(X_0)$. Equivalently, $\mathrm{NNor}(\bar{f}) \cap \overline{X}_0 \subseteq v^{-1}(\mathrm{NNor}(X_0))$ which is finite since v is finite and X_0 has an isolated singularity at x. It follows that the restriction of $\bar{f}$ on $\mathrm{NNor}(\bar{f})$ is finite. Then $\bar{f}(\mathrm{NNor}(\bar{f}))$ is closed and nowhere dense in S by [1, Theorem 2.1(3), p. 56]. The set $U :=$ $S \backslash (f(\mathrm{NRed}(f)) \cup \bar{f}(\mathrm{NNor}(\bar{f})))$ satisfies all the required properties. □

For deformations of isolated curve singularities we have the following necessary condition for their equinormalizability, in terms of the constancy of the δ-invariant of fibers. For the reader's convenience we recall the definition of the δ-invariant of isolated (not necessarily reduced) curve singularities, which is defined by Brücker and Greuel in [3].

DEFINITION 3.3. Let X be a complex curve and $x \in X$ an isolated singular point. Denote by X^{red} its reduction and let $v^{red} : \overline{X} \to X^{red}$ be the normalization of the reduced curve X^{red}. The number

$$\delta(X^{red}, x) := \dim_{\mathbf{C}} (v_*^{red} \mathcal{O}_{\overline{X}})_x / \mathcal{O}_{X^{red}, x}$$

is called the *delta-invariant of X^{red} at x*,

$$\varepsilon(X, x) := \dim_{\mathbf{C}} H^0_{\{x\}}(\mathcal{O}_X)$$

is called the *epsilon-invariant of X at x*, where $H^0_{\{x\}}(\mathcal{O}_X)$ denotes local cohomology, and

$$\delta(X, x) := \delta(X^{red}, x) - \varepsilon(X, x)$$

is called the *delta-invariant of X at x*.

If X has only finitely many singular points then the number

$$\delta(X) := \sum_{x \in \mathrm{Sing}(X)} \delta(X, x)$$

is called the *delta-invariant* of X.

It is easy to see that $\delta(X^{red}, x) \geq 0$, and $\delta(X^{red}, x) = 0$ if and only if x is an isolated point of X or the germ (X^{red}, x) is smooth. Hence, if $x \in X$ is an isolated point of X then $\delta(X, x) = -\dim_{\mathbf{C}} \mathcal{O}_{X,x} = -\varepsilon(X, x)$. In particular, $\delta(X, x) = -1$ for x an isolated and reduced (hence normal) point of X.

PROPOSITION 3.4. *Let $f : (X, x) \to (S, 0)$ be a deformation of an isolated curve singularity (X_0, x) with (X, x) pure dimensional, $(S, 0)$ normal. Suppose that there exists some representative $f : X \to S$ such that X is generically reduced over S. If f is equinormalizable, then it is δ-constant, that is, $\delta(X_s) = \delta(X_0)$ for every $s \in S$ close to 0.*

Proof (Compare to the proof of [12, Theorem 4.1 (2)]). It follows from Lemma 3.2 that there exists an open and dense set U in S such that X_s is reduced and $\overline{X}_s$ is normal for all $s \in U$.

We first show that f is δ-constant on U, i.e. $\delta(X_s) = \delta(X_0)$ for any $s \in U$. In fact, for any $s \in U$, $s \neq 0$, there exist an irreducible reduced curve singularity $C \subseteq S$ passing through 0 and s. Let $\alpha : T \to C \subseteq S$ be the normalization of this curve singularity such that $\alpha(T\backslash\{0\}) \subseteq U$, where $T \subseteq \mathbf{C}$ is a small disc with center at 0. Denote

$$X_T := X \times_S T, \quad \overline{X}_T := \overline{X} \times_S T.$$

Then we have the following Cartesian diagram:

$$
\begin{array}{ccc}
\overline{X}_T & \to & \overline{X} \\
v_T\downarrow \quad \square \quad \downarrow v \\
X_T & \to & X \\
f_T\downarrow \quad \square \quad \downarrow f \\
T & \to & S
\end{array}
$$

For any $t \in T$, $s = \alpha(t) \in S$, we have

$$(3.1) \qquad \mathcal{O}_{(X_T)_t} := \mathcal{O}_{f_T^{-1}(t)} \cong \mathcal{O}_{X_s}, \quad \mathcal{O}_{(\overline{X}_T)_t} := \mathcal{O}_{\overline{f}_T^{-1}(t)} \cong \mathcal{O}_{\overline{X}_s}.$$

Since f is flat by hypothesis and $\overline{f}$ is flat by equinormalizability, it follows from the preservation of flatness under base change (cf. [8, Prop. I.1.87]) that the induced morphisms f_T and $\overline{f}_T$ are flat over T. Hence, it follows from equinormalizability of f and (3.1) that $f_T : X_T \to T$ is equinormalizable.

For any $t \in T\backslash\{0\}$, $s = \alpha(t) \in U$, hence $(X_T)_t \cong X_s$ is reduced by the existence of U. It follows from Theorem 2.5 that X_T is reduced. On the other hand, since X and S are pure dimensional, all fibers of f, hence of f_T, are pure dimensional by the dimension formula ([7, Lemma, p. 156]). Then X_T is also pure dimensional because T is pure 1-dimensional. Therefore it follows from [3, Korollar 2.3.5] that $f_T : X_T \to T$ is δ-constant, hence $f : X \to S$ is δ-constant on U.

Let us now take $s_0 \in S \backslash U$. Since U is dense in S, $s_0 \in S$, there exists always a point $s_1 \in U$ which is close to s_0. It follows from the semi-continuity of the δ-function (cf. [12, Lemma 4.2]) that

$$\delta(X_0) \geq \delta(X_{s_0}) \geq \delta(X_{s_1}).$$

Moreover, $\delta(X_0) = \delta(X_{s_1})$ as shown above. It implies that $\delta(X_{s_0}) = \delta(X_0)$. Hence $f : X \to S$ is δ-constant. □

Remark 3.5. The complex spaces X_T and $\overline{X}_T$ appearing in the proof of Proposition 3.4 have the following properties:
  (1) X_T is reduced; $\overline{X}_T$ is reduced if $\bar{f}_T$ is flat;
  (2) they have the same normalization $\widetilde{X_T}$;
  (3) fibers of the compositions $\widetilde{X_T} \overset{\mu_T}{\to} \overline{X}_T \overset{\bar{f}_T}{\to} T$ and $\widetilde{X_T} \overset{\theta_T}{\to} X_T \overset{f_T}{\to} T$ coincide. In fact, as we have seen in the proof of Proposition 3.4, X_T is reduced. Moreover, if $\bar{f}_T$ is flat, since its generic fibers are reduced (actually normal), $\overline{X}_T$ is reduced by Theorem 2.5. Therefore we have (1).
  Now we show (2). Since finiteness and surjectivity are preserved under base change, ν_T is finite and surjective. Let us denote by $\mu_T : \widetilde{X_T} \to \overline{X}_T$ the normalization of $\overline{X}_T$. Then the composition $\theta_T := \mu_T \circ \nu_T$ is finite and surjective.
  Denote $A := \mathrm{NNor}(f_T)$. Since X_T is reduced, A is nowhere dense in X_T. Moreover, since ν_T is finite and surjective, it follows from Ritt's lemma (cf. [9, Chapter 5, §3, p. 102]) that the preimage $A' := \nu_T^{-1}(A)$ is nowhere dense in $\overline{X}_T$. Furthermore, for any $z \notin A'$, $y = \nu_T(z) \notin A$, hence the fiber $(X_T)_t$ resp. X_s is normal at y resp. $\alpha_T(y)$, where $t = f_T(y)$, $s = \alpha(t)$. Thus $(X, \alpha_T(y)) \cong (\overline{X}, \overline{\alpha}_T(z))$. It follows that $(X_T, y) \cong (\overline{X}_T, z)$. Therefore $\overline{X}_T \backslash A' \cong X_T \backslash A$. Then $(\mu_T \circ \nu_T)^{-1}(A)$ is nowhere dense in $\widetilde{X_T}$ and we have the isomorphism

$$\widetilde{X_T} \backslash (\mu_T \circ \nu_T)^{-1}(A) = \widetilde{X_T} \backslash \mu_T^{-1}(A') \cong \overline{X}_T \backslash A' \cong X_T \backslash A.$$

Therefore θ_T is bimeromorphic, whence it is the normalization of X_T. (3) is obvious.

The following theorem is the main result of this section, which asserts that under certain conditions, the δ-criterion is sufficient for equinormalizability of deformations of isolated curve singularities over smooth base spaces of dimension ≥ 1. This gives a generalization of [3, Korollar 2.3.5].

THEOREM 3.6. *Let $f : (X, x) \to (\mathbf{C}^k, 0)$, $k \geq 1$, be a deformation of an isolated curve singularity (X_0, x) with (X, x) pure dimensional. Suppose that there exists a representative $f : X \to S$ such that X is generically reduced over S. If the normalization $\overline{X}$ of X is Cohen-Macaulay[1] and f is δ-constant, then f is equinormalizable.*

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[1] This holds always for $k = 1$, since normal surfaces are Cohen-Macaulay.

Proof. First we show that Cohen-Macaulayness of $\overline{X}$ implies flatness of the composition $\overline{f}$. Since $\overline{X}$ is Cohen-Macaulay and S is smooth, it is sufficient to check that the dimension formula holds for $\overline{f}$ (cf. [7, Proposition, p. 158]). But it is always the case, since for any $z \in v^{-1}(x)$, we have

$$\dim(\overline{X}, z) = \dim(X, x) = \dim(X_0, x) + k \quad \text{by flatness of } f$$
$$= \dim(\overline{X}_0, z) + k.$$

The latter equality follows from finiteness and surjectivity of $v_0 : (\overline{X}_0, z) \to (X_0, x)$.

Let $U \subseteq S$ be the open dense set with properties described as in Lemma 3.2. For any $s \in U$, let $C \subseteq S$ be an irreducible reduced curve singularity passing through s and 0 such that $C \cap (S \backslash U) = \{0\}$. Let $\alpha : T \to C \subseteq S$ be the normalization of this curve singularity such that $\alpha(T \backslash \{0\}) \subseteq U$, where $T \subseteq \mathbf{C}$ is a small disc with center at 0. Denote X_T and $\overline{X}_T$ as in the proof of Proposition 3.4. Then, since $\overline{f}$ is flat, it follows from Remark 3.5 that X_T and $\overline{X}_T$ are reduced and they have the same normalization $\tilde{X}_T$. Consider the following Cartesian diagram:

$$
\begin{array}{c}
\widetilde{X_T} \\
\theta_T \Bigg/ \quad \Big\downarrow \mu_T \\
\overline{X}_T \xrightarrow{\bar{\alpha}_T} \overline{X} \\
\Big\downarrow v_T \;\square\; v\Big\downarrow \\
X_T \xrightarrow{\alpha_T} X \\
\Big\downarrow f_T \;\square\; f\Big\downarrow \\
T \underset{\alpha}{\to} S
\end{array}
$$

Since fibers of f and f_T are isomorphic, f_T is δ-constant and X_T is pure dimensional. Then it follows from [3, Korollar 2.3.5] that f_T is equinormalizable. Therefore, by definition, for each $t \in T$, $(\tilde{X})_t := (\tilde{f}_T)^{-1}(t)$ is normal, and it is the normalization of $(X_T)_t$.

Let us consider the flat map $\overline{f}_T : \overline{X}_T \to T$ and consider the normalization $\mu_T : \widetilde{X_T} \to \overline{X}_T$ of $\overline{X}_T$. It follows from [3, Proposition 1.2.2] that the composition $\overline{f}_T \circ \mu_T : \widetilde{X_T} \to T$ is flat. Moreover, by the same argument as given in Remark 3.5, we can show that $(X_T)_t$ and $(\overline{X}_T)_t$ have the same normalization for each $t \in T$. Hence the restriction on the fibers $(\tilde{X})_t \to (\overline{X}_T)_t$ is the normalization. Thus by definition, $\overline{f}_T$ is equinormalizable. Then $\overline{f}_T$ is δ-constant by Proposition 3.4 (or by [3, Korollar 2.3.5]). This implies that for any $t \in T \backslash \{0\}$, we have

$$\delta(\overline{X}_0) = \delta((\overline{X}_T)_0) = \delta((\overline{X}_T)_t) = 0 \quad (\text{since } (\overline{X}_T)_t \text{ is normal}).$$

Now we show that $\overline{X}_0$ is reduced. First we show that $v(\mathrm{NNor}(\overline{X}_0)) \subseteq \mathrm{NNor}(X_0)$. In fact, if $y \notin \mathrm{NNor}(X_0)$ then X_0 is normal at y. Since f is flat

and S is normal at 0, X is normal at y (cf. [8, Theorem I.1.101]). Therefore we have the isomorphism $(\overline{X}, z) \overset{\cong}{\to} (X, y)$ for every $z \in v^{-1}(y)$. It induces an isomorphism on the fibers $(\overline{X}_0, z) \overset{\cong}{\to} (X_0, y)$, hence $\overline{X}_0$ is normal at every point $z \in v^{-1}(y)$. It follows that $y \notin v(\mathrm{NNor}(\overline{X}_0))$.

Then, for any $z \in \mathrm{NNor}(\overline{X}_0)$, since $\mathrm{NNor}(X_0)$ is nowhere dense in X_0, by Ritt's lemma (cf. [9, Chapter 5, §3, 2, p. 103]) and by the dimension formula (when f is flat) we have

$$\dim(v(\mathrm{NNor}(\overline{X}_0)), v(z)) \leq \dim(\mathrm{NNor}(X_0), v(z)) < \dim(X_0, v(z))$$

$$= \dim(X, v(z)) - \dim(S, 0)$$

$$= \dim(\overline{X}, z) - \dim(S, 0) \leq \dim(\overline{X}_0, z).$$

Furthermore, the restriction $v_0 : \overline{X}_0 \to X_0$ is finite. Hence

$$\dim(v(\mathrm{NNor}(\overline{X}_0)), v(z)) = \dim(\mathrm{NNor}(\overline{X}_0), z) \quad (\text{cf. } [7, \text{Corollary, p. 141}]).$$

It follows that for any $z \in \mathrm{NNor}(\overline{X}_0)$ we have $\dim(\mathrm{NNor}(\overline{X}_0), z) < \dim(\overline{X}_0, z)$, i.e., $\mathrm{NNor}(\overline{X}_0)$ is nowhere dense in $\overline{X}_0$ by Ritt's lemma. This implies that $\overline{X}_0$ is generically normal, whence generically reduced.

Moreover, for each $z \in v^{-1}(x)$, since $\overline{f}$ is flat and $\dim(\overline{X}, z) = \dim(X, x) = k + 1$, we have

$$\mathrm{depth}(\mathcal{O}_{\overline{X}_0, z}) = \mathrm{depth}(\mathcal{O}_{\overline{X}, z}) - k \geq (k + 1) - k = 1.$$

On the other hand, we have

$$\dim(\overline{X}_0, z) = \dim(\overline{X}, z) - k = 1.$$

Hence $\mathrm{depth}(\mathcal{O}_{\overline{X}_0, z}) \geq 1 = \min\{1, \dim(\overline{X}_0, z)\}$, i.e. $\overline{X}_0$ satisfies (S_1) at every point $z \in v^{-1}(x)$. This implies that $\overline{X}_0$ is reduced at every point of $v^{-1}(x)$. Then $\overline{X}_0$ is normal, and it is the normalization of X_0. It follows that f is equinormalizable. The proof is complete. $\square$

The following example illustrates our main theorem.

Example 3.7 ([13], cf. [12, Example 4.2]). Let us consider the curve singularity $(X_0, 0) \subseteq (\mathbf{C}^4, 0)$ defined by the ideal

$$I_0 := \langle x^2 - y^3, z, w \rangle \cap \langle x, y, w \rangle \cap \langle x, y, z, w^2 \rangle \subseteq \mathbf{C}\{x, y, z, w\}.$$

The curve singularity $(X_0, 0)$ is a union of a cusp C in the plane $z = w = 0$, a straight line $L = \{x = y = w = 0\}$ and an embedded non-reduced point $O = (0, 0, 0, 0)$. Now we consider the restriction $f : (X, 0) \to (\mathbf{C}^2, 0)$ of the projection $\pi : (\mathbf{C}^6, 0) \to (\mathbf{C}^2, 0)$, $(x, y, z, w, u, v) \mapsto (u, v)$, to the complex germ $(X, 0)$ defined by the ideal

$$I = \langle x^2 - y^3 + uy^2, z, w \rangle \cap \langle x, y, w - v \rangle \subseteq \mathbf{C}\{x, y, z, w, u, v\}.$$

It is easy to check that f is flat, $f^{-1}(0, 0) = (X_0, 0)$, the total space $(X, 0)$ is reduced and pure 3-dimensional, with two 3-dimensional irreducible components.

We have $\delta((X_0)^{red}) = 2$, $\varepsilon(X_0) = 1$, hence $\delta(X_0) = 1$. Moreover, for each $u, v \in \mathbf{C}\backslash\{0\}$, we have

$$\delta(X_{(u,v)}) = \delta((X_{(u,v)})^{red}) - \varepsilon(X_{(u,v)}) = 1 - 0 = 1;$$

$$\delta(X_{(u,0)}) = 2 - 1 = 1; \quad \delta(X_{(0,v)}) = 1 - 0 = 1.$$

Hence f is δ-constant.

Moreover, the normalizations of the first component $(X_1, 0)$ and the second component $(X_2, 0)$ of $(X, 0)$ are given respectively by

$$v_1 : (\mathbf{C}^3, 0) \to (X_1, 0), \quad (T_1, T_2, T_3) \mapsto (0, 0, T_1, T_3, T_2, T_3)$$

and

$$v_2 : (\mathbf{C}^3, 0) \to (X_2, 0), \quad (T_1, T_2, T_3) \mapsto (T_3^3 + T_1 T_3, T_3^2 + T_1, 0, 0, T_1, T_2).$$

Hence the composition maps are given respectively by

$$\bar{f}_1 : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0), \quad (T_1, T_2, T_3) \mapsto (T_2, T_3)$$

and

$$\bar{f}_2 : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0), \quad (T_1, T_2, T_3) \mapsto (T_1, T_2).$$

On both components, $\bar{f}$ is flat with normal fibers, hence f is equinormalizable. Note that, in this example, the normalization of $(X, 0)$ is smooth. All the computation given above can be easily done by **SINGULAR** ([6]).

4. Topologically triviality of one-parametric families of isolated curve singularities

In this section we consider one-parametric families of isolated (not necessarily reduced) curve singularities and show that the topologically triviality of these families is equivalent to the admission of weak simultaneous resolutions ([15]).

Let $f : (X, x) \to (\mathbf{C}, 0)$ be a deformation of an isolated curve singularity (X_0, x) with (X, x) pure dimensional. Let $f : X \to T$ be a *good representative* (in the sense of [2, §2.1, p. 248]) such that X is generically reduced over T. Then X is reduced by Corollary 2.6. Let $v : \overline{X} \to X$ be the normalization of X. Denote $\bar{f} := f \circ v : \overline{X} \to T$.

DEFINITION 4.1 (cf. [3]). (1) f is said to be *topologically trivial* if there is a homeomorphism $h : X \xrightarrow{\approx} X_0 \times T$ such that $f = \pi \circ h$, where $\pi : X_0 \times T \to T$ is the projection.

(2) Assume that f admits a section $\sigma : T \to X$ such that $X_t \backslash \sigma(t)$ is smooth for all $t \in T$. Then f admits a *weak simultaneous resolution* if f is equinormalizable and

$$(v^{-1}(\sigma(T)))^{red} \cong (v^{-1}(\sigma(0)))^{red} \times T \quad (\text{over } T).$$

Remark 4.2 (cf. [15]). f admits a weak simultaneous resolution if and only if f is equinormalizable and the number of branches $r(X_t, \sigma(t))$ of $(X_t, \sigma(t))$ is constant for all $t \in T$.

Buchweitz and Greuel (1980) proved the following result for families of reduced curve singularities.

THEOREM 4.3 ([2, Theorem 5.2.2]).   *Let* $f : X \to T$ *be a good representative of a flat family of reduced curves with section* $\sigma : T \to X$ *such that* $X_t \backslash \sigma(t)$ *is smooth for each* $t \in T$.   *Then the following conditions are equivalent*:
   (1) f *admits a weak simultaneous resolution*;
   (2) *the delta number* $\delta(X_t, \sigma(t))$ *and the number of branches* $r(X_t, \sigma(t))$ *are constant for* $t \in T$;
   (3) *the Milnor number* $\mu(X_t, \sigma(t))$ *is constant for* $t \in T$;
   (4) f *is topologically trivial*.

We shall show that this result is also true for families of isolated (not necessarily reduced) curve singularities.   Due to Brücker and Greuel ([3]), we give a new definition for the *Milnor number* of a curve singulariy C at an isolated singular point $c \in C$, namely,

$$\mu(C, c) := 2\delta(C, c) - r(C, c) + 1.$$

The Milnor number of C is defined to be

$$\mu(C) := \sum_{c \in \text{Sing}(C)} \mu(C, c).$$

To state and prove a similar result to Theorem 4.3 we need the following result of Bobadilla, Snoussi and Spivakovsky (2014).

LEMMA 4.4 ([4, Theorem 4.4]).   *Let* $f : (X, x) \to (\mathbf{C}, 0)$ *be a deformation of an isolated curve singularity* (X_0, x) *with* (X, x) *reduced.   Assume that the singular locus* $\text{Sing}(X, x)$ *of* (X, x) *is smooth of dimension* 1.   *If* f *is topologically trivial, then for any* $z \in v^{-1}(x)$, $\bar{f} : (\bar{X}, z) \to (\mathbf{C}, 0)$ *is topologically trivial, and the normalization* $(\bar{X}, v^{-1}(x))$ *of* (X, x) *is smooth.*

The following theorem is the main result of this section.

THEOREM 4.5.   *Let* $f : (X, x) \to (\mathbf{C}, 0)$ *be a deformation of an isolated curve singularity* (X_0, x) *with* (X, x) *pure dimensional.   Let* $f : X \to T$ *be a good representative with section* $\sigma : T \to X$ *such that* $X_t \backslash \sigma(t)$ *is smooth for each* $t \in T$ *and* X *is generically reduced over* T.   *Assume that* $\text{Sing}(X, x)$ *is smooth of dimension* 1.   *Then the following conditions are equivalent*:

(1) f *admits a weak simultaneous resolution;*
(2) *the delta number* $\delta(X_t, \sigma(t))$ *and the number of branches* $r(X_t, \sigma(t))$ *are constant for* $t \in T$;
(3) *the Milnor number* $\mu(X_t, \sigma(t))$ *is constant for* $t \in T$;
(4) f *is topologically trivial.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.6 (for $k = 1$) and Remark 4.2. $(2) \Leftrightarrow (3)$ because of the definition of the Milnor number. The implication $(1) \Rightarrow (4)$ is proved by the same way for families of reduced curve singularities as given in the proof of the implication $(4) \Rightarrow (6)$ of [2, Theorem 5.2.2]. Now we prove that $(4) \Rightarrow (1)$.

For convenience, let us assume that $v^{-1}(x) = \{z_1, \ldots, z_r\}$. Note that $\overline{X}_0 := \overline{f}^{-1}(0)$ is reduced, $\overline{X}_t := \overline{f}^{-1}(t)$ is smooth for every $t \neq 0$ by [3, Lemma 2.1.1]. Therefore for every $i = 1, \ldots, r$, $\overline{f} : (\overline{X}, z_i) \to (\mathbf{C}, 0)$ is a family of reduced curve singularities with smooth general fibers, and there exist sections $\overline{\sigma}_1, \ldots, \overline{\sigma}_r : T \to \overline{X}$ such that $\overline{\sigma}_i(0) = z_i$, $v^{-1}(\sigma(t)) = \{\overline{\sigma}_1(t), \ldots, \overline{\sigma}_r(t)\}$, and $\overline{X}_t \backslash \overline{\sigma}_i(t)$ is smooth for every $t \in T$ and for every $i = 1, \ldots, r$.

Assume that f is topologically trivial. Then it follows from Lemma 4.4 that the deformation $\overline{f} : (\overline{X}, z_i) \to (\mathbf{C}, 0)$ of $(\overline{X}_0, z_i)$ is also topologically trivial for every $i = 1, \ldots, r$. Hence it follows from Theorem 4.3, applying for the flat family of reduced curve singularities $\overline{f} : (\overline{X}, z_i) \to (\mathbf{C}, 0)$ with section $\overline{\sigma}_i : (\mathbf{C}, 0) \to (\overline{X}, z_i)$, that the delta number $\delta(\overline{X}_t, \overline{\sigma}_i(t))$ and the number of branches $r(\overline{X}_t, \overline{\sigma}_i(t))$ are constant for $t \in T$. Then for $t \neq 0$ we have

$$\delta(\overline{X}_0) = \delta(\overline{X}_t) = 0.$$

Hence $\overline{X}_0$ is normal. It follows that f is equinormalizable. On the other hand, the equinormalizability of f over the smooth base space $(\mathbf{C}, 0)$ implies that for every $t \in T$ and for each $i = 1, \ldots, r$, the induced map of v on the fibers $v_t : (\overline{X}_t, \overline{\sigma}_i(t)) \to (X_t, \sigma(t))$ is the normalization of the corresponding irreducible component of $(X_t, \sigma(t))$. It follows that the number of irreducible components of $(X_t, \sigma(t))$ is equal to the cardinality of $v^{-1}(\sigma(t))$, which is equal to r for every $t \in T$. Hence $r(X_t, \sigma(t))$ is constant for every $t \in T$. It follows that f admits a weak simultaneous resolution, and we have (1). $\qquad \square$

Example 4.6. Let us consider again the curve singularity $(X_0, 0) \subseteq (\mathbf{C}^4, 0)$ considered in Example 3.7 which is defined by the ideal

$$I_0 := \langle x^2 - y^3, z, w \rangle \cap \langle x, y, w \rangle \cap \langle x, y, z, w^2 \rangle \subseteq \mathbf{C}\{x, y, z, w\}.$$

Now we consider the restriction $f : (X, 0) \to (\mathbf{C}, 0)$ of the projection $\pi : (\mathbf{C}^5, 0) \to (\mathbf{C}, 0)$, $(x, y, z, w, t) \mapsto t$, to the complex germ $(X, 0)$ defined by the ideal

$$I = \langle x^2 - y^3 + ty^2, z, w \rangle \cap \langle x, y, w - t \rangle \subseteq \mathbf{C}\{x, y, z, w, t\}.$$

We can check the following (all of them can be checked easily by SINGULAR):

(1) f is flat;
(2) $(X, 0)$ is reduced and pure 2-dimensional, with two 2-dimensional irreducible components;
(3) f is δ-constant with $\delta(X_t) = 1$ for all $t \in \mathbf{C}$ close to 0;
(4) $r(X_t) = 2$ for all $t \in \mathbf{C}$ close to 0;
(5) f is equinormalizable;
(6) the normalization of each component of $(X, 0)$ is $(\mathbf{C}^2, 0)$, which is smooth.

By Theorem 4.5, f is topologically trivial.

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