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KODAI MATH. J.
36 (2013), 386–395

ON MEROMORPHIC FUNCTIONS SHARING FOUR TWO-POINT SETS CM

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Abstract

We show that if two meromorphic functions sharing four two-point sets CM, then one of them is a Möbius transform of the other.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\bar{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [4] and [5], R. Nevanlinna showed the following theorems:

Theorem A. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and let $a_1, \ldots, a_4$ be four distinct points in $\bar{\boldsymbol{C}}$. If f and g share $a_1, \ldots, a_4$ CM, then f is a Möbius transform of g, i.e., $f = (ag + b)/(cg + d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$. Moreover, there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}$ and $a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

Theorem B. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing distinct five points in $\bar{\boldsymbol{C}}$ IM, then $f = g$.*

In this paper we treat some uniqueness theorems, but we do not require the conclusion that two meromorphic functions considered are identical. The conclusion required is that one of two meromorphic functions is a Möbius transform of the other. In [7], the author generalized Theorem B as following:

2010 *Mathematics Subject Classification*. Primary 30D35; Secondary 30D30.

Key words and phrases. Uniqueness theorem, Sharing sets, Nevanlinna theory.
Received September 4, 2012; revised January 31, 2013.

Theorem C. *Let $S_1, \dots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\boldsymbol{C}}$. If two nonconstant meromorphic functions f and g on $\overline{\boldsymbol{C}}$ share $S_1, \dots, S_5$ IM, then f is a Möbius transform of g.*

It is not so difficult to show that Theorem C contains Theorem B by using the little Picard theorem.

The first half of Theorem A can be generalized as following, which is a constant target version of Theorem 1 of [3]:

Theorem D. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$. Let $\xi_1, \dots, \xi_4$ be four distinct points in $\overline{\boldsymbol{C}}$ and let $\eta_1, \dots, \eta_4$ be four distinct points in $\overline{\boldsymbol{C}}$. If $f - \xi_j$ and $g - \eta_j$ share zero CM $(j = 1, \dots, 4)$, then f is a Möbius transform of g.*

On the other hand, Tohge considered two meromorphic functions sharing 1, -1, ∞ and a two-point set containing none of them, and Theorem 4 in [10] induces the following

Theorem E. *Let S_1, S_2, S_3 be one-point sets in $\overline{\boldsymbol{C}}$ and let S_4 be a two-point set in $\overline{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g.*

Also, Theorem 1.2 in [9] and its proof induce

Theorem F. *Let S_1, S_2 be one-point sets in $\overline{\boldsymbol{C}}$ and let S_3, S_4 be two two-point sets in $\overline{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g.*

Moreover, the author prove in [8]

Theorem G. *Let S_1 be one-point set in $\overline{\boldsymbol{C}}$ and let S_2, S_3, S_4 be three two-point sets in $\overline{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transform of g.*

In this paper we consider two meromorphic functions on $\boldsymbol{C}$ sharing four two-point sets in $\overline{\boldsymbol{C}}$ CM.

Theorem 1.1. *Let S_1, S_2, S_3, S_4 be four two-point sets in $\overline{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 and S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share $S_1, \dots, S_4$ CM, then f is a Möbius transform of g.*

By arranging these theorems we will get

THEOREM. *Let S_1, S_2, S_3, S_4 be four one-point or two-point sets in $\bar{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 and S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share $S_1, \dots, S_4$ CM, then f is a Möbius transform of g.*

The aim of this paper is to prove Theorem 1.1.

2. Representations of rank N and some lemmas

In this section we introduce the definition of representations of rank N. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, \dots, a_q)$ of elements a_i in G.

DEFINITION 2.1. Let N be a positive integer. We call integers μ_j representations of rank N of a_j if

$$\prod_{j=1}^{q} a_j^{\varepsilon_j} = \prod_{j=1}^{q} a_j^{\varepsilon_j'} \tag{2.1}$$

and

$$\sum_{j=1}^{q} \varepsilon_j \mu_j = \sum_{j=1}^{q} \varepsilon_j' \mu_j \tag{2.2}$$

are equivalent for any integers ε_j, ε_j' with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \le N$.

For the existence of representations of rank N, see [6].

For two entire function α and β without zeros we say that they are equivalent if α/β is constant. Then we denote $\alpha \sim \beta$. This relation "equivalent" is an equivalence relation.

We introduce following Borel's Lemma, whose proof can be found, for example, on p. 186 of [2].

LEMMA 2.2. *If entire functions $\alpha_0, \alpha_1, \dots, \alpha_n$ without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \dots, n$ there exists some $k (\ne j)$ such that $\alpha_j \sim \alpha_k$, and the sum of all elements of each equivalence class in $\{\alpha_0, \dots, \alpha_n\}$ is zero.

Now we investigate the torsion-free abelian multiplicative group $G = \mathscr{E}/\mathscr{C}$, where $\mathscr{E}$ is the abelian group of entire functions without zeros and $\mathscr{C}$ is the subgroup of all non-zero constant functions. We represent by $[\alpha]$ the element of $\mathscr{E}/\mathscr{C}$ with the representative $\alpha \in \mathscr{E}$. Let $\alpha_1, \dots, \alpha_q$ be elements in $\mathscr{E}$. Take representations μ_j of rank N of $[\alpha_j]$. For $\alpha = \prod_{j=1}^q \alpha_j^{\varepsilon_j}$ we define its index $\mathrm{Ind}(\alpha)$ by $\sum_{j=1}^q \varepsilon_j \mu_j$. The indices depend only on $[\prod_{j=1}^q \alpha_j^{\varepsilon_j}]$ under the condition $\sum_{j=1}^q |\varepsilon_j| \le N$. Trivially $\mathrm{Ind}(1) = 0$, and hence $\mathrm{Ind}(\alpha) = 0$ and the constantness of α are equivalent, and $\mathrm{Ind}(\alpha) = \mathrm{Ind}(\alpha')$ is equivalent to that α/α' is constant, where $\alpha = \prod_{j=1}^q \alpha_j^{\varepsilon_j}$ and $\alpha' = \prod_{j=1}^q \alpha_j^{\varepsilon_j'}$ with $\sum_{j=1}^q |\varepsilon_j| \le N$ and $\sum_{j=1}^q |\varepsilon_j'| \le N$.

We use the following Lemma in the proof of Theorem 1.1 which is an application of Lemma 2.2 (for the proof see Lemma 2.3 of [9]).

LEMMA 2.3. *Assume that there is a relation* $\Psi(\alpha_1, \dots, \alpha_q) \equiv 0$ *where* $\Psi(X_1, \dots, X_q) \in \boldsymbol{C}[X_1, \dots, X_q]$ *is a nonconstant polynomial of degree at most* N *of* $X_1, \dots, X_q$. *Then each term* $aX_1^{\varepsilon_1} \cdots X_q^{\varepsilon_q}$ *of* $\Psi(X_1, \dots, X_q)$ *has another term* $bX_1^{\varepsilon_1'} \cdots X_q^{\varepsilon_q'}$ *such that* $\alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}$ *and* $\alpha_1^{\varepsilon_1'} \cdots \alpha_q^{\varepsilon_q'}$ *have the same indices, where* a *and* b *are non-zero constants.*

3. A lemma from the theory of general resultants

For the proof of Theorem 1.1 a result from the theory of general resultants is represented in this section. We give it by proceeding as in Chapter 3 of [1].

Let d be a positive integer and let $F_1, \dots, F_4$ be four homogeneous polynomials of degree d of four variables X, Y, Z, W with the form $F_j(X, Y, Z, W) = P_j(X, Y) + Q_j(Z, W)$, where $P_j(X, Y)$ are homogeneous polynomials of degree d of X, Y and $Q_j(Z, W)$ are homogeneous polynomials of degree d of Z and W. Denote their Jacobian determinant by J:

$$J = \left| \frac{\partial F_j}{\partial X} \quad \frac{\partial F_j}{\partial Y} \quad \frac{\partial F_j}{\partial Z} \quad \frac{\partial F_j}{\partial W} \right|_{1 \le j \le 4}.$$

LEMMA 3.1. *Let* P *be a non-trivial common zero of* F_1, F_2, F_3, F_4. *Then* (i) J *is zero at* P; (ii) *all the partial derivatives* $\dfrac{\partial J}{\partial X}, \dfrac{\partial J}{\partial Y}, \dfrac{\partial J}{\partial Z}, \dfrac{\partial J}{\partial W}$ *are zero at* P; (iii) *the second partial derivatives* $\dfrac{\partial^2 J}{\partial X \partial Z}, \dfrac{\partial^2 J}{\partial X \partial W}, \dfrac{\partial^2 J}{\partial Y \partial Z}, \dfrac{\partial^2 J}{\partial Y \partial W}$ *have zero at* P.

Proof. By Euler's relation we have

$$(3.1) \qquad XJ = \left| X\frac{\partial F_j}{\partial X} \quad \frac{\partial F_j}{\partial Y} \quad \frac{\partial F_j}{\partial Z} \quad \frac{\partial F_j}{\partial W} \right|_{1 \le j \le 4} = d \left| F_j \quad \frac{\partial F_j}{\partial Y} \quad \frac{\partial F_j}{\partial Z} \quad \frac{\partial F_j}{\partial W} \right|_{1 \le j \le 4}$$

and, by the same way,

$$YJ = d\begin{vmatrix} \dfrac{\partial F_j}{\partial X} & F_j & \dfrac{\partial F_j}{\partial Z} & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4}, \tag{3.2}$$

$$ZJ = d\begin{vmatrix} \dfrac{\partial F_j}{\partial X} & \dfrac{\partial F_j}{\partial Y} & F_j & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4}, \tag{3.3}$$

$$WJ = d\begin{vmatrix} \dfrac{\partial F_j}{\partial X} & \dfrac{\partial F_j}{\partial Y} & \dfrac{\partial F_j}{\partial Z} & F_j \end{vmatrix}_{1\le j\le 4} \tag{3.4}$$

are obtained. Since $F_j(P) = 0$ for $j = 1,2,3,4$, all XJ, YJ, ZJ, WJ have zero at P by (3.1), (3.2), (3.3) and (3.4). Put $P = (X_0, Y_0, Z_0, W_0)$, then $J(P) = 0$ because at least one of X_0, Y_0, Z_0, W_0 is not zero. We have showed (i).

By differentiating (3.1), (3.2), (3.3) and (3.4) by X and noting $\dfrac{\partial^2 F_j}{\partial X \partial Z} = \dfrac{\partial^2 F_j}{\partial X \partial W} = \dfrac{\partial^2 F_j}{\partial Y \partial Z} = \dfrac{\partial^2 F_j}{\partial Y \partial W} = 0$, we get

$$J + X\frac{\partial J}{\partial X} = dJ + d\begin{vmatrix} F_j & \dfrac{\partial^2 F_j}{\partial X\partial Y} & \dfrac{\partial F_j}{\partial Z} & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4}, \tag{3.5}$$

$$Y\frac{\partial J}{\partial X} = d\begin{vmatrix} \dfrac{\partial^2 F_j}{\partial X^2} & F_j & \dfrac{\partial F_j}{\partial Z} & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4}, \tag{3.6}$$

$$Z\frac{\partial J}{\partial X} = d\begin{vmatrix} \dfrac{\partial^2 F_j}{\partial X^2} & \dfrac{\partial F_j}{\partial Y} & F_j & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4} + d\begin{vmatrix} \dfrac{\partial F_j}{\partial X} & \dfrac{\partial^2 F_j}{\partial X\partial Y} & F_j & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4},$$

$$W\frac{\partial J}{\partial X} = d\begin{vmatrix} \dfrac{\partial^2 F_j}{\partial X^2} & \dfrac{\partial F_j}{\partial Y} & \dfrac{\partial F_j}{\partial Z} & F_j \end{vmatrix}_{1\le j\le 4} + d\begin{vmatrix} \dfrac{\partial F_j}{\partial X} & \dfrac{\partial^2 F_j}{\partial X\partial Y} & \dfrac{\partial F_j}{\partial Z} & F_j \end{vmatrix}_{1\le j\le 4}. \tag{3.7}$$

Hence $X\dfrac{\partial J}{\partial X}$, $Y\dfrac{\partial J}{\partial X}$, $Z\dfrac{\partial J}{\partial X}$, $W\dfrac{\partial J}{\partial X}$ are all zero at P, and hence $\dfrac{\partial J}{\partial X}(P) = 0$ since some of X_0, Y_0, Z_0, W_0 are not zero. Similarly, we have $\dfrac{\partial J}{\partial Y}(P) = \dfrac{\partial J}{\partial Z}(P) = \dfrac{\partial J}{\partial W}(P) = 0$, which is (ii).

Now differentiate (3.5), (3.6) and (3.7) by Z, then we have

$$\frac{\partial J}{\partial Z} + X\frac{\partial^2 J}{\partial X\partial Z} = d\frac{\partial J}{\partial Z} + d\begin{vmatrix} F_j & \dfrac{\partial^2 F_j}{\partial X\partial Y} & \dfrac{\partial^2 F_j}{\partial Z^2} & \dfrac{\partial F_j}{\partial W} \end{vmatrix}_{1\le j\le 4} + d\begin{vmatrix} F_j & \dfrac{\partial^2 F_j}{\partial X\partial Y} & \dfrac{\partial F_j}{\partial Z} & \dfrac{\partial^2 F_j}{\partial Z\partial W} \end{vmatrix}_{1\le j\le 4}, \tag{3.8}$$

$$Y\frac{\partial^2 J}{\partial X\partial Z} = d\left|\frac{\partial^2 F_j}{\partial X^2} \quad F_j \quad \frac{\partial^2 F_j}{\partial Z^2} \quad \frac{\partial F_j}{\partial W}\right|_{1\le j\le 4} + d\left|\frac{\partial^2 F_j}{\partial X^2} \quad F_j \quad \frac{\partial F_j}{\partial Z} \quad \frac{\partial^2 F_j}{\partial Z\partial W}\right|_{1\le j\le 4},$$

$$W\frac{\partial^2 J}{\partial X\partial Z} = d\left|\frac{\partial^2 F_j}{\partial X^2} \quad \frac{\partial F_j}{\partial Y} \quad \frac{\partial^2 F_j}{\partial Z^2} \quad F_j\right|_{1\le j\le 4} + d\left|\frac{\partial F_j}{\partial X} \quad \frac{\partial^2 F_j}{\partial X\partial Y} \quad \frac{\partial^2 F_j}{\partial Z^2} \quad F_j\right|_{1\le j\le 4}$$

and hence $X\frac{\partial^2 J}{\partial X\partial Z} = Y\frac{\partial^2 J}{\partial X\partial Z} = W\frac{\partial^2 J}{\partial X\partial Z} = 0$ at P. By using the alternate equation

$$\frac{\partial J}{\partial X} + Z\frac{\partial^2 J}{\partial X\partial Z} = d\frac{\partial J}{\partial X} + d\left|\frac{\partial^2 F_j}{\partial X^2} \quad \frac{\partial F_j}{\partial Y} \quad F_j \quad \frac{\partial^2 F_j}{\partial Z\partial W}\right|_{1\le j\le 4} + d\left|\frac{\partial F_j}{\partial X} \quad \frac{\partial^2 F_j}{\partial X\partial Y} \quad F_j \quad \frac{\partial^2 F_j}{\partial Z\partial W}\right|_{1\le j\le 4}$$

of (3.8), we have $Z\frac{\partial^2 J}{\partial X\partial Z} = 0$ at P. Since some of X_0, Y_0, Z_0, W_0 are not zero, we see that $\frac{\partial^2 J}{\partial X\partial Z}$ have a zero at P. By the same way we obtain that $\frac{\partial^2 J}{\partial X\partial W}(P) = \frac{\partial^2 J}{\partial Y\partial Z}(P) = \frac{\partial^2 J}{\partial Y\partial W}(P) = 0$, as desired. □

Let

$$F_j(X, Y, Z, W) = a_{j1}X^2 + a_{j2}XY + a_{j3}Y^2 + a_{j4}Z^2 + a_{j5}ZW + a_{j6}W^2 \quad (j = 1, 2, 3, 4)$$

be four quadratic homogeneous polynomials. Then we have

$$\begin{aligned} J &= |2a_{j1}X + a_{j2}Y \quad a_{j2}X + 2a_{j3}Y \quad 2a_{j4}Z + a_{j5}W \quad a_{j5}Z + 2a_{j6}W|_{1\le j\le 4} \\ &= 4D_1X^2Z^2 + 8D_2X^2ZW + 4D_3X^2W^2 + 8D_4XYZ^2 + 16D_5XYZW \\ &\quad + 8D_6XYW^2 + 4D_7Y^2Z^2 + 8D_8Y^2ZW + 4D_9Y^2W^2, \end{aligned}$$

where

$$D_1 = |a_{j1} \quad a_{j2} \quad a_{j4} \quad a_{j5}|_{1 \le j \le 4}, \quad D_2 = |a_{j1} \quad a_{j2} \quad a_{j4} \quad a_{j6}|_{1 \le j \le 4},$$
$$D_3 = |a_{j1} \quad a_{j2} \quad a_{j5} \quad a_{j6}|_{1 \le j \le 4}, \quad D_4 = |a_{j1} \quad a_{j3} \quad a_{j4} \quad a_{j5}|_{1 \le j \le 4},$$
$$D_5 = |a_{j1} \quad a_{j3} \quad a_{j4} \quad a_{j6}|_{1 \le j \le 4}, \quad D_6 = |a_{j1} \quad a_{j3} \quad a_{j5} \quad a_{j6}|_{1 \le j \le 4},$$
$$D_7 = |a_{j2} \quad a_{j3} \quad a_{j4} \quad a_{j5}|_{1 \le j \le 4}, \quad D_8 = |a_{j2} \quad a_{j3} \quad a_{j4} \quad a_{j6}|_{1 \le j \le 4},$$
$$D_9 = |a_{j2} \quad a_{j3} \quad a_{j5} \quad a_{j6}|_{1 \le j \le 4},$$

and we get

$$\frac{\partial^2 J}{\partial X \partial Z} = 16(D_1 XZ + D_2 XW + D_4 YZ + D_5 YW),$$
$$\frac{\partial^2 J}{\partial X \partial W} = 16(D_2 XZ + D_3 XW + D_5 YZ + D_6 YW),$$
$$\frac{\partial^2 J}{\partial Y \partial Z} = 16(D_4 XZ + D_5 XW + D_7 YZ + D_8 YW),$$
$$\frac{\partial^2 J}{\partial Y \partial W} = 16(D_5 XZ + D_6 XW + D_8 YZ + D_9 YW).$$

Therefore, if there exists a common zero $P = (X_0, Y_0, Z_0, W_0)$ of F_1, F_2, F_3, F_4 such that some of $X_0 Z_0$, $X_0 W_0$, $Y_0 Z_0$, $Y_0 W_0$ are not zero, then we see, by considering Lemma 3.1, that

$$\Delta := \begin{vmatrix} D_1 & D_2 & D_4 & D_5 \\ D_2 & D_3 & D_5 & D_6 \\ D_4 & D_5 & D_7 & D_8 \\ D_5 & D_6 & D_8 & D_9 \end{vmatrix} = 0. \tag{3.9}$$

4. The key theorem and the proof of Theorem 1.1

THEOREM 4.1. *Let $f = f_1/f_0$ and $g = g_1/g_0$ be nonconstant meromorphic functions on $\boldsymbol{C}$, where f_0 and f_1 are entire functions without common zero and so are g_0 and g_1. Let $P_j(z) = z^2 + a_j z + b_j$ $(j = 1, 2, 3, 4)$ be polynomials such that $P_j(z)$ and $P_k(z)$ have no common zero for distinct j, k and let $Q_j(z) = z^2 + p_j z + q_j$ $(j = 1, 2, 3, 4)$ be polynomials such that $Q_j(z)$ and $Q_k(z)$ have no common zero for distinct j, k. Assume that there exist entire functions α_j without zeros such that*

$$f_1^2 + a_j f_1 f_0 + b_j f_0^2 = \alpha_j (g_1^2 + p_j g_1 g_0 + q_j g_0^2) \quad (j = 1, 2, 3, 4).$$

Then there exist distinct j_1 and j_2 such that $\alpha_{j_1}/\alpha_{j_2}$ is constant, and hence

$$\frac{f^2 + a_{j_1} f + b_{j_1}}{f^2 + a_{j_2} f + b_{j_2}} = c \frac{g^2 + a_{j_1} g + b_{j_1}}{g^2 + a_{j_2} g + b_{j_2}},$$

where c is a non-zero constant.

Proof. Take $z \in \boldsymbol{C}$ which is not zero of any of f_1, f_0, g_1, g_1. Then $(f_1(z), f_0(z), g_1(z), g_0(z))$ is a common zero of

$$X^2 + a_j XY + b_j Y^2 + A_j(Z^2 + p_j ZW + q_j W^2) \quad (j = 1, 2, 3, 4),$$

where $A_j = -\alpha_j(z)$. By (3.9), we have $\Delta = 0$ for any $z \in \boldsymbol{C}$ since the zero sets of f_1, f_0, g_1, g_0 are discrete. Since each D_j is a quadratic homogeneous polynomial of α_1, α_2, α_3, α_4 which consists of terms $\alpha_k \alpha_l$ $(k \neq l)$, Δ is a homogeneous polynomial of degree eight whose terms are $\prod_{m=1}^{4} \alpha_{j_m} \alpha_{k_m}$, where $j_m \neq k_m$, $m = 1, 2, 3, 4$, with complex coefficients. Now take representations μ_1, μ_2, μ_3, μ_4 of $[\alpha_1]$, $[\alpha_2]$, $[\alpha_3]$, $[\alpha_4]$ of rank 8. Suppose that any α_j/α_k is not constant for $j \neq k$. Then we have that $\mu_j \neq \mu_k$ for $j \neq k$ and we may assume that $\mu_1 > \mu_2 > \mu_3 > \mu_4$. In terms of the expansion of Δ, only the term $(\alpha_1\alpha_2)^4$ has the maximal index. By Lemma 2.3, its coefficient must be zero, and so we calculate the coefficient. Now, we have

$$D_1 = |1 \quad a_j \quad \alpha_j \quad p_j\alpha_j|_{1 \le j \le 4}, \qquad D_2 = |1 \quad a_j \quad \alpha_j \quad q_j\alpha_j|_{1 \le j \le 4},$$

$$D_3 = |1 \quad a_j \quad p_j\alpha_j \quad q_j\alpha_j|_{1 \le j \le 4}, \qquad D_4 = |1 \quad b_j \quad \alpha_j \quad p_j\alpha_j|_{1 \le j \le 4},$$

$$D_5 = |1 \quad b_j \quad \alpha_j \quad q_j\alpha_j|_{1 \le j \le 4}, \qquad D_6 = |1 \quad b_j \quad p_j\alpha_j \quad q_j\alpha_j|_{1 \le j \le 4},$$

$$D_7 = |a_j \quad b_j \quad \alpha_j \quad p_j\alpha_j|_{1 \le j \le 4}, \qquad D_8 = |a_j \quad b_j \quad \alpha_j \quad q_j\alpha_j|_{1 \le j \le 4},$$

$$D_9 = |a_j \quad b_j \quad p_j\alpha_j \quad q_j\alpha_j|_{1 \le j \le 4}.$$

Put $a_{jk} = a_j - a_k$, $b_{jk} = b_j - b_k$, $c_{jk} = a_k b_j - a_j b_k$, $p_{jk} = p_j - p_k$, $q_{jk} = q_j - q_k$, $r_{jk} = p_k q_j - p_j q_k$, then in their expansions $\alpha_1\alpha_2$ has the coefficients $a_{34}p_{21}$, $a_{43}q_{21}$, $a_{43}r_{21}$, $b_{43}p_{21}$, $b_{43}q_{21}$, $b_{43}r_{21}$, $c_{43}p_{21}$, $c_{43}q_{21}$, $c_{43}r_{21}$, respectively. Hence in the expansion of Δ the term $(\alpha_1\alpha_2)^4$ has the coefficient

$$\begin{vmatrix} a_{43}p_{21} & a_{43}q_{21} & b_{43}p_{21} & b_{43}q_{21} \\ a_{43}q_{21} & a_{43}r_{21} & b_{43}q_{21} & b_{43}r_{21} \\ b_{43}p_{21} & b_{43}q_{21} & c_{43}p_{21} & c_{43}q_{21} \\ b_{43}q_{21} & b_{43}r_{21} & c_{43}q_{21} & c_{43}r_{21} \end{vmatrix} = \begin{vmatrix} p_{21} & q_{21} & 0 & 0 \\ q_{21} & r_{21} & 0 & 0 \\ 0 & 0 & p_{21} & q_{21} \\ 0 & 0 & q_{21} & r_{21} \end{vmatrix} \cdot \begin{vmatrix} a_{43} & 0 & b_{43} & 0 \\ 0 & a_{43} & 0 & b_{43} \\ b_{43} & 0 & c_{43} & 0 \\ 0 & b_{43} & 0 & c_{43} \end{vmatrix}$$

$$= (p_{21}r_{21} - q_{21}^2)^2 \begin{vmatrix} a_{43} & b_{43} & 0 & 0 \\ b_{43} & c_{43} & 0 & 0 \\ 0 & 0 & a_{43} & b_{43} \\ 0 & 0 & b_{43} & c_{43} \end{vmatrix}$$

$$= (p_{21}r_{21} - q_{21}^2)^2 (a_{43}c_{43} - b_{43}^2)^2$$

$$= \{R(Q_1, Q_2) R(P_3, P_4)\}^2,$$

where $R(P, Q)$ denotes the resultant of two polynomials P and Q. By assumption $R(P_3, P_4) \neq 0$, $R(Q_1, Q_2) \neq 0$, which is a contradiction. Therefore we conclude that some α_j/α_k $(j \neq k)$ are constant. $\square$

Note that, in Theorem 4.1, we did not assume that P_j, Q_j have no double zeros. We mention that it is not so difficult to prove Theorem D, Theorem E and Theorem F by Theorem 4.1.

Proof of Theorem 1.1. We set

$$S_j = \{\xi_j, \eta_j\} = \{z; z^2 + a_j z + b_j = 0\} \quad (j = 1, 2, 3, 4)$$

and we can take $P_j(z) = Q_j(z) = z^2 + a_j z + b_j$ in Theorem 4.1 with some entire functions α_j without zeros such that

$$f_1^2 + a_j f_1 f_0 + b_j f_0^2 = \alpha_j (g_1^2 + a_j g_1 g_0 + b_j g_0^2) \quad (j = 1, 2, 3, 4).$$

Then we may assume, by Theorem 4.1, that α_1/α_2 is constant. Put $c = \alpha_1/\alpha_2$. Then we have

$$\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = c \frac{g^2 + a_1 g + b_1}{g^2 + a_2 g + b_2}.$$

If there exists a point $z \in \boldsymbol{C}$ such that $f(z) = g(z) \notin S_1 \cup S_2$, then we get $c = 1$ and hence we see that f is a Möbius transform of g. Otherwise, by assumption we have $f^{-1}(\xi_j) = g^{-1}(\eta_j)$, $f^{-1}(\eta_j) = g^{-1}(\xi_j)$ $(j = 3, 4)$, and by Theorem D we conclude that f is a Möbius transform of g.

Acknowledgements. The author thanks the referee for his or her valuable comments.

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