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ON COMPLEX n-FOLDS POLARIZED BY AN AMPLE LINE BUNDLE L WITH $\mathrm{Bs}|L| = \emptyset$, $g(X,L) = q(X) + m$ AND $h^0(L) = n + m - 1$*

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Abstract

Let (X,L) be a polarized manifold with $\dim X = n \geq 3$ and $\mathrm{Bs}|L| = \emptyset$. In this paper, we classify (X,L) with $g(X,L) = q(X) + m$ and $h^0(L) = n + m - 1$.

1. Introduction

This is a continuation of [13]. Let X be a smooth projective variety over the field of complex numbers $\mathbf{C}$, and let L be an ample (resp. a nef and big) line bundle on X. Then we call the pair (X,L) a *polarized* (resp. *quasi-polarized*) *manifold*. The sectional genus $g(X,L)$ of (X,L) is defined as follows:

$$g(X,L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical line bundle of X. A classification of (X,L) with small value of sectional genus was obtained by several authors. On the other hand, as a problem of a lower bound for the sectional genus, Fujita proposed the following conjecture:

Conjecture 1.1 (Fujita). *Let (X,L) be a polarized manifold. Then $g(X,L) \geq q(X)$, where $q(X) = h^1(\mathcal{O}_X)$ is the irregularity of X.*

This conjecture is very difficult and it is unknown even for the case in which X is a surface.

If $\dim \mathrm{Bs}|L| \leq 1$, then we can prove that $g(X,L) \geq q(X)$ (see [9, Theorem 3.2] and [20, Theorem 3.3]). Furthermore the author proved that if (X,L) is a quasi-polarized manifold with $\dim X = 3$ and $h^0(L) := \dim H^0(L) \geq 2$, then

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$g(X, L) \geq q(X)$ (see [11, Theorem 2.1]). Moreover the author obtained the classification of polarized 3-folds (X, L) with the following types:

(1) $g(X, L) = q(X)$ and $h^0(L) \geq 3$ ([11]).
(2) $g(X, L) = q(X) + 1$ and $h^0(L) \geq 4$ ([8]).
(3) $g(X, L) = q(X) + 2$ and $h^0(L) \geq 5$ ([12]).

On the other hand, if $|L|$ is base point free, then we can get the classification of polarized n-folds (X, L) with $g(X, L) = q(X)$, $g(X, L) = q(X) + 1$, or $g(X, L) = q(X) + 2$ (see [8], [11], and [12]).

By considering the result of 3-dimensional case, it is natural to consider the following problem:

Problem 1.1. Let (X, L) be a polarized manifold with $\dim X = n$ and $g(X, L) = q(X) + m$, where m is a nonnegative integer. Assume that $h^0(L) \geq n + m$. Then classify (X, L) with these properties.

In [13], we obtained an answer of this problem if $\dim \mathrm{Bs}|L| \leq 0$.

The next problem it might be considered is a classification of (X, L) with $h^0(L) = n + m - 1$. In this paper, under the assumption that $\mathrm{Bs}|L| = \emptyset$, we get a classification of (X, L) with $h^0(L) = n + m - 1$, that is, we obtain the following main theorem.

Theorem 1.1. *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$. *Assume that* $\mathrm{Bs}|L| = \emptyset$, $h^0(L) = n + m - 1$, *where* $m = g(X, L) - q(X)$. *Let* (M, A) *be a reduction of* (X, L) (*see Definition* 2.1 (2)). *Then* (X, L) *is one of the following types*:

(1) (X, L) *is a hyperquadric fibration over a smooth projective curve with*

$$q(X) \leq \frac{3}{2} + \frac{2m - 1}{2n}.$$

(2) (X, L) *is a classical scroll over a smooth projective surface* Y *with* $\kappa(Y) = -\infty$.
If $q(X) > 0$ *and* Y *is relatively minimal, then*

$$q(X) \leq 1 + \frac{2m - n + 1}{n^2 - 3n + 4}.$$

If $q(X) > 0$ *and* Y *is not relatively minimal, then*

$$q(X) \leq 1 + \frac{4m - 1}{8n^2 - 20n + 16}.$$

(3) $(M, A) = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2))$ *and* (X, L) *is obtained by* 7 *times simple blowing ups of* (M, A). *In this case* $m = 5$.
(4) $(M, A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$ *and* (X, L) *is obtained by* 7 *times simple blowing ups of* (M, A). *In this case* $m = 5$.
(5) $(M, A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$ *and* (X, L) *is obtained by* 8 *times simple blowing ups of* (M, A). *In this case* $m = 10$.

(6) *M is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ with $(F, A|_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ for any fiber F of it, and (X, L) is obtained by 7 times simple blowing ups of (M, A). In this case $m \geq 6$.*

(7) *(X, L) is a Mukai manifold with $L^n = 2m - 2$ and $\Delta(X, L) = m - 1$.*

Remark 1.1. In general, $m \geq 2$ holds because L is ample and $\mathrm{Bs}|L| = \emptyset$ with $h^0(L) = n + m - 1$.

By using Theorem 1.1, we can classify polarized manifolds (X, L) such that L is very ample and $3 \leq g(X, L) - q(X) \leq 5$. We are planning to write a paper on this.

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Notation and conventions

In this paper, we shall study mainly a smooth projective variety X over the field of complex numbers $\mathbf{C}$. We will use the customary notation in Algebraic Geometry.

2. Preliminaries

Definition 2.1. (1) Let (X, L) and (X', L') be polarized manifolds with $\dim X = \dim X' = n$. Then (X, L) is called a *simple blowing up of* (X', L') if X is the blowing up of X' at a point of X' and $L = \pi^*(L') - E$, where $\pi : X \to X'$ denotes its blowing up and E is the exceptional divisor.

(2) Let X (resp. M) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. M). Then we say that (M, A) is a *reduction of* (X, L) if there exists a birational morphism $\mu : X \to M$ such that μ is a composition of simple blowing ups and (M, A) is not obtained by a simple blowing up of any polarized manifold. In this case the map μ is called the *reduction map*.

Remark 2.1. Let (X, L) be a polarized manifold and let (M, A) be a reduction of (X, L). Then the following hold.

(1) $g(X, L) = g(M, A)$.

(2) $A^n - L^n = t$, where t is the number of simple blowing ups of (M, A).

Proposition 2.1. *Let X and M be smooth projective varieties with* $\dim X = \dim M = n \geq 3$, *and let L and A be ample line bundles on X and M respectively such that (X, L) is obtained by a finite number of simple blowing ups of (M, A) and $K_M + (n - 2)A$ is nef. Let $\mu : X \to M$ be its birational morphism. Assume that* $\mathrm{Bs}|L| = \emptyset$. *Then* $\dim \mathrm{Bs}|A| \leq 0$ *holds and for any general member $D \in |L|$ we have* $\mathrm{Bs}|L_D| = \emptyset$, *$\mu(D) \in |A|$ is smooth, (D, L_D) is obtained by a finite number of* *simple blowing ups of* $(\mu(D), A_{\mu(D)})$ *via* $\mu|_D : D \to \mu(D)$, $\dim \mathrm{Bs}|A_{\mu(D)}| \le 0$ *and* $K_{\mu(D)} + (n-3)A_{\mu(D)}$ *is nef.*

Proof. We note that $\dim \mathrm{Bs}|A| \le 0$ since $\mathrm{Bs}|L| = \emptyset$ and (X, L) is obtained by a finite number of simple blowing ups of (M, A). By [19, Proposition 2.1] we see that $\mu(D) \in |A|$ is smooth, (D, L_D) is obtained by a finite number of simple blowing ups of $(\mu(D), A_{\mu(D)})$ via $\mu|_D : D \to \mu(D)$. Because $\mathrm{Bs}|L_D| = \emptyset$, we have $\dim \mathrm{Bs}|A_{\mu(D)}| \le 0$. Since $K_M + (n-2)A$ is nef, so is $K_{\mu(D)} + (n-3)A_{\mu(D)}$ by adjunction formula. □

Proposition 2.2. *Let* (X, L) *and* (M, A) *be polarized manifolds with* $n = \dim X = \dim M \ge 3$. *Assume that* $\mathrm{Bs}|L| = \emptyset$, (X, L) *is obtained by a finite number of simple blowing ups of* (M, A), *and* $K_M + (n-2)A$ *is nef. Then there exists a polarized surface* (S, H) *such that* S *is smooth,* $A^n = H^2$, $\dim \mathrm{Bs}|H| \le 0$, $g(S, H) = g(X, L)$, $q(X) = q(S)$, K_S *is nef, and* $h^0(H) \ge h^0(A) - (n-2)$.

Proof. By Proposition 2.1 we see that there exist smooth projective varieties X_i and M_i of dimension $n-i$ for $1 \le i \le n-2$ and ample line bundles L_i and A_i on X_i and M_i respectively such that $X_i \in |L_{i-1}|$ and $M_i \in |A_{i-1}|$, and (X_i, L_i) is obtained by a finite number of simple blowing ups of (M_i, A_i), where $X_0 := X$, $M_0 := M$, $L_0 := L$, $A_0 := A$, $L_i := L_{i-1}|_{X_i}$ and $A_i = A_{i-1}|_{M_i}$. We set $S := M_{n-2}$ and $H := A_{n-2}$. Then we see from Proposition 2.1 that S is smooth, $A^n = H^2$, $\dim \mathrm{Bs}|H| \le 0$, $g(X, L) = g(M, A) = g(S, H)$ and $q(X) = q(M) = q(S)$ hold. Finally we see from the exact sequence

$$0 \to \mathcal{O}_{X_i} \to A_i \to A_{i+1} \to 0$$

that

$$\begin{aligned} h^0(H) &= h^0(A_{n-2}) \ge h^0(A_{n-3}) - 1 \ge h^0(A_{n-4}) - 2 \ge \cdots \ge h^0(A_0) - (n-2) \\ &= h^0(A) - (n-2). \end{aligned}$$

So we get the assertion. □

Proposition 2.3. *Let* (X, L) *and* (M, A) *be polarized manifolds with* $n = \dim X = \dim M \ge 3$. *Assume that* $\mathrm{Bs}|L| = \emptyset$, (X, L) *is obtained by a finite number of simple blowing ups of* (M, A), *and* $K_M + (n-2)A$ *is nef. Then* $\chi_2^H(X, L) = \chi_2^H(M, A) \ge 1$, *where* $\chi_2^H(X, L)$ (*resp.* $\chi_2^H(M, A)$) *denotes the second sectional* H*-arithmetic genus of* (X, L) (*resp.* (M, A)) (*see* [17, Definition 2.1]).

Proof. Since $K_M + (n-2)A$ is nef, we have $\kappa(K_M + (n-2)A) \ge 0$ by the nonvanishing theorem [26, (0.2)].

If $0 \le \kappa(K_M + (n-2)A) \le 1$, then by [17, Theorem 3.2.1] we get $\chi_2^H(M, A) \ge 1$.

Next we consider the case of $\kappa(K_M + (n-2)A) \ge 2$. By the argument in [17, Remark 2.2] and the proof of Proposition 2.2, there exists a smooth projective surface S such that $\chi_2^H(M,A) = \chi(\mathcal{O}_S)$. Furthermore we see from [17, Proposition 2.1 (1)] that $\kappa(S) = 2$. Hence by Castelnuovo's theorem $\chi(\mathcal{O}_S) \geq 1$ holds. Therefore we have $\chi_2^H(M,A) \geq 1$.

Since $\chi_2^H(X,L) = \chi_2^H(M,A)$ by [17, Remark 2.1 (5)], we get the assertion. □

Definition 2.2. Let (X,L) be a polarized manifold of dimension n.

(1) We say that (X,L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$) for some ample line bundle A on Y.

(2) (X,L) is called a *classical scroll over a normal variety* Y if there exists a vector bundle $\mathcal{E}$ on Y such that $X \cong \mathbf{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle.

(3) We say that (X,L) is a *hyperquadric fibration over a smooth projective curve* C if (X,L) is a quadric fibration over C such that the morphism $f : X \to C$ is the contraction morphism of an extremal ray. In this case, if $n \geq 3$, then $(F, L_F) \cong (\mathbf{Q}^{n-1}, \mathcal{O}_{\mathbf{Q}^{n-1}}(1))$ for any general fiber F of f, every fiber of f is irreducible and reduced (see [22] or [5, Claim (3.1)]) and $h^2(X,\mathbf{C}) = 2$.

Theorem 2.1. *Let* (X,L) *be a polarized manifold with* $n = \dim X \geq 3$. *Then* (X,L) *is one of the following types*:

(1) $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$.

(2) *A scroll over a smooth projective curve.*

(3) $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$.

(4) $K_X \sim -(n-1)L$, *that is,* (X,L) *is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth projective curve.*

(6) *A classical scroll over a smooth projective surface.*

(7) *Let* (M,A) *be a reduction of* (X,L).

 (7.1) $n = 4$, $(M,A) = (\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(2))$.

 (7.2) $n = 3$, $(M,A) = (\mathbf{Q}^3, \mathcal{O}_{\mathbf{Q}^3}(2))$.

 (7.3) $n = 3$, $(M,A) = (\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(3))$.

 (7.4) $n = 3$, M *is a* $\mathbf{P}^2$*-bundle over a smooth projective curve* C *with* $(F, A|_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ *for any fiber* F *of it.*

 (7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [2, Proposition 7.2.2, Theorem 7.2.3, Theorem 7.2.4, Theorem 7.3.2, and Theorem 7.3.4]. See also [6, (11.2), (11.7) and (11.8) in Chapter II] or [22, Theorem in Section 1]. □

Remark 2.2. (1) A polarized manifold (X,L) in the case 2 in [2, Theorem 7.3.2] is a quadric fibration over a smooth curve. If (X,L) is a quadric fibration over a smooth curve C with $\dim X \geq 3$, then by [2, Theorem

14.2.1] and the proof of [22, Lemma (c) in Section 1], we see that (X, L) is one of the following:
(a) A hyperquadric fibration over C.
(b) A classical scroll over a smooth surface with $\dim X = 3$.

(2) A polarized manifold (X, L) in the case 3 in [2, Theorem 7.3.2] is a scroll over a normal surface. If (X, L) is a scroll over a normal surface S, then we can prove that S is smooth and (X, L) is a classical scroll over S (see [2, Theorem 11.1.1]).

(3) In the case 4 in [2, Theorem 7.3.4], the reduction (M, A) of (X, L) has the property that there exist a smooth curve C and a surjective morphism $f: M \to C$ with connected fibers such that $(F, A|_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ for a *general* fiber F. However, in this case we see from [6, (11.8.5), (5-i) in the proof of (11.8) Theorem] that $(F, A|_F) = (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2))$ for *any* fiber F and (M, A) is the case (7.4) in Theorem 2.1 (see also [6, (13.10)]).

Definition 2.3 (See [7, Definition 1.9].).
(1) Let (X, L) be a quasi-polarized surface. Then (X, L) is called *L-minimal* if $LE > 0$ for any (-1)-curve E on X.
(2) For any quasi-polarized surface (X, L), there is a quasi-polarized surface (S, A) and a birational morphism $\mu: X \to S$ such that $L = \mu^*(A)$ and (S, A) is A-minimal. Then we call (S, A) an *L-minimalization of* (X, L).

Remark 2.3. If (X, L) is a polarized surface, then (X, L) is L-minimal.

Theorem 2.2. *Let (X, L) be a quasi-polarized surface with $h^0(L) \geq 2$ and $\kappa(X) = 2$. Assume that $g(X, L) = q(X) + m$ for $m \geq 0$. Then $L^2 \leq 2m$. Moreover if $L^2 = 2m$ and (X, L) is L-minimal, then $X \cong C_1 \times C_2$ and $L \equiv C_1 + 2C_2$, where C_1 and C_2 are smooth curves with $g(C_1) \geq 2$ and $g(C_2) = 2$.*

Proof. We obtain this assertion by using [10, Theorem 3.1] and the fact that $L^2 \leq 2m$ is equivalent to $K_X L \geq 2q(X) - 2$. □

Theorem 2.3. *Let (X, L) be a quasi-polarized surface with $\kappa(X) = 0$ or 1. Assume that $g(X, L) = q(X) + m$. Then $L^2 \leq 2m + 2$.*

If this equality holds and (X, L) is L-minimal, then (X, L) is one of the following;
(1) *The case where $\kappa(X) = 0$.*
X is an Abelian surface and L is any nef and big divisor.
(2) *The case where $\kappa(X) = 1$.*
$X \cong F \times C$ and $L \equiv C + (m+1)F$, where F and C are smooth projective curves with $g(C) \geq 2$ and $g(F) = 1$.

Proof. We get the assertion by using [10, Theorem 2.1] and the fact that $L^2 \leq 2m + 2$ is equivalent to $K_X L \geq 2q(X) - 4$. □

THEOREM 2.4. *Let X be a smooth projective surface, and let $\mathscr{E}$ be an ample and spanned vector bundle of rank $r \geq 2$ on X.*

(1) *If $c_2(\mathscr{E}) = 1$, then $(X, \mathscr{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2})$.*

(2) *If $c_2(\mathscr{E}) = 2$, then $r = 2$ and $(X, \mathscr{E})$ is one of the following pairs:*

(2.1) $(X, \mathscr{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}(2))$.

(2.2) $(X, \mathscr{E}) \cong (\mathbf{Q}^2, \mathcal{O}_{\mathbf{Q}^2}(1) \oplus \mathcal{O}_{\mathbf{Q}^2}(1))$.

(2.3) *X is isomorphic to a geometrically ruled surface $\mathbf{P}_C(\mathscr{F})$ over an elliptic curve C with the projection $\pi : \mathbf{P}_C(\mathscr{F}) \to C$ and with the tautological line bundle $H(\mathscr{F})$, and $\mathscr{E} \cong \pi^*(\mathscr{G}) \otimes H(\mathscr{F})$, where $\mathscr{F}$ and $\mathscr{G}$ are indecomposable rank two vector bundles on C of degree* 1.

(2.4) *There exists a finite morphism $f : X \to \mathbf{P}^2$ of degree* 2 *and $\mathscr{E} \cong f^*(\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2})$.*

Proof. (1) See [2, Theorem 11.1.3].

(2) See [25, Theorem 6.1] and [23, Corollary]. □

LEMMA 2.1. *Let (X, L) be a polarized manifold with* $\dim X = n \geq 3$, $\mathrm{Bs}|L| = \emptyset$, *$g(X, L) = q(X) + m$, and $h^0(L) = n + m - 1$. If $L^n \leq 2m - 2$, then $q(X) = 0$, $g(X, L) = m = \Delta(X, L) + 1$ and $L^n = 2\Delta(X, L) = 2m - 2$, where $\Delta(X, L)$ is the Δ-genus of (X, L) (see* [6]).

Proof. Then $\Delta(X, L) = n + L^n - h^0(L) \leq n + 2m - 2 - (n + m - 1) = m - 1$. Hence we get

$$g(X, L) \geq q(X) + \Delta(X, L) + 1 > \Delta(X, L). \tag{1}$$

We set $t := m - 1 - \Delta(X, L)$, where t is a non-negative integer. Then

$$\begin{aligned} m - t = \Delta(X, L) + 1 &= n + L^n - h^0(L) + 1 \\ &= n + L^n - n - m + 2 \\ &= L^n - m + 2. \end{aligned}$$

So we obtain $L^n = 2m - 2 - t = 2\Delta(X, L) + t \geq 2\Delta(X, L)$.

If $t > 0$, then $L^n \geq 2\Delta(X, L) + 1$ and we get $q(X) = 0$ and $m = g(X, L) = \Delta(X, L)$ by [6, (I.3.5)], but this is a contradiction by (1). Hence $t = 0$, $\Delta(X, L) = m - 1$, and $L^n = 2\Delta(X, L) = 2m - 2$.

Next we prove that $q(X) = 0$. If $q(X) > 0$, then $g(X, L) \geq m + 1 = \Delta(X, L) + 2$. Hence by [4, Corollary (1.10)] (X, L) is a hyperelliptic polarized manifold. By [4, (6.1) Table II] we see that (X, L) is a scroll over a smooth curve. Then $m = 0$ holds because $g(X, L) = q(X)$. However, since L is ample and spanned with $h^0(L) = n + m - 1$, we have $m \geq 2$ and this is impossible. □

LEMMA 2.2. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Let $m = g(X, L) - q(X)$ and let (M, A) be a reduction of (X, L). Assume that*

$\mathrm{Pic}(M) \cong \mathbf{Z}$, $\mathrm{Bs}|L| = \emptyset$, $h^0(L) = n + m - 1$ *and* $L^n \leq 2m - 2$. *Then* $(X, L) = (M, A)$.

Proof. By Lemma 2.1 we see that $q(X) = 0$, $g(X, L) = \Delta(X, L) + 1 = m$ and $L^n = 2m - 2 = 2\Delta(X, L)$. Then we see from [4, (1.4) Theorem] that (X, L) is either a Mukai manifold[1] or a hyperelliptic polarized manifold.

Assume that $(X, L) \neq (M, A)$. Then there exist polarized manifolds (X_i, L_i) and birational morphisms $\mu_i : X_{i-1} \to X_i$ for $i = 1, \ldots, s$ such that $(X_0, L_0) = (X, L)$, $(X_s, L_s) = (M, A)$ and μ_i is a simple blowing up of (X_i, L_i). We set $\mu := \mu_s \circ \cdots \circ \mu_1$. Let E_i be the exceptional divisor of μ_i.

(A) If (X, L) is a Mukai manifold, then $\mathcal{O}_X(K_X + (n-2)L) = \mathcal{O}_X$. Since $K_X = \mu_1^*(K_{X_1}) + (n-1)E_1$ and $L = \mu_1^*(L_1) - E_1$, we have $K_X + (n-2)L = \mu_1^*(K_{X_1} + (n-2)L_1) + E_1$. So we get

$$\mu_1^*(K_{X_1} + (n-2)L_1) = -E_1 \tag{2}$$

by assumption. Since $0 < h^0(\mu_1^*(K_{X_1} + (n-2)L_1) + E_1) = h^0(K_{X_1} + (n-2)L_1)$, we infer that $\mu_1^*(K_{X_1} + (n-2)L_1)H^{n-1} \geq 0$ for any ample line bundle H on X. On the other hand, since H is ample and E_1 is a nonzero effective divisor, we have $(-E_1)H^{n-1} < 0$. So we get a contradiction from (2). Therefore the case where (X, L) is a Mukai manifold is impossible.

(B) Next we consider the case where (X, L) is a hyperelliptic polarized manifold. First we note that $b_2(X) \geq 2$ since we assume that $(X, L) \neq (M, A)$. So by [4, (6.1) Table II] we see that (X, L) is the type $(\Sigma^n(\delta)^+_{a,b})$.

In this case, X is a double covering of a projective bundle $\mathbf{P}_{\mathbf{P}^1}(\mathscr{E})$ over $\mathbf{P}^1$, where $\mathscr{E}$ is a vector bundle on $\mathbf{P}^1$. Let $\pi : X \to \mathbf{P}_{\mathbf{P}^1}(\mathscr{E})$ be the morphism of the double covering and let $p : \mathbf{P}_{\mathbf{P}^1}(\mathscr{E}) \to \mathbf{P}^1$ be the projection. Then $f := p \circ \pi : X \to \mathbf{P}^1$ is a surjective morphism. Since $E_1 \subset X$ and $\mathbf{P}^{n-1} \cong E_1$, we see that $f(E_1)$ is a point. Therefore by using [2, Lemma 4.1.13] we infer that there exists a surjective morphism $f_1 : X_1 \to \mathbf{P}^1$ such that $f = f_1 \circ \mu_1$. By iterating this process, there exists a surjective morphism $f_i : X_i \to \mathbf{P}^1$ with connected fibers such that $f_i = f_{i+1} \circ \mu_{i+1}$ for each i. In particular there exists a surjective morphism $f_s : M = X_s \to \mathbf{P}^1$ with connected fibers such that $f = f_s \circ \mu_s \circ \cdots \circ \mu_1 = f_s \circ \mu$, but this is impossible because $\mathrm{Pic}(M) \cong \mathbf{Z}$. Therefore we get the assertion. □

LEMMA 2.3. *Let a polarized manifold* (X, L) *be a classical scroll over a smooth projective surface* Y *with* $n = \dim X \geq 3$ *and* $q(X) > 0$. *Assume that* $\mathrm{Bs}|L| = \emptyset$, $g(X, L) = q(X) + m$, $h^0(L) = n + m - 1$, *and* $\kappa(Y) = -\infty$. *Then*

(i) *If* Y *is relatively minimal, then*

$$q(X) \leq 1 + \frac{2m - n + 1}{n^2 - 3n + 4}.$$

[1] A polarized manifold (X, L) is called a *Mukai manifold* if $\mathcal{O}_X(K_X + (n-2)L) = \mathcal{O}_X$. In [4] Fujita used the terminology "Fano-K3 variety".

(ii) *If Y is not relatively minimal, then*

$$q(X) \leq 1 + \frac{4m-1}{8n^2 - 20n + 16}.$$

Proof (See also [8, Lemma 1.21] and [13, Lemma 1.7].). Let $\mathscr{E}$ be an ample vector bundle of rank $n-1$ on Y such that $X = \mathbf{P}_Y(\mathscr{E})$ and $L = H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle on $\mathbf{P}_Y(\mathscr{E})$. Let $N = c_1(\mathscr{E})$. We note that N is ample and spanned. Since (Y, N) is not scroll over a smooth curve, we have $g(Y, N) \geq 2q(Y)$ by $\kappa(Y) = -\infty$ and [8, Lemma 1.16]. Moreover we note that $c_2(\mathscr{E}) \geq 1$ holds since $\mathscr{E}$ is ample.

(a) The case in which Y is relatively minimal.

(a.1) The case in which $c_2(\mathscr{E}) \geq 3$.

Since $q(Y) > 0$ by assumption, we see that Y is a $\mathbf{P}^1$-bundle over a smooth curve C with $g(C) > 0$. Let $\mathscr{F}$ be a vector bundle of rank two on C such that $Y = \mathbf{P}_C(\mathscr{F})$. Let $\pi : Y \to C$ be the projection and let C_0 be a minimal section of π with $e = -C_0^2$. Let F_π be a fiber of π. We put $N \equiv aC_0 + bF_\pi$. Then

$$n - 1 = \operatorname{rank}(\mathscr{E}) \leq a = NF_\pi. \tag{3}$$

On the other hand, we get

$$K_Y N = 2q(Y) - 2 + (a-1)(2q(Y) - 2) + ae - 2b.$$

Claim 2.1. *$2b - ae \geq (a-1)(2q(Y) - 2) + 2$ holds.*

Proof. If $2b - ae \leq (a-1)(2q(Y) - 2) + 1$, then $K_Y N \geq 2q(Y) - 3$ by the above equality, that is, $N^2 \leq 2m + 1$. (We note that $g(Y, N) = g(X, L) = q(X) + m = q(Y) + m$.)

On the other hand, since $c_2(\mathscr{E}) \geq 3$ and $L^n = N^2 - c_2(\mathscr{E})$, we get $L^n \leq 2m - 2$. By Lemma 2.1, we get $q(Y) = q(X) = 0$ and this is a contradiction. □

Therefore by (3) and Claim 2.1 we get the following:

$$\begin{aligned} N^2 &= 2ab - a^2 e \\ &= a(2b - ae) \\ &\geq a(a-1)(2q(Y) - 2) + 2a \\ &\geq 2(n-1)(n-2)(q(Y) - 1) + 2(n-1). \end{aligned} \tag{4}$$

Furthermore since $q(Y) > 0$ and (Y, N) is not scroll, $K_Y + N$ is nef. Hence we have the following:

$$\begin{aligned} 0 \leq (K_Y + N)^2 &= 4(g(Y, N) - 2q(Y) + 1) - N^2 \\ &= 4(m - q(Y) + 1) - N^2. \end{aligned} \tag{5}$$

By the inequalities (4) and (5) we get

$$q(X) = q(Y) \leq 1 + \frac{2m - n + 1}{n^2 - 3n + 4}.$$

(a.2) The case in which $c_2(\mathscr{E}) \leq 2$.

Then by $c_2(\mathscr{E}) \leq 2$ we get $q(Y) \leq 1$ by Theorem 2.4. Since $q(Y) > 0$, we have $q(Y) = 1$, and by Theorem 2.4 we have $n = 3$. On the other hand since L is ample with $\mathrm{Bs}|L| = \emptyset$ and $h^0(L) = n + m - 1$, we get $m \geq 2$. So in this case we obtain

$$1 = q(Y) = q(X) < 1 + \frac{2m - 2}{4} = 1 + \frac{2m - n + 1}{n^2 - 3n + 4}.$$

(b) The case in which Y is not relatively minimal.

Let Y' be the relatively minimal model of Y and let $\mu : Y \to Y'$ be its birational morphism. Since $q(Y) = q(X) > 0$, Y' is a $\mathbf{P}^1$-bundle over a smooth curve C. Let $\pi' : Y' \to C$ be the projection. Here we note that $q(Y) = q(Y')$. We put $\mu = \mu_t \circ \cdots \circ \mu_1$, where $\mu_i : Y_{i-1} \to Y_i$ is one point blowing up, $Y_0 := Y$ and $Y_t := Y'$. Let E_i be the μ_i-exceptional curve. Let $N_0 := N$ and $N_i = (\mu_i)_*(N_{i-1})$. Then $N_{i-1} = (\mu_i)^*(N_i) - n_i E_i$ for some positive integer n_i. We put $N' := N_t$. Let $N' \equiv aC_0 + bF_{\pi'}$, where C_0 is a minimal section of π' and $F_{\pi'}$ is a fiber of π'. Then we obtain

$$K_Y N = K_{Y'} N' + \sum_{i=1}^{t} n_i$$

and

$$K_{Y'} N' = 2q(Y') - 2 + (a - 1)(2q(Y') - 2) + ae - 2b.$$

CLAIM 2.2. *$2b - ae \geq (a - 1)(2q(Y') - 2) + \sum_{i=1}^{t} n_i$ holds.*

Proof. If $2b - ae + 1 \leq (a - 1)(2q(Y') - 2) + \sum_{i=1}^{t} n_i$, then $K_Y N \geq 2q(Y') - 1 = 2q(Y) - 1$, that is, $N^2 \leq 2m - 1$ because $g(Y, N) = q(Y) + m$. So we get

$$L^n + 1 \leq L^n + c_2(\mathscr{E}) = N^2 \leq 2m - 1.$$

Hence $L^n \leq 2m - 2$. By Lemma 2.1 we get $q(X) = 0$ and this is a contradiction by hypothesis. □

By Claim 2.2 we get

$$(N')^2 = a(2b - ae) \geq a(a - 1)(2q(Y') - 2) + a \sum_{i=1}^{t} n_i.$$

On the other hand, we get $a = NF = N_{i-1}F_{i-1} \geq N_{i-1}E_i = n_i$ for each i, where F (resp. F_{i-1}) is a fiber of $\pi' \circ \mu : Y \to C$ (resp. $\pi' \circ \mu_t \circ \cdots \circ \mu_i : Y_{i-1} \to C$). Hence

$$(N')^2 \geq a(a-1)(2q(Y')-2) + \sum_{i=1}^{t} n_i^2. \tag{6}$$

Furthermore since $q(Y) > 0$ and (Y, N) is not scroll, $K_Y + N$ is nef and we have

$$\begin{aligned}
0 &\leq (K_Y + N)^2 \\
&= (K_{Y'} + N')^2 - \sum_{i=1}^{t} (n_i - 1)^2 \\
&= 4(g(Y', N') - 2q(Y') + 1) - (N')^2 - \sum_{i=1}^{t} (n_i - 1)^2 \\
&= 4(m - q(Y') + 1) - (N')^2 + \sum_{i=1}^{t} 2n_i(n_i - 1) - \sum_{i=1}^{t} (n_i - 1)^2
\end{aligned}$$

because $q(Y') + m = q(Y) + m = g(Y, N) = g(Y', N') - \sum_{i=1}^{t} \frac{1}{2} n_i(n_i - 1)$. Hence

$$(N')^2 \leq 4(m - q(Y') + 1) + \sum_{i=1}^{t} (n_i^2 - 1). \tag{7}$$

Since Y is not minimal, we get $a = N'F_{\pi'} = NF \geq 2\,\mathrm{rank}(\mathscr{E}) = 2(n-1)$, where F is a fiber of $\pi' \circ \mu : Y \to C$. Hence by (6) we get

$$(N')^2 \geq 2(n-1)(2n-3)(2q(Y')-2) + \sum_{i=1}^{t} n_i^2. \tag{8}$$

Therefore by the above inequalities (8) and (7), we see that

$$\begin{aligned}
2(n-1)(2n-3)(2q(Y')-2) &\leq 4(m - q(Y') + 1) - t \\
&\leq 4m + 3 - 4q(Y').
\end{aligned}$$

So we obtain

$$q(X) = q(Y) = q(Y') \leq 1 + \frac{4m - 1}{8n^2 - 20n + 16}.$$

This completes the proof of Lemma 2.3. □

3. The proof of Theorem 1.1

In this section we are going to give a proof of Theorem 1.1.

Proof. (A) The case in which (X, L) is not any type from (1) to (7.4) in Theorem 2.1.

Let (M, A) be a reduction of (X, L). In this case, we see from the assumption that $K_M + (n-2)A$ is nef (see Theorem 2.1). Then by Proposition 2.2 there exists a polarized surface (S, H) such that S is smooth, $\dim \mathrm{Bs}|H| \leq 0$, $q(X) = q(S)$, $g(S, H) = g(X, L) = q(X) + m = q(S) + m$, K_S is nef and $h^0(H) \geq h^0(A) - (n-2) \geq h^0(L) - (n-2) = m+1$. Since K_S is nef, we see that S is minimal with $\kappa(S) \geq 0$.

(A.1) The case in which $h^0(H) = m+1$.

We use [14, Theorem 2.1]. Then (S, H) is one of the types from (M-1) to (M-3-6) in [14, Theorem 2.1].

First we consider the case where (S, H) is the type (M-1) in [14, Theorem 2.1]. In this case, $A^n = 1$ because $H^2 = 1$. Hence $(M, A) = (X, L)$ and $L^n = 1$. Since $\mathrm{Bs}|L| = \emptyset$, we have $\Delta(X, L) = 0$ and $K_X + (n-2)L$ is not nef. However, this is impossible by the assumption in (A).

Next we consider the case where (S, H) is the type (M-3-6) in [14, Theorem 2.1]. Then S is an abelian surface, but this is impossible because any abelian surface cannot be an ample divisor of a smooth projective 3-fold (see [3, Proposition (2.2)]).

Next we consider the case where (S, H) is the types from (M-2-1) to (M-2-6) in [14, Theorem 2.1]. Then S is a relatively minimal elliptic fibration over a smooth curve C with $\chi(\mathcal{O}_S) = 0$. On the other hand, by the argument in [17, Remark 2.2] we see that $\chi(\mathcal{O}_S) = \chi_2^H(M, A)$, and by Proposition 2.3 we have $\chi_2^H(M, A) \geq 1$. So we get $\chi(\mathcal{O}_S) \geq 1$, but this is a contradiction.

Next we consider the case where (S, H) is the types from (M-3-1) to (M-3-5) in [14, Theorem 2.1]. Then $\kappa(S) = 0$. By using results in [4], we get $h^0(K_S) = 1$. (For example, see [4, (6.1)].) Since S is minimal, we have $K_S = \mathcal{O}_S$. By the Lefschetz theorem for Picard groups (see e.g. [6, (7.1) Theorem 5)]) we see that $K_M + (n-2)A = \mathcal{O}_M$, that is, (M, A) is a Mukai manifold. In particular, $q(X) = q(M) = 0$ and $g(X, L) = g(M, A) = 1 + (1/2)A^n$. So we get $L^n \leq A^n = 2m-2$. We see from $h^0(L) = n+m-1$ and $\mathrm{Bs}|L| = \emptyset$ that $L^n = 2m-2$ holds by Lemma 2.1. Hence $L^n = A^n$ and so $(X, L) = (M, A)$. This is the type (7) in Theorem 1.1.

(A.2) The case where $h^0(H) \geq m+2$.

Since $\dim \mathrm{Bs}|H| \leq 0$ and $g(S, H) = q(S) + m$, we get $\kappa(S) = -\infty$ by [15, Theorem 2.1], but this is impossible because K_S is nef.

(B) The case in which (X, L) is one of the types from (1) to (7.4) in Theorem 2.1.

(B.1) If (X, L) is one of the types (1), (2), (3) in Theorem 2.1, then we have $g(X, L) = q(X)$. Hence $h^0(L) = n+m-1 = n-1$ by our hypothesis. However, since L is ample with $\mathrm{Bs}|L| = \emptyset$, we get $h^0(L) \geq n+1$ and this is impossible.

(B.2) If (X, L) is the type (4) in Theorem 2.1, then $m = g(X, L) - q(X) = 1 - 0 = 1$ and $h^0(L) = n+m-1 = n$. However, this is impossible because L is ample and $\mathrm{Bs}|L| = \emptyset$.

(B.3) Assume that (X, L) is the type (5) in Theorem 2.1. Let $f : X \to W$ be a hyperquadric fibration over a smooth curve W. We put $\mathscr{E} := f_*(L)$. Then $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on W. Let $\pi : \mathbf{P}_W(\mathscr{E}) \to W$ be the projective bundle map. Then there exists an embedding $\iota : X \hookrightarrow \mathbf{P}_W(\mathscr{E})$ such that $X \in |2H(\mathscr{E}) + \pi^*B|$ for some $B \in \mathrm{Pic}(W)$, where $H(\mathscr{E})$ is the tautological line bundle on $\mathbf{P}_W(\mathscr{E})$. Let $b := \deg B$, $e := c_1(\mathscr{E})$, $d := L^n$, and $s := 2e + (n+1)b$. Then by easy calculations we get $d = 2e + b$ and $q(X) = m + 1 - e - b$. So we have

$$(n-1)d + s + 2nq(X) = 2n(m+1). \tag{9}$$

By [5, (3.3)], we have $s \geq 0$.

If $d \leq 2m - 2$, then we see from Lemma 2.1 that $q(X) = 0 < \dfrac{3}{2} + \dfrac{2m-1}{2n}$.

If $d \geq 2m - 1$, then by (9) we get

$$(2m-1)(n-1) + s + 2nq(X) \leq 2n(m+1).$$

Since $s \geq 0$, we obtain

$$\begin{aligned} 2nq(X) &\leq 2n(m+1) - (2m-1)(n-1) \\ &= 3n + 2m - 1. \end{aligned}$$

Therefore

$$q(X) \leq \frac{3}{2} + \frac{2m-1}{2n}.$$

This case is the type (1) in Theorem 1.1.

(B.4) Assume that (X, L) is the type (6) in Theorem 2.1. Let $\pi : X \to Y$ be its $\mathbf{P}^{n-2}$-bundle, where Y is a smooth projective surface. Let $\mathscr{E}$ be an ample vector bundle on Y of rank $n-1$ such that $X = \mathbf{P}_Y(\mathscr{E})$ and $L = H(\mathscr{E})$, where $H(\mathscr{E})$ is the tautological line bundle on $\mathbf{P}_Y(\mathscr{E})$. Then we can prove the following.

Claim 3.1. *$\kappa(Y) = -\infty$ holds.*

Proof. Here we put $B := c_1(\mathscr{E})$. Then B is ample and spanned.

(α) The case in which $c_2(\mathscr{E}) \geq 3$.

(α.1) The case in which $\kappa(Y) = 2$.

Then $B^2 \leq 2m$ by Theorem 2.2 because $g(Y, B) = g(X, L) = q(X) + m = q(Y) + m$. Since $L^n + c_2(\mathscr{E}) = B^2$, we have $L^n = B^2 - c_2(\mathscr{E}) \leq 2m - 3$. So by Lemma 2.1 this is impossible.

(α.2) The case in which $\kappa(Y) = 0$ or 1.

In this case $B^2 \leq 2m + 2$ by Theorem 2.3.

(α.2.1) If $B^2 \leq 2m + 1$, then $L^n = B^2 - c_2(\mathscr{E}) \leq 2m - 2$ and so by Lemma 2.1 we have $g(X, L) = m$ and $L^n = 2m - 2$. In particular, $(K_X + (n-2)L)L^{n-1} = 0$ holds. So we get

$$\begin{aligned} 0 &= (K_X + (n-2)L)L^{n-1} \\ &= (-H(\mathscr{E}) + \pi^*(K_Y + c_1(\mathscr{E})))H(\mathscr{E})^{n-1} \\ &= c_2(\mathscr{E}) + K_Y c_1(\mathscr{E}). \end{aligned}$$

Since $\kappa(Y) \geq 0$, we have $K_Y c_1(\mathscr{E}) \geq 0$. On the other hand $c_2(\mathscr{E}) > 0$ because $\mathscr{E}$ is ample. Hence $c_2(\mathscr{E}) + K_Y c_1(\mathscr{E}) > 0$ and this is impossible.

(α.2.2) Next we consider the case where $B^2 = 2m+2$.

(α.2.2.1) First we assume that $\kappa(Y) = 1$. Then Theorem 2.3 (2) we have $Y \cong F \times C$ and $B \equiv C + (m+1)F$, where F is a smooth elliptic curve and C is a smooth projective curve of genus $g(C) \geq 2$. Then $BF = 1$. Moreover since B is generated by its global sections, so is B_F. Hence we see that $F \cong \mathbf{P}^1$, but this is a contradiction. Therefore this case cannot occur.

(α.2.2.2) Next we consider the case where $\kappa(Y) = 0$. Then we see from Theorem 2.3 (1) that Y is an Abelian surface. In this case, $h^0(B) = m+1$ because $h^0(B) = B^2/2$. We note that $K_Y + B$ is ample. Hence (X, L) is a scroll over Y in the sense of Definition 2.2 (1) because $K_X + (n-1)L = \pi^*(K_Y + B)$. Since L is base point free, there exists a ladder $X =: X_0 \supset X_1 \supset \cdots \supset X_{n-2}$ such that X_i is a smooth projective variety of dimension $n-i$ with $X_i \in |L|_{X_{i-1}}|$ for $1 \leq i \leq n-2$. Then we note that

$$h^0(L|_{X_{n-2}}) \geq m+1 \tag{10}$$

because $h^0(L) = n+m-1$ from the assumption. By [2, Theorem 11.1.2] and the proof of [2, Theorem 11.1.1], we see that (Y, B) is a reduction of $(X_{n-2}, L|_{X_{n-2}})$. Since B is spanned by its global sections, we have $h^0(L|_{X_{n-2}}) < h^0(B) = m+1$, but this contradicts to (10). Therefore this case also cannot occur.

We see from (α.1) and (α.2) that $\kappa(Y) = -\infty$ holds if $c_2(\mathscr{E}) \geq 3$.

(β) The case in which $c_2(\mathscr{E}) \leq 2$.

Assume that $\kappa(Y) \geq 0$. Then by Theorem 2.4, we have $\operatorname{rank}(\mathscr{E}) = 2$, $n = 3$, and (Y, B) is the following type: There exists a finite morphism $f : Y \to \mathbf{P}^2$ of degree 2 such that $\mathscr{E} \cong f^*(\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2})$. Since $\kappa(Y) \geq 0$, we see that the branch locus is an element of the complete linear system of $\mathcal{O}_{\mathbf{P}^2}(2a)$, where a is an integer with $a \geq 3$. So in particular we get

$$\begin{aligned} h^0(L) = h^0(\mathscr{E}) &= h^0(f^*(\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2})) \\ &= 2(h^0(\mathcal{O}_{\mathbf{P}^2}(1)) + h^0(\mathcal{O}_{\mathbf{P}^2}(1) \otimes \mathcal{O}_{\mathbf{P}^2}(-a))) \\ &= 6. \end{aligned} \tag{11}$$

On the other hand, since $q(X) = q(Y) = 0$,

$$\begin{aligned} m = q(X) + m = g(X, L) &= g(Y, B) \\ &= 1 + \frac{1}{2}(f^*(K_{\mathbf{P}^2} + \mathcal{O}_{\mathbf{P}^2}(a)) + f^*(\mathcal{O}_{\mathbf{P}^2}(2)))f^*\mathcal{O}_{\mathbf{P}^2}(2) \\ &= 1 + (\mathcal{O}_{\mathbf{P}^2}(-3) + \mathcal{O}_{\mathbf{P}^2}(a) + \mathcal{O}_{\mathbf{P}^2}(2))\mathcal{O}_{\mathbf{P}^2}(2) \\ &= 2a - 1. \end{aligned}$$

However, since $h^0(L) = n + m - 1 = m + 2 = 2a + 1$, by (11) we have $a = 5/2$ and this is a contradiction. Hence $\kappa(Y) = -\infty$ holds if $c_2(\mathscr{E}) \le 2$.

By (α) and (β), we get the assertion of Claim 3.1. □

Hence we get the type (2) in Theorem 1.1 by Claim 3.1 and Lemma 2.3.

(B.5) Next we consider the types from (7.1) to (7.4) in Theorem 2.1. Let t be the number of simple blowing ups of (M, A). For $1 \le i \le t$, let (X_i, L_i) be a polarized manifold, let $\mu_i : X_{i-1} \to X_i$ be the birational morphism of a simple blowing up of (X_i, L_i), and let E_i be the exceptional divisor of μ_i. Here we set $(X_0, L_0) := (X, L)$ and $(X_t, L_t) = (M, A)$.

(B.5.1) Assume that (M, A) is the types (7.1) (resp. (7.2), (7.3)) in Theorem 2.1, that is, $(M, A) = (\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^4}(2))$ (resp. $(\mathbf{Q}^3, \mathscr{O}_{\mathbf{Q}^3}(2))$, $(\mathbf{P}^3, \mathscr{O}_{\mathbf{P}^3}(3))$). Then $g(M, A) = 5$ (resp. 5, 10), $A^4 = 16$ (resp. $A^3 = 16$, $A^3 = 27$). Since $q(M) = 0$ (resp. 0, 0), we have $m = 5$ (resp. 5, 10). Hence by assumption we have $h^0(L) = n + m - 1 = 8$ (resp. 7, 12). On the other hand $h^0(A) = 15$ (resp. 14, 20). Here we note that we have $h^0(L_i) - 1 \le h^0(L_{i-1}) \le h^0(L_i)$ for each i by the following exact sequence

$$0 \to L_{i-1} \to \mu_i^*(L_i) \to \mathscr{O}_{E_i} \to 0. \tag{12}$$

Hence $t \ge 7$ (resp. $t \ge 7$, $t \ge 8$) because $h^0(A) - h^0(L) = 7$ (resp. 7, 8).

If $t \ge 8$ (resp. $t \ge 8$, $t \ge 9$), then $L^4 = A^4 - t \le 8 = 2m - 2$ (resp. $L^3 = A^3 - t \le 8 = 2m - 2$, $L^3 = A^3 - t \le 18 = 2m - 2$). However, by Lemma 2.2, this case cannot occur. Hence we have $t = 7$ (resp. $t = 7$, $t = 8$). Therefore we get the type (3) (resp. (4), (5)) in Theorem 1.1.

(B.5.2) Assume that (M, A) is the type (7.4) in Theorem 2.1. Let $H = K_M + 2A$. Then (M, H) is a scroll over C. So there exists an ample vector bundle $\mathscr{E}$ of rank three on C such that $M = \mathbf{P}_C(\mathscr{E})$ and $H = H(\mathscr{E})$. Furthermore there exists a line bundle B on C such that $A = 2H(\mathscr{E}) + f^*(B)$, where $f : M = \mathbf{P}_C(\mathscr{E}) \to C$ is the projection. We set $e := \deg \mathscr{E}$ and $b := \deg B$. Then we get the following equations (see e.g. [6, (13.10)]).

$$g(X, L) = g(M, A) = 1 + 2e + 2b, \tag{13}$$

$$A^3 = 8e + 12b, \tag{14}$$

$$2q(M) - 2 + e + 2b = 0. \tag{15}$$

We see from (13) and (15) that

$$\begin{aligned} g(X, L) = g(M, A) &= 1 + 2e + 2b \\ &= q(M) + \frac{1}{2}(e + 2b) + 2(e + b) \\ &= q(X) + \frac{5}{2}e + 3b. \end{aligned}$$

Therefore

$$m = \frac{5}{2}e + 3b, \tag{16}$$

and by (15) we have

$$q(X) = 1 - \frac{1}{2}(e + 2b) = 1 - \frac{1}{3}m + \frac{1}{3}e, \tag{17}$$

that is, $m = 3(1 - q(X)) + e$. Noting that

$$e = \deg \mathscr{E} = H(\mathscr{E})^3 = (K_M + 2A)^3,$$

we have

$$\begin{aligned} m &= 3(1 - q(X)) + e \\ &= 3(1 - q(X)) + (K_M + 2A)^3. \end{aligned} \tag{18}$$

By (14), (15), (16) and (18), we have

$$\begin{aligned} A^3 &= 8e + 12b \\ &= 3(e + 2b) + (5e + 6b) \\ &= 3(2 - 2q(M)) + 2m \\ &= 4m - 2(K_M + 2A)^3. \end{aligned} \tag{19}$$

We note that

$$\Delta(M, K_M + 2A) = 3 + (K_M + 2A)^3 - h^0(K_M + 2A),$$

and by [18, Corollary 3.1][2] we have

$$h^0(K_M + 2A) = g(M, A) - q(M) = g(X, L) - q(X) = m.$$

Hence we have

$$(K_M + 2A)^3 = m - 3 + \Delta(M, K_M + 2A). \tag{20}$$

By (19) and (20) we have

$$\begin{aligned} A^3 &= 4m - 2(K_M + 2A)^3 \\ &= 2m + 6 - 2\Delta(M, K_M + 2A). \end{aligned} \tag{21}$$

Since $2m - 2 \le L^3$ by Lemma 2.1, we see from (21) that

$$2m - 2 \le L^3 \le A^3 = 2m + 6 - 2\Delta(M, K_M + 2A). \tag{22}$$

[2] In this case, $g_2(M, A) = 0$, $h^2(\mathcal{O}_M) = 0$ and $g_3(M, A) = h^3(\mathcal{O}_M) = 0$, where $g_i(M, A)$ denotes the ith sectional geometric genus of (M, A) (see [16, Example 2.10 (11)] for details).

Namely we have $\Delta(M, K_M + 2A) \leq 4$. On the other hand, we have $\Delta(M, K_M + 2A) \geq 0$ since $K_M + 2A$ is ample. Therefore $\Delta(M, K_M + 2A) = 0, 1, 2, 3, 4$.

Next we study (M, A) for each value of $\Delta(M, K_M + 2A)$. Before that, we note the following.

Lemma 3.1. *If* $q(M) = 0$, *then* $t = 7$.

Proof. First we prove the following claim.

Claim 3.2. $0 \leq t \leq 7$ *holds.*

Proof. Since $q(M) = 0$, we have $g(M, A) = q(M) + m = m$ and by (13) we have

$$1 + 2e + 2b = m. \tag{23}$$

On the other hand, by (15) we have

$$e + 2b = 2. \tag{24}$$

By (23) and (24) we have

$$e = m - 3, \tag{25}$$

$$b = \frac{1}{2}(5 - m). \tag{26}$$

Therefore $A^3 = 8e + 12b = 2m + 6$. Since $L^3 \geq 2m - 2$ and $t = A^3 - L^3$, we see that $0 \leq t \leq 8$.

Assume that $t = 8$. In particular $(X, L) \neq (M, A)$. In this case $L^3 = 2m - 2$. So by Lemma 2.1 we have $L^3 = 2\Delta(X, L) = 2m - 2$ and $q(X) = 0$. We note that $g(X, L) = m > \Delta(X, L)$. Hence by [4, (1.4) Theorem] we see that (X, L) is either a Mukai manifold or a hyperelliptic polarized manifold. Since $(X, L) \neq (M, A)$, we see from the proof of Lemma 2.2 that (X, L) is not a Mukai manifold but a hyperelliptic polarized manifold of type $(\Sigma^n(\delta)^+_{a,b})$. In particular X is a double covering of a projective bundle $\mathbf{P}_{\mathbf{P}^1}(\mathscr{F})$ over $\mathbf{P}^1$, where $\mathscr{F}$ is a vector bundle of rank three on $\mathbf{P}^1$. Let $\pi : X \to \mathbf{P}_{\mathbf{P}^1}(\mathscr{F})$ be the double covering map and let $p : \mathbf{P}_{\mathbf{P}^1}(\mathscr{F}) \to \mathbf{P}^1$ be the projection. Here we note that $L = \pi^*(H(\mathscr{F}))$. Since $t = 8$, there exists a divisor E on X such that E is the exceptional divisor of a simple blowing up $\mu_1 : X \to X_1$ and $L = \mu_1^* L_1 - E$, where X_1 is a smooth projective variety and L_1 is an ample line bundle on X_1. Because $E \cong \mathbf{P}^2$, we have $(p \circ \pi)(E)$ is a point. Therefore E is contained in a fiber of $p \circ \pi$. Let $F_r = (p \circ \pi)^{-1}((p \circ \pi)(E))$. Then F_r has at least two components because $E^3 = 1$. We note that $\pi|_{F_r} : F_r \to F_p$ is surjective, where $F_p = p^{-1}((p \circ \pi)(E))$. Since π is a double covering and $F_p \cong \mathbf{P}^2$, F_r has two components and $\pi|_E : E \to F_p$ is birational. Since E and F_p are isomorphic to

$\mathbf{P}^2$ and $\pi|_E$ is a finite morphism, we see that $\pi|_E$ is an isomorphism by the Zariski Main Theorem. Note that

$$K_X E^2 = (\mu_1^*(K_{X_1}) + 2E)E^2 = 2. \tag{27}$$

Here we note that $K_X = \pi^*(K_{\mathbf{P}_{\mathbf{P}^1}(\mathscr{F})} + D)$ for some divisor D on $\mathbf{P}_{\mathbf{P}^1}(\mathscr{F})$. Hence

$$\begin{aligned} K_X E^2 &= \pi^*(K_{\mathbf{P}_{\mathbf{P}^1}(\mathscr{F})} + D)E^2 \\ &= (\pi|_E)^*(K_{F_p} + D|_{F_p})E|_E \\ &= (K_{F_p} + D|_{F_p})\mathscr{O}_{\mathbf{P}^2}(-1) \\ &= \mathscr{O}_{\mathbf{P}^2}(-3+b)\mathscr{O}_{\mathbf{P}^2}(-1) \\ &= 3 - b, \end{aligned} \tag{28}$$

where we set $\mathscr{O}_{\mathbf{P}^2}(b) = D|_{F_p}$. Therefore (27) and (28) imply $b = 1$. Hence $K_X = \pi^*(-2H(\mathscr{F}) + f^*(P))$ for some $P \in \operatorname{Pic}(\mathbf{P}^1)$. So we get $K_X + 2L = \pi^* \circ f^*(P)$, but then $K_X + 2L$ is not big and this is impossible because $K_X + 2L = \mu^*(K_M + 2A)$ and $K_M + 2A$ is ample[3]. Therefore the case where $t = 8$ cannot occur, and we get the assertion of Claim 3.2. □

Next we prove the following.

CLAIM 3.3. $h^0(A) \geq m + 9$.

Proof. We note that $\deg(S^2(\mathscr{E}) \otimes B) = 4e + 6b = m + 3$ by (25) and (26). Hence by the Riemann-Roch theorem we have

$$\begin{aligned} h^0(A) = h^0(2H(\mathscr{E}) + f^*(B)) &= h^0(S^2(\mathscr{E}) \otimes B) \\ &\geq \deg(S^2(\mathscr{E}) \otimes B) + \operatorname{rank}(S^2(\mathscr{E}) \otimes B)(1 - g(C)) \\ &= \deg(S^2(\mathscr{E}) \otimes B) + 6 \\ &= m + 9. \end{aligned}$$ □

Note that we see from the exact sequence (12) that

$$h^0(A) - h^0(L) \leq t \tag{29}$$

holds. Since $h^0(L) = m + 2$, by (29) and Claims 3.2 and 3.3 we have $t = 7$. Therefore we get the assertion of Lemma 3.1. □

(B.5.2.i) The case where $\Delta(M, K_M + 2A) = 4$. Then by (22) we have $L^3 = A^3 = 2m - 2$. So by Lemma 2.1 we get $q(M) = q(X) = 0$. Since $(M, K_M + 2A)$ is a scroll over C, we have $g(M, K_M + 2A) = q(M) = 0$. Hence by [6, (12.1) Theorem] we infer that $\Delta(K_M + 2A) = 0$ and this is impossible.

[3] Here $\mu : X \to M$ denotes the reduction map.

(B.5.2.ii) The case where $\Delta(M, K_M + 2A) = 0$. Then we see from $\Delta(M, K_M + 2A) = 0$ that $q(M) = 0$ by [6, (5.10) Theorem]. So by Lemma 3.1 we see that $t = 7$.

(B.5.2.iii) The case where $\Delta(M, K_M + 2A) = 1$. Then by (21) we have $A^3 = 2m + 4$. Since $L^3 \geq 2m - 2$ by Lemma 2.1 we see that $t = A^3 - L^3 \leq 6$, and by Lemma 3.1 we have $q(M) \geq 1$. So we have

$$g(M, K_M + 2A) = q(M) \geq 1 = \Delta(M, K_M + 2A). \tag{30}$$

Here we note that e is a positive even integer by (15). Moreover since $\Delta(M, K_M + 2A) = 1$, we see that $(M, K_M + 2A)$ has a ladder by [6, Theorems (4.2) and (4.15)].

Claim 3.4. *The case* (B.5.2.iii) *cannot occur.*

Proof. (a) Assume that $e \geq 4$. Then

$$(K_M + 2A)^3 = e \geq 4 = 2\Delta(M, K_M + 2A) + 2.$$

However, then by [6, (3.5) Theorem 3)] we have $q(M) = 0$ and this is impossible.

(b) Assume that $e = 2$. Then by (15) we have $b = -q(M)$, and by (14) we get

$$0 < A^3 = 8e + 12b = 16 - 12q(M),$$

that is, $q(M) \leq 1$. So we get $q(M) = 1$, $b = -q(M) = -1$ and $A^3 = 4$. On the other hand, since $L^3 \geq 2m - 2$, we have $2m - 2 \leq L^3 \leq A^3 = 4$. Namely we get $m \leq 3$. We note that $m \geq 2$ since L is ample and spanned with $h^0(L) = n + m - 1$. Hence $m = 2$ or 3. By (16) we get $(e, b, m) = (2, -1, 2)$ because b is an integer. Then we note that $L^3 = 2, 3$ or 4 because $2 = 2m - 2 \leq L^3 \leq A^3 = 4$. Since $q(X) = q(M) = 1$, $g(X, L) = g(M, A) = q(M) + m = 3$ and L is generated by its global sections, we see from the classification of polarized manifolds with sectional genus three [21] that this case cannot occur.

Therefore these complete the proof of Claim 3.4. □

(B.5.2.iv) The case where $\Delta(M, K_M + 2A) = 2$. Then by (21) we have $A^3 = 2m + 2$. Since $L^3 \geq 2m - 2$ by Lemma 2.1 we see that $t = A^3 - L^3 \leq 4$, and $q(M) \geq 1$ by Lemma 3.1. Moreover since $h^0(K_M + 2A) = g(M, A) - q(M) = m$ and $\Delta(M, K_M + 2A) = 2$ in this case, we have

$$(K_M + 2A)^3 = m - 1. \tag{31}$$

Claim 3.5. $L^3 \geq 5$ *holds.*

Proof. Assume that $L^3 \leq 4$. Then $A^3 \leq 8$ because $t \leq 4$ in this case. Hence by (19) and (31) we have

$$8 \geq A^3 = 4m - 2(K_M + 2A)^3 = 2m + 2,$$

that is, $m \le 3$. Because L is ample and spanned with $h^0(L) = n + m - 1$, we get $m \ge 2$. Therefore $m = 2$ or 3.

If $m = 2$, then $A^3 = 6$ and by (14) and (16) we get

$$2 = m = \frac{5}{2}e + 3b$$
$$6 = A^3 = 8e + 12b.$$

So we get $e = 1$ and $b = -\frac{1}{6}$, but this is impossible becaue b is an integer.

If $m = 3$, then $A^3 = 8$ and by (14) and (16) we get

$$3 = m = \frac{5}{2}e + 3b$$
$$8 = A^3 = 8e + 12b.$$

So we get $e = 2$ and $b = -\frac{2}{3}$ and this is also impossible. Therefore we get the assertion of Claim 3.5. □

Claim 3.6. *$K_M + 2A$ is generated by its global sections.*

Proof. Since L is generated by its global sections with $L^3 \ge 5$ and $g(X, L) = q(X) + m > q(X)$, we see from [2, Theorem 9.2.1] that $K_X + 2L$ is generated by its global sections. Since $K_X + 2L = \pi^*(K_M + 2A)$, where $\pi : X \to M$ is the reduction map, we infer that $K_M + 2A$ is also generated by its global sections. □

Claim 3.7. *$m \ge 4$ holds.*

Proof. First we note that $h^0(K_M + 2A) = g(M, A) - q(M) = m$. Since $K_M + 2A$ is ample, we see from Claim 3.6 that $h^0(K_M + 2A) \ge 4$. Therefore we get the assertion. □

Claim 3.8. *The case* (B.5.2.iv) *cannot occur.*

Proof. (a) Assume that $e \ge 6$.

(a.1) If $q(M) \ge 2$, then $g(M, K_M + 2A) = q(M) \ge 2 = \Delta(M, K_M + 2A)$. By Claim 3.6 $(M, K_M + 2A)$ has a ladder. Moreover

$$(K_M + 2A)^3 = e \ge 6 = 2\Delta(M, K_M + 2A) + 2.$$

Hence by [6, (3.5) Theorem 3)] we have $q(M) = 0$ and this is impossible.

(a.2) If $q(M) = 1$, then by (15) we have $e + 2b = 0$. Furthermore by (13) we have

$$1 + m = g(X, L) = g(M, A) = 1 + 2e + 2b = 1 + e.$$

Namely $m = e$. Thereore by (14)

$$2e + 2 = 2m + 2 = A^3 = 8e + 12b.$$

However, then $6e + 12b = 2$ and this is impossible because e and b are integer.

(b) Assume that $e = 4$. Then

$$\begin{aligned}\Delta(M, K_M + 2A) &= 3 + (K_M + 2A)^3 - h^0(K_M + 2A)\\ &= 3 + e - m = 7 - m.\end{aligned}$$

Here we note that $4 \le m$ by Claim 3.7.

(b.1) If $m \ge 6$, then $\Delta(M, K_M + 2A) \le 1$, but this contradicts the assumption that $\Delta(M, K_M + 2A) = 2$.

(b.2) If $m = 5$, then by $e = 4$ and (16) we have

$$5 = m = \frac{5}{2}e + 3b = 10 + 3b.$$

However, this is impossible because b is an integer.

(b.3) If $m = 4$, then by $e = 4$ and (16) we have

$$4 = m = \frac{5}{2}e + 3b = 10 + 3b.$$

Hence we have $b = -2$. By (14) we have $A^3 = 8e + 12b = 8$. On the other hand $A^3 = 2m + 2 = 10$ and this is impossible.

(c) Assume that $e = 2$. Then

$$\begin{aligned}\Delta(M, K_M + 2A) &= 3 + (K_M + 2A)^3 - h^0(K_M + 2A)\\ &= 3 + e - m = 5 - m.\end{aligned}$$

So by Claim 3.7 we have $\Delta(M, K_M + 2A) \le 1$, but this contradicts the assumption that $\Delta(M, K_M + 2A) = 2$.

These complete the proof of Claim 3.8. □

(B.5.2.v) The case where $\Delta(M, K_M + 2A) = 3$. By (21) we have $A^3 = 2m$. Since $L^3 \ge 2m - 2$ by Lemma 2.1 we see that $t = A^3 - L^3 \le 2$, and $q(M) \ge 1$ by Lemma 3.1. First we prove the following.

Claim 3.9. $0 \le t \le 1$ *holds.*

Proof. If $t = 2$, then $L^3 = 2m - 2$ and by Lemma 2.1 we have $q(M) = 0$, but this is impossible. □

Moreover since $h^0(K_M + 2A) = g(M, A) - q(M) = m$ and $\Delta(M, K_M + 2A) = 3$ in this case, we have

$$(K_M + 2A)^3 = m. \tag{32}$$

CLAIM 3.10. *$L^3 \geq 5$ holds.*

Proof. Assume that $L^3 \leq 4$. Then $A^3 \leq 5$ because $0 \leq t \leq 1$ holds by Claim 3.9. Hence by (19) and (32) we have

$$5 \geq A^3 = 4m - 2(K_M + 2A)^3 = 2m,$$

that is, $m \leq 2$. Because L is ample and spanned with $h^0(L) = n + m - 1$, we get $m \geq 2$. Therefore $m = 2$ and $A^3 = 2m = 4$.

Then by (16) and (14) we get

$$2 = m = \frac{5}{2}e + 3b$$
$$4 = A^3 = 8e + 12b.$$

So we get $e = 2$ and $b = -1$. By (15) we have $q(M) = 1$. Therefore $g(X, L) = g(M, A) = q(M) + m = 3$. Here we note that $L^3 = 3$ or 4 because $t \leq 1$. Since L is ample and spanned, by the classification of polarized manifolds with sectional genus three, this case cannot occur (see [21, (3) Case (C) (3-2) or the proof of (5-1-2)]). Therefore we get the assertion. □

By the same argument as the proof of Claims 3.6 and 3.7 we get the following.

CLAIM 3.11. *$K_M + 2A$ is generated by its global sections and $m \geq 4$.*

Here we prove the following.

CLAIM 3.12. *The case* (B.5.2.v) *cannot occur.*

Proof. (a) Assume that $e \geq 8$.

(a.1) If $q(M) \geq 3$, then $g(M, K_M + 2A) = q(M) \geq 3 = \Delta(M, K_M + 2A)$. Moreover

$$(K_M + 2A)^3 = e \geq 8 = 2\Delta(M, K_M + 2A) + 2.$$

Hence by Claim 3.11 and [6, (3.5) Theorem 3)] we have $q(M) = 0$, but this is impossible.

(a.2) If $q(M) = 2$, then by (15) we have

$$e + 2b = -2. \tag{33}$$

Furthermore by (13) we have

$$2 + m = q(X) + m = g(X, L) = g(M, A) = 1 + 2e + 2b = e - 1.$$

Namely $m = e - 3$. Therefore by (14)

$$2e - 6 = 2m = A^3 = 8e + 12b.$$

Namely

$$e + 2b = -1. \tag{34}$$

However, by (33) and (34) we get a contradiction.

(a.3) If $q(M) = 1$, then by (15) we have

$$e + 2b = 0. \tag{35}$$

Furthermore by (13) we have

$$1 + m = g(X, L) = g(M, A) = 1 + 2e + 2b = 1 + e.$$

Namely $m = e$. So we get $q(M) = 1$ and $m = e = -2b$ by (35).

(b) Assume that $e = 6$. Then

$$\begin{aligned}\Delta(M, K_M + 2A) &= 3 + (K_M + 2A)^3 - h^0(K_M + 2A)\\ &= 3 + e - m = 9 - m.\end{aligned}$$

Since $\Delta(M, K_M + 2A) \geq 0$, we have $m \leq 9$. Here we note that $4 \leq m$ by Claim 3.11. By using (16) we get the following types.

m	e	b	$\Delta(M, K_M + 2A)$
6	6	-3	3
9	6	-2	0

If $m = 9$, then $q(M) = 0$ by [6, (5.10) Theorem], but this is impossible. If $m = 6$, then by (15) we have $q(M) = 1$.

(c) Assume that $e = 4$. Then

$$\begin{aligned}\Delta(M, K_M + 2A) &= 3 + (K_M + 2A)^3 - h^0(K_M + 2A)\\ &= 3 + e - m = 7 - m.\end{aligned}$$

By $\Delta(M, K_M + 2A) \geq 0$ and Claim 3.11, we have $4 \leq m \leq 7$. By (16) we get the following types.

m	e	b	$\Delta(M, K_M + 2A)$
4	4	-2	3
7	4	-1	0

If $m = 7$, then $q(M) = 0$ by [6, (5.10) Theorem], but this is impossible. If $m = 4$, then by (15) we have $q(M) = 1$.

(d) Assume that $e = 2$. Then

$$\begin{aligned}\Delta(M, K_M + 2A) &= 3 + (K_M + 2A)^3 - h^0(K_M + 2A)\\ &= 3 + e - m = 5 - m.\end{aligned}$$

By $\Delta(M, K_M + 2A) \geq 0$ and Claim 3.11, we have $4 \leq m \leq 5$. By (16) we get the following types.

m	e	b	$\Delta(M, K_M + 2A)$
5	2	0	0

However, if $m = 5$, then $q(M) = 0$ by [6, (5.10) Theorem], and this is impossible.

We see from the above argument that the following type possibly occurs.

(36) $\qquad q(M) = 1$, e is even with $e \geq 4$ and $m = e = -2b$.

(e) Finally we will prove that (36) cannot occur.

First we calculate $h^0(A)$. We note that $h^0(A) = h^0(2H(\mathscr{E}) + f^*(B)) = h^0(S^2(\mathscr{E}) \otimes B)$. Since $2H(\mathscr{E}) + f^*(B)$ is ample, so is $\mathscr{E}\langle \frac{1}{2}B \rangle$ by the definition of ampleness of $\mathscr{E}\langle \frac{1}{2}B \rangle$ (see [24, Definition 6.2.3]), where $\mathscr{E}\langle \frac{1}{2}B \rangle$ denotes a **Q**-twisted bundle (see [24, Definition 6.2.1]). By [24, Lemma 6.2.8] we infer that $S^2(\mathscr{E}\langle \frac{1}{2}B \rangle)$ is ample. On the other hand, $S^2(\mathscr{E}) \otimes B \cong_{\mathbf{Q}} S^2(\mathscr{E}\langle \frac{1}{2}B \rangle)$, we see that $S^2(\mathscr{E}) \otimes B$ is ample, where $\cong_{\mathbf{Q}}$ denotes **Q**-isomorphism (see [24, Definition 6.2.2]). Since $g(C) = 1$, we see from [1, Lemma 15] that

$$h^0(S^2(\mathscr{E}) \otimes B) = \deg(S^2(\mathscr{E}) \otimes B) = 4e + 6b = e.$$

Therefore $h^0(A) = e$. Since $h^0(L) = n + m - 1 = e + 2$, we have $h^0(L) > h^0(A)$, but this is impossible because (M, A) is a reduction of (X, L). Hence we get the assertion of Claim 3.12. □

Therefore the case where $\Delta(M, K_M + 2A) = 0$ possibly occurs, and then $q(M) = 0$, that is, $g(C) = 0$ and $t = 7$ if (M, A) is the type (7.4) in Theorem 2.1. In this case by (17) we have $m = e + 3$. Since $\mathscr{E}$ is an ample vector bundle of rank three on $\mathbf{P}^1$, we have $e \geq 3$. Hence we get $m \geq 6$. So we get the type (6) in Theorem 1.1.

These complete the proof of Theorem 1.1. □

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