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ON HERMITIAN MODULAR FORMS OF SMALL WEIGHT OVER IMAGINARY QUADRATIC FIELDS

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Abstract

In this paper, we prove that an Hermitian modular form with small weight over the quadratic field with class number one is a linear combination of theta series associated with Hermitian quadratic forms.

Introduction

Resnikoff and Freitag proved that a Siegel modular form with small weight is a singular form. Shimura [8] generalized these results for more general modular forms. In [5], Freitag proved that a singular Siegel modular form is a linear combination of theta series.

The purpose of this note is to discuss analogous results in the case of Hermitian modular forms over the quadratic fields. By virture of [8], we can see that Hermitian modular forms with small weight are singular forms. Using this theorem and the results in [4], we deduce that an Hermitian modular form with small weight over the quadratic field with class number one is a linear combination of theta series. We mention that we can not remove the condition that the quadratic field is class number one.

§1. Notation and preliminaries

We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers. For a ring A, we denote by A_m^n the set of all $n \times m$ matrices with entries in A and, we put $A_1^n = A^n$ (resp. $A_n^n = M_n(A)$). Let $K = \mathbf{Q}(\sqrt{-D})$ be the imaginary quadratic field of discriminant $-D$ and $\mathfrak{O}$ the ring of integers in K. Put $GL_n(\mathfrak{O}) = \{g \in M_n(\mathfrak{O}) \mid \det g \in \mathfrak{O}^{\times}\}$, where $\mathfrak{O}^{\times}$ means the group of all invertible

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elements in $\mathfrak{O}$. Let $\Gamma^s_{\mathfrak{H}}(K)$ be the Hermitian modular group of degree s over K, i.e.,

$$(1.1)\quad \Gamma^s_{\mathfrak{H}}(K) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2s}(\mathfrak{O}) \,\middle|\, M^* \begin{pmatrix} 0 & E_s \\ -E_s & 0 \end{pmatrix} M = \begin{pmatrix} 0 & E_s \\ -E_s & 0 \end{pmatrix} \right\},$$

where $M^* = {}^t(\overline{M})$ and E_s means the unity of $GL_s(\mathfrak{O})$. Let $\mathfrak{Z}^s_{\mathfrak{H}}$ be the complex Hermitian half space of degree s, i.e.,

$$(1.2)\qquad \mathfrak{Z}^s_{\mathfrak{H}} = \left\{ Z \in M_s(\mathbf{C}) \,\middle|\, \frac{1}{2\sqrt{-1}}(Z - Z^*) > 0 \right\}.$$

We define an action of $\Gamma^s_{\mathfrak{H}}(K)$ on $\mathfrak{Z}^s_{\mathfrak{H}}$ by

$$(1.3)\qquad Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$$

for all $Z \in \mathfrak{Z}^s_{\mathfrak{H}}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^s_{\mathfrak{H}}(K)$. A holomorphic function F on $\mathfrak{Z}^s_{\mathfrak{H}}$ is called an Hermitian modular form of weight γ and of degree s over K, if the following condition is satisfied

$$(1.4)\qquad F(M\langle Z\rangle) = \det(CZ + D)^{\gamma} F(Z).$$

We denote by $M(\Gamma^s_{\mathfrak{H}}(K), \gamma)$ the space of such all forms $F(Z)$ (cf. [1, 2, 3, 4]).

Here we introduce theta series (cf. [7]). Let H be a positive Hermitian matrix of degree γ and let $\mathfrak{L}$ stand for a lattice in $\mathbf{C}^{\gamma}_s$ considered as a real vector space. Then we define the theta series on $\mathfrak{Z}^s_{\mathfrak{H}}$ associated with H and $\mathfrak{L}$ by

$$(1.5)\quad \mathbf{\Theta}_{\mathfrak{L}}(Z, H) = (\mathrm{vol}\ \mathfrak{L})^{1/2} \sum_{N \in \mathfrak{L}} \exp(\pi\sqrt{-1}\ \mathrm{tr}(ZN^*HN)) \quad \text{for all } Z \in \mathfrak{Z}^s_{\mathfrak{H}}.$$

By [7], we obtain

$$(1.6)\qquad \mathbf{\Theta}_{\hat{\mathfrak{L}}}(-Z^{-1}, H^{-1}) = (\det(-\sqrt{-1}Z))^{\gamma} (\det H)^s \Theta_{\mathfrak{L}}(Z, H)$$

where $\hat{\mathfrak{L}} = \{\hat{N} \in \mathbf{C}^{\gamma}_s \mid \mathrm{tr}(\hat{N}N) \in \mathbf{Z} \text{ for all } N \in \mathfrak{L}.\}$. We take $\mathfrak{L} = \mathfrak{O}^{\gamma}_s$. Since $\hat{\mathfrak{L}} = \frac{2}{\sqrt{-D}}\mathfrak{L}$, we see that

$$\mathbf{\Theta}_{\hat{\mathfrak{L}}}(Z, H) = \frac{1}{\mathrm{vol}(\mathfrak{L})} \mathbf{\Theta}_{\mathfrak{L}}\left(Z, \frac{4}{D}H\right).$$

Therefore

$$(1.7)\quad \mathbf{\Theta}_{\mathfrak{L}}\left(-Z^{-1}, \frac{4}{D}H^{-1}\right) = (-\sqrt{-1})^{\gamma s} (\det Z)^{\gamma} (\det H)^s\ \mathrm{vol}(\mathfrak{L}) \mathbf{\Theta}_{\mathfrak{L}}(Z, H).$$

Suppose that $\mathbf{\Theta}_{\mathfrak{L}}(Z, H)$ is an Hermitian modular form of weight γ and of degree s. Then we see that

$$\mathbf{\Theta}_{\mathfrak{L}}(-Z^{-1}, H) = (-\sqrt{-1})^{\gamma s} (\det Z)^{\gamma} \mathbf{\Theta}_{\mathfrak{L}}(Z, H),$$

which yields that

$$\Theta_{\mathfrak{L}}\left(Z, \frac{4}{D}H^{-1}\right) = (\mathrm{vol}\ \mathfrak{L})(\det H)^s \Theta_{\mathfrak{L}}(Z, H). \tag{1.8}$$

Put $Z = X + \sqrt{-1}Y$ with X and Y Hermitian, and compute the limit as the eigenvalue of Y approach infinity (and X remains in some compact set). Then, by [4, (8)], we see that

$$\det H = 2^{\gamma} D^{-\gamma/2}. \tag{1.9}$$

We refer to [4, (23)] for the existence of Hermitian matrices satisfying (1.9). Consider an Hermitian matrix $H = (h_{ij})$ of degree γ such that

$$H \in \frac{2}{\sqrt{-D}} M_{\gamma}(\mathfrak{O}), \quad h_{ii} \in 2\mathfrak{O} \quad (1 \le i \le \gamma).$$

We call H an even integral Hermitian matrix of degree γ. The following proposition is proved in [4].

Proposition 1. *Let H be a positive even integral Hermitian matrix of degree γ of determinant $2^{\gamma} D^{-\gamma/2}$. Then 4 divides γ and the theta series*

$$\Theta(Z, H) = \sum_{N \in \mathfrak{O}_s^{\gamma}} \exp(\pi\sqrt{-1}\ \mathrm{tr}(ZN^*HN)) \tag{1.10}$$

belongs to $M(\Gamma_{\mathfrak{H}}^s(K), \gamma)$.

§2. Main theorem

The purpose of this section is to investigate the space $M(\Gamma_{\mathfrak{H}}^s(K), \gamma)$ where $s > \gamma$. If $s > \gamma$ and 4 does not divide γ, then $M(\Gamma_{\mathfrak{H}}^s(K), \gamma) = 0$ (cf. [4, Theorem 3]). We deduce the following theorem.

Theorem 2. *Suppose that K is class number one and $s > 4k$. Then $M(\Gamma_{\mathfrak{H}}^s(K), 4k)$ is spanned by theta series of the type discribed in Proposition 1.*

Proof. Let $F(Z)$ be an element of $M(\Gamma_{\mathfrak{H}}^s(K), 4k)$. Then $F(Z)$ has a Fourier expansion of the form

$$F(Z) = \sum_{H \in L(s)} a(H) \exp(\pi\sqrt{-1}\ \mathrm{tr}(HZ)), \tag{2.1}$$

where $L(s) = \{H \mid H$ is even integral Hermitian matrix of degree s and $H \ge 0\}$. From [8], we see that $F(Z)$ is a singular form. Using this and the property that $F(U^*ZU) = F(Z)$ for every $U \in GL_s(\mathfrak{O})$, we obtain that

$$a(H) \ne 0 \Rightarrow \det H = 0 \quad \text{and} \quad U^*HU \in L(s) \quad \text{for every } U \in GL_s(\mathfrak{O}). \tag{2.2}$$

First we prove the following assertion: Suppose that $F(Z)$ is a non-zero element of $M(\Gamma_{\mathfrak{H}}^{s}(K), 4k)$. Then there exists a matrix $H_0 \in L(4k)$ such that

$$a\left(\begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix}\right) \neq 0 \quad \text{and} \quad \det H_0 = 2^{4k} D^{-2k} \tag{2.3}$$

To verify this fact, let ρ be the maximal rank of those H for which $a(H) \neq 0$. Then $0 < \rho < s$; any H of rank ρ with $a(H) \neq 0$ can be represented as

$$H = U^* \begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix} U$$

with $H_0 \in L(\rho)$, $H_0 > 0$ and $U \in GL_s(\mathfrak{O})$ because of class number one of K and (2.2). Choose H and U such that $\det H_0$ becomes minimal under these conditions and fix H_0 from now on. Then

$$a\left(\begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix}\right) \neq 0. \tag{2.4}$$

We consider the restriction F onto $\mathfrak{Z}_{\mathfrak{H}}^{s-\rho} \times \mathfrak{Z}_{\mathfrak{H}}^{\rho}$,

$$F\left(\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right) = \sum_{H_1 \in L(\rho),\, H_1 \geq 0} \alpha_{H_1}(w) \exp(\pi\sqrt{-1}\, \mathrm{tr}(H_1 z)) \tag{2.5}$$

for all $z \in \mathfrak{Z}_{\mathfrak{H}}^{\rho}$, $w \in \mathfrak{Z}_{\mathfrak{H}}^{s-\rho}$. We see that

$$\alpha_{H_1}(w) = \sum_{H} a(H) \exp(\pi\sqrt{-1}\, \mathrm{tr}(H_2 w))$$

belongs to $M(\Gamma_{\mathfrak{H}}^{s-\rho}(K), 4k)$, where the summation is taken over all positive semi-definite matrices $H = \begin{pmatrix} H_2 & t_2 \\ t_2^* & H_1 \end{pmatrix}$ in $L(s)$. If $a\left(\begin{pmatrix} H_2 & t_2 \\ t_2^* & H_0 \end{pmatrix}\right) \neq 0$, then $\begin{pmatrix} H_2 & t_2 \\ t_2^* & H_0 \end{pmatrix}$ is of rank ρ because of the maximal condition for the rank. Therefore

$$\begin{pmatrix} H_2 & t_2 \\ t_2^* & H_0 \end{pmatrix} = V^* \begin{pmatrix} 0 & 0 \\ 0 & H' \end{pmatrix} V \tag{2.6}$$

with $V \in GL_s(\mathfrak{O})$, $H' \in L(\rho)$ and $H' > 0$, which implies that $\det H' \leq \det H_0$. We obtain $\det H' = \det H_0$ because of the minimal condition for $\det H_0$. Hence we have $H_0 = (V')^* H' V'$ for some $V' \in GL_\rho(\mathfrak{O})$ and

$$\alpha_{H_0}(w) = a\left(\begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix}\right) \sum_{H} \exp(\pi\sqrt{-1}\, \mathrm{tr}(H_2 w)),$$

where $H = \begin{pmatrix} H_2 & t_2 \\ t_2^* & H_0 \end{pmatrix}$ runs over $L(s)$ such that $H \geq 0$, which is represented as $H = W^* \begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix} W$ with $W \in GL_s(\mathfrak{O})$. We can check that this condition is equivalent to

$$H = \begin{pmatrix} E_{s-\rho} & 0 \\ g & E_\rho \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix} \begin{pmatrix} E_{s-\rho} & 0 \\ g & E_\rho \end{pmatrix}, \tag{2.7}$$

where g runs over the matrices in $\mathfrak{O}_{s-\rho}^{\rho}$. Hence

$$\alpha_{H_0}(w) = a\left(\begin{pmatrix} 0 & 0 \\ 0 & H_0 \end{pmatrix}\right) \sum_{g \in \mathfrak{O}_{s-\rho}^{\rho}} \exp(\pi\sqrt{-1}\,\mathrm{tr}(wg^* H_0 g)) \tag{2.8}$$

belongs to $M(\Gamma_{\mathfrak{H}}^{s-\rho}(K), 4k)$. Comparing the weight, we see that $\rho = 4k$. Moreover, by virtue of (1.9), we see that $\det H_0 = 2^{4k} D^{-2k}$. Therefore, we have the first assertion.

Next we prove our theorem. Take a complete set $H_1, \ldots, H_\ell$ of representatives of the classes of all positive Hermitian matrices of degree $4k$ which are even integral and of determinant $2^{4k} D^{-2k}$ (cf. [6]). We put

$$F^*(Z) = F(Z) - \sum_{i=1}^{\ell} c_i \boldsymbol{\Theta}(Z, H_i) = \sum_{H \in L(s), H \geq 0} a^*(H) \exp(\pi\sqrt{-1}\,\mathrm{tr}(HZ)). \tag{2.9}$$

We obtain

$$a^*\left(\begin{pmatrix} 0 & 0 \\ 0 & H_i \end{pmatrix}\right) = a\left(\begin{pmatrix} 0 & 0 \\ 0 & H_i \end{pmatrix}\right) - c_i \alpha(H_i, H_i)$$

for $i = 1, 2, \ldots, \ell$, where $\alpha(H_i, H_i)$ is the number of units of H_i. Now, c_i can be determined by

$$a^*\left(\begin{pmatrix} 0 & 0 \\ 0 & H_i \end{pmatrix}\right) = 0 \tag{2.10}$$

for $i = 1, 2, \ldots, \ell$. Applying the above arguments for a singular form $F^*(Z)$, we obtain $F^*(Z) \equiv 0$. Hence we deduce that

$$F(Z) = \sum_{i=1}^{\ell} c_i \boldsymbol{\Theta}(Z, H_i). \tag{2.11}$$

This completes our proof of the theorem.

References

[1] H. Braun, Hermitian modular functions 1, Ann. of Math. **50** (1949), 827–855.
[2] H. Braun, Hermitian modular functions 2, Ann. of Math. **51** (1950), 82–104.
[3] H. Braun, Hermitian modular functions 3, Ann. of Math. **53** (1951), 143–160.

[4] D. M. Cohen and H. L. Resnikoff, Hermitian quadratic forms and Hermitian modudar forms, Pacific J. of Math. **76** (1978), 329–337.
[5] F. Freitag, Stabile modulformen, Math. Ann. **230** (1977), 197–211.
[6] P. Humbert, Théorie de la réduction des formes quadratique définies positives dans un corps algébrique K fini, Comment. Math. Helv. **12** (1939/40), 263–306.
[7] H. L. Resnikoff, Theta functions for Jordan algebra, Invent. Math. **31** (1975), 87–104.
[8] G. Shimura, Differential operators, holomorphic projection and singular forms, Duke Math. J. **76** (1994), 141–173.

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