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THE IDEALS OF THE HOMOLOGICAL GOLDMAN LIE ALGEBRA

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Abstract

We determine all the ideals of the homological Goldman Lie algebra, which reflects the structure of an oriented surface.

1. Introduction

By a *surface*, we mean an oriented two-dimensional smooth manifold possibly with boundary. It is well known that the first homology group and the intersection form of a surface reflect the topological structure of the surface. For example, they have information about the genus and the boundary components of the surface.

To study them in detail, we consider a Lie algebra coming from them. We call it the homological Goldman Lie algebra. Goldman introduced the Lie algebra for study of the moduli space of $GL_1(\mathbf{R})$-flat bundles over the surface [G] p. 295–p. 297. We define the Lie algebra in more general setting. Let H be a $\mathbf{Z}$-module, i.e., an abelian group, which is not necessarily finitely generated, and $\langle -, - \rangle : H \times H \to \mathbf{Z}$, $(x, y) \mapsto \langle x, y \rangle$, an alternating $\mathbf{Z}$-bilinear form. For example, we consider that H is the first homology group, and $\langle -, - \rangle$ is the intersection form of a surface. We define a $\mathbf{Z}$-linear map $\mu : H \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ by $\mu(x)(y) = \langle x, y \rangle$. Denote by $\mathbf{Q}[H]$ the $\mathbf{Q}$-vector space with basis the set H;

$$\mathbf{Q}[H] := \left\{ \sum_{i=1}^{n} c_i [x_i] \mid n \in \mathbf{N}, c_i \in \mathbf{Q}, x_i \in H \right\},$$

where $[-] : H \to \mathbf{Q}[H]$ is the embedding as basis. Here, we remark that $c[x] \neq [cx]$ for any $c \neq 1$ and $x \in H$. We define a $\mathbf{Q}$-bilinear map $[-, -] : \mathbf{Q}[H] \times \mathbf{Q}[H] \to \mathbf{Q}[H]$ by $[[x], [y]] := \langle x, y \rangle [x + y]$ for $x, y \in H$. It is easy to see that this bilinear map is skew and satisfies the Jacobi identity [G] p. 295–p. 297. The Lie algebra $(\mathbf{Q}[H], [-, -])$ is called the *homological Goldman Lie algebra of* $(H, \langle -, - \rangle)$. The Lie algebra $\mathbf{Q}[H]$ is equipped with a product $\mathbf{Q}[H] \times \mathbf{Q}[H] \to \mathbf{Q}[H]$, $([x], [y]) \mapsto [x][y] = [x + y]$ as a group ring. Then $\mathbf{Q}[H]$ is a Poisson algebra.

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The homological Goldman Lie algebra is infinite dimensional and we can define this Lie algebra only from algebraic information. So, it is interesting in an algebraic context. The homological Goldman Lie algebra comes from the first homology group and its intersection form. So, it is also interesting in a geometric context. For example, the homological Goldman Lie algebra is exactly the subalgebra of all Fourier polynomials in the Poisson algebra on the symplectic torus if the surface is closed [G] p. 295–p. 297. Namely, the correspondence $[\sum_{i=1}^{g}(a_iA_i + b_iB_i)] \mapsto \exp(\sum_{i=1}^{g}(a_ip_i + b_iq_i))$ is an injective homomorphism of Poisson algebras, where g is the genus of the surface, $\{A_i, B_i\}_{i=1}^{g}$ a symplectic basis of the first homology group of the surface, and $\{p_i, q_i\}_{i=1}^{g}$ symplectic coordinates of $\mathbf{R}^{2g}$. Moreover, we can consider a more complicated Lie algebra coming from free loops and the intersection form. Take two free loops α and β on the surface in general position. We define $[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)\alpha \cdot_p \beta$, where $\varepsilon(p; \alpha, \beta)$ is the local intersection number of α and β at p, and $\alpha \cdot_p \beta$ is the free homotopy class of the product in the fundamental group with base point p. The bracket induces a well-defined operator in the free module with basis the set of homotopy classes of free loops, and it is easy to show that the bracket is skew and satisfies the Jacobi identity. We call this Lie algebra the Goldman Lie algebra. We have a surjective Lie algebra homomorphism from the Goldman Lie algebra onto the homological Goldman Lie algebra [G] p. 295–p. 297.

The purpose of this paper is to study the algebraic structure of the homological Goldman Lie algebra. More precisely, we determine the ideals of the homological Goldman Lie algebra. Here an ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ with $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, namely, $[X, Y] \in \mathfrak{h}$ for $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. In particular, we will show the following. If the surface is closed, the number of all the ideals of the homological Goldman Lie algebra is finite (see Corollary 4.4). In the forthcoming papers, we determine the minimal number of generators of the Lie algebra [K], and compute its second cohomology [T].

To state our result, let us prepare some notations. For $x \in H$, we denote $\bar{x} := x + \ker \mu$. The Lie algebra $\mathbf{Q}[H]$ is a graded Lie algebra of type $\overline{H} := H/\ker \mu$ [B] Chapter II §11. Namely, we have $\mathbf{Q}[H] = \bigoplus_{\bar{x} \in \overline{H}} \mathbf{Q}[\bar{x}]$ and $[\mathbf{Q}[\bar{x}], \mathbf{Q}[\bar{y}]] \subset \mathbf{Q}[\overline{x + y}]$. Here $\mathbf{Q}[\bar{x}] = \{\sum_{i=1}^{n} c_i[x_i] \in \mathbf{Q}[H] \mid n \in \mathbf{N}, c_i \in \mathbf{Q}, x_i \in \bar{x}\}$. The ker μ-*degree* is the degree induced by grading of type $\overline{H}$. For $X \in \mathbf{Q}[\bar{x}]$, we denote $\deg X = \bar{x}$.

For $x \in H$, we define $T(x) : \mathbf{Q}[H] \to \mathbf{Q}[H]$ by $T(x)([y]) = [x + y]$, where $y \in H$. The map $T : H \to GL(\mathbf{Q}[H])$ induces an action of H on $\mathbf{Q}[H]$. For $x, y \in H$ and $Y \in \mathbf{Q}[\bar{y}]$, we have $[[x], Y] = \langle x, y \rangle T(x)(Y) = \langle x, y \rangle [x]Y$.

Let A be a subgroup of H, and V a subspace of $\mathbf{Q}[H]$. We say that V is *A-stable* if $T(A)(V) \subset V$.

Our main theorem is the following.

THEOREM 1.1 (Classification of the ideals of the homological Goldman Lie algebra). *For any ideal* $\mathfrak{h}$ *of* $\mathbf{Q}[H]$, *there exists a unique pair* (V_0, V) *such that*

(1) *V_0 and V are subspaces of* $\mathbf{Q}[\bar{0}] = \mathbf{Q}[\ker \mu]$,
(2) *V is* $\ker \mu$*-stable, and*
(3) $\mathfrak{h} = V_0 \oplus \bigoplus_{\bar{x} \in \overline{H}\setminus\{\bar{0}\}} T(\bar{x})(V)$,
where $T(\bar{x})(V) := T(x)(V)$, *which is well-defined by* (2). *If* $\mu = 0$, *we define* $V = 0$.

Conversely, if a pair (V_0, V) *satisfies* (1) *and* (2), *then the subspace* $\mathfrak{h}$ *of* $\mathbf{Q}[H]$ *defined by* (3) *is an ideal of* $\mathbf{Q}[H]$.

This means that any ideal of $\mathbf{Q}[H]$ is a graded Lie algebra of type $\overline{H}$.

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2. Preparations for our main theorem

LEMMA 2.1 (Key lemma). *If* $x_1, \ldots, x_n \in H\setminus\ker \mu$, *there exists* $z \in H$ *that satisfies* $\langle x_1, z\rangle \neq 0, \ldots,$ *and* $\langle x_n, z\rangle \neq 0$.

Proof. We prove this by induction on n. It is clear in the case $n = 1$. Consider the case $n > 1$. By the inductive assumption, we can take $u \in H$ satisfying $\langle x_i, u\rangle \neq 0$ for $i = 1, \ldots, n-1$. If $\langle x_n, u\rangle \neq 0$, the element u is a desired one. Suppose $\langle x_n, u\rangle = 0$. We can choose $v \in H$ such that $\langle x_n, v\rangle \neq 0$, since $x_n \notin \ker \mu$. We shall prove that

$$z := u + (1 + |\langle x_1, u\rangle| + \cdots + |\langle x_{n-1}, u\rangle|)v$$

is a desired one. We have

$$\langle x_n, z\rangle = (1 + |\langle x_1, u\rangle| + \cdots + |\langle x_{n-1}, u\rangle|)\langle x_n, v\rangle \neq 0.$$

For $k < n$, $\langle x_k, z\rangle = \langle x_k, u\rangle \neq 0$ if $\langle x_k, v\rangle = 0$.
If $\langle x_k, v\rangle \neq 0$, we also have $\langle x_k, z\rangle \neq 0$, because

$$|\langle x_k, z\rangle| \geq (1 + |\langle x_1, u\rangle| + \cdots + |\langle x_{n-1}, u\rangle|)|\langle x_k, v\rangle| - |\langle x_k, u\rangle| > 0. \qquad \square$$

Let $\mathfrak{h}$ be an ideal of $\mathbf{Q}[H]$. Then we have

PROPOSITION 2.2 (Decomposition of an ideal with respect to $\ker \mu$-degree).

$$\mathfrak{h} = \bigoplus_{\bar{x} \in \overline{H}} (\mathfrak{h} \cap \mathbf{Q}[\bar{x}])$$

Proof. It is clear that the sum $\Sigma(\mathfrak{h} \cap \mathbf{Q}[\bar{x}])$ is a direct sum and $\mathfrak{h}$ includes $\bigoplus_{\bar{x} \in \overline{H}}(\mathfrak{h} \cap \mathbf{Q}[\bar{x}])$. Let $X \in \mathfrak{h}\setminus\{0\}$. Since $X \in \mathfrak{h} \subset \mathbf{Q}[H] = \bigoplus_{\bar{x} \in \overline{H}} \mathbf{Q}[\bar{x}]$, there exist $n \geq 1$, $x_i \in H$ and $X_i \in (\mathbf{Q}[\overline{x_i}])\setminus\{0\}$ for $i = 1, \ldots, n$ such that $\overline{x_i} \neq \overline{x_j}$ if $i \neq j$ and $X = X_1 + \cdots + X_n$. It suffices to show $X_i \in \mathfrak{h}$ for all $i = 1, \ldots, n$.

STEP 1. *If* $\overline{x_i} \neq 0$ *for all* $i = 1, \ldots, n$, *then* $X_k \in \mathfrak{h}$ *for all* $k \in \{1, \ldots, n\}$.

We show this case by induction on n, the number of the non-zero components.

CLAIM. *Suppose* $n > 1$. *For any* $k \in \{1, \dots, n\}$, *there exist* $c_1, \dots, c_n \in \mathbf{Q}$ *such that* (1) $c_k \neq 0$, (2) *there exists some* $j \in \{1, \dots, n\}$ *with* $c_j = 0$, *and* (3) $c_1 X_1 + \cdots + c_n X_n \in \mathfrak{h}$.

Proof of Claim. We may assume $k = 1$. First we consider the case $\langle x_2, x_1 \rangle \neq 0$. Set

$$Y := ad([-x_2])ad([x_2])(X) = \sum_{i=1}^{n} \langle x_2, x_i \rangle \langle -x_2, x_i + x_2 \rangle X_i.$$

Since $\langle x_2, x_1 \rangle \langle -x_2, x_1 + x_2 \rangle \neq 0$, $\langle x_2, x_2 \rangle = 0$, and $Y \in \mathfrak{h}$, this claim holds.

Second we consider the other case, i.e., $\langle x_2, x_1 \rangle = 0$. Since $\overline{x_1} \neq \overline{0}$, $\overline{x_2} \neq 0$, and $\overline{x_1} \neq \overline{x_2}$, by Lemma 2.1, we can choose $z \in H$ that satisfies $\langle x_1, z \rangle \neq 0$, $\langle x_2, z \rangle \neq 0$, and $\langle x_1 - x_2, z \rangle \neq 0$. Set

$$\begin{aligned} Y &:= ad([x_2 - z])ad([z])ad([-z])ad([z - x_2])(X) \\ &= \sum_{i=1}^{n} \langle z - x_2, x_i \rangle \langle -z, x_i + z - x_2 \rangle \langle z, x_i - x_2 \rangle \langle x_2 - z, x_i - x_2 + z \rangle X_i. \end{aligned}$$

Since $\langle z - x_2, x_1 \rangle \langle -z, x_1 + z - x_2 \rangle \langle z, x_1 - x_2 \rangle \langle x_2 - z, x_1 - x_2 + z \rangle \neq 0$, $\langle z, x_2 - x_2 \rangle = 0$, and $Y \in \mathfrak{h}$, this claim holds. □

Proof of Step 1. Induction on n. If $n = 1$, $X_1 = X \in \mathfrak{h}$. Suppose $n > 1$. For any $k \in \{1, \dots, n\}$, we can take $c_1, \dots, c_n \in \mathbf{Q}$ as in Claim. By the assertions (2) and (3) of Claim, we can apply the inductive assumption to $c_1 X_1 + \cdots + c_n X_n$. Then we have $c_k X_k \in \mathfrak{h}$. By the assertion (1) of Claim, we have $X_k \in \mathfrak{h}$. This completes the induction and proves Step 1. □

STEP 2. *If* $\overline{x_i} = \overline{0}$ *for some* $i = 1, \dots, n$, *then* $X_j \in \mathfrak{h}$ *for all* $j = 1, \dots, n$.

Proof of Step 2. The index i with $\overline{x_i} = \overline{0}$ is unique since $\overline{x_j} \neq \overline{0}$ if $j \neq i$. We can assume $i = 1$. If $n = 1$, we have $X_1 = X \in \mathfrak{h}$. Suppose $n > 1$. Since $\overline{x_2} \neq \overline{0}, \dots,$ and $\overline{x_n} \neq \overline{0}$, by Lemma 2.1, there exists $z \in H$ such that $\langle x_2, z \rangle \neq 0, \dots,$ and $\langle x_n, z \rangle \neq 0$. Set $Y := ad([-z])ad([z])(X) = \sum_{j=1}^{n} \langle z, x_j \rangle \langle -z, x_j + z \rangle X_j$. Since $\langle z, x_1 \rangle = 0$, we have $Y = \sum_{j=2}^{n} \langle z, x_j \rangle \langle -z, x_j + z \rangle X_j$. Thus we can apply Step 1 to Y, so we have $\langle z, x_j \rangle \langle -z, x_j + z \rangle X_j \in \mathfrak{h}$ for all $j = 2, \dots, n$. Since $\langle z, x_j \rangle \langle -z, x_j + z \rangle \neq 0$, we obtain $X_j \in \mathfrak{h}$ for all $j = 2, \dots, n$. Moreover, $X_1 = X - X_2 - \cdots - X_n \in \mathfrak{h}$.

This proves Step 2. □

Step 1 and Step 2 complete the proof Proposition 2.2. □

PROPOSITION 2.3 (Homogeneity of an ideal). *For* $x, y \in H\backslash\ker\mu$,

$$T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}]) = T(-y)(\mathfrak{h}\cap\mathbf{Q}[\bar{y}])$$

Proof. Let $X' \in T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}])$. Then there exists $X \in \mathfrak{h}\cap\mathbf{Q}[\bar{x}]$ with $X' = T(-x)(X)$. Since $X' = T(-y)(T(y-x)(X))$ and $\deg T(y-x)(X) = \overline{y-x}+\bar{x} = \bar{y}$, it is sufficient to prove $T(y-x)(X) \in \mathfrak{h}$.

By $x, y \in H\backslash\ker\mu$ and Lemma 2.1, we can choose $z \in H$ with $\langle x, z\rangle \neq 0$ and $\langle y, z\rangle \neq 0$. We obtain $T(y-x)(X) \in \mathfrak{h}$ because

$$ad([-z])ad([y])ad([-x])ad([z])(X) = (\langle z, x\rangle\langle y, z\rangle)^2 T(y-x)(X).$$

We can prove the other inclusion by replacing the role of x and y. □

COROLLARY 2.4 (ker μ-stability of an ideal). *For* $x \in H\backslash\ker\mu$ *and* $v \in \ker\mu$,

$$T(v)(T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}])) = T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}]), \quad \textit{that is,}\ T(v)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}]) = \mathfrak{h}\cap\mathbf{Q}[\bar{x}].$$

Proof. We apply Proposition 2.3 to x and $x-v$. Then

$$T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}]) = T(-(x-v))(\mathfrak{h}\cap\mathbf{Q}[\overline{x-v}]) = T(v)(T(-x)(\mathfrak{h}\cap\mathbf{Q}[\bar{x}])).$$ □

3. Proof of Theorem 1.1

Proof. **Existence:** When $\mu = 0$, $\bar{0} = H$. So we can define $V_0 = \mathfrak{h}$ and $V = 0$. Assume $\mu \neq 0$. Then we can choose $x_0 \in H\backslash\ker\mu$.

By Proposition 2.2, we have

$$\mathfrak{h} = \bigoplus_{\bar{x}\in\bar{H}} (\mathfrak{h}\cap\mathbf{Q}[\bar{x}]).$$

Let $V_0 := \mathfrak{h}\cap\mathbf{Q}[\bar{0}]$ and $V := T(-x_0)(\mathfrak{h}\cap\mathbf{Q}[\overline{x_0}])$. By Corollary 2.4, V is $\ker\mu$-stable. For all $y \in H\backslash\ker\mu$, we have

$$\mathfrak{h}\cap\mathbf{Q}[\bar{y}] = T(y)(T(-y)(\mathfrak{h}\cap\mathbf{Q}[\bar{y}])) = T(y)(T(-x_0)(\mathfrak{h}\cap\mathbf{Q}[\overline{x_0}])) = T(y)(V)$$

by Proposition 2.3. So we obtain

$$\mathfrak{h} = V_0 \oplus \bigoplus_{\bar{x}\in\bar{H}\backslash\{\bar{0}\}} T(\bar{x})(V).$$

Uniqueness: We assume that (V_0, V) and (W_0, W) satisfy (1), (2), and

$$V_0 \oplus \bigoplus_{\bar{x}\in\bar{H}\backslash\{\bar{0}\}} T(\bar{x})(V) = W_0 \oplus \bigoplus_{\bar{x}\in\bar{H}\backslash\{\bar{0}\}} T(\bar{x})(W).$$

Then we obtain

$$V_0 = \mathbf{Q}[\bar{0}] \cap \left(V_0 \oplus \bigoplus_{\bar{x} \in \bar{H} \setminus \{\bar{0}\}} T(\bar{x})(V) \right)$$

$$= \mathbf{Q}[\bar{0}] \cap \left(W_0 \oplus \bigoplus_{\bar{x} \in \bar{H} \setminus \{\bar{0}\}} T(\bar{x})(W) \right) = W_0.$$

If $\mu = 0$, then $V = 0 = W$ by definition. Suppose $\mu \neq 0$. Take $\overline{x_0} \in \bar{H} \setminus \{\bar{0}\}$. We obtain $V = W$ because

$$T(\overline{x_0})(V) = \mathbf{Q}[\overline{x_0}] \cap \left(V_0 \oplus \bigoplus_{\bar{x} \in \bar{H} \setminus \{\bar{0}\}} T(\bar{x})(V) \right)$$

$$= \mathbf{Q}[\overline{x_0}] \cap \left(W_0 \oplus \bigoplus_{\bar{x} \in \bar{H} \setminus \{\bar{0}\}} T(\bar{x})(W) \right) = T(\overline{x_0})(W).$$

Converse: Assume that (V_0, V) satisfies (1) and (2), and let $\mathfrak{h}$ be the subspace of $\mathbf{Q}[H]$ defined by (3).

For $X \in \mathbf{Q}[H]$ and $Y \in V_0$, we have $[X, Y] = 0 \in \mathfrak{h}$ since $Y \in V_0 \subset \mathbf{Q}[\bar{0}]$.

For $x \in H$, $v \in V$ and $y \in H \setminus \ker \mu$, we define $Z := [[x], T(y)(v)]$. We have $Z = \langle x, y \rangle T(x + y)(v)$ since $v \in V \subset \mathbf{Q}[\bar{0}]$. If $\overline{x + y} \neq \bar{0}$, $Z \in T(\overline{x + y})(V) \subset \mathfrak{h}$. If $\overline{x + y} = \bar{0}$, we have $Z = 0 \in \mathfrak{h}$ because $0 = \langle x + y, y \rangle = \langle x, y \rangle$. Hence, we obtain $[\mathbf{Q}[H], \mathfrak{h}] \subset \mathfrak{h}$. □

4. Corollaries

LEMMA 4.1. *For $X \in \mathbf{Q}[H]$, $X = 0$ if and only if $X^2 = 0$.*

Proof. It is enough to show the lemma for $X \in \mathbf{C}[H]$ since $\mathbf{Q}[H] \subset \mathbf{C}[H]$. We can take a finitely generated subgroup A of H with $X \in \mathbf{C}[A]$. Hence we may assume H is a finitely generated abelian group. Then there exist a finitely generated free abelian group F and a finite abelian group T with $H \cong F \times T$. Since the group T is a finite abelian group and $\mathbf{C}$ is algebraically closed, we have an isomorphism $\mathbf{C}[T] \cong \mathbf{C}^{\#T}$ of $\mathbf{C}$-algebras [S] p. 48, Proposition 10. We may assume H is free because

$$\mathbf{C}[H] \cong \mathbf{C}[F] \otimes_{\mathbf{C}} \mathbf{C}[T] \cong \mathbf{C}[F] \otimes_{\mathbf{C}} \mathbf{C}^{\#T} \cong (\mathbf{C}[F])^{\#T}.$$

Let $H = \mathbf{Z}^r$. Then we have an isomorphism $\mathbf{C}[H] \to \mathbf{C}[Z_1^{\pm 1}, \ldots, Z_r^{\pm 1}]$ of $\mathbf{C}$-algebras by $[(x_1, \ldots, x_r)] \mapsto Z_1^{x_1} \cdots Z_r^{x_r}$ for $x_i \in \mathbf{Z}$. The Laurent polynomial ring $\mathbf{C}[Z_1^{\pm 1}, \ldots, Z_r^{\pm 1}]$ is an integral domain. Hence the lemma holds. □

PROPOSITION 4.2. *Let* $\mathfrak{h}$ *be an ideal of* $\mathbf{Q}[H]$, *and* (V_0, V) *the pair of* $\mathfrak{h}$ *in Theorem* 1.1. *Then,* $\mathfrak{h}$ *is abelian if and only if* $V = 0$.

Proof. Suppose $V = 0$. Then, since $\mathfrak{h} = V_0 \subset \mathbf{Q}[\ker \mu]$, $\mathfrak{h}$ is abelian.

Conversely, suppose $V \neq 0$. Then, since $\mu \neq 0$, there exist $x, y \in H$ with $\langle x, y \rangle \neq 0$. Take $Z \in V \backslash \{0\}$. Then, $\mathfrak{h}$ is not abelian, since $T(x)(Z)$, $T(y)(Z) \in \mathfrak{h}$ and $[T(x)(Z), T(y)(Z)] = \langle x, y \rangle T(x+y)(Z^2) \neq 0$ by Lemma 4.1. □

We define the descent sequences $\mathbf{Q}[H]^{(m)}$ and $\mathbf{Q}[H]_{(m)}$ by $\mathbf{Q}[H]^{(1)} = \mathbf{Q}[H]_{(1)} = [\mathbf{Q}[H], \mathbf{Q}[H]]$, $\mathbf{Q}[H]^{(m)} = [\mathbf{Q}[H]^{(m-1)}, \mathbf{Q}[H]^{(m-1)}]$, and $\mathbf{Q}[H]_{(m)} = [\mathbf{Q}[H], \mathbf{Q}[H]_{(m-1)}]$, respectively. We can calculate the center $\mathfrak{z}(\mathbf{Q}[H])$, the derived subalgebra $[\mathbf{Q}[H], \mathbf{Q}[H]]$ and the descent sequences $\mathbf{Q}[H]^{(m)}$ and $\mathbf{Q}[H]_{(m)} (m > 0)$; $\mathfrak{z}(\mathbf{Q}[H]) = \mathbf{Q}[\ker \mu]$, $[\mathbf{Q}[H], \mathbf{Q}[H]] = \mathbf{Q}[H \backslash \ker \mu]$, and $\mathbf{Q}[H]^{(m)} = \mathbf{Q}[H]_{(m)} = \mathbf{Q}[H \backslash \ker \mu]$. In particular, we can decompose $\mathbf{Q}[H]$ into the center and the derived Lie algebra; $\mathbf{Q}[H] = \mathbf{Q}[\ker \mu] \oplus \mathbf{Q}[H \backslash \ker \mu] = \mathfrak{z}(\mathbf{Q}[H]) \oplus [\mathbf{Q}[H], \mathbf{Q}[H]]$. Moreover, we have the abelianization of $\mathbf{Q}[H]$; $\mathbf{Q}[H]^{ab} = \mathbf{Q}[H]/[\mathbf{Q}[H], \mathbf{Q}[H]] \cong \mathbf{Q}[\ker \mu]$.

We say $\langle -, - \rangle$ is *non-degenerate* if $\ker \mu = 0$.

Example 4.3. Let Σ be a surface with $\#\pi_0(\partial\Sigma) \leq 1$. We consider $H = H_1(\Sigma, \mathbf{Z})$ and the intersection form $\langle -, - \rangle$ on H. Then, H is a free $\mathbf{Z}$-module, and $\langle -, - \rangle$ is a non-degenerate alternating $\mathbf{Z}$-bilinear form.

COROLLARY 4.4. *If* $\langle -, - \rangle$ *is non-degenerate and* $H \neq 0$, *all the ideals of* $\mathbf{Q}[H]$ *are*

$$0, \quad \mathbf{Q}[0], \quad \mathbf{Q}[H \backslash 0], \quad \textit{and} \quad \mathbf{Q}[H].$$

Proof. Since $\langle -, - \rangle$ is non-degenerate, $\ker \mu = 0$. So all the subspace of $\mathbf{Q}\bar{0}$ are 0 and $\mathbf{Q}[0]$, and they are $\ker \mu$-stable.

Since $T(x)(\mathbf{Q}[0]) = \mathbf{Q}[x]$, we obtain this Corollary. □

Remark 4.5. We can define the homological Goldman Lie algebra $R[H]$ over an arbitrary commutative ring R instead of $\mathbf{Q}$. We can prove all the results in this paper for $R[H]$ if R is a commutative ring including $\mathbf{Q}$ except for Lemma 4.1. Lemma 4.1 holds if R is an ideal domain.

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