M. MURATA AND T. TSUCHIDA
KODAI MATH. J.
37 (2014), 1–15

UNIQUENESS OF L^1 HARMONIC FUNCTIONS ON ROTATIONALLY SYMMETRIC RIEMANNIAN MANIFOLDS

Minoru Murata and Tetsuo Tsuchida

Abstract

We show that any rotationally symmetric Riemannian manifold has the L^1-Liouville property for harmonic functions, i.e., any integrable harmonic function on it must be identically constant. We also give a characterization of a manifold which carries a non-constant L^1 nonnegative subharmonic function.

1. Introduction

The main purpose of this paper is to show that any rotationally symmetric Riemannian manifold has the L^1-Liouville property for harmonic functions, i.e., any integrable harmonic function on it must be identically constant.

There is a vast literature related to the L^p-Liouville property for $1 < p \leq \infty$ (see [1], [2], [6], [7], [8], [12], [14], [15], [17], [24], [18], [20], [21], [25], [26], [27]). For $1 < p < \infty$, it follows from an L^p-Liouville theorem of Yau [27] that any complete non-compact Riemannian manifold has the L^p-Liouville property, i.e., any p-th integrable harmonic function on it must be identically constant. The L^1-Liouville or L^∞-Liouville property, however, depends on the geometry of manifolds. In particular, for the L^1-Liouville property, the major question of its geometric background is still open; although several sufficient conditions for it and counterexamples to it are given (see [3], [5], [13], [16]). Among several counterexamples, Example 1 in Section 3 of [16] seems to be a unique example of a manifold which has only one end and does not have the L^1-Liouville property. The proof of this example, however, is not correct because these exists no Green's function $G(0, x)$ on a compact surface which has the properties they require. Thus, as for the L^1-Liouville property of manifolds having only one end, there exist no counterexamples. On the other hand, any Cartan-Hadamard manifold has the L^1-Liouville property (see Theorem 2.2 (a) of [16]). We suspect that any manifold with only one end has the L^1-Liouville property. The present paper is an initial step toward this speculation.

2000 *Mathematics Subject Classification*. 35J05, 31C12, 31B05, 31A05, 58J05, 34C11.

Key words and phrases. harmonic function, rotationally symmetric Riemannian manifold, L^1-Liouville property, nonnegative subharmonic function.

Received December 13, 2012.

Since the absolute value of a harmonic function is subharmonic, a natural problem related to the L^1-Liouville property for harmonic functions is whether a manifold has the L^1-Liouville property for nonnegative subharmonic functions. In this paper we also give a characterization of a manifold which carries a non-constant L^1 nonnegative subharmonic function.

Now, in order to state our main results, we fix notations. Let M be an n-dimensional ($n \geq 2$) smooth Riemannian manifold with pole p which is rotationally symmetric at p. Then the Riemannian metric in terms of geodesic polar coordinates at p is given by

$$ds^2 = dr^2 + f(r)^2 \, d\Theta^2,$$

where $d\Theta^2$ is the standard metric of the unit sphere S^{n-1} and f is a nonnegative smooth function on $[0, \infty)$ such that $f > 0$ in $(0, \infty)$, $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. The Laplace-Beltrami operator Δ on M is represented by

$$\Delta = f^{1-n}\partial/\partial r(f^{n-1}\partial/\partial r) + f^{-2}\Lambda,$$

where Λ is the standard Laplace-Beltrami operator on S^{n-1}. The Riemannian measure v on M is given by $dv = f^{n-1}(r)\, dr d\sigma$, where $d\sigma$ is the standard area element on S^{n-1}. We denote by $L^1(M)$ the set of integrable functions on M with respect to v. In what follows, we shall identify M and the pole p with $\mathbf{R}^n$ and the origin 0 of $\mathbf{R}^n$, respectively.

Theorem 1.1. *Any $L^1(M)$ harmonic function on M must be identically constant.*

We put emphasis on that this theorem is curvature condition free. Theorem 1.1 is a direct consequence of Theorems 2.1 and 2.2 to be shown in Section 2, which are more quantitative and precise than Theorem 1.1.

Here we recall, for comparison, that the L^∞-Liouville theorem holds if and only if

$$\int_1^\infty f(r)^{n-3}\left(\int_r^\infty f(s)^{1-n}\, ds\right) dr = \infty$$

(see [18] and [21]).

Next, let us consider $L^1(M)$ nonnegative subharmonic functions. Recall that a function u on M is said to be subharmonic on M if

(i) $-\infty \leq u(x) < \infty$, $u(x) \not\equiv -\infty$ on M;

(ii) u is upper semi-continuous on M;

(iii) if D is a relatively compact domain of M, and if w is a real-valued continuous function on $\bar{D}$ such that w is harmonic in D and satisfies $w(x) \geq u(x)$ on ∂D, then

$$w(x) \geq u(x) \quad \text{in } D.$$

It is known (see [11] and [19]) that u is subharmonic on M if and only if u is locally integrable on M and $\Delta u \geq 0$ in the distribution sense. Put

$$J = \int_1^\infty f(r)^{n-1} \left(\int_1^r f(s)^{1-n} \, ds \right) dr. \tag{1.1}$$

Example 1.2. Suppose $J < \infty$. Then there exists an $L^1(M)$ positive smooth subharmonic function on M which is not identically constant. Indeed, consider the equation

$$(-\Delta + \Psi(|x|))v = 0 \quad \text{in } M,$$

where $\Psi \in C_0^\infty((0, 1/2))$ is a nonnegative function which is not identically zero. Then we see that this equation has a positive smooth radial solution v such that

$$v(x) = \int_1^{|x|} f(s)^{1-n} \, ds(1 + o(1)) \quad \text{as } |x| \to \infty$$

(see Theorem 1.2 (i) and the proof of Lemma 2.3 of [23]). Clearly, v is a desired subharmonic function.

As for the condition $J < \infty$, it has played a crucial role in studying the structure of nonnegative solutions to the heat equation on M (see [23]). A typical example satisfying it is $f(r) = \exp(-r^\alpha)$ for $r > 1$ with $\alpha > 2$. Note that the condition $J < \infty$ implies that $I = \infty$, where

$$I = \int_1^\infty f(r)^{1-n} \, dr, \tag{1.2}$$

since

$$\int_2^\infty f(r)^{n-1} \left(\int_1^2 f(s)^{1-n} \, ds \right) dr \leq J < \infty, \tag{1.3}$$

$$R - 1 = \int_1^R dr \leq \left[\int_1^R f(r)^{n-1} \, dr \right]^{1/2} \left[\int_1^R f(r)^{1-n} \, dr \right]^{1/2}, \quad R > 1. \tag{1.4}$$

It is well-known that $-\Delta$ on M is critical (i.e., there is no positive Green function for it) if and only if $I = \infty$. Geometrically, J may be regarded as an quantity to determine whether the constriction rate at infinity of M is big enough.

A manifold satisfying $J < \infty$ is interesting for several reasons: (1) It does not carry a non-constant nonnegative superharmonic function because it is parabolic (i.e., critical) for $I = \infty$; (2) it does not carry a non-constant $L^1(M)$ harmonic function because of Theorem 1.1; but (3) it carries a non-constant $L^1(M)$ positive subharmonic function.

Finally we give a characterization of a manifold which carries a non-constant $L^1(M)$ nonnegative subharmonic function.

THEOREM 1.3. *There exists a non-constant $L^1(M)$ nonnegative subharmonic function on M if and only if $J < \infty$.*

This theorem follows from Example 1.2 and Theorem 3.3 to be shown in Section 3. It gives an implicit geometric meaning of the condition $J < \infty$.

2. Harmonic functions

In this section we give growth estimates at infinity of non-constant harmonic functions, Theorems 2.1 and 2.2, which imply Theorem 1.1.

For $R > 0$, we denote by $B(R)$ the geodesic ball with center 0 and radius R.

THEOREM 2.1. *Let $n = 2$. Then, for any non-constant harmonic function u on M there exists a positive constant C such that*

$$\int_{B(R)} |u|\, dv \geq CR^2, \quad R > 2. \tag{2.1}$$

THEOREM 2.2. (i) *Let $n \geq 2$. Suppose $J = \infty$. Then, for any non-constant harmonic function u on M there exists a positive constant C such that*

$$\int_{B(R)} |u|\, dv \geq CJ(R), \quad R > 2, \tag{2.2}$$

where

$$J(R) = \int_1^R f(r)^{n-1} \left(\int_1^r f(s)^{1-n}\, ds \right) dr. \tag{2.3}$$

(ii) *Let $n \geq 3$. Suppose $J < \infty$. Then, for any non-constant harmonic function u on M there exists a positive constant C such that*

$$\int_{B(R)} |u|\, dv \geq CR^n, \quad R > 2. \tag{2.4}$$

For $R > 0$, we denote by $|B(R)|$ the volume of $B(R)$:

$$|B(R)| = \sigma_n \int_0^R f(r)^{n-1}\, dr,$$

where σ_n is the area of S^{n-1}. The volume of M is denoted by $\mathrm{Vol}(M)$. Note that there exists a positive constant C_1 such that

$$J(R) \geq C_1 |B(R)|, \quad R > 2. \tag{2.5}$$

Thus $\mathrm{Vol}(M) = \infty$ implies $J = \infty$. Furthermore, if $I < \infty$, then there exist positive constants C_2 and C_3 such that

$$J(R) \leq C_2 |B(R)|, \quad R > 2, \tag{2.6}$$

$$|B(R)| \geq C_3 R^2, \quad R > 2. \tag{2.7}$$

Indeed, (2.6) follows from (2.3) and (1.2); while (2.7) follows from (1.4).

Let us show Theorems 2.1 and 2.2. Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $-\Lambda$ on S^{n-1} repeated according to multiplicity, and ϕ_j $(j = 0, 1, 2, \dots)$ be corresponding eigenfunctions such that $\{\phi_j\}_{j=0}^{\infty}$ is a complete orthonormal system of $L^2(S^{n-1})$. In particular, $\phi_0 = \sigma_n^{-1/2}$ and $\lambda_1 = n-1$. Furthermore, it is known that for any j there exists a unique nonnegative integer k such that $\lambda_j = k(k+n-2)$. For $j = 0, 1, 2, \dots,$ let $\lambda_j = k(k+n-2)$ and g_j be a unique solution of the initial value problem

$$f(r)^{1-n}(f(r)^{n-1}w'(r))' - \lambda_j f(r)^{-2}w(r) = 0 \quad \text{in } (0,\infty), \tag{2.8}$$

$$w(r) = r^k(1+o(1)) \quad \text{as } r \to 0. \tag{2.9}$$

In particular, $g_0(r) = 1$.

Lemma 2.3. *For any $j \ge 1$, g_j and g_j' are positive in $(0,\infty)$. Furthermore,*

$$g_j(r) \ge C_j \int_1^r f(s)^{1-n}\, ds, \quad r > 1, \tag{2.10}$$

where C_j is a positive constant.

Proof. We have by (2.8) and (2.9)

$$f(r)^{n-1}g_j'(r) = \lambda_j \int_0^r f(s)^{n-3}g_j(s)\, ds. \tag{2.11}$$

This implies that $g_j(r) > 0$ and $g_j'(r) > 0$ for $r > 0$. Furthermore, (2.10) holds with

$$C_j = \lambda_j \int_0^1 f(s)^{n-3}g_j(s)\, ds. \qquad \square$$

Lemma 2.4. *Let $n = 2$. Then*

$$\inf_{R>2} R^{-2}\int_0^R g_j(r)f(r)\, dr > 0, \quad j \ge 1. \tag{2.12}$$

Proof. In the subcritical case, i.e., $I < \infty$, (2.12) directly follows from (2.10) and (2.7). Let us treat the critical case, i.e., $I = \infty$. We have

$$f(r)(f(r)g_j'(r))' - \lambda_j g_j(r) = 0 \quad \text{in } (0,\infty).$$

Change the variable r to

$$t = \int_1^r f(s)^{-1}\, ds, \tag{2.13}$$

and set $h_j(t) = g_j(r)$. Then

$$h_j''(t) = \lambda_j h_j(t) \quad \text{in } (-\infty,\infty),$$

$h_j(0) > 0$ and $h_j'(0) > 0$. Thus there exist constants $a > 0$ and $b \in \mathbf{R}$ such that $-a < b < a$ and

$$h_j(t) = ae^{\mu t} + be^{-\mu t} \quad \text{in } (0, \infty),$$

where $\mu = \sqrt{\lambda_j}$. Hence $h_j(t) \geq ce^{\mu t}$ in $(0, \infty)$ for some $c > 0$. We have

$$R - 1 = \int_1^R dr \leq \left[\int_1^R g_j(r) f(r)\, dr\right]^{1/2} \left[\int_1^R g_j(r)^{-1} f(r)^{-1}\, dr\right]^{1/2}.$$

On the other hand,

$$\int_1^\infty g_j(r)^{-1} f(r)^{-1}\, dr = \int_0^\infty h_j(t)^{-1}\, dt \leq \int_0^\infty (ce^{\mu t})^{-1}\, dt < \infty.$$

This implies (2.12). □

Lemma 2.5. *Let $n \geq 2$. Suppose $J = \infty$. Then*

$$\inf_{R>2} J(R)^{-1} \int_0^R g_j(r) f(r)^{n-1}\, dr > 0, \quad j \geq 1. \tag{2.14}$$

Proof. The assertion follows from (2.10). □

Proposition 2.6. *Let $n \geq 3$ and $J < \infty$. Then*

$$\inf_{R>2} R^{-n} \int_0^R g_j(r) f(r)^{n-1}\, dr > 0, \quad j \geq 1. \tag{2.15}$$

The proof of this proposition is decomposed into the following 3 lemmas, where we assume that $n \geq 3$ and $J < \infty$.

Lemma 2.7. *For any $\alpha \in (0, n-1]$, $f^\alpha \in L^1((0, \infty); dr)$.*

Proof. Choose $S > 1$ so large that

$$\int_1^S f(r)^{1-n}\, dr > 1.$$

For any $R > S$, we have

$$\begin{aligned}(R - S)^2 &\leq \int_S^R f(r)^{n-1} \left(\int_1^r f(s)^{1-n}\, ds\right) dr \\ &\quad \times \int_S^R f(r)^{1-n} \left(\int_1^r f(s)^{1-n}\, ds\right)^{-1} dr \\ &\leq J\left[\log\left(\int_1^R f(r)^{1-n}\, dr\right) - \log\left(\int_1^S f(r)^{1-n}\, dr\right)\right].\end{aligned}$$

Thus

$$\exp[(R-S)^2/J] \le \int_1^R f(r)^{1-n}\,dr, \quad R > S. \tag{2.16}$$

Hence

$$\int_S^R f(r)^{n-1} e^{(r-S)^2/J}\,dr \le \int_S^R f(r)^{n-1}\left(\int_1^r f(s)^{1-n}\,ds\right)dr \le J < \infty.$$

This shows that $f^{n-1} \in L^1((0,\infty); dr)$. Next, let $0 < \alpha < n-1$. For any $R > S$, we have

$$\int_S^R f(r)^\alpha\,dr \le \left\{\int_S^R f(r)^{n-1}\exp\left[\frac{(r-S)^2}{J}\right]dr\right\}^{\alpha/(n-1)} \times \left\{\int_S^R \exp\left[-\frac{(r-S)^2}{J}\frac{\alpha}{n-1-\alpha}\right]dr\right\}^{(n-1-\alpha)/(n-1)} \le C,$$

where C is a positive constant depending only on J and α. Hence $f^\alpha \in L^1((0,\infty); dr)$. □

Here we note that Lemma 2.7 implies that

$$\int_1^R f(r)^{-1}\,dr \ge C(R-1)^2, \quad R > 1, \tag{2.17}$$

where C is a positive constant independent of R. Indeed,

$$(R-1)^2 \le \left(\int_1^\infty f(s)\,ds\right)\left(\int_1^R f(s)^{-1}\,ds\right).$$

LEMMA 2.8. *For any* $\alpha, \beta > 0$,

$$\int_1^R f(r)^\alpha \exp\left(\beta\int_1^r f(s)^{-1}\,ds\right)dr \ge \left(\frac{\beta}{\alpha}\right)^\alpha (R-1)^{\alpha+1}, \quad R > 1. \tag{2.18}$$

Proof. By Hölder's inequality,

$$R-1 \le \left\{\int_1^R f(r)^\alpha \exp\left(\beta\int_1^r f(s)^{-1}\,ds\right)dr\right\}^{1/(1+\alpha)} \times \left\{\int_1^R f(r)^{-1}\exp\left(-\frac{\beta}{\alpha}\int_1^r f(s)^{-1}\,ds\right)dr\right\}^{\alpha/(1+\alpha)}.$$

This together with

$$\int_1^\infty f(r)^{-1} \exp\left(-\frac{\beta}{\alpha}\int_1^r f(s)^{-1}\, ds\right) dr = \frac{\alpha}{\beta}$$

shows (2.18). □

LEMMA 2.9. *For any $j \geq 1$, there exists a positive constant C_j such that*

$$g_j(r) \geq C_j \exp\left(\sqrt{\lambda_j}\int_1^r f(s)^{-1}\, ds\right), \quad r \geq 1. \tag{2.19}$$

Proof. We write $g = g_j$, $\lambda = \lambda_j$, and $h = f^{n-1}g_j'$. Then $h' = \lambda f^{n-3}g$, $g(0) = h(0) = 0$, $g(r) > 0$ and $h(r) > 0$ for $r > 0$. We have

$$\begin{aligned}(gh)' &= f^{1-n}h^2 + \lambda f^{n-3}g^2\\ &\geq 2\sqrt{\lambda}f^{-1}gh.\end{aligned}$$

Thus

$$(gh)(r) \geq (gh)(1)\exp\left(2\sqrt{\lambda}\int_1^r f(s)^{-1}\, ds\right), \quad r \geq 1.$$

Therefore, with $A = 2(gh)(1)$,

$$\frac{1}{2}f(r)^{n-1}[g(r)^2]' \geq \frac{A}{2}\exp\left(2\sqrt{\lambda}\int_1^r f(s)^{-1}\, ds\right).$$

Hence

$$g(r)^2 \geq A\int_1^r f(s)^{1-n}\exp\left(2\sqrt{\lambda}\int_1^s f(t)^{-1}\, dt\right) ds, \quad r \geq 1. \tag{2.20}$$

By Hölder's inequality, for any $\delta > 0$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$,

$$\begin{aligned}&\int_1^r f(s)^{-1}\exp\left(\delta\int_1^s f(t)^{-1}\, dt\right) ds\\ &\quad \leq \left\{\int_1^r f(s)^{p/2}\, ds\right\}^{1/p}\left\{\int_1^r f(s)^{-(3/2)q}\exp\left(\delta q\int_1^s f(t)^{-1}\, dt\right) ds\right\}^{1/q}.\end{aligned}$$

Put $q = (2/3)(n-1)$ and $\delta = 2\sqrt{\lambda}/q = 3\sqrt{\lambda}/(n-1)$. Then $q > 1$ and $p/2 = (n-1)/(2n-5) \leq n-1$ because $n \geq 3$. Thus, by Lemma 2.7,

$$\begin{aligned}&\int_1^r f(s)^{-1}\exp\left(\delta\int_1^s f(t)^{-1}\, dt\right) ds\\ &\quad \leq C\left\{\int_1^r f(s)^{1-n}\exp\left(2\sqrt{\lambda}\int_1^s f(t)^{-1}\, dt\right) ds\right\}^{1/q}\end{aligned}$$

for any $r \geq 1$, where C is a positive constant independent of r. Choose $S > 1$ so large that

$$\exp\left(\delta \int_1^S f(t)^{-1}\, dt\right) > 2.$$

Then, for any $r \geq S$,

$$\begin{aligned}\int_1^r f(s)^{-1} \exp\left(\delta \int_1^s f(t)^{-1}\, dt\right) ds &= \frac{1}{\delta}\left\{\exp\left(\delta \int_1^r f(t)^{-1}\, dt\right) - 1\right\} \\ &\geq \frac{1}{2\delta} \exp\left(\delta \int_1^r f(t)^{-1}\, dt\right).\end{aligned}$$

Hence, in view of $\delta q = 2\sqrt{\lambda}$,

$$\int_1^r f(s)^{1-n} \exp\left(2\sqrt{\lambda} \int_1^s f(t)^{-1}\, dt\right) ds \geq (2\delta C)^{-q} \exp\left(2\sqrt{\lambda} \int_1^r f(t)^{-1}\, dt\right) \tag{2.21}$$

for any $r \geq S$. This together with (2.20) implies (2.19). □

We are now ready to show Proposition 2.6.

Proof of Proposition 2.6. By Lemmas 2.8 and 2.9,

$$\begin{aligned}\int_0^R g_j(r) f(r)^{n-1}\, dr &\geq C_j \int_1^R f(r)^{n-1} \exp\left(\sqrt{\lambda_j} \int_1^r f(s)^{-1}\, ds\right) dr \\ &\geq C_j'(R-1)^n\end{aligned}$$

for any $R > 1$, where C_j and C_j' are positive constants independent of R. □

Let us complete the proof of Theorem 2.1. We show only Theorem 2.1, since Theorem 2.2 can be shown similarly by using Lemma 2.5 and Proposition 2.6 instead of Lemma 2.4.

Proof of Theorem 2.1. Let u be a harmonic function on M. It suffices to show that if

$$\liminf_{R \to \infty} R^{-2} \int_{B(R)} |u|\, dv = 0,$$

then u must be a constant. Put

$$u_j(r) = \int_{S^{n-1}} u(r\omega)\phi_j(\omega)\, d\sigma(\omega), \quad r > 0, \quad j = 0, 1, 2, \ldots.$$

Then u_j satisfies the equation (2.8),

$$\lim_{r\to 0} u_0(r) = \sqrt{\sigma_n}u(0),$$

$$\lim_{r\to 0} u_j(r) = 0, \quad j = 1, 2, \dots.$$

Thus $u_j(r) = a_j g_j(r)$ for some constants a_j $(j = 0, 1, 2, \dots)$. We have

$$\begin{aligned}\int_0^R |u_j(r)| f(r)^{n-1}\, dr &\le \int_0^R \int_{S^{n-1}} |u(r\omega)|\, |\phi_j(\omega)|\, d\sigma(\omega) f(r)^{n-1}\, dr \\ &\le \sup_{|\omega|=1} |\phi_j(\omega)| \int_{B(R)} |u|\, dv.\end{aligned}$$

Thus

$$|a_j|\left\{\liminf_{R\to\infty} R^{-2}\int_0^R g_j(r) f(r)^{n-1}\, dr\right\} = 0.$$

This together with Lemma 2.4 shows that $a_j = 0$ for $j \ge 1$. Since $\{\phi_j\}_{j=0}^{\infty}$ is a complete orthonormal system of $L^2(S^{n-1})$, we conclude that u is a constant. □

3. Nonnegative subharmonic functions

In this section we give growth estimates at infinity of non-constant nonnegative subharmonic functions.

We begin with an estimate based upon a simple mean-value inequality. The following proposition directly implies Theorem 1.1 in the case $\mathrm{Vol}(M) = \infty$.

PROPOSITION 3.1. *Suppose* $\mathrm{Vol}(M) = \infty$. *Then any nonnegative subharmonic function u on M which is not identically zero satisfies*

$$\liminf_{R\to\infty} \frac{1}{|B(R)|}\int_{B(R)} u\, dv > 0. \tag{3.1}$$

For the proof of this proposition, we prepare a lemma on the Poisson kernel K_R of $-\Delta$ on $B(R)$ with respect to the area element $f(R)^{n-1}\, d\sigma$ of $\partial B(R)$.

LEMMA 3.2. *The Poisson kernel K_R is represented by*

$$K_R(x,\omega) = f(R)^{1-n}\sum_{j=0}^{\infty}\frac{g_j(|x|)}{g_j(R)}\phi_j\left(\frac{x}{|x|}\right)\phi_j(\omega), \quad |x| < R, \ \omega \in S^{n-1}, \tag{3.2}$$

where the series converges uniformly on the product of any compact subset of $B(R)$ and S^{n-1}. In particular, $K_R(0,\omega) = f(R)^{1-n}\sigma_n^{-1}$.

Proof. Put

$$h_{j,R}(r) = g_j(r)\int_r^R f(s)^{1-n} g_j(s)^{-2}\,ds, \tag{3.3}$$

where g_j is the solution of (2.8) and (2.9). Then the Green function G_R of $-\Delta$ on $B(R)$ is represented by

$$G_R(x,y) = \sum_{j=0}^{\infty} g_j(|x|)h_{j,R}(|y|)\phi_j\left(\frac{x}{|x|}\right)\phi_j\left(\frac{y}{|y|}\right) \tag{3.4}$$

for x, y with $|x| < |y| < R$, and $G_R(y,x) = G_R(x,y)$ (see (3.20) of [20] and Lemma 8.3 of [22]). Since

$$K_R(x,\omega) = -\frac{\partial G_R}{\partial |y|}(x,y)\bigg|_{|y|=R}, \quad |x| < R, \ \omega \in S^{n-1},$$

we get (3.2) (see Lemma 8.9 of [22]). □

Proof of Proposition 3.1. Let u be a nonnegative subharmonic function on M. From definition, we have

$$u(x) \le \int_{S^{n-1}} K_R(x,\omega)u(R\omega)f(R)^{n-1}\,d\sigma(\omega), \quad x \in B(R).$$

Fix $x \in M$. Since $K_R(0,\omega) = f(R)^{1-n}\sigma_n^{-1}$, the Harnack inequality shows that there exists a positive constant C such that

$$K_R(x,\omega) \le Cf(R)^{1-n}$$

for any $R > |x| + 1$. Thus

$$u(x) \le C\int_{S^{n-1}} u(r\omega)\,d\sigma(\omega), \quad r > |x| + 1.$$

We have

$$\begin{aligned} u(x)\int_{|x|+1}^R f(r)^{n-1}\,dr &\le C\int_{|x|+1}^R \left(\int_{S^{n-1}} u(r\omega)\,d\sigma(\omega)\right) f(r)^{n-1}\,dr \\ &\le C\int_{B(R)} u\,dv. \end{aligned}$$

Thus, for sufficiently large R

$$u(x) \le \frac{C}{|B(R)|}\int_{B(R)} u\,dv \tag{3.5}$$

with another positive constant C. Suppose (3.1) does not hold. Then it follows from (3.5) that $u(x) = 0$. Hence u must be identically zero. □

We conclude this section with an estimate via flux. The following theorem together with Example 1.2 implies Theorem 1.3 immediately.

THEOREM 3.3. *Let* $n \geq 2$. *Suppose* $J = \infty$. *Then, for any non-constant nonnegative subharmonic function* u *on* M *there exist positive constants* C *and* R_0 *such that*

$$\int_{B(R)} u\, dv \geq CJ(R), \quad R > R_0. \tag{3.6}$$

Proof. By virtue of Theorem 2.2(i), it suffices to show the theorem in the case where u is a nonnegative subharmonic function which is not harmonic on the whole space M. Choose $S > 0$ so large that u is not harmonic in $B(S/2)$. We first show (3.6) with $R_0 = 2S$ in the case where u is smooth in $B(R)$. For $r > 0$, put

$$[u]_r = \int_{S^{n-1}} u(r\omega)\, d\sigma(\omega).$$

Fix $R > 2S$. We claim that

$$[u]_r \geq A \int_S^r f(s)^{1-n}\, ds, \quad S \leq r < R, \tag{3.7}$$

for some positive constant A independent of R. Since $\Delta u \geq 0$ in $B(R)$, we have

$$\begin{aligned} f(r)^{1-n}\partial_r(f(r)^{n-1}\partial_r[u]_r) &= \int_{S^{n-1}} f(r)^{1-n}\partial_r(f(r)^{n-1}\partial_r u(r\omega))\, d\sigma(\omega) \\ &\geq -\int_{S^{n-1}} f(r)^{-2}\Lambda u(r\omega)\, d\sigma(\omega) = 0. \end{aligned}$$

Thus $f(r)^{n-1}\partial_r[u]_r$ is increasing on $(0, R)$. On the other hand,

$$\begin{aligned} f(r)^{n-1}\partial_r[u]_r &= \int_{S^{n-1}} f(r)^{n-1}\partial_r u(r\omega)\, d\sigma(\omega) \\ &= \int_{B(r)} \Delta u\, dv. \end{aligned}$$

Hence, for $r \in [S, R)$

$$f(r)^{n-1}\partial_r[u]_r \geq \int_{B(S)} \varphi\Delta u\, dv,$$

where $\varphi \in C_0^\infty(B(S))$ is a function such that $0 \leq \varphi(x) \leq 1$ on $B(S)$ and $\varphi(x) = 1$ on $B(S/2)$. This implies (3.7) with

$$A \equiv \int_{B(S)} \varphi\Delta u\, dv > 0, \tag{3.8}$$

since $\Delta u \not\equiv 0$ in $B(S/2)$. The proof of the claim is now complete. From (3.7), we obtain that

$$\int_{B(R)} u\, dv \geq \sigma_n \int_S^R [u]_r f(r)^{n-1}\, dr \tag{3.9}$$
$$\geq \sigma_n A \int_S^R f(r)^{n-1} \left(\int_S^r f(s)^{1-n}\, ds \right) dr$$
$$\geq ABJ(R),$$

where B is a positive constant independent of $R > 2S$ and u.

We next show (3.6) with $R_0 = 2S$ in the general case. By the Riesz decomposition theorem (see [11], [19], [9], [10], [4]), for a subharmonic function u on M there exists a unique Borel measure μ on M with $\mu(K) < \infty$ for any compact subset K of M such that for any $T > 0$

$$u(x) = -\int_{B(T)} G_T(x, y)\, d\mu(y) - v_T(x), \quad x \in B(T),$$

where G_T is the Green function of $-\Delta$ on $B(T)$ and v_T is harmonic on $B(T)$. With $\lambda = \mu|_{B(R)}$, we have

$$u(x) = -\int_{B(2R)} G_{2R}(x, y)\, d\lambda(y) + w(x) \quad \text{in } B(2R), \tag{3.10}$$

where

$$w(x) = -\int_{B(2R)\setminus B(R)} G_{2R}(x, y)\, d\mu(y) - v_{2R}(x)$$

is harmonic in $B(R)$. Let ψ be a canonical diffeomorphism from M to $\mathbf{R}^n$ which maps the pole p to the origin. Let $\tilde{\lambda}$ be the induced measure on $\mathbf{R}^n$ by ψ from λ: $\tilde{\lambda}(B) = \lambda(\psi^{-1}(B))$ for any Borel set B of $\mathbf{R}^n$. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be a nonnegative function satisfying $\operatorname{supp} \rho \subset \{|x| \leq 1\}$ and $\int \rho(x)\, dx = 1$ with the Lebesque measure dx. For $j = 1, 2, \ldots,$ put $\tilde{\lambda}_j(x) = j^n \int \rho(j(x - z))\, d\tilde{\lambda}(z)$. Set

$$m_j(y) = \tilde{\lambda}_j(\psi(y))|\psi(y)|^{n-1} f(|\psi(y)|)^{1-n}, \quad y \in B(2R),$$
$$d\lambda_j = m_j\, dv,$$
$$v_j(x) = -\int_{B(2R)} G_{2R}(x, y)\, d\lambda_j(y), \quad x \in B(2R),$$

and $u_j = v_j + w$. Then u_j are smooth and $\Delta u_j = m_j \geq 0$ in $B(R)$. Thus, by (3.8) and (3.9)

$$\int_{B(R)} u_j\, dv \geq A_j BJ(R),$$
$$A_j = \int_{B(S)} \varphi\, d\lambda_j = \int_{B(2R)} \varphi\, d\lambda_j.$$

We see that as $j \to \infty$,

$$A_j = \int_{B(2R)} \varphi m_j \, dv = \int_{\psi(B(2R))} \varphi(\psi^{-1}(x))\tilde{\lambda}_j(x) \, dx$$
$$\to \int_{\psi(B(2R))} \varphi(\psi^{-1}(x)) \, d\tilde{\lambda}(x) = \int_{B(2R)} \varphi \, d\lambda \equiv A > 0.$$

Since the function

$$\int_{B(R)} G_{2R}(x, y) \, dv(x)$$

of $y \in B(2R)$ is bounded continuous and supp λ_j are compact in $B(2R)$ for sufficiently large j, we obtain

$$\lim_{j \to \infty} \int_{B(R)} u_j \, dv = \int_{B(R)} u \, dv$$

(see also Lemma 4 of [10]). Hence

$$\int_{B(R)} u \, dv \geq ABJ(R).$$

This completes the proof. □

Acknowledgment. The first author thanks Eiji Yanagida for his suggestion to use the change of variable (2.13) in the proof of Lemma 2.4.

References

[1] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, Ann. Math. **125** (1987), 429–461.
[2] M. T. Anderson and R. Schoen, Positive harmonic functions on complete manifolds of negative curvature, Ann. Math. **121** (1985), 429–461.
[3] L. O. Chung, Existence of harmonic L^1 functions in complete Riemannian manifolds, Proc. Amer. Math. Soc. **88** (1983), 531–532.
[4] C. Constantinescu and A. Cornea, Potential theory on harmonic spaces, Springer, Berlin, 1972.
[5] A. Grigor'yan, Stochastically complete manifolds and summable harmonic functions, Math. USSR Izvestiya **33** (1989), 425–432.
[6] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bulletin of Amer. Math. Soc. **36** (1999), 135–249.
[7] A. Grigor'yan and W. Hansen, A Liouville property for Schrödinger operators, Math. Ann. **312** (1998), 659–716.
[8] A. Grigor'yan and N. S. Nadirashvili, Liouville theorems and exterior boundary value problems, Soviet Math. (Izv. VUZ) **31**, no. 5 (1987), 31–42.
[9] R. M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potential, Ann. Inst. Fourier **12** (1962), 415–571.

[10] R. M. Hervé and M. Hervé, Les fonctions surharmoniques associées à un opérateur elliptique du second ordre à coefficients discontinus, Ann. Inst. Fourier **19** (1969), 305–359.
[11] S. Itô, On definitions of superharmonic functions, Ann. Inst. Fourier **25** (1975), 309–316.
[12] Y. T. Kuzmenko and S. A. Molchanov, Counterexamples to Liouville-type theorems, Moscow Univ. Math. Bull. **34** (1979), 35–39.
[13] P. Li, Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds, J. Differential Geometry **20** (1984), 447–457.
[14] P. Li, Curvature and function theory on Riemannian manifold, Surveys in Diff. Geom. **VII** (2000), 375–432.
[15] P. Li, Harmonic functions on complete Riemannian manifolds, Handbook of geometric analysis, No. 1, Adv. Lect. Math. **7**, Int. Press, Somerville, MA, 2008, 195–227.
[16] P. Li and R. Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds, Acta Math. **153** (1984), 279–301.
[17] A. G. Losev, On some Liouville theorems on noncompact Riemannian manifolds, Siberian Math. J. **39** (1998), 74–80.
[18] P. March, Brownian motion and harmonic function on rotationally symmetric manifolds, Ann. Prob. **14** (1986), 793–801.
[19] K. Masuda, A remark on superharmonic functions, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **25** (1978), 61–66.
[20] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbf{R}^n$, Duke Math. J. **53** (1986), 869–943.
[21] M. Murata, Positive harmonic functions on rotationary symmetric Riemannian manifolds, Potential theory (M. Kishi, ed.), Walter de Gruyter and Co., Berlin-New York, 1991, 251–259.
[22] M. Murata, Martin boundaries of elliptic skew products, semismall perturbations, and fundamental solutions of parabolic equations, J. Funct. Anal. **194** (2002), 53–141.
[23] M. Murata, Nonnegative solutions of the heat equation on rotationally symmetric Riemannian manifolds and semismall perturbations, Revista Mat. Iberoamericana **27** (2011), 885–907.
[24] N. S. Nadirashvili, A theorem of Liouville type on a Riemannian manifold, Russian Math. Surveys **40** (1986), 235–236.
[25] K.-Th. Sturm, Analysis on local Dirichlet spaces I. Recurrence, conservativeness and L^p-Liouville properties, J. Reine Angew. Math. **456** (1994), 173–196.
[26] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. **28** (1975), 201–228.
[27] S. T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J. **25** (1976), 659–670.

Minoru Murata
5-30-1 Shinyoshida-higashi
Kohoku-ku, Yokohama, 223-0058
Japan
E-mail: minoru3@math.titech.ac.jp

Tetsuo Tsuchida
Department of Mathematics
Meijo University
Shiogamaguchi, Tenpaku-ku
Nagoya, 468-8502
Japan
E-mail: tsuchida@meijo-u.ac.jp