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ISOSPECTRAL KÄHLER GRAPHS

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Abstract

We give some basic ways to construct Kähler graphs which are compound graphs having principal and auxiliary graphs. By use of these methods we give some examples of isospectral pairs of Kähler graphs.

1. Introduction

Graphs which are pairs of a set of vertices and a set of edges are considered as discrete models of Riemannian manifolds. Considering them as 1-dimensional CW-complexes we regard paths on them which are chains of edges as geodesics. In his paper [2] the second author introduced the notion of a Kähler graph to give a discrete model of a Riemannian manifold admitting a magnetic field. As a generalization of a static magnetic field on a Euclidean 3-space $\mathbf{R}^3$, a closed 2-form on a Riemannian manifold is said to be a magnetic field (see [7], for example). As typical examples of magnetic fields we have constant multiples of the Kähler form on a Kähler manifold. They are called Kähler magnetic fields ([1]). We consider that geodesics are trajectories of electric charged particles without the action of magnetic fields. Under the influence of a magnetic field motions of electric charged particles have their accelerations by getting the Lorentz force. If we adopt graphs as discrete models of Kähler manifolds, as graphs does not have 2-simplexes, we need a system to show complex structure. For this sake we consider decompound graphs having two kinds of edges, principal edges and auxiliary edges. We consider paths consisting of principal edges as geodesics and consider paths consisting of both principal and auxiliary edges as trajectories under an action of a magnetic field.

In this paper we study Laplacians for Kähler graphs corresponding to paths where principal and auxiliary edges appear alternatively. Since the Laplacian of

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a graph is the generating operator of the random walk defined by paths of graphs, we may say that its eigenvalues show some properties of the graph. We show basic ways of constructing Kähler graphs, which are to take complement graphs and to take product graphs. By investigating the relationship between eigenvalues of Laplacians for Kähler graphs and eigenvalues of discrete Laplacians for their principal and auxiliary graphs, we give examples of pairs of isospectral Kähler graphs.

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2. Laplacians for a Kähler graphs

A graph $G = (V, E)$ is a 1-dimensional CW-complex which consists of a set V of vertices and a set E of edges. We assume it has no loops and multiple edges. Also we assume that it is not directed and is locally finite. We call a graph $G = (V, E)$ *Kähler* if the set E of edges is divided into two disjoint subsets $E^{(p)}$, $E^{(a)}$ and satisfies the following condition: At each vertex v there are at least four edges emanating from v; two are contained in $E^{(p)}$ and two are contained in $E^{(a)}$. For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$, we call the graphs $(V, E^{(p)})$ and $(V, E^{(a)})$ its *principal* graph and *auxiliary* graph, respectively. For vertices $v, w \in V$ we denote as $v \sim_p w$ if they are adjacent to each other in the principal graph $(V, E^{(p)})$, and denote as $v \sim_a w$ if they are adjacent to each other in the auxiliary graph $(V, E^{(a)})$. For a vertex $v \in V$ we set $d^{(p)}(v) = \#\{w \in V \mid w \sim_p v\}$ and $d^{(a)}(v) = \#\{w \in V \mid w \sim_a v\}$, and call them the principal degree and the auxiliary degree at v, respectively. Here, for a set X we denote by $\#X$ its cardinality.

A *bicolored path* $\gamma = (v_0, \dots, v_{2m})$ on a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is a $2m$-step path satisfying $v_{2k} \sim_p v_{2k+1}$ and $v_{2k+1} \sim_a v_{2k+2}$ for $k = 0, \dots, m-1$. Roughly speaking bicolored paths show trajectories of charged particles under the action of a magnetic field of strength 1. We consider that an edge (v_{2k}, v_{2k+1}) shows a motion of a charged particle without actions of magnetic fields, and that if it gets a Lorentz force it is bended and reaches to v_{2k+2}. More generally, for a pair (p, q) of relatively prime positive integers, we can consider paths corresponding to trajectories of charged particles under the action of a magnetic field of strength q/p by use of p-step paths in the principal graph and q-step paths in the auxiliary graph. But as their treatment is a bit complicated we shall only consider paths under the action of a magnetic field of strength 1. Moreover, we note that if we only consider paths under the action of a magnetic field of strength 1, the condition $d^{(p)}(v), d^{(a)}(v) \geq 2$ at each $v \in V$ in the definition of Kähler graphs can be weaken to the condition $d^{(p)}(v), d^{(a)}(v) \geq 1$ at each $v \in V$. For a bicolored path $\gamma = (v_0, \dots, v_{2m})$, we define its probabilistic weight $\omega(\gamma)$ as $\omega(\gamma) = \prod_{k=0}^{m} \{d^{(a)}(v_{2k+1})\}^{-1}$. Since a graph is a 1-dimensional CW-complex, we can not show the direction of the action of a magnetic field, therefore we treat the position of the terminus of a trajectory probabilistically.

For a finite ordinary graph $G = (V, E)$ we set $d_G(v) = \#\{w \in V \mid w \sim v\}$ and call it the degree at $v \in V$. We define the adjacency operator A_G and the transition operator P_G acting on the space $C(V)$ of functions on V by

$$A_G f(v) = \sum_{w \sim v} f(w), \quad P_G f(v) = \frac{1}{d_G(v)} \sum_{w \sim v} f(w).$$

The combinatorial Laplacian Δ_{A_G} and the transitional Laplacian Δ_{P_G} acting on $C(V)$ are defined by $\Delta_{A_G} = D_G - A_G$ and by $\Delta_{P_G} = I - P_G$, respectively, where the degree operator D_G is given as $D_G f(v) = d_G(v) f(v)$. When G is regular, that is, its degree-function d_G does not depend on the choice of vertices, these Laplacians are related with each other as $\Delta_{A_G} = d_G \Delta_{P_G}$ because $A_G = d_G P_G$ (see [3, 5, 9] for more on Laplacians).

Corresponding to these operators we define Laplacians (or more precisely $(1,1)$-Laplacians) for finite Kähler graphs in the following manner. Let $G = (V, E^{(p)} \cup E^{(a)})$ be a finite Kähler graph. We denote by $A^{(p)}, P^{(p)}$ the adjacency operator and transition operator of the principal graph $(V, E^{(p)})$ acting on the space $C(V, \mathbf{C})$ of all (complex valued) functions on V. Similarly, we denote by $P^{(a)}$ the transition operator of the auxiliary graph $(V, E^{(a)})$. According to the lines in the previous paragraph, we set the adjacency and the transition operators of G corresponding to bicolored paths as $A = A^{(p)} P^{(a)}$ and $P = P^{(p)} P^{(a)}$. We should note that these operators are not symmetric. Denoting $D^{(p)}$ the degree operator of the principal graph, we set Δ_A, Δ_P as $\Delta_A = D^{(p)} - A$, $\Delta_P = I - P$, and call them the combinatorial and the transitional Laplacians for a finite Kähler graph, respectively. When the principal graph is regular as an ordinary graph, we find $A = d^{(p)} P$ and $\Delta_A = d^{(p)} \Delta_P$.

We explain adjacency and transition operators in another way. For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$, we can define a derived directed graph $G_{(1,1)}$ by use of 2-step bicolored paths in the following manner. The set of vertices is V. We say a vertex v is joined to a vertex w by a derived directed edge if there is a 2-step bicolored path γ with origin $o(\gamma) = v$ and terminus $t(\gamma) = w$. We note that this graph $G_{(1,1)}$ may have loops and multiple edges. For each edge of $G_{(1,1)}$, which is a 2-step bicolored path, we attach it its probabilistic weight. Then the adjacency and the transition operators of a Kähler graph are those of the derived directed graph with weights on edges:

$$Af(v) = \sum_{\gamma} \omega(\gamma) f(t(\gamma)),$$

$$Pf(v) = \frac{1}{d^{(p)}(v)} \sum_{\gamma} \omega(\gamma) f(t(\gamma)) = \frac{1}{\sum_{\gamma} \omega(\gamma)} \sum_{\gamma} \omega(\gamma) f(t(\gamma)),$$

where γ runs over the set of all 2-step bicolored paths with $o(\gamma) = v$.

In this paper we only treat Laplacians of finite graphs. As the spaces of functions on their sets of vertices are of finite dimensional, we frequently identify operators acting on these spaces with matrices.

3. Kähler graphs with complement auxiliary edges

One of the most typical way to construct Kähler graphs is to take complements of graphs. We take an ordinary finite graph $G = (V, E)$ and consider its complement graph $G^c = (V, E^c)$. Here, the complement graph is given as follows: Two distinct vertices are adjacent to each other in G^c if and only if they are not adjacent in G. When the degree-function d_G of the finite graph G satisfies $\min_{v \in V} d_G(v) \geq 2$ and $\max_{v \in V} d_G(v) \leq \#V - 3$, we obtain a Kähler graph $G^K = (V, E \cup E^c)$ by taking its complement as the auxiliary graph. We call this the *complement-filled* Kähler graph of G. In this section we study eigenvalues of Laplacians for complement-filled Kähler graphs.

We call a Kähler graph *regular* if both of its principal and auxiliary graphs are regular. When the degree d_G of finite regular graph G satisfies $2 \leq d_G \leq \#V - 3$, we see its complement graph is reglar and of degree $d_{G^c} = \#V - d_G - 1$, hence find that $G^K = (V, E \cup E^c)$ is a regular Kähler graph.

THEOREM 1. *Let $G = (V, E)$ be a connected regular finite graph whose degree satisfies $2 \leq d_G \leq \#V - 3$. If we denote the eigenvalues of $\Delta_{A_G} = D_G - A_G$ for G as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{\#V}$, then the eigenvalues of $\Delta_A = D_G - A$ for the complement-filled Kähler graph G^K are $\hat{\lambda}_1 = 0$, $\hat{\lambda}_i = \{\lambda_i^2 - \lambda_i(2d_G + 1) + d_G\#V\} \cdot (\#V - d_G - 1)^{-1}$ $(i = 2, \ldots, \#V)$. Moreover, if $f_i : V \to \mathbf{R}$ is an eigenfunction corresponding to λ_i, then it is an eigenfunction corresponding to $\hat{\lambda}_i$.*

Proof. If we put

$$N = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix},$$

then the adjacency matrix A_{G^c} is given as $A_{G^c} = N - A_G$.

We note that the condition $(D_G - A_G)f_i = \lambda_i f_i$ is equivalent to the condition $A_G f_i = (d_G - \lambda_i) f_i$. For $\lambda_1 = 0$, the eigenfunction f_1 is a non-zero constant function. Therefore we have $A_{G^c} f_1 = N f_1 - A_G f_1 = (\#V - 1 - d_G) f_1$. Hence we obtain

$$\Delta_A f_1 = D_G f_1 - A_G P_{G^c} f_1 = d_G f_1 - \frac{1}{\#V - 1 - d_G} A_G A_{G^c} f_1 = d_G f_1 - A_G f_1 = 0.$$

For λ_i $(i \geq 2)$, the eigenfunction f_i is orthogonal to f_1. That is, $\langle f_i, f_1 \rangle = \sum_{v \in V} f_1(v) f_i(v) = 0$. Hence we have $\sum_{v \in V} f_i(v) = 0$. Therefore we get

$$N f_i(v) = \sum_{w \in V, w \neq v} f_i(w) = -f_i(v).$$

Thus we have $A_{G^c}f_i = Nf_i - A_G f_i = (\lambda_i - d_G - 1)f_i$ and obtain

$$\Delta_A f_i = \left\{ d_G - \frac{(d_G - \lambda_i)(\lambda_i - d_G - 1)}{\#V - 1 - d_G} \right\} f_i = \frac{\lambda_i^2 - \lambda_i(2d_G + 1) + d_G\#V}{\#V - 1 - d_G} f_i.$$

This completes the proof. □

Remark 1. (1) The eigenvalues of $\Delta_P = I - P$ for our Kähler graph G^K are 0, $\{\lambda_i^2 - \lambda_i(2d_G + 1) + d_G\#V\}\{d_G(\#V - d_G - 1)\}^{-1}$ $(i = 2, \dots, \#V)$.
(2) These operators A, P are symmetric because $NA_G = A_G N$, hence so are Δ_A, Δ_P.

Two finite graphs are said to be combinatorially isospectral (resp. transitionary isospectral) if their combinatorial Laplacians (resp. transitional Laplacians) have the same eigenvalues by taking account of their multiplicities. Clearly these notions are equivalent when these graphs are regular. So in this case we just say that these graphs are isospectral. It is well known that there exist many pairs of isospectral regular graphs (see [3]). We here study Kähler graphs from this point of view. We call a pair of Kähler graphs combinatorially isospectral if their principal graphs are combinatorially isospectral and their combinatorial Laplacians as Kähler graphs have the same eigenvalues by taking account of their multiplicities. Also, we call a pair of Kähler graphs transitionary isospectral if their principal graphs are transitionary isospectral and their transitional Laplacians as Kähler graphs have the same eigenvalues by taking account of their multiplicities. When the principal graphs of two Kähler graphs are regular, they are combinatorially isospectral if and only if they are transitionary isospectral. In such a case we just call them isospectral.

Corollary 1. *If two finite connected regular graphs G_1, G_2 are isospectral, then their complement-filled Kähler graphs G_1^K, G_2^K are isospectral as Kähler graphs.*

We here give some examples following to [3] and [4].

Example 1. The following figures show an isospectral pair of Kähler graphs consisted by isospectral regular graphs and their complements. In these figures we show principal and auxiliary graphs separately to get their feature clearly. We draw auxiliary edges by dotted lines. Their eigenvalues of combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{A_G}) = \{0, 3, 5, 5, 5, 5, 4 - \sqrt{5}, 4 + \sqrt{5}, (9 - \sqrt{17})/2, (9 + \sqrt{17})/2\},$$
$$\mathrm{Spec}(\Delta_A) = \{0, 4, 4, 4, 4, 22/5, 24/5, 24/5, (25 - \sqrt{5})/5, (25 + \sqrt{5})/5\}.$$

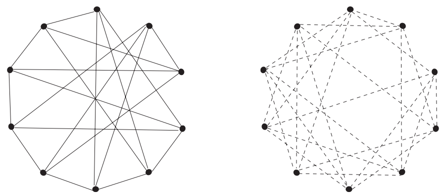

FIGURE 1

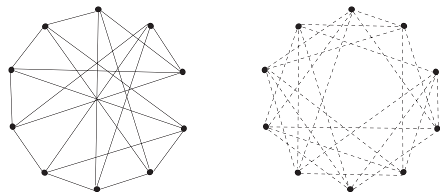

FIGURE 2

If we denote the eigenvalues of Δ_{A_G} for a connected regular finite graph G as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{\#V}$, then the eigenvalues of $\Delta_{A_{G^c}}$ of its complement graph G^c are 0 and $\#V - \lambda_i$ $(i = 2, \dots, \#V)$. Therefore we can reverse the principal and the auxiliary graphs of an isospectral pair of Kähler graphs.

Example 2. If we reverse the principal and the auxiliary graphs of the Kähler graphs in Example 1, their eigenvalues of combinatorial Laplacians are

$$\mathrm{Spec}(\Delta_{A_G}) = \{0, 5, 5, 5, 5, 7, 6 - \sqrt{5}, 6 + \sqrt{5}, (11 - \sqrt{17})/2, (11 + \sqrt{17})/2\},$$
$$\mathrm{Spec}(\Delta_{A}) = \{0, 5, 5, 5, 5, 11/2, 6, 6, (25 - \sqrt{5})/4, (25 + \sqrt{5})/4\}.$$

Example 3. The following figures show another isospectral pair of Kähler graphs consisted by isospectral regular graphs and their complements. Their eigenvalues of combinatorial Laplacians are

$$\begin{aligned}\mathrm{Spec}(\Delta_{A_G}) = \{&0, 5, 5, (9 - \sqrt{5})/2, (9 + \sqrt{5})/2, \text{ solutions of the equation}\\ &t^5 - 21t^4 + 167t^3 - 624t^2 + 1092t - 716 = 0\},\end{aligned}$$
$$\begin{aligned}\mathrm{Spec}(\Delta_{A}) = \{&0, 4, 4, 21/5, 21/5, \text{ solutions of the equation } 5^5t^5 - 5^4 \cdot 118t^4\\ &+ 5^3 \cdot 5557t^3 - 5^2 \cdot 130552t^2 + 5 \cdot 1530052t - 7156316 = 0\}.\end{aligned}$$

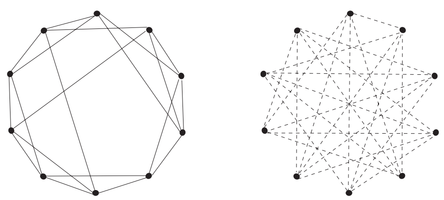

FIGURE 3

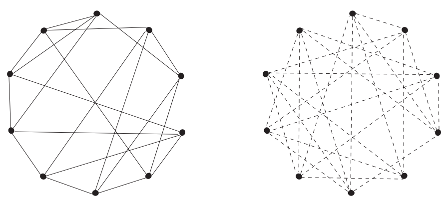

FIGURE 4

It is known that the pairs in Examples 1, 3 are the only pairs of isospectral regular graphs whose numbers of vertices are not greater than ten (see [3, 4]).

Therefore we have only four pairs of isospectral regular complete Kähler graphs having ten vertices.

If we consider only A for the adjacency by 2-step bicolored paths and do not consider principal graphs, we have many examples (c.f. [10]).

Example 4. The following vertex-transitive Kähler graphs are not isomorphic, but their derived graphs by 2-step bicolored paths are isomorphic, hence their Δ_A and Δ_P have the same eigenvalues

$$\begin{aligned}\mathrm{Spec}(\Delta_A) = \Big\{0, \frac{9}{2}, \frac{9}{2}, \frac{1}{2}\left(9 + \sqrt{3}\cos\frac{\pi}{18}\right), \frac{1}{2}\left(9 + \sqrt{3}\cos\frac{\pi}{18}\right),\\ \frac{1}{2}\left(9 - \sqrt{3}\cos\frac{5}{18}\pi\right), \frac{1}{2}\left(9 - \sqrt{3}\cos\frac{5}{18}\pi\right),\\ \frac{1}{2}\left(9 - \sqrt{3}\cos\frac{7}{18}\pi\right), \frac{1}{2}\left(9 - \sqrt{3}\cos\frac{7}{18}\pi\right)\Big\}.\end{aligned}$$

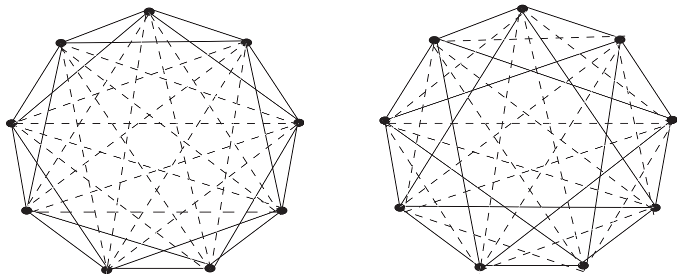

FIGURE 5. non-isomorphic vertex-transitive Kähler graphs

Since we treat connected regular graphs in Theorem 1, we here study operations of graphs to treat Kähler graphs induced by non-connected graphs. Given two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ we set $V = V_1 \cup V_2$ and $E^{(p)} = E_1 \cup E_2$. We define $E^{(a)}$ in the following manner: Arbitrary $v \in V_1$ and $w \in V_2$ are adjacent to each other, two vertices in V_1 are not adjacent to each other, and nor are two vertices in V_2. We call $(V, E^{(p)} \cup E^{(a)})$ the *joined* Kähler graph of G_1 and G_2, and denote it by $G_1 \hat{+} G_2$.

PROPOSITION 1. *The eigenvalues* Δ_P *of the joined Kähler graph* $G_1 \hat{+} G_2$ *of graphs* $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ *are* $0, 1, \dots, 1, 2$, *where the multiplicity of* 1 *is* $\#V_1 + \#V_2 - 2$.

Proof. We put $m_i = \#V_i$ $(i = 1, 2)$. We denote by M_{ij} an (m_i, m_j)-matrix all of whose entries are 1. The adjacency matrix $A^{(p)}$ for the principal graph and the transition matrix $P^{(a)}$ for the auxiliary graph of $G_1 \hat{+} G_2$ are

$$A^{(p)} = \begin{pmatrix} A_{G_1} & O \\ O & A_{G_2} \end{pmatrix}, \quad P^{(a)} = \begin{pmatrix} O & \dfrac{1}{m_2} M_{12} \\ \dfrac{1}{m_1} M_{21} & O \end{pmatrix}.$$

Thus, if we take functions $f : V_1 \to \mathbf{R}$ and $g : V_2 \to \mathbf{R}$ then for the function $(f, g) : V = V_1 \cup V_2 \to \mathbf{R}$ we have $P^{(a)}((f, g)) = \left(\frac{1}{m_2} \sum_{w \in V_2} g(w), \frac{1}{m_1} \sum_{v \in V_1} f(v)\right)$. We denote the function of degree on G_i also by $d_{G_i} : V_i \to \mathbf{R}$. We then have

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} d_{G_1} \\ d_{G_2} \end{pmatrix}, \quad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -d_{G_1} \\ d_{G_2} \end{pmatrix},$$
$$A\begin{pmatrix} \delta_{v_1} - \delta_{v_i} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} 0 \\ \delta_{w_1} - \delta_{w_j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $V_1 = \{v_1, \ldots, v_{m_1}\}$, $V_2 = \{w_1, \ldots, w_{m_2}\}$ and $\delta_v : V_1 \to \mathbf{R}$, $\delta_w : V_2 \to \mathbf{R}$ denote characteristic functions. Thus we get the conclusion. □

By the proof of the above proposition we obtain the following.

Proposition 2. *The eigenvalues Δ_A of the joined Kähler graph $G_1 \hat{+} G_2$ of regular graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are $0, d_{G_1}, \ldots, d_{G_1}, d_{G_2}, \ldots, d_{G_2}, d_{G_1} + d_{G_2}$, where d_{G_i} appears $\# V_i - 1$ times.*

Proof. We only need to change $(1, -1)$ to $(d_{G_1}, -d_{G_2})$. We then have

$$A\begin{pmatrix} d_{G_1} \\ -d_{G_2} \end{pmatrix} = A^{(p)}\begin{pmatrix} -d_{G_2} \\ d_{G_1} \end{pmatrix} = \begin{pmatrix} -d_{G_1} d_{G_2} \\ d_{G_1} d_{G_2} \end{pmatrix},$$

hence get

$$\Delta_A\begin{pmatrix} d_{G_1} \\ -d_{G_2} \end{pmatrix} = \begin{pmatrix} d_{G_1}^2 + d_{G_1} d_{G_2} \\ -d_{G_2}^2 - d_{G_1} d_{G_2} \end{pmatrix} = (d_{G_1} + d_{G_2})\begin{pmatrix} d_{G_1} \\ -d_{G_2} \end{pmatrix}.$$

We therefore obtain the conclusion. □

Given two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ we consider a Kähler graph constructed by taking their complements and their join. One can easily find that it is the Kähler graph $(G_1 \cup G_2)^K$ by taking the complement of $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Proposition 3. *Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be connected regular graphs. We denote the eigenvalues of $\Delta_{A_{G_1}}$ as $\lambda_1 < \lambda_2 \le \cdots \le \lambda_{\# V_1}$ and the eigenvalues of $\Delta_{A_{G_2}}$ as $\eta_1 < \eta_2 \le \cdots \le \eta_{\# V_2}$.*

(1) *The eigenvalues of* Δ_A *of* $(G_1 \cup G_2)^K$ *are*

$$0, d_{G_1} - \hat{d}_{G_1}^{-1}(d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1) \quad (j = 2, \dots, \#V_1),$$
$$d_{G_2} - \hat{d}_{G_2}^{-1}(d_{G_2} - \eta_k)(\eta_k - d_{G_2} - 1) \quad (k = 2, \dots, \#V_2),\ d_{G_1}\hat{d}_{G_1}^{-1}\#V_2 + d_{G_2}\hat{d}_{G_2}^{-1}\#V_1.$$

where $\hat{d}_{G_i} = \#V_1 + \#V_2 - d_{G_i} - 1.$

(2) *The eigenvalues of* Δ_P *of* $(G_1 \cup G_2)^K$ *are*

$$0, 1 - (d_{G_1}\hat{d}_{G_1})^{-1}(d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1) \quad (j = 2, \dots, \#V_1),$$
$$1 - (d_{G_2}\hat{d}_{G_2})^{-1}(d_{G_2} - \eta_k)(\eta_k - d_{G_2} - 1) \quad (k = 2, \dots, \#V_2),\ \hat{d}_{G_1}^{-1}\#V_2 + \hat{d}_{G_2}^{-1}\#V_1.$$

Proof. By using the notation in the proof of Proposition 1, we see the transition matrix $P^{(a)}$ of the auxiliary graph of $(G_1 \cup G_2)^K$ is

$$P^{(a)} = \begin{pmatrix} \hat{d}_{G_1}^{-1}A_{G_1^c} & \hat{d}_{G_1}^{-1}M_{12} \\ \hat{d}_{G_2}^{-1}M_{21} & \hat{d}_{G_2}^{-1}A_{G_2^c} \end{pmatrix}.$$

Therefore we have

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} d_{G_1} \\ d_{G_2} \end{pmatrix},$$

$$A\begin{pmatrix} 1 \\ -\dfrac{d_{G_2}\hat{d}_{G_1}\#V_1}{d_{G_1}\hat{d}_{G_2}\#V_2} \end{pmatrix} = \begin{pmatrix} \dfrac{d_{G_1}}{\hat{d}_{G_1}}\left\{\#V_1 - d_{G_1} - 1 - \dfrac{d_{G_2}\hat{d}_{G_1}\#V_1}{d_{G_1}\hat{d}_{G_2}}\right\} \\ \dfrac{d_{G_2}}{\hat{d}_{G_2}}\left\{\#V_1 - \dfrac{d_{G_2}\hat{d}_{G_1}\#V_1}{d_{G_1}\hat{d}_{G_2}\#V_2}(\#V_2 - d_{G_2} - 1)\right\} \end{pmatrix}.$$

Also, if we take eigenfunctions f_j and g_k for λ_j and η_k with $j \geq 2$, $k \geq 2$, respectively, as we have $\sum_{v \in V_1} f_j(v) = 0$ and $\sum_{w \in V_2} g_k(w) = 0$, we find

$$A\begin{pmatrix} f_j \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{d}_{G_1}^{-1}(d_{G_1} - \lambda_j)(\lambda_j - d_{G_1} - 1)f_j \\ 0 \end{pmatrix},$$

$$A\begin{pmatrix} 0 \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{d}_{G_2}^{-1}(d_{G_2} - \eta_k)(\eta_k - d_{G_2} - 1)g_k \end{pmatrix}.$$

Thus we get our conclusion for Δ_A.

As we have

$$P\begin{pmatrix} 1 \\ -\dfrac{\hat{d}_{G_1}\#V_1}{\hat{d}_{G_2}\#V_2} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\hat{d}_{G_1}}\left\{\#V_1 - d_{G_1} - 1 - \dfrac{\hat{d}_{G_1}\#V_1}{\hat{d}_{G_2}}\right\} \\ \dfrac{1}{\hat{d}_{G_2}}\left\{\#V_1 - \dfrac{\hat{d}_{G_1}\#V_1}{\hat{d}_{G_2}\#V_2}(\#V_2 - d_{G_2} - 1)\right\} \end{pmatrix},$$

we get our conclusion for Δ_P. $\square$

4. Kähler graphs of product type

We give typical examples of Kähler graphs. Let $G = (V, E)$ and $H = (W, F)$ be two (non-oriented) graphs. We define their Kähler graph of Cartesian product type $G \mathbin{\hat{\square}} H$ as follows:

i) its set of vertices is the product $V \times W$;
ii) two vertices $(v, w), (v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if v, v' are adjacent to each other in G and $w = w'$;
iii) two vertices $(v, w), (v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if $v = v'$ and w, w' are adjacent to each other in H.

Example 5. If G and H are graphs of real lines, which are non-circuit regular graphs of degree 2, then their Kähler graph of Cartesian product type is a graph of complex line.

FIGURE 6. $G = H$

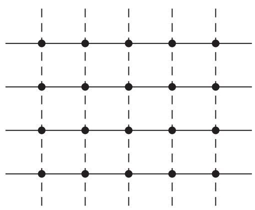

FIGURE 7. $G \mathbin{\hat{\square}} H$

When G, H are graphs of finite degrees, their Kähler graph of Cartesian product type is of finite degree. At a vertex $(v, w) \in V \times W$, we have $d^{(p)}_{G \hat{\square} H}(v, w) = d_G(v)$ and $d^{(a)}_{G \hat{\square} H}(v, w) = d_H(w)$.

THEOREM 2. *Let $G = (V, E)$, $H = (W, F)$ be finite graphs whose eignevalues of Δ_{P_G} and Δ_{P_H} are μ_i $(1 \le i \le m (= \#V))$ and ν_α $(1 \le \alpha \le n (= \#W))$, respectively. Then the eigenvalues of Δ_P for their Kähler graph $G \mathbin{\hat{\square}} H$ of Cartesian product type are $\mu_i + \nu_\alpha - \mu_i \nu_\alpha$ $(1 \le i \le m, 1 \le \alpha \le n)$.*

Moreover, if $\Delta_{P_G} f = \mu f$, $\Delta_{P_H} g = \nu g$ and if we set a function $\varphi_{f,g}$ on $V \times W$ by $\varphi_{f,g}(v, w) = f(v) g(w)$, then we have $\Delta_P \varphi_{f,g} = (\mu + \nu - \mu\nu) \varphi_{f,g}$.

Proof. We denote by $A_G = (a^G_{ij})$ the adjacency matrix of G and by $P_H = (p^H_{\alpha\beta})$ the transition matrix of H. Then the adjacency matrix $A^{(p)} = (a^{(p)}_{(i,\alpha),(j,\beta)})$ of the principal graph of $G \mathbin{\hat{\square}} H$ and the transition matrix $P^{(a)} = (p^{(a)}_{(i,\alpha),(j,\beta)})$ of the auxiliary graph of $G \mathbin{\hat{\square}} H$ are of the form

$$A^{(p)} = \begin{pmatrix} O & a^G_{12}I & \cdots & a^G_{1m}I \\ a^G_{21}I & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^G_{m-1m}I \\ a^G_{m1}I & \cdots & a^G_{mm-1}I & O \end{pmatrix}, \quad P^{(a)} = \begin{pmatrix} P_H & O & \cdots & O \\ O & P_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & O & P_H \end{pmatrix},$$

where I denotes the unit matrix and the components of $A^{(p)}$ and $P^{(a)}$ are expressed according to lexicographical order. If we denote their components, we have $a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta}$ and $p^{(a)}_{(i,\alpha),(j,\beta)} = \delta_{ij}p^H_{\alpha\beta}$. Corresponding to these if we denote $f = {}^t(f_1,\dots,f_m)$, $g = {}^t(g_1,\dots,g_n)$ and $\varphi_{f,g} = {}^t(f_1g_1,\dots,f_1g_n,\dots,f_mg_1,\dots,f_mg_n)$, we then have

$$A^{(p)}P^{(a)}\varphi_{f,g} = A^{(p)}\begin{pmatrix} f_1P_Hg \\ \vdots \\ f_mP_Hg \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m a^G_{1j}f_jP_Hg \\ \vdots \\ \sum_{j=1}^m a^G_{mj}f_jP_Hg \end{pmatrix}$$
$$= (1-\nu)(1-\mu)\begin{pmatrix} d_G(v_1)f_1g \\ \vdots \\ d_G(v_m)f_mg \end{pmatrix},$$

where $d_G(v_i)$ denotes the degree of G at the vertex v_i. Thus we have

$$\Delta_P\varphi_{f,g} = \{1-(1-\nu)(1-\mu)\}\varphi_{f,g} = (\mu+\nu-\mu\nu)\varphi_{f,g}.$$

This completes the proof. □

For a finite Kähler graph $G \,\hat{\square}\, H$ of Cartesian product given by $G = (V,E)$ and $H = (W,F)$, its principal graph is a $\#W$-copies of G. Therefore if two graphs G_1, G_2 are combinatorially (resp. transitively) isospectral and if the cardinalities of the sets of verticies of graphs H_1, H_2 are the same, then the principal graphs of the Kähler graphs $G_1 \,\hat{\square}\, H_1$, $G_2 \,\hat{\square}\, H_2$ are combinatorially (resp. transitively) isospectral. Hence we get the following.

Corollary 2. *If G_1, G_2 are transitionary isospectral graphs and H_1, H_2 are also transitionary isospectral graphs, then their Kähler graphs $G_1 \,\hat{\square}\, H_1$, $G_2 \,\hat{\square}\, H_2$ of Cartesian product type are transitionary isospectral.*

Remark 2. For Kähler graphs of Cartesian product type in Corollary 2, their auxiliary graphs are also transitionary isospectral.

By the proof of Theorem 2 we have the following.

Proposition 4. *Let G be a regular finite graph of degree d_G and H be a finite graph. If $\Delta_{P_G} f = \mu f$, $\Delta_{P_H} g = \nu g$ and if we set a function $\varphi_{f,g}$ on $V \times W$ by $\varphi_{f,g}(v,w) = f(v)g(w)$, then the combinatorial Laplacian of $G \mathbin{\hat{\Box}} H$ satisfies $\Delta_A \varphi_{f,g} = d_G(\mu + \nu - \mu\nu)\varphi_{f,g}$.*

We note that there are pairs of isospectral regular graphs of same degree (see for example [4, 5]).

Corollary 3. *Let G_1, G_2 be isospectral regular graphs of same degree and H_1, H_2 be transitionary isospectral graphs. Then their Kähler graphs $G_1 \mathbin{\hat{\Box}} H_1$, $G_2 \mathbin{\hat{\Box}} H_2$ of Cartesian product type are combinatorially isospectral.*

Most typical Kähler graphs of product type are Kähler graphs of Cartesian product type. But for the sake of contrast, we here give definitions of other Kähler graphs of product type which correspond to ordinary product operations of graphs.

Given two graphs $G = (V,E)$ and $H = (W,F)$, we define their Kähler graph of strong product type $G \mathbin{\hat{\boxtimes}} H$ as follows:

i) its set of vertices is the product $V \times W$;
ii) two vertices $(v,w), (v',w') \in V \times W$ are adjacent to each other by a principal edge if and only if v, v' are adjacent to each other in G and $w = w'$;
iii) two vertices $(v,w), (v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if either $v = v'$ and w, w' are adjacent to each other in H or v, v' are adjacent to each other in G and w, w' are adjacent to each other in H.

We define their Kähler graph of semi-tensor product type $G \mathbin{\hat{\otimes}} H$ as follows:

i) its set of vertices is the product $V \times W$;
ii) two vertices $(v,w), (v',w') \in V \times W$ are adjacent to each other by a principal edge if and only if v, v' are adjacent to each other in G and $w = w'$;
iii) two vertices $(v,w), (v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if v, v' are adjacent to each other in G and w, w' are adjacent to each other in H.

Example 6. If G and H are graphs of real lines, then their Kähler graphs of strong product type and of semi-tensor product type are as Figures 8 and 9 in the next page.

When G, H are graphs of finite degrees, their Kähler graph of strong product type is of finite degrees. At a vertex $(v,w) \in V \times W$, we have $d^{(p)}_{G \hat{\boxtimes} H}(v,w) = d_G(v)$ and $d^{(a)}_{G \hat{\boxtimes} H}(v,w) = d_H(w)\{d_G(v) + 1\}$. Therefore we have the following property on eigenvalues of Kähler graphs of strong product type.

Theorem 3. *Let $G = (V,E)$, $H = (W,F)$ be finite graphs whose eignevalues of Δ_{P_G} and Δ_{P_H} are μ_i $(1 \le i \le m(= \#V))$ and ν_α $(1 \le \alpha \le n(= \#W))$, respec-*

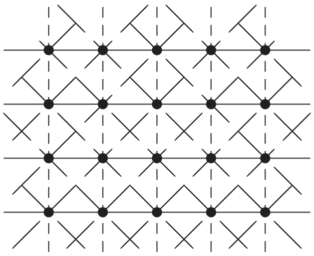

FIGURE 8. $G \hat{\boxtimes} H$

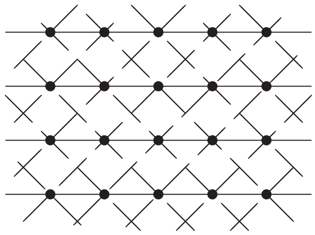

FIGURE 9. $G \hat{\otimes} H$

tively. Suppose G is regular of degree d_G. Then the eigenvalues of Δ_P of their Kähler graph $G \hat{\boxtimes} H$ of strong product type are

$$\{(1 + d_G - d_G\mu_i)(\mu_i + \nu_\alpha - \mu_i\nu_\alpha) + d_G\mu_i\}/\{d_G + 1\} \quad (1 \le i \le m, 1 \le \alpha \le n).$$

Moreover, if $\Delta_{P_G} f = \mu f$, $\Delta_{P_H} g = \nu g$ and if we set a function $\varphi_{f,g}$ on $V \times W$ by $\varphi_{f,g}(v, w) = f(v)g(w)$, then we have

$$\Delta_P \varphi_{f,g} = \frac{(1 + d_G - d_G\mu)(\mu + \nu - \mu\nu) + d_G\mu}{d_G + 1}\varphi_{f,g}.$$

Proof. For arbitrary finite graphs G, H, the adjacency matrix $A^{(p)}$ for the principal graph of $G \hat{\boxtimes} H$ is the same as of $G \hat{\square} H$, and the transition matrix $P^{(a)}$ for the auxiliary graph of $G \hat{\boxtimes} H$ is given as

$$P^{(a)} = \begin{pmatrix} \frac{1}{d_G(v_1)+1}P_H & \frac{a_{12}^G}{d_G(v_1)+1}P_H & \cdots & \frac{a_{1m}^G}{d_G(v_1)+1}P_H \\ \frac{a_{21}^G}{d_G(v_2)+1}P_H & \frac{1}{d_G(v_2)+1}P_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{a_{m-1m}^G}{d_G(v_{m-1})+1}P_H \\ \frac{a_{m1}^G}{d_G(v_m)+1}P_H & \cdots & \frac{a_{mm-1}^G}{d_G(v_m)+1}P_H & \frac{1}{d_G(v_m)+1}P_H \end{pmatrix}.$$

That is, their components are given as $a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta}$ and $p^{(a)}_{(i,\alpha),(j,\beta)} = p^H_{\alpha\beta}(\delta_{ij} + a^G_{ij})/\{d_G(v_i) + 1\}$. Thus we have

$$A^{(p)}P^{(a)} = \left(p^H_{\alpha\beta}\left(a^G_{ij} + \sum_{k=1}^m a^G_{ik}a^G_{kj}\right)\Big/\{d_G(v_i) + 1\}\right).$$

We now compute eigenvalues of $G \hat{\boxtimes} H$ when G is regular. We have

$$
\begin{aligned}
\Delta_P \varphi_{f,g} &= \varphi_{f,g} - \frac{1}{d_G} A^{(p)} P^{(a)} \varphi_{f,g} \\
&= \left(f_i g_\alpha - \frac{1}{d_G(d_G+1)} \left(\sum_{\beta=1}^{n} p_{\alpha\beta}^{H} g_\beta \right) \left(\sum_{j=1}^{m} a_{ij}^{G} f_j + \sum_{j=1}^{m} \sum_{k=1}^{m} a_{ik}^{G} a_{kj}^{G} f_j \right) \right) \\
&= \left(f_i g_\alpha - \frac{(1-v)(1-\mu)}{d_G+1} g_\alpha \left(f_i + \sum_{k=1}^{m} a_{ik}^{G} f_k \right) \right) \\
&= \left(f_i g_\alpha - \frac{(1-v)(1-\mu)}{d_G+1} \{1 + d_G(1-\mu)\} f_i g_\alpha \right) \\
&= \frac{(1 + d_G - d_G\mu)(\mu + v - \mu v) + d_G\mu}{d_G+1} \varphi_{f,g}.
\end{aligned}
$$

Thus we get the conclusion. □

Corollary 4. *Let G_1, G_2 be isospectral regular finite graphs of same degree and H_1, H_2 be transitionary isospectral finite graphs. Then their Kähler graphs $G_1 \hat{\boxtimes} H_1$, $G_2 \hat{\boxtimes} H_2$ of strong product type are isospectral.*

Remark 3. For Kähler graphs of strong product type in Corollary 4, their auxiliary graphs are also transitionary isospectral.

When G, H are graphs of finite degrees, it is clear that their Kähler graph of semi-tensor product type is of finite degrees. At a vertex $(v,w) \in V \times W$, we have $d^{(p)}_{G\hat{\otimes}H}(v,w) = d_G(v)$ and $d^{(a)}_{G\hat{\otimes}H}(v,w) = d_H(w)d_G(v)$. Therefore we have the following property on eigenvalues of Kähler graphs of semi-tensor product type.

Theorem 4. *Let $G = (V,E)$, $H = (W,F)$ be finite graphs whose eignevalues of Δ_{P_G} and Δ_{P_H} are μ_i $(1 \le i \le m(= \#V))$ and v_α $(1 \le \alpha \le n(= \#W))$. Then the eigenvalues of Δ_P of their Kähler graph $G \hat{\otimes} H$ of semi-tensor product type are*

$$\{(1-\mu_i)(\mu_i + v_\alpha - \mu_i v_\alpha) + \mu_i\} \quad (1 \le i \le m, 1 \le \alpha \le n).$$

Moreover, if $\Delta_{P_G} f = \mu f$, $\Delta_{P_H} g = vg$ and if we set a function $\varphi_{f,g}$ on $V \times W$ by $\varphi_{f,g}(v,w) = f(v)g(w)$, then we have

$$\Delta_P \varphi_{f,g} = \{(1-\mu)(\mu + v - \mu v) + \mu\} \varphi_{f,g}.$$

Proof. For arbitrary finite graphs G, H, the adjacency matrix $A^{(p)}$ for the principal graph of $G \hat{\otimes} H$ is the same as of $G \hat{\square} H$, and the transition matrix $P^{(a)}$ for its auxiliary graph is given as

$$P^{(a)} = \begin{pmatrix} O & \dfrac{a_{12}^G}{d_G(v_1)} P_H & \cdots & \dfrac{a_{1m}^G}{d_G(v_1)} P_H \\ \dfrac{a_{21}^G}{d_G(v_2)} P_H & O & \ddots & \vdots \\ \vdots & \ddots & \ddots & \dfrac{a_{m-1m}^G}{d_G(v_{m-1})} P_H \\ \dfrac{a_{m1}^G}{d_G(v_m)} P_H & \cdots & \dfrac{a_{mm-1}^G}{d_G(v_m)} P_H & O \end{pmatrix}.$$

That is, their components are given as $a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta}$ and $p^{(a)}_{(i,\alpha),(j,\beta)} = a^G_{ij} p^H_{\alpha\beta}/d_G(v_i) = p^G_{ij} p^H_{\alpha\beta}$. Here, we denote by $P_G = (p^G_{ij})$ the transition matrix of G. Thus we have

$$A^{(p)} P^{(a)} = \left(p^H_{\alpha\beta} \sum_{k=1}^m a^G_{ik} p^G_{kj} \right).$$

Therefore, we have

$$\begin{aligned} \Delta_P \varphi_{f,g} &= \left(f_i g_\alpha - \frac{1}{d_G(v_i)} \left(\sum_{\beta=1}^n p^H_{\alpha\beta} g_\beta \right) \left(\sum_{j=1}^m \sum_{k=1}^m a^G_{ik} p^G_{kj} f_j \right) \right) \\ &= \left(f_i g_\alpha - \frac{(1-\mu)(1-\nu)}{d_G(v_i)} g_\alpha \left(\sum_{k=1}^m a^G_{ik} f_k \right) \right) \\ &= (f_i g_\alpha - (1-\nu)(1-\mu)^2 f_i g_\alpha) = \{1 - (1-\nu)(1-\mu)^2\} \varphi_{f,g}. \end{aligned}$$

Thus we get the conclusion. □

PROPOSITION 5. *Let G be a regular finite graph of degree d_G and H be a finite graph. If $\Delta_{P_G} f = \mu f$, $\Delta_{P_H} g = \nu g$ and if we set a function $\varphi_{f,g}$ on $V \times W$ by $\varphi_{f,g}(v,w) = f(v)g(w)$, then the combinatorial Laplacian Δ_A of $G \hat{\otimes} H$ satisfies $\Delta_A \varphi_{f,g} = d_G\{(1-\mu)(\mu+\nu-\mu\nu)+\mu\}\varphi_{f,g}$.*

COROLLARY 5. *Let G_1, G_2 be transitionary isospectral finite graphs and H_1, H_2 be also transitionary isospectral finite graphs.*

(1) *Their Kähler graphs $G_1 \hat{\otimes} H_1$, $G_2 \hat{\otimes} H_2$ of semi-tensor product type are transitionary isospectral.*
(2) *If G_1, G_2 are regular and of same degree, then their Kähler graphs $G_1 \hat{\otimes} H_1$, $G_2 \hat{\otimes} H_2$ of semi-tensor product type are (combinatorially) isospectral.*

Remark 4. For Kähler graphs of semi-tensor product type in Corollary 5 (2), their auxiliary graphs are also transitionary isospectral.

We can consider lexicographical products. Given two graphs $G = (V, E)$ and $H = (W, F)$, we define their Kähler graph of lexicographical product type $G \triangleright H$ as follows:

i) its set of vertices is the product $V \times W$;

ii) two vertices $(v, w), (v', w') \in V \times W$ are adjacent to each other by a principal edge if and only if v, v' are adjacent to each other in G and $w = w'$;

iii) two vertices $(v, w), (v', w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if w, w' are adjacent to each other in H.

Example 7. If we take G and H to graphs of real lines, then their Kähler graph of lexicographical product type is as Figure 10.

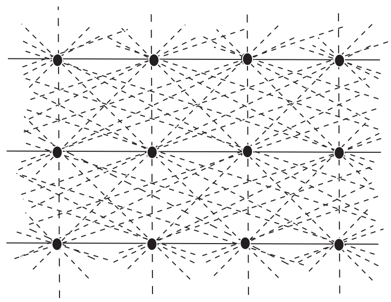

FIGURE 10. $G \triangleright H$

When $G = (V, E)$ is a finite graph and $H = (W, F)$ is a graph of finite degree, their Kähler graph of lexicographical product type is of finite degrees. At a vertex $(v, w) \in V \times W$, we have $d^{(p)}_{G \triangleright H}(v, w) = d_G(v)$ and $d^{(a)}_{G \triangleright H}(v, w) = \#V d_H(w)$. In order to study eigenvalues of $G \triangleright H$, we consider the mean M on V which is given by $Mf(v) = \sum_{u \in V} f(u)$. Eigenvalues of M are $0, \dots, 0, m$, where $m = \#V$. We denote as $V = \{v_1, \dots, v_m\}$ and define a function ε_1 by $\varepsilon_1(v) = 1$ for all $v \in V$, and define functions $\varepsilon_2, \dots, \varepsilon_m$ by $\varepsilon_k = \delta_{v_1} - \delta_{v_k}$ with characteristic functions δ_v. Then ε_1 is an eigenfunction corresponding to m, and $\varepsilon_2, \dots, \varepsilon_m$ are linearly independent eigenfunctions corresponding to 0. Therefore we have the following property on eigenvalues of Kähler graphs of lexicographical product type.

THEOREM 5. *Let $G = (V, E)$ be a finite graph and $H = (W, F)$ be a finite graph whose eignevalues of Δ_{P_H} are ν_α $(1 \le \alpha \le n(= \#W))$. Then the eigenvalues of Δ_P of their Kähler graph $G \triangleright H$ of lexicographical product type are $1, \dots, 1$, $\nu_1, \dots, \nu_n$, where 1 appears $(\#V - 1)\#W$ times.*

Moreover, for k $(k = 1, \dots, m(= \#V))$ *and for a function* g *with* $\Delta_{P_H} g = \nu g$ *we define a function* $\psi_{k,g}$ *on* $V \times W$ *by* $\psi_{k,g}(v, w) = \varepsilon_k(v) g(w)$. *Then we have*

$$\Delta_P \psi_{1,g} = \nu \psi_{1,g} \quad \textit{and} \quad \Delta_P \psi_{k,g} = \psi_{k,g} \quad (k = 2, \dots, m).$$

Proof. For arbitrary finite graphs G, H, the adjacency matrix $A^{(p)}$ for the principal graph of $G \triangleright H$ is the same as of $G \,\hat{\square}\, H$, and the transition matrix $P^{(a)}$ for its auxiliary graph is given as

$$P^{(a)} = \begin{pmatrix} \frac{1}{m} P_H & \cdots & \frac{1}{m} P_H \\ \vdots & & \vdots \\ \frac{1}{m} P_H & \cdots & \frac{1}{m} P_H \end{pmatrix}.$$

That is, $a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij} \delta_{\alpha\beta}$ and $p^{(a)}_{(i,\alpha),(j,\beta)} = p^H_{\alpha\beta}/m$. Thus we have

$$A^{(p)} P^{(a)} = \left(\frac{1}{m} p^H_{\alpha\beta} \sum_{k=1}^{m} a^G_{ik} \right) = \left(\frac{d_G(i)}{m} p^H_{\alpha\beta} \right).$$

We therefore have

$$\begin{aligned} \Delta_P \psi_{k,g} &= \left(\varepsilon_k(v_i) g_\alpha - \frac{1}{m} \left(\sum_{\beta=1}^{n} p^H_{\alpha\beta} g_\beta \right) \left(\sum_{j=1}^{m} \varepsilon_k(v_j) \right) \right) \\ &= \begin{cases} (g_\alpha - (1-\nu) g_\alpha) = \nu \varphi_{1,g}, & k = 1, \\ (\varepsilon_k(v_i) g_\alpha) = \varphi_{k,g}, & k \neq 1. \end{cases} \end{aligned}$$

Thus we get the conclusion. □

Corollary 6. *Let* G_1, G_2 *be finite graphs and* H_1, H_2 *be transitionary isospectral finite graphs. Suppose cardinalities of the sets of vertices of* G_1 *and of* G_2 *coincide. Then their Kähler graphs* $G_1 \triangleright H_1$, $G_2 \triangleright H_2$ *of lexicographical product type are transitionary isospectral.*

For construction of Kähler graphs, we can generalize the notion of Kähler graphs of lexicographical product type to the following. Let $H = (W, F)$ be a finite graph and $G_\alpha = (V_\alpha, E_\alpha)$ $(\alpha = 1, \dots, n = \#W)$ be finite graphs. We denote $W = \{w_1, \dots, w_n\}$. We set $V = V_1 \cup \cdots \cup V_n$ and $E^{(p)} = E_1 \cup \cdots \cup E_n$. We define $E^{(a)}$ in the following manner: Two vertices $v, v' \in V$ are adjacent to each other in auxiliary graph if and only if $v \in V_i$, $v' \in V_j$ and w_i, w_j are adjacent to each other in H. We denote this Kähler graph $(V, E^{(p)} \cup E^{(a)})$ by

$H^K(G_1,\dots,G_n)$ and call it a *Kähler extension* of H by $G_1,\dots,G_n$. It is clear that $G \triangleright H = H^K(G,\dots,G)$ and that $G_1 \hat{+} G_2 = H^K(G_1,G_2)$ with $H = (\{w_1,w_2\}, \{(w_1,w_2)\})$.

We here give another way to construct Kähler graphs. Since the operations of taking complement graphs and of taking products are independent, we may do both of them. For example, given two graphs $G=(V,E)$, $H=(W,F)$ we define a Kähler graph $G\,\hat{\square}^K H$ as follows:

i) its set of vertices is the product $V \times W$;
ii) two vertices $(v,w),(v',w') \in V \times W$ are adjacent to each other by a principal edge if and only if v, v' are adjacent to each other in G and $w = w'$;
iii) two vertices $(v,w),(v',w') \in V \times W$ are adjacent to each other by an auxiliary edge if and only if either $v = v'$ and w, w' are adjacent to each other in H or $w = w'$, $v \neq v'$ and v, v' are not adjacent to each other in G.

For the sake that this definition makes sense, we suppose G is not complete. One may easily find the definitions of $G\,\hat{\boxtimes}^K H$, $G\,\hat{\otimes}^K H$ and $G \triangleright^K H$.

Example 8. When G and H are graphs of real lines, principal and auxiliary edges emanating from a vertex of their Kähler graphs $G\,\hat{\square}^K H$, $G\,\hat{\boxtimes}^K H$, $G\,\hat{\otimes}^K H$ and $G \triangleright^K H$ are like the following figures.

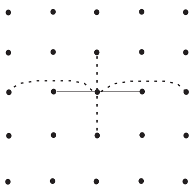
FIGURE 11. $G\,\hat{\square}^K H$

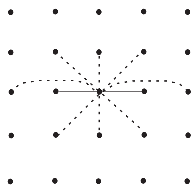
FIGURE 12. $G\,\hat{\boxtimes}^K H$

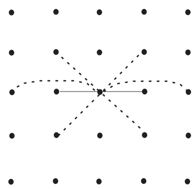
FIGURE 13. $G\,\hat{\otimes}^K H$

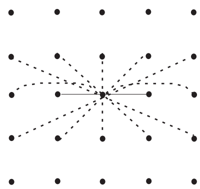
FIGURE 14. $G \triangleright^K H$

PROPOSITION 6. *Let $G=(V,E)$, $H=(W,F)$ be finite regular graphs. Suppose G is connected. We denote the eignevalues of Δ_{P_G} and Δ_{P_H} by μ_i $(1 \le i \le m(=\#V), \mu_1 = 0)$ and ν_α $(1 \le \alpha \le n(=\#W))$, respectively. Then the eigenvalues of Δ_P of the regular Kähler graph $G\,\hat{\square}^K H$ are*

$$\frac{d_H}{\mathscr{D}}\nu_\alpha, 1 - \frac{1}{\mathscr{D}}(1-\mu_i)(d_G\mu_i - d_H\nu_\alpha - d_G + d_H - 1) \quad (2 \le i \le m, 1 \le \alpha \le n),$$

where $\mathscr{D} = m - d_G + d_H - 1$.

Proof. The principal graph of $G\,\hat{\square}^K H$ is the same as of $G\,\hat{\square}\,H$, and the transition matrix $P^{(a)}$ for its auxiliary graph is given as

$$P^{(a)} = \begin{pmatrix} \frac{1}{\mathscr{D}} A_H & \frac{a_{12}^{G^c}}{\mathscr{D}} I & \cdots & \frac{a_{1m}^{G^c}}{\mathscr{D}} I \\ \frac{a_{21}^{G^c}}{\mathscr{D}} I & \frac{1}{\mathscr{D}} A_H & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{a_{m-1m}^{G^c}}{\mathscr{D}} I \\ \frac{a_{m1}^{G^c}}{\mathscr{D}} I & \cdots & \frac{a_{mm-1}^{G^c}}{\mathscr{D}} I & \frac{1}{\mathscr{D}} A_H \end{pmatrix}.$$

That is, their components are given as $a^{(p)}_{(i,\alpha),(j,\beta)} = a^G_{ij}\delta_{\alpha\beta}$ and $p^{(a)}_{(i,\alpha),(j,\beta)} = (a^{G^c}_{ij}\delta_{\alpha\beta} + \delta_{ij}a^H_{\alpha\beta})/\mathscr{D}$. Therefore we have

$$A^{(p)}P^{(a)} = \left(\frac{1}{\mathscr{D}}\left\{a^G_{ij}a^H_{\alpha\beta} + \sum_{k=1}^m a^G_{ik}a^{G^c}_{kj}\delta_{\alpha\beta}\right\}\right).$$

For functions f, g satisfying $\Delta_{P_G} f = \mu f$ and $\Delta_{P_H} g = \nu g$ we define a function $\varphi_{f,g}$ as in the proof of Theorem 2. Since we have $A_G f = d_G(1-\mu)f$ and G is connected, we see that $A_{G^c} f = \{m - d_G - 1\}f$ when $\mu = 0$ and $A_{G^c} f = \{d_G(\mu - 1) - 1\}f$ when $\mu \neq 0$ (see the proof of Theorem 1). As $A_H g = d_H(1-\nu)g$, we find

$$\Delta_P \varphi_{f,g} = \begin{cases} \left\{1 - \frac{1}{\mathscr{D}}(m - d_G - 1 + d_H - d_H\nu)\right\}\varphi_{f,g}, & \text{when } \mu = 0, \\ \left\{1 - \frac{1}{\mathscr{D}}(1-\mu)(d_G\mu - d_G - 1 + d_H - d_H\nu)\right\}\varphi_{f,g}, & \text{when } \mu \neq 0, \end{cases}$$

and get the conclusion. □

By the same argument we have the following.

Proposition 7. *Let $G = (V, E)$, $H = (W, F)$ be finite regular graphs. Suppose G is connected. We denote the eignevalues of Δ_{P_G} and Δ_{P_H} by μ_i $(1 \le i \le m(= \#V), \mu_1 = 0)$ and ν_α $(1 \le \alpha \le n(= \#W))$, respectively. Then the eigenvalues of Δ_P of the regular Kähler graph $G \,\hat{\boxtimes}^K H$ are*

$$\frac{d_H(d_G+1)}{\mathscr{D}}\nu_\alpha, \quad 1 - \frac{1}{\mathscr{D}}(1-\mu_i)(d_G\mu_i - d_G - 1)(d_H\nu_\alpha - d_H + 1)$$
$$(2 \le i \le m, 1 \le \alpha \le n),$$

where $\mathscr{D} = m + d_G d_H - d_G + d_H - 1$.

Proof. The components of the transition matrix $P^{(a)}$ for its auxiliary graph is given as $p^{(a)}_{(i,\alpha),(j,\beta)} = (a^{G^c}_{ij}\delta_{\alpha\beta} + (\delta_{ij} + a^G_{ij})a^H_{\alpha\beta})/\mathscr{D}$. For functions f, g satisfying

$\Delta_{P_G} f = \mu f$ and $\Delta_{P_H} g = \nu g$ we define a function $\varphi_{f,g}$ as in the proof of Theorem 2. By the same argument as in the proof of Proposition 6, we find

$$P\varphi_{f,g} = \begin{cases} \dfrac{1}{\mathscr{D}}(d_H(1-\nu)(d_G+1)+m-d_G-1)\varphi_{f,g}, & \text{when } \mu = 0, \\ \dfrac{1}{\mathscr{D}}(1-\mu)(d_G\mu - d_G - 1)(d_H\nu - d_H + 1)\varphi_{f,g}, & \text{when } \mu \neq 0, \end{cases}$$

hence get the conclusion. □

Proposition 8. *Let $G = (V, E)$, $H = (W, F)$ be finite regular graphs. Suppose G is connected. We denote the eignevalues of Δ_{P_G} and Δ_{P_H} by μ_i $(1 \le i \le m(= \#V), \mu_1 = 0)$ and ν_α $(1 \le \alpha \le n(= \#W))$, respectively. Then the eigenvalues of Δ_P of the regular Kähler graph $G \hat{\otimes}^K H$ are*

$$\frac{d_G d_H}{\mathscr{D}}\nu_\alpha, \quad 1 - \frac{1}{\mathscr{D}}(1-\mu_i)\{d_G d_H(1-\mu_i)(1-\nu_\alpha) + d_G\mu_i - d_G - 1\}$$
$$(2 \le i \le m, 1 \le \alpha \le n),$$

where $\mathscr{D} = m + d_G d_H - d_G - 1$.

Proof. The components of the transition matrix $P^{(a)}$ for its auxiliary graph is given as $p^{(a)}_{(i,\alpha),(j,\beta)} = (a^{G^c}_{ij}\delta_{\alpha\beta} + a^G_{ij}a^H_{\alpha\beta})/\mathscr{D}$. For functions f, g satisfying $\Delta_{P_G} f = \mu f$ and $\Delta_{P_H} g = \nu g$ we define a function $\varphi_{f,g}$ as in the proof of Theorem 2. By the same argument as in the proof of Proposition 6, we find

$$P\varphi_{f,g} = \begin{cases} \dfrac{1}{\mathscr{D}}(d_G d_H(1-\nu) + \#V - d_G - 1)\varphi_{f,g}, & \text{when } \mu = 0, \\ \dfrac{1}{\mathscr{D}}(1-\mu)(d_G d_H(1-\mu)(1-\nu) + d_G\mu - d_G - 1)\varphi_{f,g}, & \text{when } \mu \neq 0, \end{cases}$$

hence get the conclusion. □

Proposition 9. *Let $G = (V, E)$, $H = (W, F)$ be finite regular graphs. Suppose G is connected. We denote the eignevalues of Δ_{P_G} and Δ_{P_H} by μ_i $(1 \le i \le m(= \#V), \mu_1 = 0)$ and ν_α $(1 \le \alpha \le n(= \#W))$, respectively. Then the eigenvalues of Δ_P of the Kähler graph $G \triangleright^K H$ are*

$$\frac{m d_H}{\mathscr{D}}\nu_\alpha, \quad 1 - \frac{1}{\mathscr{D}}(1-\mu_i)(d_G\mu_i - d_G - 1) \quad (2 \le i \le m, 1 \le \alpha \le n),$$

where $\mathscr{D} = m(d_H + 1) - d_G - 1$ and the latter appears n times.

Proof. The components of the transition matrix $P^{(a)}$ for its auxiliary graph is given as $p^{(a)}_{(i,\alpha),(j,\beta)} = (a^{G^c}_{ij}\delta_{\alpha\beta} + a^H_{\alpha\beta})/\mathscr{D}$. For functions f, g satisfying $\Delta_{P_G} f = \mu f$ and $\Delta_{P_H} g = \nu g$ we define a function $\varphi_{f,g}$ as in the proof of Theorem 2. Since

$\sum_{v \in V} f(v) = 0$ when $\mu \neq 0$, by the same argument as in the proof of Proposition 6, we find

$$P\varphi_{f,g} = \begin{cases} \dfrac{1}{\mathscr{D}}(md_H(1-v) + m - d_G - 1)\varphi_{f,g}, & \text{when } \mu = 0, \\ \dfrac{1}{\mathscr{D}}(1-\mu)(d_G\mu - d_G - 1)\varphi_{f,g}, & \text{when } \mu \neq 0, \end{cases}$$

hence get the conclusion. □

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