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# ON TRIANGLES IN THE UNIVERSAL TEICHMÜLLER SPACE

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**Abstract**

Let $\mathscr{T}(\Delta)$ be the universal Teichmüller space, viewed as the set of all Teichmüller equivalent classes $[f]$ of quasiconformal mappings $f$ of $\Delta$ onto itself. The notion of completing triangles was introduced by F. P. Gardiner. Three points $[f]$, $[g]$ and $[h]$ are called to form a completing triangle if each pair of them has a unique geodesic segment joining them. Otherwise, they form a non-completing triangle. In this paper, we construct two Strebel points $[f]$ and $[g]$ such that $[f]$, $[g]$ and $[id]$ form a non-completing triangle. A sufficient condition for points $[f]$, $[g]$ and $[id]$ to form a completing triangle is also given.

## §1. Introduction

Let $\Delta$ be the unit disc on the complex plane $\mathbf{C}$. By $\mathscr{QC}(\Delta)$ we denote the set of all quasiconformal mappings of $\Delta$ onto itself that keep $1$, $-1$ and $i$ fixed. Two elements $f$ and $\tilde{f}$ of $\mathscr{QC}(\Delta)$ are said to be *Teichmüller equivalent*, denoted by $f \sim \tilde{f}$ or $\mu \sim \tilde{\mu}$, if and only if ([1], [7], [9], [10])

$$f|_{\partial\Delta} = \tilde{f}|_{\partial\Delta},$$

where $\mu$ and $\tilde{\mu}$ are the complex dilatations of $f$ and $\tilde{f}$ respectively.

We denote by $Bel(\Delta)$ the Banach space of Beltrami coefficients $\mu(z)$ on $\Delta$ with finite $L^\infty$-norm and denote by $M(\Delta)$ the open unit ball in $Bel(\Delta)$. For any $\mu \in M(\Delta)$, there exists a quasiconformal mapping $f$ from $\Delta$ onto itself with Beltrami coefficient $\mu$ as its complex dilatation and keeps $1$, $-1$ and $i$ fixed.

The Teichmüller equivalent class of a quasiconformal mapping $f \in \mathscr{QC}(\Delta)$ with $\mu$ as its complex dilatation is denoted by $[f]$ or $[\mu]$. Then the universal Teichmüller space of $\Delta$ is defined as

$$\mathscr{T}(\Delta) := \{[f] : f \in \mathscr{QC}(\Delta)\} = \{[\mu],\ \mu \text{ is the complex dilatation of } f \in \mathscr{QC}(\Delta)\},$$

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or equivalently,

$$\mathcal{T}(\Delta) := \mathcal{QC}(\Delta)/\sim.$$

Let $id : \Delta \to \Delta$ be the identity map. We call $[id]$ the *base-point* of $\mathcal{T}(\Delta)$. A quasiconformal mapping $f \in \mathcal{QC}(\Delta)$ or $\mu$ is said to be *extremal*, if

$$K(f) \leq K(\tilde{f}) : \text{for each} \quad \tilde{f} \in [f],$$

where $K(\tilde{f})$ is the maximal dilatation of the quasiconformal mapping $\tilde{f}$ and $\mu$ is the complex dilatation of $f$. $f$ is said to be uniquely extremal if it is extremal and if

$$K(\tilde{f}) > K(f)$$

holds for any $\tilde{f} \in [f]$ other than $f$.

For a given point $[f]$ of $\mathcal{T}(\Delta)$, we define the quantity

$$K_0([f]) := \inf\{K(\tilde{f}) : \tilde{f} \in [f]\},$$

which is called the *extremal maximal dilatation* of the point $[f]$.

We also need another quantity of $[f]$:

$$H([f]) := \inf_{\tilde{f} \in [f];\, E \subset \Delta} \{K(\tilde{f}|_{\Delta \setminus E})\},$$

where $E$ ranges over all compact subsets of $\Delta$. $H([f])$ is called the *boundary dilatation* of $[f]$.

Following [3], a point $[f]$ of $\mathcal{T}(\Delta)$ is called a *Strebel point*, if $H([f]) < K_0([f])$. Otherwise, it is called a *non-Strebel point*.

For every point $[f]$, we have $H([f]) \leq K_0([f])$. So $[f]$ is a non-Strebel point, if and only if $H([f]) = K_0([f])$.

Let $\zeta$ be a point in the boundary $\partial\Delta$ of $\Delta$ and let $\mu \in M(\Delta)$. Denote

$$h^*_\zeta(\mu) = \inf\{\|\mu|_U\|_\infty \,|\, U \text{ is an open disk in } \mathbf{C} \text{ containing } \zeta\},$$

where $\mu$ is equal to 0 outside of $\Delta$.

Let

$$h_\zeta([\mu]) = \inf\{h^*_\zeta(\nu) \,|\, \nu \in [\mu]\}.$$

Then the local boundary dilatations at $\zeta$ of $\mu \in M(\Delta)$ and $\tau = [\mu] \in \mathcal{T}(\Delta)$ are defined as

$$H^*_\zeta(\mu) = \frac{1 + h^*_\zeta(\mu)}{1 - h^*_\zeta(\mu)}$$

and

$$H_\zeta([\mu]) = \frac{1 + h_\zeta([\mu])}{1 - h_\zeta([\mu])},$$

respectively [15].

If there exists a point $\zeta \in \partial\Delta$ such that

$$H_\zeta([\mu]) = K_0([f^\mu]),$$

then we call $\zeta$ a essential boundary point.

Let $[f]$ and $[g]$ be any two points of $\mathscr{T}(\Delta)$. The *Teichmüller distance* between them is defined as

$$d_T([f],[g]) := \frac{1}{2} \inf\{\log K(h) : h \sim f \circ g^{-1}\}$$
$$\equiv \frac{1}{2} \log K_0([f \circ g^{-1}]).$$

It is well-known that for any Beltrami coefficient $\mu$ in $M(\Delta)$ which is extremal, the image of the map from hyperbolic disc to $\mathscr{T}(\Delta)$,

$$\Gamma_\mu : \Delta \to \mathscr{T}(\Delta); \quad t \to \left[\frac{t}{\|\mu\|_\infty}\mu\right]$$

is a holomorphic isometry [2]. We call this image a Teichmüller disc in $\mathscr{T}(\Delta)$.

A curve $\gamma$ in $\mathscr{T}(\Delta)$ with initial point $\tau_1$ and terminal point $\tau_2$ is called a *geodesic segment* joining $\tau_1$ and $\tau_2$, if $\gamma$ is the isometric image of $[a,b]$ into $\mathscr{T}(\Delta)$ with respect to the Euclidian metric of $[a,b]$ and the Teichmüller metric of $\mathscr{T}(\Delta)$, respectively.

It is a well-known fact that, if $\tau$ $(\tau \neq [id])$ is a Strebel point, then the geodesic segment joining $[id]$ and $\tau$ is unique. While if $\tau$ is a non-Strebel point that contains an extremal mapping of landslide type ([11], [21]),[1] then there are infinitely many geodesic segments joining $[id]$ and $\tau$ ([3] or [2], [12], [13], [20]).

Let $\tau_0$, $\tau_1$ and $\tau_2$ be three distinct points in $\mathscr{T}(\Delta)$. According to Frederick P. Gardiner ([6]), they form a "*completing triangle*", if for each pair of them, there is only one geodesic segment joining them. Otherwise, they form a "*non-completing triangle*".

Now we introduce some background and motivation of our study. We first give some definitions. By definition, a geodesic disc in a metric space $M$ is the image of an isometric embedding $I : \Delta \to M$ of $\Delta$ into $M$ with respect to the Poincaré metric and the metric of $M$, respectively. And a totally geodesic set $S$ of a metric space $M$ is the set such that for any two points $p$ and $q$ in $S$, all the geodesic segments connecting $p$ and $q$ are contained in $S$. For a geodesic disc, if it is also a totally geodesic set, then it is called a totally geodesic disk.

An unresolved problem is to describe geodesic discs and totally geodesic discs in Teichmüller space. It is well-known that all Teichmüller discs are totally geodesics. But we do not know much about the geodesic discs and totally geodesic discs in Teichmüller spaces. For example, many people believe a

[1] An extremal quasiconformal mapping $f : \Delta \to \Delta$ is called of landslide type if there is a constant $\delta > 0$ and an open set $U \subset \bar{U} \subset \Delta$ such that $|\mu_f(z)|_U \leq \|\mu_f\|_\infty - \delta$, where $\mu_f$ is the Beltrami coefficient of $f$.

geodesic disc in finite dimensional Teichmüller space should be a Teichmüller disc. This is an open problem for a long time. The referee told the authors that a graduate student of McMullen recently solves this problem affirmatively. And we don't know any details for this result. It is proved [14] that, in infinite dimensional Teichmüller spaces, there exist infinite many geodesic discs such that the intersection set of these geodesic discs is a closed set. And a geodesic disc should not be a holomorphic disc in infinite dimensional Teichmüller spaces.

But there are still many questions relating to this. For example, can we find a totally geodesic disc in Teichmüller space which is not a Teichmüller disc? And if all the points in a geodesic disc are Strebel points, is this geodesic disk a totally geodesic disk? Here a related question is, for two Strebel points $p$ and $q$, is the geodesic segment connecting them unique? Actually this question is equivalent to whether the three points $[id]$, $p$, $q$ form a completing triangle.

Then it is natural to ask the following questions:

Question $\mathscr{A}$. For arbitrarily given two Strebel points $\tau_1$ and $\tau_2$, do the three points $\tau_1$, $\tau_2$ and $[id]$ always form a completing triangle?

If the answer of this question is negative, then we may consider:

Question $\mathscr{B}$. Suppose both $\tau_1$ and $\tau_2$ are two Strebel points. What are the conditions for the three points $\tau_1$, $\tau_2$ and $[id]$ to form a completing triangle?

In this paper, it is shown that the answer to Question $A$ is negative, and a sufficient condition for $\tau_1$, $\tau_2$ and $[id]$ to form a completing triangle is provided.

Theorem 1. *There are two Strebel points $\tau_1$ and $\tau_2$ with $\tau_1 \neq \tau_2$ such that $\tau_1$, $\tau_2$ and $[id]$ do not form a completing triangle.*

Theorem 2. *Suppose both $[f]$ and $[g_K]$ are Strebel points. Moreover, $g_K$ is a Teichmüller mapping whose Beltrami coefficient is*

$$\mu_K = \frac{K-1}{K+1}\frac{\bar{\phi}}{|\phi|} \quad (K > 1),$$

*where $\phi$ is an integrable holomorphic quadratic differential on $\Delta$. If $K$ is sufficiently closed to 1, then the three points $\tau = [f]$, $\tau_K = [g_K \circ f]$ and $[id]$ form a completing triangle.*

We will prove Theorem 1 and Theorem 2 in §2.

## §2. Proof of Theorems

Now we are going to prove Theorem 1, that is to construct a counter example for Question $\mathscr{A}$.

*Proof of Theorem* 1. Take a strip:

$$Q := \{x + iy : 0 < x < +\infty; 0 < y < 1\}.$$

With the Caratheodory prime-endpoint topology, $\overline{Q}$ is conformally equivalent to $\overline{\Delta}$. In what follows, by $+\infty$ we denote the prime endpoint of $\partial Q$, which is the limit of the points $x + iy \in Q$ as $x$ tends to $+\infty$, with respect to the prime-endpoint topology.

Let $\mathscr{QC}(Q)$ be the set of all quasiconformal mappings of $Q$ onto itself that keep 0, $i$ and $+\infty$ fixed. Similarly as before, we can define the Teichmüller equivalent class $[f]$ of $f \in \mathscr{QC}(Q)$ and the Teichmüller space

$$\mathscr{T}(Q) := \{[f] : f \in \mathscr{QC}(Q)\}.$$

All of other terminologies and notations in §1, such as $K_0[f]$, $H[f]$ and the concepts of Strebel points or non-Strebel points, can be established for the space $\mathscr{T}(Q)$.

We will construct our counter examples with $\mathscr{T}(Q)$ instead of $\mathscr{T}(\Delta)$ for convenience.

Let $K$ be a real number with $K > 1$. We define a function $\xi_K(x)$ on $[0, +\infty)$ as following:

$$\xi_K(x) = 1, \quad \text{as } 0 \le x \le 1;$$

$$\xi_K(x) = (2 - x) + (x - 1)K, \quad \text{as } 1 < x \le 2;$$

$$\xi_K(x) = K, \quad \text{as } 2 < x \le 3;$$

$$\xi_K(x) = (4 - x)K + (x - 3), \quad \text{as } 3 < x \le 4;$$

$$\xi_K(x) = 1, \quad \text{as } x > 4.$$

Let

$$\Lambda_K(x) := \int_0^x \xi_K(t)\, dt.$$

Then we have a quasiconformal mapping $F_K$ of $Q$ onto itself:

$$F_K : x + iy \mapsto \Lambda_K(x) + iy, \quad \forall x + iy \in Q.$$

By $\mu_K$ we denote the Beltrami coefficient of the mapping $F_K(z)$. A simple computation shows

$$\mu_K(z) = \frac{\xi_K(x) - 1}{\xi_K(x) + 1}, \quad \forall z = x + iy \in Q.$$

Hence $F_K(z)$ is a conformal mapping in $(0, 1) \times (0, 1)$ and $(4, \infty) \times (0, 1)$.

Now we claim that, for any $K > 1$, the boundary dilatation of $[F_K]$ must be 1, namely

$$H([F_K]) = 1. \tag{2.1}$$

Indeed, since $F_K|_{(0,+\infty)}$ is $C^1$-smooth at any boundary point $\zeta = x$ with $0 < x < +\infty$ of $\partial Q$, the local boundary dilatation of $F_K|_{\partial Q}$ at $\zeta = x$ is 1 (see [15]). The same discussion and the same conclusion hold for any boundary point $\zeta = x + i$ with $0 < x < +\infty$. On the other hand, by the definition of the local boundary dilatation, the fact that $F_K|_{(0,1)\times(0,1)}$ is a conformal mapping implies that the local dilatation of $F_K|_{\partial Q}$ at the boundary point $\zeta = iy$ with $0 < y < 1$ is equal to 1, and so dose it at $\zeta = 0$ and $\zeta = i$. The local boundary dilation of $F_K|_{\partial Q}$ at $\zeta = +\infty$ is also equal to 1, because $F_K|_{(4,+\infty)\times(0,1)}$ is conformal. Now we conclude that the local boundary dilatation of $F_K|_{\partial Q}$ at any boundary point is 1. By the Fehlmann's theorem ([4], [5]), we get $H([F_K]) = 1$.

By the definition of $F_K$, it is easy to check that $K_0([F_K]) > 1$. Combining with (2.1) we know that $[F_K]$ is a Strebel point.

Let $\tau_1 = [F_K]$, the point that we need in Theorem 1. Now we want to find another Strebel point $\tau_2$ that we need in Theorem 1.

Now we define a map $\Upsilon : Q \to Q$ as follows:

$$\Upsilon(x + iy) = x + iy, \quad \text{as } 0 < x < 1, 0 < y < 1; \quad \text{and}$$
$$\Upsilon(x + iy) = 1 + K_0(x - 1) + iy, \quad \text{as } x \geq 1,\ 0 < y < 1,$$

where $K_0 > 1$ is a constant.

Based on the result ([19]) of K. Strebel, we know that $\Upsilon$ is an extremal quasiconformal mapping with the maximal dilatation $K_0$ and $+\infty$ is an essential boundary point. The local boundary dilatations of $\Upsilon|_{\partial Q}$ at both points 1 and $1 + i$ are equal to ([15])

$$\ell_0 := 1 + \frac{\log^2 K_0}{2\pi^2} + \frac{\log K_0}{\pi}\sqrt{1 + \frac{\log^2 K_0}{4\pi^2}}.$$

While the local boundary dilatation of $\Upsilon|_{\partial Q}$ at any boundary point $\zeta$ ($\zeta \neq 1$, $1 + i, +\infty$) is 1. Noting the fact that $\ell_0 < K_0$ when $K_0$ is large enough, we see $+\infty$ is the unique essential boundary point of $\Upsilon|_{\partial Q}$.

Let $\Phi$ be a conformal mapping of $Q$ onto itself with the following boundary correspondance:

$$\Phi(+\infty) = 0, \quad \Phi(0) = i, \quad \Phi(i) = +\infty.$$

We define $G$ as $\Phi \circ \Upsilon \circ \Phi^{-1}$. Then $G$ belongs to $\mathscr{QC}$ and is an extremal mapping with $K(G) = K_0$. The local boundary dilatation of $G|_{\partial Q}$ at 0 is equal to $K_0$. The local boundary dilatations of $G|_{\partial Q}$ at both points $\Phi(1)$ and $\Phi(1 + i)$ are equal to $\ell_0$. At any other point, it is equal to 1.

Recalling $K_0 > \ell_0$ again, we know that $[G]$ is a non-Strebel point of $\mathscr{T}(Q)$.

Let $\mu_G$ be the Beltrami coefficient of $G$. Then $\mu_G(z)|_U = 0$, where $U := \{x + iy : x > N, 0 < y < \delta\}$ for some $\delta$ with $0 < \delta < 1$ and a sufficiently large $N$. By the known results (for example [13] or [20]), there are infinitely many geodesic segments joining $[G]$ and $[id]$.

Now we suppose $K > K_0$ and let $f_K = G \circ F_K$. Recalling the properties of the local boundary dilatation of $G$ and $F_K$, it is clear that

$$H([f_K]) = K_0.$$

Now we fix $K_0$ and let $K$ change. We claim that, when $K$ is sufficiently large, the point $[f_K]$ is a Strebel point of $\mathscr{T}(Q)$.

To prove our claim, we focus on the rectangle $R = [0,3] \times [0,1]$. Since $F_K|_{[2,3]\times[0,1]}$ is an affine mapping with a factor $K$, we know that

$$\lim_{K\to\infty} \frac{\mathrm{Mod}(f_K(R))}{\mathrm{Mod}(G(R))} = +\infty,$$

which implies

$$\lim_{K\to\infty} f_K(3) = +\infty. \tag{2.2}$$

For the domains $Q[i, +\infty, 3, 0]$ and $Q[i, +\infty, f_K(3), 0]$, it follows from (2.2) that

$$\lim_{K\to+\infty} \frac{\mathrm{Mod}(Q[i, +\infty, f_K(3), 0])}{\mathrm{Mod}(Q[i, +\infty, 3, 0])} = +\infty$$

Therefore, when $K$ is sufficiently large, we have

$$\frac{\mathrm{Mod}(Q[i, +\infty, f_K(3), 0])}{\mathrm{Mod}(Q[i, +\infty, 3, 0])} > K_0. \tag{2.3}$$

From now on we suppose $K$ is large enough so that (2.3) holds.

Let $\tilde{f}_K$ be any element in $[f_K]$, namely $\tilde{f}_K|_{\partial Q} = f_K|_{\partial Q}$. We have

$$\frac{\mathrm{Mod}(Q[i, +\infty, \tilde{f}_K(3), 0])}{\mathrm{Mod}(Q[i, +\infty, 3, 0])} > K_0, \tag{2.4}$$

then it follows from (2.4) that

$$K_0[f_K] > K_0. \tag{2.5}$$

On the other hand, $H([F_K]) = 1$ implies $H([f_K]) = H([G]) = K_0$. From (2.5) we get

$$K_0([f_K]) > H([f_K]),$$

which means that $[f_K]$ is a Strebel point of $\mathscr{T}(Q)$.

Let $\tau_1 = [F_K]$ and $\tau_2 = [f_K]$. Then $\tau_1$ and $\tau_2$ are the points we desired in Theorem 1.

To prove this, we need to show that there are infinitely many geodesic segments joining $\tau_1$ and $\tau_2$.

It is clear that $f_K \circ (F_K)^{-1} = G$. We have known that there are infinitely many geodesic segments joining $[id]$ and $[G]$.

Suppose $\gamma : [0, t_0] \to \mathcal{T}(Q)$ is a geodesic segment with $\gamma(0) = [id]$ and $\gamma(t_0) = [G]$. This means

$$d_T(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|, \quad \forall t_1, t_2 \in [0, t_0].$$

Suppose $\gamma_{(t)} = [\mathcal{G}_t]$, where $G_t \in \mathcal{QC}(Q)$. Then, by the definition of $d_T$, we have

$$\begin{aligned} |t_1 - t_2| &= d_T(\gamma(t_1), \gamma(t_1)) = d_T([\mathcal{G}_{t_1}], [\mathcal{G}_{t_2}]) \\ &= d_T([\mathcal{G}_{t_1} \circ F_K], [\mathcal{G}_{t_2} \circ F_K]). \end{aligned}$$

This means that $[\mathcal{G}_t \circ F_K] : [0, t_0] \to \mathcal{T}(Q)$ is a geodesic segment, which joins $[F_K] = \tau_1$ and $[G \circ F_K] = [f_K] = \tau_2$. We denote this geodesic segment by $\Gamma_\gamma$. It is easy to check, if $\gamma_1$ and $\gamma_2$ are distinct geodesic segments joining $[id]$ and $[G]$, then $\Gamma_{\gamma_1}$ is different from $\Gamma_{\gamma_2}$. We get infinitely many geodesic segments joining $[F_K] = \tau_1$ and $[f_K] = \tau_2$.

This is the counter example that we need for Question $\mathcal{A}$. Then the proof of Theorem 1 is completed. □

*Remark* 1. By the proof of Theorem 1, we know that there are two Strebel points $\tau_1$ and $\tau_2$ such that there exist infinitely many geodesic segments joining them. Next we will prove the following proposition:

PROPOSITION. *There exist two non-Strebel points $[\mu_1]$ and $[\mu_2]$ such that there is only one geodesic segment joining them.*

To prove Proposition, we need a notation and a lemma as follows:

The notion of non-decreasable dilatation for quasiconformal mappings was introduced by Edgar Reich ([16]). An element $g$ in $[f]$ has a non-decreasable dilatation (or its Beltrami coefficient $\nu$ is called non-decreasable), if for any $h$ in $[f]$ together with the condition

$$|\omega| \le |\nu| \text{ almost everywhere in } D,$$

then $g = h$, where $\omega$ is the Beltrami coefficients of $h$.

LEMMA ([18]). *Let $\varphi$ be a holomorphic function on $\Delta$. If Beltrami coefficient $k\frac{|\varphi|}{\varphi}$ $(0 < k < 1)$ is uniquely extremal, then for any non-negative measurable function $k(z)$, $\|k(z)\|_\infty < 1$, the inverse of the mapping with complex dilatation $\mu(z) = k(z)\frac{|\varphi|}{\varphi}$ has non-decreasable dilatation.*

*Proof of Proposition.* Let $Q$ be defined as before and

$$Q_1 := \left\{ x + iy : 1 < x < 2; \frac{1}{4} < y < \frac{3}{4} \right\}.$$

We define $\mu_1(z)$ and $\mu_2(z)$ on $Q$ by

$$\mu_1(z) := \begin{cases} 2k, & \text{as } z \in Q - Q_1; \\ \dfrac{3k}{2}, & \text{as } z \in Q_1. \end{cases};$$

$$\mu_2(z) := \begin{cases} k, & \text{as } z \in Q - Q_1; \\ 0, & \text{as } z \in Q_1. \end{cases},$$

where $0 < k < \dfrac{\sqrt{6}}{6}$.

It is easy to prove that ([17])

$$K_0[\mu_1] = H[\mu_1] = \frac{1+2k}{1-2k}$$

and

$$K_0[\mu_2] = H[\mu_2] = \frac{1+k}{1-k}.$$

Hence $\mu_1$ and $\mu_2$ are not Strebel points.

Let $f_1$ and $f_2$ be two quasiconformal mappings of $Q$ onto itself with $\mu_1(z)$ and $\mu_2(z)$ as their Beltrami coefficients respectively and keeping 0, $i$ and $+\infty$ fixed.

There exists a conformal mapping $\varphi$ from $\Delta$ onto $Q$ keeping 1, $-1$ and $i$ fixed. Let

$$\tilde{f}_j = \varphi^{-1} \circ f_j \circ \varphi \quad (j = 1, 2).$$

Then the complex dilatation $\tilde{\mu}$ of $\tilde{g} = \varphi^{-1} \circ \tilde{f}_1 \circ \tilde{f}_2^{-1} \circ \varphi$ is

$$\tilde{\mu}(\zeta) := \begin{cases} \dfrac{k}{1-2k^2} \dfrac{|\varphi'|^2}{(\varphi')^2}, & \text{as } z \in \varphi^{-1}(Q - Q_1); \\ \dfrac{3k}{2} \dfrac{|\varphi'|^2}{(\varphi')^2}, & \text{as } z \in \varphi^{-1}(Q_1). \end{cases},$$

where $\zeta = \varphi^{-1} \circ \tilde{f}_2 \circ \varphi(z)$. It is well-known that $k\dfrac{|\varphi'|^2}{(\varphi')^2}$ is uniquely extremal ([19]).

By Lemma, we obtain that $\tilde{g}^{-1}$ has a non-decreasable dilatation. If

$$K_0[\tilde{g}^{-1}] \leq \left(1 + \frac{k}{1-2k^2}\right) \Big/ \left(1 - \frac{k}{1-2k^2}\right),$$

then there exists $\nu_1 \in [\mu_{\tilde{g}^{-1}}]$ such that $\|\nu_1\|_\infty \leq \dfrac{k}{1-2k^2}$.

It is easy to know that when $0 < k < \frac{\sqrt{6}}{6}$, $\frac{k}{1-2k^2} < \frac{3k}{2}$. Combining with the fact $|\mu_{\tilde{g}^{-1}}| = \frac{3k}{2}$ for $z \in \varphi^{-1}(Q_1)$, we conclude that $|\nu_1| \le |\mu_{\tilde{g}^{-1}}|$ for any $z \in \Delta$. So $\tilde{g}^{-1}$ does not have a non-decreasable dilatation. A contradiction appears. Then we have

$$K_0[\tilde{g}^{-1}] > \left(1 + \frac{k}{1-2k^2}\right) \Big/ \left(1 - \frac{k}{1-2k^2}\right).$$

Since

$$K_0[\tilde{g}] = K_0[\tilde{g}^{-1}].$$

We get

$$K_0[\tilde{g}] > \left(1 + \frac{k}{1-2k^2}\right) \Big/ \left(1 - \frac{k}{1-2k^2}\right).$$

Moreover, we have ([19])

$$H[\tilde{g}] = \left(1 + \frac{k}{1-2k^2}\right) \Big/ \left(1 - \frac{k}{1-2k^2}\right).$$

We obtain that $[\tilde{g}]$ is a Strebel point. So $[f_1 \circ f_2^{-1}]$ is a Strebel point. We conclude that there is only one geodesic segment joining $[f_1]$ and $[f_2]$.

The proof of Proposition is completed. □

*Proof of Theorem* 2. Suppose $\tau = [f]$ and $g_K$ are given in Theorem 2. It is known that the set of all Strebel points in $\mathscr{T}(\Delta)$ is an open set (see [8]). So for any given Strebel point $[f]$, there is a $\delta = \delta([f]) > 0$ such that any point $[\tilde{f}] \neq [f]$ with $d_T([f], [\tilde{f}]) < \delta$ must be a Strebel point. It is clear that when $K$ is sufficiently closed to 1, $d_T([f], [g_K \circ f]) < \delta$ and hence $\tau_K = [g_K \circ f]$ is a Strebel point.

On the other hand, from the result of [3], we know that for any $K > 1$, $[g_K]$ is a Strebel point. So there is only one geodesic segment joining $\tau = [f]$ and $\tau_K = [g_K \circ f]$.

Therefore, when $K > 1$ is sufficiently closed to 1, for instance, $d_T(\tau, \tau_K) < \delta$, the three points $\tau$, $\tau_K$ and $[id]$ form a good triangle.

The proof of Theorem 2 is completed. □

*Remark* 2. We have the following question:

Question $\mathscr{C}$. For $[f]$ and $g_K$ as in Theorem 2, whether or not for all $K > 1$, $[f \circ g_K]$ is always a Strebel point?

We conjecture that the answer to this question is negative in general.

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