H. TOKUNAGA
KODAI MATH. J.
35 (2012), 78–104

SOME SECTIONS ON RATIONAL ELLIPTIC SURFACES AND CERTAIN SPECIAL CONIC-QUARTIC CONFIGURATIONS

Hiro-o Tokunaga

Introduction

In this article, we continue to study quadratic residue conics to an irreducible quartic, which was our main subject in [17]. We first recall some of results in [17]. Note that all varieties throughout this article are defined over the field of complex numbers, $\mathbf{C}$.

Let C be a smooth conic in $\mathbf{P}^2$ and let $f_C : Z_C \to \mathbf{P}^2$ be a double cover with branch locus $\Delta_{f_C} = C$. We denote the covering transformation of f_C by σ_{f_C}. Let D be an irreducible curve on $\mathbf{P}^2$, which is different from C. The pull back f_C^*D is either irreducible or reducible with two irreducible components D^+ and D^- such that $\sigma_{f_C}^* D^+ = D^-$. Following to [17], we say that C is a "*quadratic residue conic* mod D if f_C^*D is reducible. In [17], we introduce notation (C/D) such that

- $(C/D) = 1$ if C is a quadratic residue conic mod D, and
- $(C/D) = -1$ if C is not a quadratic residue conic mod D

We first remark the following: Let $I_x(C,D)$ denotes the intersection multiplicity at $x \in C \cap D$. If there exists a point $x \in C \cap D$ such that $I_x(C,D)$ is odd, then $(C/D) = -1$. In fact, if such a point x exists, then f induces a double cover on the normalization of D which has the non empty branch set.

Hence if $(C/D) = 1$, then $I_x(C,D)$ is always even. In the following, we always assume that

($*$) For $\forall x \in C \cap D$, $I_x(C,D)$ is even and D is smooth at x.

Under the condition ($*$), as we see in the Introduction of [17], one can easily determine (C/D) if $\deg D \le 3$, and the first interesting case is $\deg D = 4$. In fact, in [17], we obtain the following

Theorem 0.1. *Let C be a smooth conic, let Q be an irreducible quartic satisfying ($*$), and Ξ_Q denotes the set of types of singularities of Q. Here we use the notation in [3] in order to describe the types of singularity.*

Then we have the following:

- *If $\Xi_Q \neq \{2A_1\}, \{A_3\}$, then (C/Q) is determined by Ξ_Q.*
- *There exist smooth conics C_1, C_2 and irreducible quartics Q_1, Q_2 such that*

Received December 17, 2010; revised May 24, 2011.

(i) C_i *and* Q_i $(i = 1, 2)$ *satisfy* $(*)$,
(ii) $\Xi_{Q_1} = \Xi_{Q_2} = \{2A_1\}, \{A_3\}$, *and*
(iii) $(C_1/Q_1) = 1$, $(C_2/Q_2) = -1$.

Moreover, in [17], we also show that the topological fundamental group $\pi_1(\mathbf{P}^2 \backslash (C_1 + Q_1), \star)$ is not isomorphic to $\pi_1(\mathbf{P}^2 \backslash (C_2 + Q_2), \star)$.

We here introduce a terminology for later use:

Definition 0.1. (i) Let C and Q be a smooth conic and an irreducible quartic as in Theorem 0.1. We call such $C + Q$ *a conic-quartic configuration* (a CQ-configuration for short).

(ii) A CQ-configuration such that $\Xi_Q = \{2A_1\}$ (resp. $= \{A_3\}$) is said to be type I (resp. type II).

In [17], however, we do not care about how many points are in $C \cap Q$. In this paper, we consider this problem.

Put $C \cap Q = \{x_1, \dots, x_r\}$ and we define a r-ple of natural numbers $I(C, Q)$ to be $(I_{x_1}(C, Q), \dots, I_{x_r}(C, Q))$. We call $I(C, Q)$ *the intersection multiplicity sequence* between C and Q. Without loss of generality, we may assume that $I_{x_1}(C, Q) \geq \cdots \geq I_{x_r}(C, Q)$. There are five possible cases for $I(C, Q)$: $(2, 2, 2, 2)$, $(4, 2, 2), (4, 4), (6, 2), (8)$.

Now we state our main result in this article:

Theorem 0.2. *Let* $(e_1, \dots, e_r)$ *be any* r*-ple of natural numbers such that* $e_1 \geq \cdots \geq e_r$, e_i $(i = 1, \dots, r)$: *even and* $\sum_i e_i = 8$. *There exist pairs of CQ-configurations* $(C + Q, C' + Q')$ *of types I and II satisfying the following properties*:

- $I(C, Q) = I(C', Q') = (e_1, \dots, e_r)$.
- $(C/Q) = 1$ *and* $(C'/Q') = -1$.

Note that the pairs $(C + Q, C' + Q')$ are all Zariski pairs (see [1] for Zariski pairs). All of Zariski pairs in Theorem 0.2 can be found in [12]. However, our method to see that they are Zariski pairs is totally different from that in [12], which is our justification.

We now give a brief explanation of our strategy to obtain the CQ-configurations in Theorem 0.2, which is main ingredient of this paper.

Let B_1 and B_2 be plane curves in $\mathbf{P}^2$. Let Σ be a rational surface such that there exists a birational map $\Phi : \mathbf{P}^2 \dashrightarrow \Sigma$ so that the proper tansforms $\tilde{B}_1$ and $\tilde{B}_2$ of B_1 and B_2, resepectively, are linearly equivalent. Let $\Lambda_{B_1+B_2}$ be a pencil on Σ generated by $\tilde{B}_1$ and $\tilde{B}_2$. Let $\nu : W \to \Sigma$ be the resolution of the indeterminancy and we denote the induced fibration by $\varphi_{B_1+B_2} : W \to \mathbf{P}^1$. Note that

(i) the proper transforms $\nu^{-1}\tilde{B}_1$ and $\nu^{-1}\tilde{B}_2$ are contained in some fibers of $\varphi_{B_1+B_2}$, and
(ii) the way how $\tilde{B}_1$ and $\tilde{B}_2$ intesect reflects the configuration of singular fibers of $\varphi_{B_1+B_2}$.

Conversely, suppose that a fibered rational surface $\varphi : W \to \mathbf{P}^1$ and a birational morphism $\nu : W \to \Sigma$ are given in such a way that some part of fibers F_1 and F_2

give rise to $\tilde{B}_1$ and $\tilde{B}_2$ as above. By considering the proper transforms of $\tilde{B}_1$ and $\tilde{B}_2$ by Φ^{-1}, we obtain $B_1 + B_2$.

In this article, we apply the above idea to the case when $B_1 = C$ and $B_2 = Q$, where $C + Q$ is a CQ-configuration of either type I or II.

As we see in §3, $\Sigma = \mathbf{P}^1 \times \mathbf{P}^1$ in the case when $\Xi_Q = \{2A_1\}$, while $\Sigma =$ the Hirzebruch surface of degree 2 in the case when $\Xi_Q = \{A_3\}$. For both cases, we consider a pencil of curves of genus 1. Hence $\varphi_{C+Q} : W \to \mathbf{P}^1$ is a rational elliptic surfaces.

The group of sections, $\mathrm{MW}(X)$, of $\varphi : X \to \mathbf{P}^1$ is called the Mordell-Weil group. $\mathrm{MW}(X)$ has been studied by many mathematicians mainly from the viewpoint of arithmetic interest. In this article, however, we make use of the group structure of $\mathrm{MW}(X)$ in order to find sections which play essential roles to construct prescribed CQ configurations. This is a feature of this article. As for rational elliptic surfaces, many detail results about the configurations of singular fibers, the groups of sections called the Mordell-Weil groups are well-known (see [9], [10], [11] and [14], for example). These results make the author possible to consider the above application of $\mathrm{MW}(X)$.

We hope our method to construct curves with prescribed conditions can be considered as another new application of theory of elliptic surfaces.

This article consists of 6 sections. In §1, we summarize some basic facts on elliptic surfaces. We show that the existence of CQ-configurations of types I and II is reduced to that of pencils of genus 1 on $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ in §2. In §3, we consider some rational elliptic surfaces and certain special sections, which play important roles in constructing CQ-configurations with prescribed $I(C, Q)$. We prove Theorem 0.2 in §§4 and 5. We construct Zariski triples given in [12] via our method in §6.

1. Preliminaries from the theory of elliptic surfaces

As for details on the results in this section, we refer to [6], [7], [8], [9] and [13].

1.1. General facts

Throughout this article, an elliptic surface always means a smooth projective surface X with a fibration $\varphi : X \to C$ over a smooth projective curve, C, such that (i) $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C$ except no empty finite points $\mathrm{Sing}(\varphi) \subset C$, (ii) there exists a section $O : C \to X$ (we identify O with its image in X), and (iii) there is no exceptional curve of the first kind in any fiber.

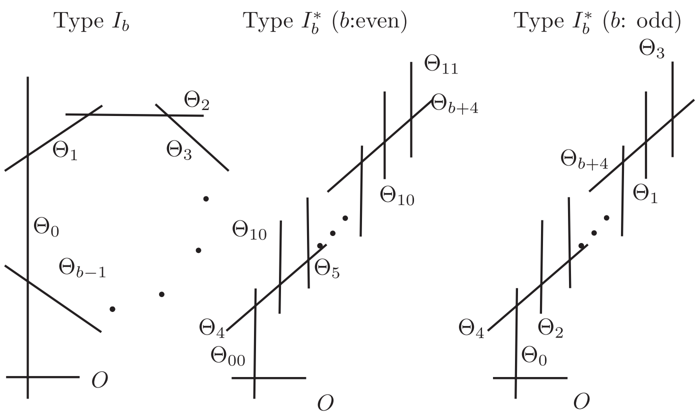
We call $F_v = \varphi^{-1}(v)(v \in \mathrm{Sing}(\varphi))$ a singular fiber over v. We denote the irreducible decomposition of F_v by

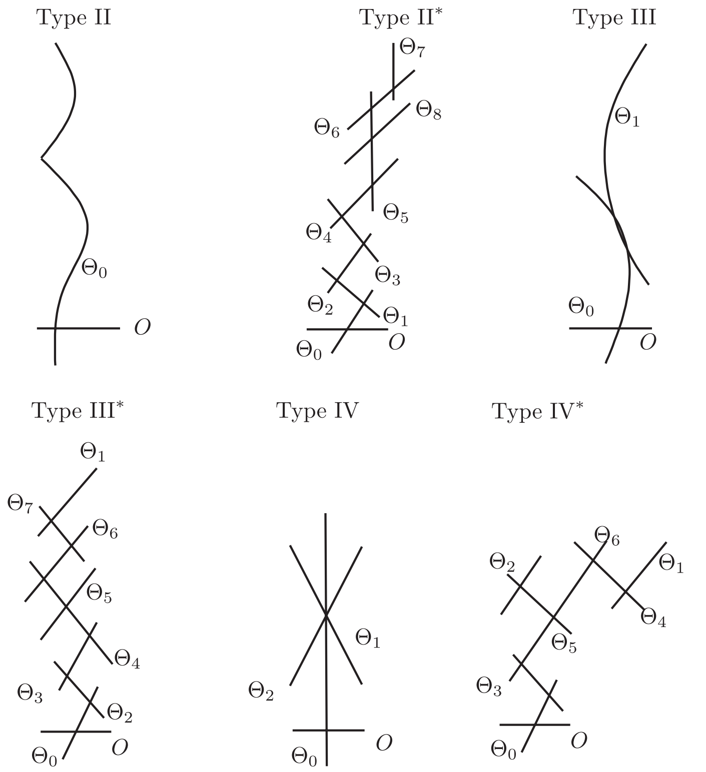
$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i} \Theta_{v,i},$$

where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0} O = 1$. We call $\Theta_{v,0}$ *the identity component*. We also define a subset $\mathrm{Red}(\varphi)$ of $\mathrm{Sing}(\varphi)$ to be $\mathrm{Red}(\varphi) := \{v \in \mathrm{Sing}(\varphi) \mid F_v \text{ is reducible}\}$.

Let $\mathrm{MW}(X)$ be the set of sections of $\varphi : X \to C$. By our assumption, $\mathrm{MW}(X) \neq \emptyset$. On a smooth fiber F of φ, by regarding $F \cap O$ as the zero element, we can consider the abelian group structure on F. Hence for $s_1, s_2 \in \mathrm{MW}(X)$, one can define $s_1 + s_2$ on $C \setminus \mathrm{Sing}(\varphi)$. By [6, Theorem 9.1], $s_1 + s_2$ can be extended over C, and we can consider $\mathrm{MW}(X)$ as an abelian group. On the other hand, we can regard the generic fiber X_η of X as a curve of genus 1 over $\mathbf{C}(C)$, the rational function field of C. The restriction of O to X_η gives rise to a $\mathbf{C}(C)$-rational point of X_η, and one can regard X_η as an elliptic curve over $\mathbf{C}(C)$, O being the zero element. By considering the restriction to the generic fiber for each sections, $\mathrm{MW}(X)$ can be identified with the set of $\mathbf{C}(C)$-rational points of X_η. For $s \in \mathrm{MW}(X)$, s is said to be *integral* if $sO = 0$. It is known that any torsion element in $\mathrm{MW}(X)$ is integral (*cf.* [8]). In the following, we call $\mathrm{MW}(X)$ the Mordell-Weil group of X. As for later use, we see how $s_1 + s_2$ on $C \setminus \mathrm{Sing}(\varphi)$ is extended briefly. For details, see [6], §9. For a singular fiber $F_v = \sum_i a_{v,i}\Theta_{v,i}$, $v \in \mathrm{Sing}(\varphi)$, we put $F_v^\# = \bigcup_{a_{v,i}=1} \Theta_{v,i}^\#$, where $\Theta_{v,i}^\# := \Theta_{v,i} \setminus$(singular points of $(F_v)_{red}$). For $s \in \mathrm{MW}(X)$, $sF_v = 1$. Hence $s \cap F_v^\# \neq \emptyset$. Note that we have the following table for $F_v^\#$, where we label the irreducible components of F_v as below:

Type of F_v	$F_v^\#$
I_b	$\bigcup_{i=0}^{b-1} \Theta_i^\#$
I_b^* (b: even)	$\Theta_{00}^\# \cup \Theta_{10}^\# \cup \Theta_{01}^\# \cup \Theta_{11}^\#$
I_b^* (b: odd)	$\Theta_0^\# \cup \Theta_1^\# \cup \Theta_2^\# \cup \Theta_3^\#$
$\mathrm{II}, \mathrm{II}^*$	$\Theta_0^\#$
$\mathrm{III}, \mathrm{III}^*$	$\Theta_0^\# \cup \Theta_1^\#$
$\mathrm{IV}, \mathrm{IV}^*$	$\Theta_0^\# \cup \Theta_1^\# \cup \Theta_2^\#$





Under these labeling, we have the following isomorphisms of abelian groups and we define a finite abelian group $G_{F_v^\#}$ as follows (see [6] for details):

Type of F_v	Group structure	$G_{F_v^\#}$
I_b	$F_v^\# \cong \mathbf{C}^\times \times \mathbf{Z}/b\mathbf{Z}$ $t_k \mapsto (t_k, k)$, t_k: a local coordinate of $\Theta_k^\# \cong \mathbf{C}^\times$	$\mathbf{Z}/b\mathbf{Z}$
I_b^* (b: even)	$F_v^\# \cong \mathbf{C} \times (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ $t_{kl} \mapsto (t_{kl}, k, l)$, t_{kl}: a local coordinate of $\Theta_{kl}^\# \cong \mathbf{C}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$
I_b^* (b: odd)	$F_v^\# \cong \mathbf{C} \times \mathbf{Z}/4\mathbf{Z}$ $t_k \mapsto (t_k, k)$, t_k: a local coordinate of $\Theta_k^\# \cong \mathbf{C}^\times$	$\mathbf{Z}/4\mathbf{Z}$
$\mathrm{II}, \mathrm{II}^*$	$F_v^\# \cong \mathbf{C}$ $t_0 \mapsto t_0$, t_0: a local coordinate of $\Theta_k^\# \cong \mathbf{C}$	$\{0\}$
$\mathrm{III}, \mathrm{III}^*$	$F_v^\# \cong \mathbf{C} \times \mathbf{Z}/2\mathbf{Z}$ $t_k \mapsto (t_k, k)$, t_k: a local coordinate of $\Theta_k^\# \cong \mathbf{C}$	$\mathbf{Z}/2\mathbf{Z}$
$\mathrm{IV}, \mathrm{IV}^*$	$F_v^\# \cong \mathbf{C} \times \mathbf{Z}/3\mathbf{Z}$ $t_k \mapsto (t_k, k)$, t_k: a local coordinate of $\Theta_k^\# \cong \mathbf{C}$	$\mathbf{Z}/3\mathbf{Z}$

Put $G_{\mathrm{Sing}(\varphi)} := \bigoplus_{v \in \mathrm{Sing}(\varphi)} G_{F_v^\#}$. Now we define a homomorphism $\gamma : \mathrm{MW}(X) \to G_{\mathrm{Sing}(\varphi)}$ to be the composition of the restriction morphism $\mathrm{MW}(X) \to \bigoplus_{v \in \mathrm{Sing}(\varphi)} F_v^\#$ and the natural morphism $\bigoplus_{v \in \mathrm{Sing}(\varphi)} F_v^\# \to G_{\mathrm{Sing}(\varphi)}$. Note that $\gamma(s)$ describes at which irreducible component s meets on F_v.

We next summarize some results on the theory of the Mordell-Weil lattices studied by Shioda in [13]. In [13], a **Q**-valued bilinear form $\langle\, , \rangle$ called the height pairing on $\mathrm{MW}(X)$ with the following property is defined:

- $\langle s, s \rangle \geq 0$ for $\forall s \in \mathrm{MW}(X)$ and the equality holds if and only if s is an element of finite order in $\mathrm{MW}(X)$.
- More explicitly, $\langle s_1, s_2 \rangle$ $(s_1, s_2 \in \mathrm{MW}(X))$ is given as follows:

$$\langle s_1, s_2 \rangle = \chi(\mathcal{O}_X) + s_1 O + s_2 O - s_1 s_2 - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{Corr}_v(s_1, s_2),$$

where $\mathrm{Corr}_v(s_1, s_2)$ is given by

$$\mathrm{Corr}_v(s_1, s_2) = (s_1\Theta_{v,1}, \dots s_1\Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix} s_2\Theta_{v,1} \\ \cdot \\ s_2\Theta_{v,m_v-1} \end{pmatrix}.$$

Here $\Theta_{v,1}, \dots, \Theta_{v,m_v-1}$ are irreduicble components of F_v $(v \in \mathrm{Red}(\varphi))$ and A_v is the intersection matrix $(\Theta_{v,i}\Theta_{v,j})_{1 \leq i,j \leq m_v-1}$. As for explicit values of $\mathrm{Corr}_v(s_1, s_2)$, we refer to [13, (8.16)].

The following lemma is also immediate from the explicit formula:

Lemma 1.1. *If $\gamma(s) = 0$, then $\mathrm{Corr}_v(s, s) = 0$ for $\forall v \in \mathrm{Sing}(\varphi)$. In particular, if $\gamma(s) = 0$, then $\langle s, s \rangle \geq 2\chi(\mathcal{O}_X)$ unless $s = O$.*

Corollary 1.1. *Let s be a torsion of order n in $\mathrm{MW}(X)$. Then the order of $\gamma(s)$ is n.*

Proof. Suppse that $m\gamma(s) = \gamma(ms) = 0$ for some $m < n$. As $\langle ms, ms \rangle = 0$, we have $ms = O$ by Lemma 1.1, but this contradicts to our assumption. □

1.2. Rational elliptic surface

An elliptic surface $\varphi : X \to C$ is said to be rational if X is a rational surface. Note that $C = \mathbf{P}^1$ if $\varphi : X \to C$ is a rational elliptic surface. Also it is well-known that X is obtained as the resolution of the base points of a pencil of cubic curves in $\mathbf{P}^2$, i.e., X is obtained from $\mathbf{P}^2$ by 9-time blowing-ups. As for more properties, we refer to [9]. Let us start with the following lemma:

Lemma 1.2. *Let $\varphi : X \to \mathbf{P}^1$ be a rational elliptic surface. If C is a smooth irreducible curve on X with $C^2 < 0$, then either $C^2 = -1$ and C is a section of φ or $C^2 = -2$ and C is an irreducible component of some reducible singular fiber.*

Proof. By the canonical bundle formula for an elliptic surface, $K_X \sim -F$, F being a fiber of φ. Hence $K_X C \leq 0$. If $K_X C = 0$, i.e., $FC = 0$, then C is an irreducible component of some reducible singular fiber. If $K_X C < 0$, as $C^2 + K_X C \geq -2$, we have $C^2 = -1$ and $K_X C = -1$, i.e., $FC = 1$. Hence C is a section of φ. □

Corollary 1.2. *Let $\varphi : X \to \mathbf{P}^1$ be a rational elliptic surface and let $\nu : X \to \overline{X}$ be a composition of 8-times blowing downs. Then $\overline{X}$ is either $\mathbf{P}^1 \times \mathbf{P}^1$, the Hirzebruch surface of degree 2, Σ_2, or one point blowing up $\mathbf{P}^2$, Σ_1.*

Proof. Since the Picard number of $\overline{X}$ is 2, $\overline{X}$ is either minimal or Σ_1. By Lemma 1.2, we infer that $\overline{X}$ is either $\mathbf{P}^1 \times \mathbf{P}^1$ or Σ_2 if $\overline{X}$ is not minimal. □

For a rational elliptic surface $\varphi : X \to \mathbf{P}^1$ and $s_1, s_2 \in \mathrm{MW}(X)$, we have

$$\langle s_1, s_2 \rangle = 1 + s_1 O + s_2 O - s_1 s_2 - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{Corr}_v(s_1, s_2).$$

In particular,

$$\langle s_1, s_1 \rangle = 2 + 2 s_1 O - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{Corr}_v(s_1, s_1).$$

By these formulas, we easily obtain the following corollaries:

Corollary 1.3. *If $\sum_{v \in \mathrm{Red}(\varphi)} \mathrm{Corr}_v(s, s) \leq 2$, then every $s \in \mathrm{MW}(X)$ with $\langle s, s \rangle < 2$ is integral.*

Corollary 1.4. *Let s_1 and s_2 be integral sections. If $\langle s_1, s_2 \rangle > 0$, $s_1 s_2 = 0$.*

Proof. As $\mathrm{Corr}_v(s_1, s_2) \geq 0$ for any F_v, our statement is immediate. □

The following theorem is fundamental for $\mathrm{MW}(X)$ of a rational elliptic surface.

Theorem 1.1 [13, Theorem 10.8]. *The Mordell-Weil group of a rational elliptic surface is generated by integral sections.*

2. Rational elliptic surfaces and CQ-configurations of type I and II

In this section, we show that pencils of curves of genus 1 on $\mathbf{P}^1 \times \mathbf{P}^1$ and the Hirzebruch surface of degree 2, Σ_2, canonically arise from CQ-configurations of types I and II, respectively. Let us start with type I.

2.1. CQ-configurations of type I

We denote two nodes of Q by P_1 and P_2, and let L be the line through P_1 and P_2. Let $\mu_1 : \widehat{\mathbf{P}^2} \to \mathbf{P}^2$ be a composition of blowing-ups at P_1 and P_2. We denote the proper transform of L, the exceptional curves arising from P_1 and P_2 by $\overline{L}$, E_1 and E_2, respectively. Let $\mu_2 : \widehat{\mathbf{P}^2} \to \mathbf{P}^1 \times \mathbf{P}^1$ be the blowing down of $\overline{L}$. We denote the image of E_1 and E_2 by l_1 and l_2, respectively. We also denote the linear equivalence class of divisors $al_1 + bl_2$ by (a, b). Under the birational map $\Phi_{\mathrm{I}} := \mu_2 \circ \mu_1^{-1} : \mathbf{P}^2 \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1$, we easily see the followings:

- C is mapped to an irrreducible curve $\tilde{C}$ with one node and $\tilde{C} \sim (2, 2)$.
- Q is mapped to a smooth irreducible curve $\tilde{Q}$ and $\tilde{Q} \sim (2, 2)$.
- $\tilde{C}$ and $\tilde{Q}$ intersect in the same way as that of C and Q.

Let Λ_{C+Q} be a pencil generated by $\tilde{C}$ and $\tilde{Q}$. By resolving base points of Λ_{C+Q}, we have a rational surface $\varphi_{C+Q} : X_{C+Q} \to \mathbf{P}^1$ with a section. Note that $\tilde{C}$ gives rise to a singular fiber of type I_1.

Conversely, if we choose a suitable rational elliptic surface $\varphi : X \to \mathbf{P}^1$ so that (i) φ has at least one I_1-fiber F_o and (ii) we can blow down X to $\mathbf{P}^1 \times \mathbf{P}^1$ so that the images of F_o and a general fiber intersect the same way as in $\tilde{C}$ and $\tilde{Q}$. Then by considering Φ_{I}^{-1}, we have a CQ-configuration of type I.

2.2. CQ-configurations of type II

Let Σ_2 be the Hirzebruch surface of degree 2, and let Δ_∞ be a section of Σ_2 with $\Delta_\infty^2 = 2$. Let P_1 be the A_3 singular point of Q and let L be the maximal tangent line at P_1. Let $\mu_{1,1} : (\mathbf{P}^2)_{P_1} \to \mathbf{P}^2$ be a blowing up at P_1. We denote $\mu_{1,1}^{-1}L$ and E_1 be the proper transform of L and the exceptional divisor of $\mu_{1,1}$. Let $\mu_{1,2} : \widehat{\mathbf{P}^2} \to (\mathbf{P})_{P_1}$ be a blowing up at $\mu_{1,1}^{-1}L \cap E_1$, and we put $\mu_1 := \mu_{1,1} \circ \mu_{1,2} : \widehat{\mathbf{P}^2} \to \mathbf{P}^2$. We denote the proper transforms of $\mu_{1,1}^{-1}L$ and E_1 by $\overline{L}$ and $\overline{E}_1$, respectively. By blowing down $\overline{L}$, we obtain Σ_2, and we denote it by $\mu_2 : \widehat{\mathbf{P}^2} \to \Sigma_2$. Under the birational map $\Phi_{\mathrm{II}} := \mu_2 \circ \mu_1^{-1}$, we infer that both C and Q are mapped to irreducible curves both of which are linear equivalent to $2\Delta_\infty$, which we denote by $\tilde{C}$ and $\tilde{Q}$, respectively. Let Λ_{C+Q} be the pencil given by $\tilde{C}$ and $\tilde{Q}$. By resolving base points of Λ_{C+Q}, we obtain a rational elliptic surface $\varphi_{C+Q} : X_{C+Q} \to \mathbf{P}^1$ with a section. Conversely, if we choose a suitable rational elliptic surface $\varphi : X \to \mathbf{P}^1$ so that (i) φ has at least one I_1-fiber F_o and (ii) we can blow down X to Σ_2 so that the images of F_o and a general fiber intersect the same manner as in $\tilde{C}$ and $\tilde{Q}$. Then by considering Φ_{II}^{-1}, we have a CQ-configuration of type II.

We make use of our observation in this section to find CQ-configurations with prescribed $I(C, Q)$ in §4.

3. Some special sections on certain rational elliptic surfaces

We keep the notation introduced in §1. In this section, we look into existence or non-existence of sections for certain rational elliptic surfaces $\varphi : X \to \mathbf{P}^1$.

In order to obtain CQ-configurations of type I and II, we blow down X to either $\mathbf{P}^1 \times \mathbf{P}^1$ or Σ_2. Then, by considering birational maps from $\mathbf{P}^1 \times \mathbf{P}^1$ or Σ_2 to $\mathbf{P}^2$ considered in §2, we see that a smooth fiber and an I_1-fiber of φ give rise to the desired CQ-configuration.

Let $\varphi_n : X_n \to \mathbf{P}^1$ be a rational elliptic surface whose structure of the Mordell-Weil lattice is the type No. n in [10]. Our proof of Theorem 0.2 is done by case-by-case consideration. For this purpose, we choose 20 type rational elliptic surfaces as in the table below. As for their existence, we refer to [10] and [11]. By [11], we can asssume that the configuration of singular fibers of X_n is as follows:

No	Singular fibers	No	Singular fibers	No	Singular fibers	No	Singular fibers
9	$\mathrm{I}_0^*, 6\mathrm{I}_1$	24	$5\mathrm{I}_2, 2\mathrm{I}_1$	35	$2\mathrm{I}_4, 4\mathrm{I}_1$	49	$\mathrm{IV}^*, \mathrm{I}_2, 2\mathrm{I}_1$
13	$4\mathrm{I}_2, 4\mathrm{I}_1$	26	$\mathrm{I}_2^*, 4\mathrm{I}_1$	38	$\mathrm{I}_4, 3\mathrm{I}_2, 2\mathrm{I}_1$	53	$\mathrm{I}_6, 2\mathrm{I}_2, 2\mathrm{I}_1$
16	$\mathrm{I}_1^*, 5\mathrm{I}_1$	27	$\mathrm{IV}^*, 4\mathrm{I}_1$	43	$\mathrm{III}^*, 3\mathrm{I}_1$	58	$2\mathrm{I}_4, \mathrm{I}_2, 2\mathrm{I}_1$
18	$\mathrm{I}_0^*, \mathrm{I}_2, 4\mathrm{I}_1$	28	$\mathrm{I}_6, \mathrm{I}_2, 4\mathrm{I}_1$	44	$\mathrm{I}_8, 4\mathrm{I}_1$	65	$\mathrm{III}^*, \mathrm{I}_2, \mathrm{I}_1$
21	$\mathrm{I}_4, 2\mathrm{I}_2, 4\mathrm{I}_1$	30	$\mathrm{I}_1^*, \mathrm{I}_2, 3\mathrm{I}_1$	48	$\mathrm{I}_2^*, \mathrm{I}_2, 2\mathrm{I}_1$	70	$\mathrm{I}_8, \mathrm{I}_2, 2\mathrm{I}_1$

By [10], we see the structures of $\mathrm{MW}(X_n)$ in the above table are as follows:

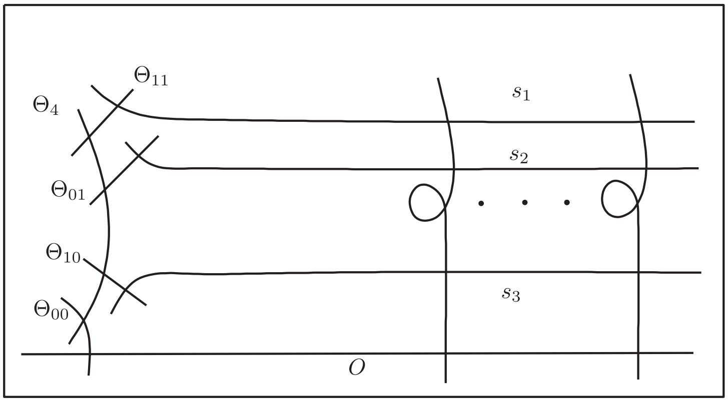
No	$\mathrm{MW}(X_n)$	No	$\mathrm{MW}(X_n)$
9	D_4^*	35	$(A_1^*)^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z}$
13	$D_4^* \oplus \mathbf{Z}/2\mathbf{Z}$	38	$A_1^* \oplus \langle 1/4 \rangle \oplus \mathbf{Z}/2\mathbf{Z}$
16	A_3^*	43	A_1^*
18	$(A_1^*)^{\oplus 3}$	44	$A_1^* \oplus \mathbf{Z}/2\mathbf{Z}$
21	$A_3^* \oplus \mathbf{Z}/2\mathbf{Z}$	48	$A_1^* \oplus \mathbf{Z}/2\mathbf{Z}$
24	$(A_1^*)^{\oplus 3} \oplus \mathbf{Z}/2\mathbf{Z}$	49	$\langle 1/6 \rangle$
26	$(A_1^*)^{\oplus 2}$	53	$\langle 1/6 \rangle \oplus \mathbf{Z}/2\mathbf{Z}$
27	A_2^*	58	$A_1^* \oplus \mathbf{Z}/4\mathbf{Z}$
28	$A_2^* \oplus \mathbf{Z}/2\mathbf{Z}$	65	$\mathbf{Z}/2\mathbf{Z}$
30	$A_1^* \oplus \langle 1/4 \rangle$	70	$\mathbf{Z}/4\mathbf{Z}$

In the above table, we use the same terminology as that in [10] in order to describe the structure of $\mathrm{MW}(X_n)$. For $s \in \mathrm{MW}(X_n)$, $-s$ denotes the inverse element of s with respect to the group law on $\mathrm{MW}(X_n)$. We denote the addition on $\mathrm{MW}(X_n)$ by $\dot{+}$. Also $\gamma_n : \mathrm{MW}(X_n) \to G_{\mathrm{Sing}(\varphi_n)}$ denotes the homomorphism introduced in the previous section.

In order to obtain conic-quartic configurations in the Introduction with prescribed properties, we consider a pencil of curves genus 1 on $\mathbf{P}^1 \times \mathbf{P}^1$ and Σ_2 as in §2. This can be done by 8-time blowing downs from rational elliptic surfaces X_n as above to $\mathbf{P}^1 \times \mathbf{P}^1$ or Σ_2 in special manners. This means that we need to find special configurations of 8 rational curves on X_n, which willl be done in the rest of this section for each case of X_n in the above table.

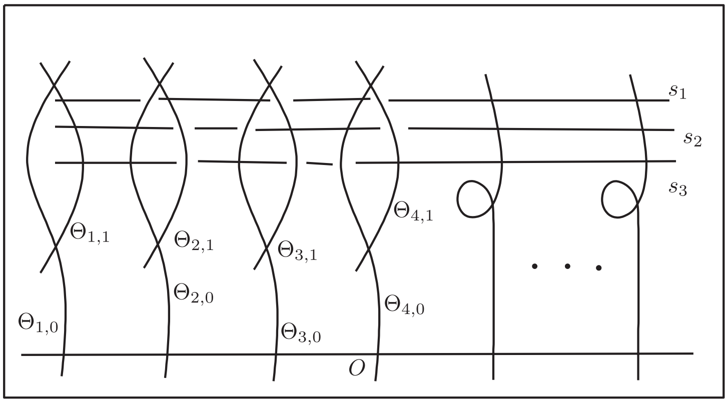
No. 9: Since $\mathrm{MW}(X_9) \cong D_4^*$, by Theorem 1.1, we infer that there exist integral sections s_1, s_2 and s_3 such that $\langle s_i, s_i \rangle = 1$, $(i = 1, 2, 3)$ and $\langle s_i, s_j \rangle = 1/2$ $(i \neq j)$.

By the correction term of the explicit formula for $\langle\, , \rangle$, we infer that s_1, s_2, s_3 and irreducible components of singular fiber intersect as in the following figure of No. 9.



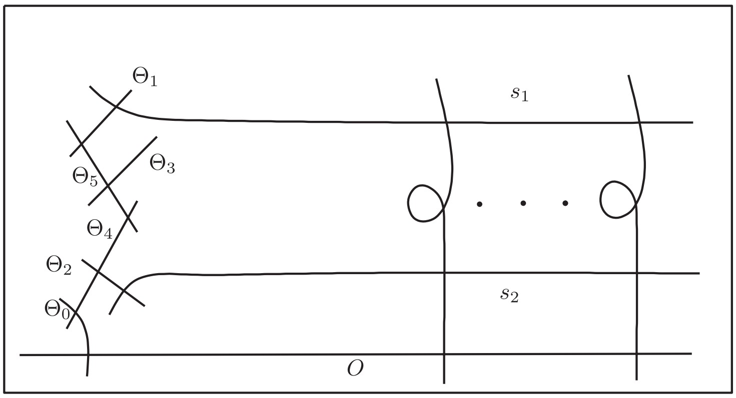
No. 9

No. 13: $\mathrm{MW}(X_{13}) \cong D_4^* \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{13})} \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 4}$. By Theorem 1.1, there exist integral sections s_1, s_2, s_3 such that $\langle s_i, s_i \rangle = 1$, $(i = 1, 2, 3)$ and $\langle s_i, s_j \rangle = 1/2$ $(i \neq j)$. We also denote a 2-torsion by τ. Let $F_i = \Theta_{i,0} + \Theta_{i,1}$ $(i = 1, 2, 3, 4)$ denote the irreducible decomposition of singular fibers. Here we label each irreducible component as in §1. From possible values of the correction terms of the explicit formula for $\langle s_i, s_j \rangle$ $(i, j = 1, 2, 3)$, we may assume that $s_i s_j = 0$ $(i \neq j)$, $\gamma_{13}(s_1) = (1, 1, 0, 0)$, $\gamma_{13}(s_2) = (1, 0, 1, 0)$ and $\gamma_{13}(s_3)$ is either $(1, 0, 0, 1)$ or $(0, 1, 1, 0)$. As $\gamma_{13}(\tau) = (1, 1, 1, 1)$, by replacing s_3 by $s_3 \dot{+} \tau$, if necessary, we may assume $\gamma_{13} = (1, 0, 0, 1)$. Thus we obtain the following figure for No. 13.



No. 13

No. 16: Since $\mathrm{MW}(X_{16}) \cong A_3^*$, by Theorem 1.1, there exist integral sections s_1 and s_2 such that $\langle s_1, s_1 \rangle = 3/4$, $\langle s_2, s_2 \rangle = 1$, $\langle s_1, s_2 \rangle = 1/2$. From possible values of the correction terms of the explicit formula for $\langle s_i, s_i \rangle$ $(i = 1, 2)$, we infer that s_1 and s_2 meet the I_1^*-fiber as in the figure for No. 16.



No. 16

No. 18: As $\mathrm{MW}(X_{18}) \cong (A_1^*)^{\oplus 3}$, by Theorem 1.1, we infer that there exist integral sections s_1, s_2 and s_3 such that $\langle s_i, s_i \rangle = 1/2$ $(i = 1, 2, 3)$, $\langle s_i, s_j \rangle = 0$ $(i \neq j)$. Put $s_4 := s_1 \dot{+} s_2$, $s_5 := s_2 \dot{+} s_3$, $s_6 := s_3 \dot{+} s_1$. By the explicit formula for $\langle \, , \rangle$, we have

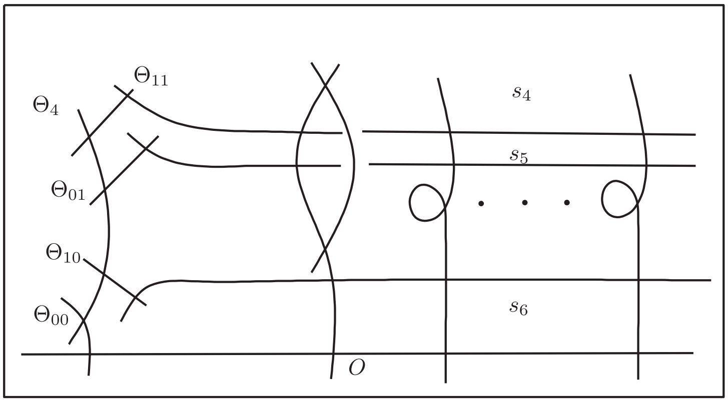
$$\langle s_i, s_i \rangle = 2 + 2 s_i O - a_i - b_i = 1 \quad (i = 4, 5, 6)$$

$$\langle s_i, s_j \rangle = 1 + s_i O + s_j O - s_i s_j - a_{ij} - b_{ij} = \frac{1}{2} \quad (4 \leq i < j \leq 6)$$

where $a_i \in \{0, 1\}$, $a_{ij} \in \{0, 1, 1/2\}$, $b_i, b_{ij} \in \{0, 1/2\}$. Hence we have $a_i = 1$, $b_i = 0$

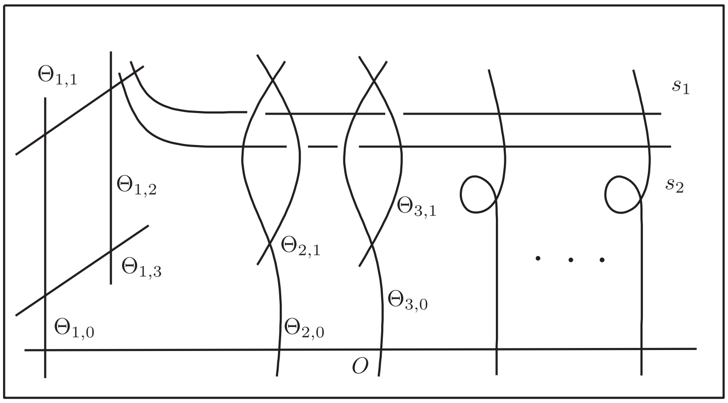
and $s_i O = 0$ $(i = 4, 5, 6)$, and this implies that $s_i s_j = 0$, $a_{ij} = 1/2$, $b_{ij} = 0$ $(4 \le i < j \le 6)$ by Corollaries 1.3 and 1.4.

Now by labeling irreducible components of the I_0^* fiber suitably, we have the figure for No. 18 as below:



No. 18

No. 21: $\mathrm{MW}(X_{21}) \cong A_3^* \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{21})} \cong \mathbf{Z}/4\mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. By Theorem 1.1, there exist integral sections s_1 and s_2 such that $\langle s_i, s_i \rangle = 3/4$ $(i = 1, 2)$ and $\langle s_1, s_2 \rangle = 1/4$. We also denote a 2-torsion by τ. By Corollary 1.4, $s_1 s_2 = 0$. Let $F_1 = \Theta_{1,0} + \Theta_{1,1} + \Theta_{1,2} + \Theta_{1,3}$ be the I_4 fiber and let $F_i = \Theta_{i,0} + \Theta_{i,1}$ $(i = 2, 3)$ be I_2-fibers. By labeling the irreducible components of these singular fibers as in §1, and the possible values of the correction terms of the explicit formula for $\langle s_i, s_i \rangle$ $(i = 1, 2)$, we may assume that $\gamma_{21}(s_1) = (1, 1, 0)$ and $\gamma_{21}(s_2)$ is either $(3, 1, 0)$ or $(1, 0, 1)$. As $\gamma_{21}(\tau) = (2, 1, 1)$, by replacing s_2 by $s_2 \dot{+} \tau$, if necessary, we may assume that $\gamma_{21}(s_2) = (1, 0, 1)$. Thus we have the figure for No. 21 as below:



No. 21

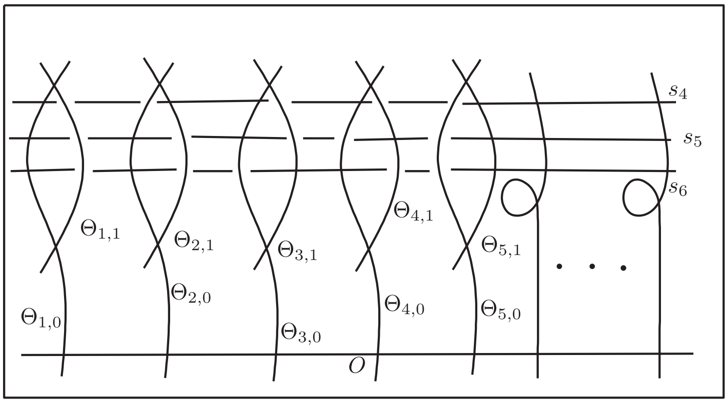
No. 24: $\mathrm{MW}(X_{24}) \cong (A_1^*)^{\oplus 3} \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{24})} \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 5}$. By Theorem 1.1, there exist integral sections s_1, s_2 and s_3 with $\langle s_i, s_i \rangle = 1/2$ $(i = 1, 2, 3)$ and $\langle s_i, s_j \rangle = 0$ $(1 \le i < j \le 3)$. By the explicit formula for $\langle s, s' \rangle$, $s, s' \in \mathrm{MW}(X_{24})$, we have

$$\langle s, s' \rangle = 1 + sO + s'O - ss' - \frac{k}{2}, \quad 0 \le k \le 5.$$

Also we denote a 2-torsion by τ. Note that the integer k in the above formula is equal to the number of common non-zero entries of $\gamma_{24}(s)$ and $\gamma_{24}(s')$. In particular, for $\langle s, s \rangle$, the integer k is equal to the number of non-zero entries of $\gamma_{24}(s)$. Without loss of generality, we may assume that $\gamma_{24}(\tau) = (0, 1, 1, 1, 1)$. As $\langle s_1, s_1 \rangle = \langle s_1 \dotplus \tau, s_1 \dotplus \tau \rangle = 1/2$, three of five entries of $\gamma_{24}(s_1)$ and $\gamma(s_1 \dotplus \tau)$ are 1. Hence we may assume that $\gamma_{24}(s_1) = (1, 1, 1, 0, 0)$. Similarly, three of five entries of $\gamma_{24}(s_2)$ are 1 and the first entry is 1. Hence by Corollaries 1.3 and 1.4, we infer that $s_1 s_2 = 0$ and the integer k in the above formula for $\langle s_1, s_2 \rangle$ is 2. Therefore $\gamma_{24}(s_1)$ and $\gamma_{24}(s_2)$ have two non-zero common entries, and we may assume that $\gamma_{24}(s_2) = (1, 1, 0, 1, 0)$. Under these circumstances, we infer that $\gamma_{24}(s_3)$ is either $(1, 1, 0, 0, 1)$ or $(1, 0, 1, 1, 0)$. If $\gamma_{24}(s_3) = (1, 1, 0, 0, 1)$, we replace s_3 by $s_3 \dotplus \tau$. Thus we may assume $\gamma_{24}(s_3) = (1, 0, 1, 1, 0)$. Now put

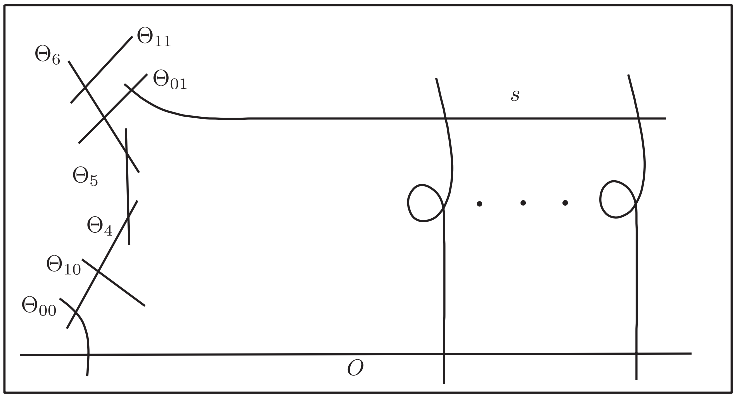
$$s_4 := s_1 \dotplus s_2 \dotplus \tau, \quad s_5 := s_2 \dotplus s_3 \dotplus \tau, \quad s_6 := s_3 \dotplus s_1 \dotplus \tau,$$

and we have $\gamma_{24}(s_4) = (0, 1, 0, 0, 1)$, $\gamma_{24}(s_5) = (0, 0, 0, 1, 1)$, $\gamma_{24}(s_6) = (0, 0, 1, 0, 1)$. As $\langle s_i, s_i \rangle = 1$ $(i = 4, 5, 6)$ and $\langle s_i, s_j \rangle = 1/2$ $(4 \le i < j \le 6)$, s_i $(i = 4, 5, 6)$ are integral and $s_i s_j = 0$ $(4 \le i < j \le 6)$ by Corollaries 1.3 and 1.4. Thus we obtain the following figure for No. 24:



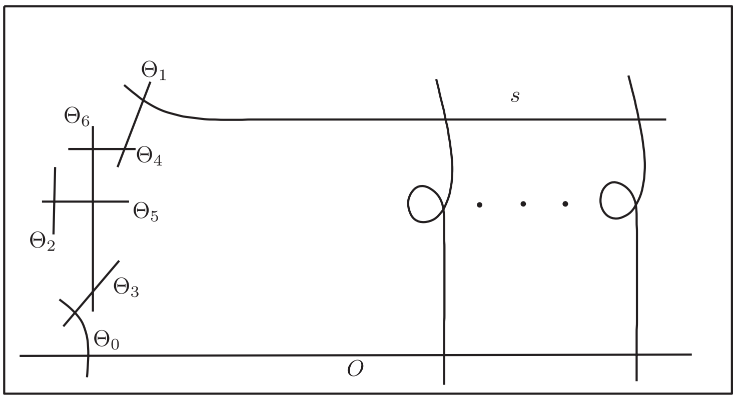
No. 24

No. 26: As $\mathrm{MW}(X_{26}) \cong (A_1^*)^{\oplus 2}$, by Theorem 1.1, there exists an integral section s with $\langle s, s\rangle = 1/2$. Hence the unique correction term of $\langle s, s\rangle$ is $3/2$. Thus we have the following figure for No. 26 below:



No. 26

No. 27: As $\mathrm{MW}(X_{27}) \cong A_2^*$, by Theorem 1.1, there exists a section s with $\langle s, s\rangle = 2/3$. The correction term of $\langle s, s\rangle$ is $4/3$. Thus we have the following figure for No. 27:



No. 27

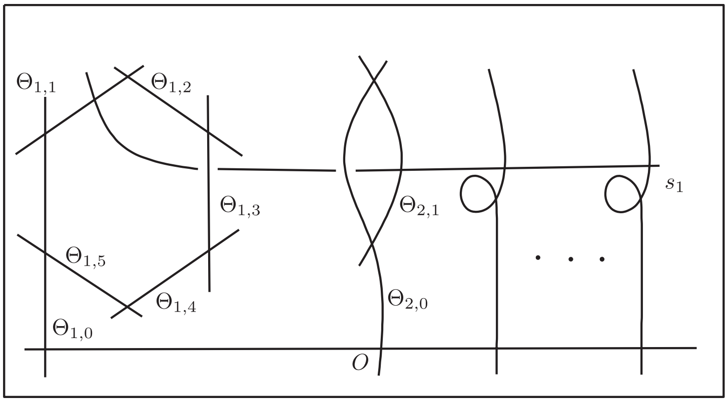
No. 28: $\mathrm{MW}(X_{28}) \cong A_2^* \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{28})} \cong \mathbf{Z}/6\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. By Theorem 1.1, there exists an integral section s_0 with $\langle s_0, s_0 \rangle = 2/3$. Also we denote a 2-torsion by τ. By the explicit formula for $\langle s, s \rangle$ for $s \in \mathrm{MW}(X_{28})$, we have

$$\langle s, s \rangle = 2 + 2sO - \frac{k_1}{6} - \frac{k_2}{2}, \quad k_1 \in \{0, 5, 8, 9\}, \quad k_2 \in \{0, 1\}$$

By the above formula for $\langle s, s \rangle$, $\gamma(s_0)$ is either $(2,0)$, $(4,0)$, $(1,1)$ or $(5,1)$ and $\gamma(\tau) = (3,1)$. Note that we can choose an integral section s_1 in such a way that $\gamma(s_1) = (1,1)$. In fact, we define s_1 as follows:

$$s_1 = \begin{cases} s_0 & \text{if } \gamma(s_0) = (1,1), \\ -s_0 \dotplus \tau & \text{if } \gamma(s_0) = (2,0), \\ s_0 \dotplus \tau & \text{if } \gamma(s_0) = (4,0), \\ -s_0 & \text{if } \gamma(s_0) = (5,1). \end{cases}$$

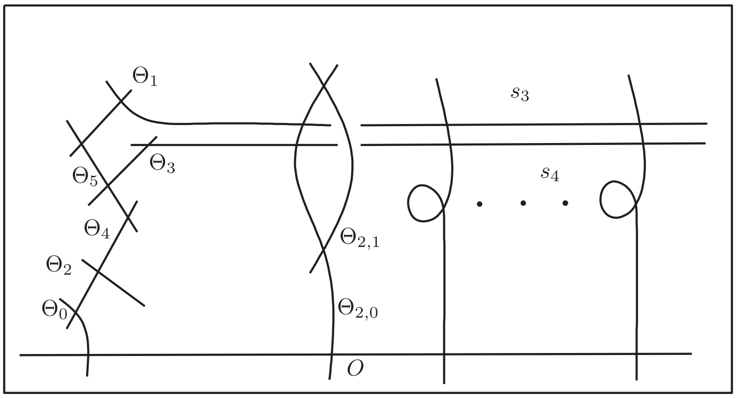
One see that s_1 is integral for every case as above by $\langle s_1, s_1 \rangle = 2/3$ and $\gamma_{28}(s_1) = (1,1)$. Thus we have the following figure for No. 28 as below:



No. 28

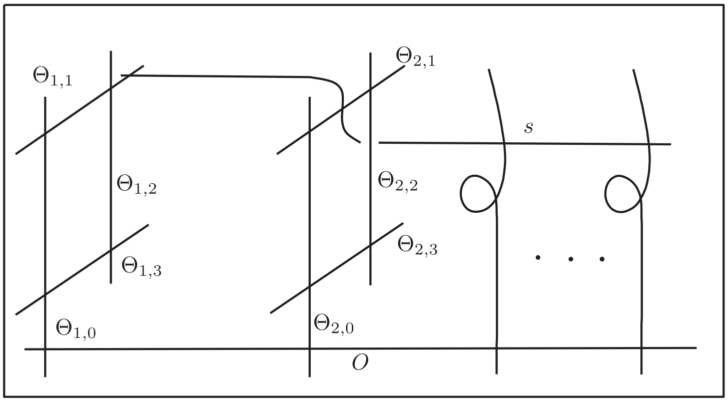
No. 30: $\mathrm{MW}(X_{30}) \cong A_1^* \oplus \langle 1/4 \rangle$ and $G_{\mathrm{Sing}(\varphi_{30})} \cong \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. By Theorem 1.1, there exist integral sections s_1 and s_2 with $\langle s_1, s_1 \rangle = 1/2$, $\langle s_2, s_2 \rangle = 1/4$ and $\langle s_1, s_2 \rangle = 0$. By considering possible values of the explicit formula for $\langle s, s \rangle$, $s \in \mathrm{MW}(X_{30})$, $\gamma_{30}(s_1) = (2,1)$ and $\gamma_{30}(s_2)$ is either $(1,1)$ or $(3,1)$. By considering $-s_2$, if necessarily, we may assume that $\gamma_{30}(s_2) = (3,1)$. Now put $s_3 := s_1 \dotplus s_2$ and $s_4 := s_1 \dotplus (-s_2)$. Then we have $\gamma_{30}(s_3) = (1,0)$ and $\gamma_{30}(s_4) = (3,0)$. Since $\langle s_3, s_3 \rangle = \langle s_4, s_4 \rangle = 3/4$ and $\langle s_3, s_4 \rangle = 1/4$, by Corollaries 1.3 and

1.4, we infer that both s_1 and s_2 are integral and $s_1 s_2 = 0$. Thus we have the following figure for No. 30:



No. 30

No. 35: $\mathrm{MW}(X_{35}) \cong (A_1^*)^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{35})} \cong (\mathbf{Z}/4\mathbf{Z})^{\oplus 2}$. By Theorem 1.1, there exists an integral section s such that $\langle s, s \rangle = 1/2$. After suitable lableling of irreducible components of I_4-fibers, we may assume that $\gamma_{35}(s) = (1, 1)$. Thus we have the following figure for No. 35:



No. 35

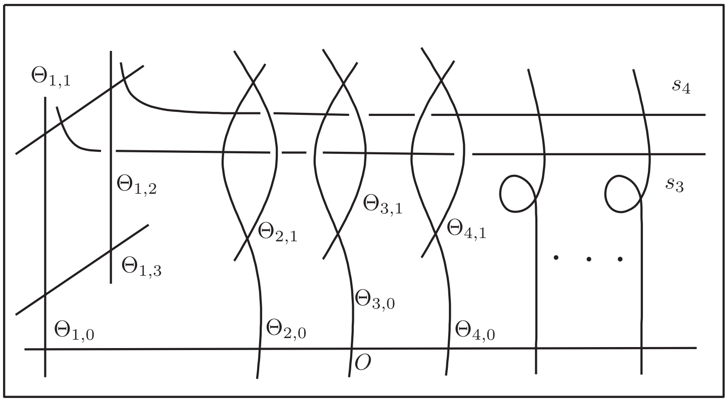
No. 38: $\mathrm{MW}(X_{38}) \cong A_1^* \oplus \langle 1/4 \rangle \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{38})} \cong \mathbf{Z}/4\mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. By Theorem 1.1, there exist a 2 torsion τ and integral sections s_1 and s_2 with $\langle s_1, s_1 \rangle = 1/2$, $\langle s_2, s_2 \rangle = 1/4$ and $\langle s_1, s_2 \rangle = 0$. By possible values of the correction terms of $\langle s_2, s_2\rangle$, we may assume that $\gamma_{38}(s_2) = (1,1,1,0)$. Since $\langle \tau, \tau\rangle = 0$ and $\langle s_2 \dotplus \tau, s_2 \dotplus \tau\rangle = 1/4$, $\gamma_{38}(\tau)$ is either $(2,1,0,1)$ or $(2,0,1,1)$. If $\gamma_{38}(\tau) = (2,1,0,1)$, by $\langle s_1, s_1\rangle = \langle s_1 \dotplus \tau, s_1 \dotplus \tau\rangle = 1/2$, $\gamma_{38}(s_1)$ is either $(2,0,1,0)$ or $(0,1,1,1)$. Similarly, if $\gamma_{38}(\tau) = (2,0,1,1)$, $\gamma_{38}(s_1)$ is either $(2,1,0,0)$ or $(0,1,1,1)$. Thus, by replacing s_1 by $s_1 + \tau$ if necessary, we may assume that

$$\gamma_{38}(s_1) = (0,1,1,1), \quad \gamma_{38}(s_2) = (1,1,1,0), \quad \gamma_{38}(\tau) = (2,0,1,1).$$

Now put

$$s_3 := s_1 \dotplus (-s_2) \dotplus \tau, \quad s_4 := s_1 \dotplus s_2,$$

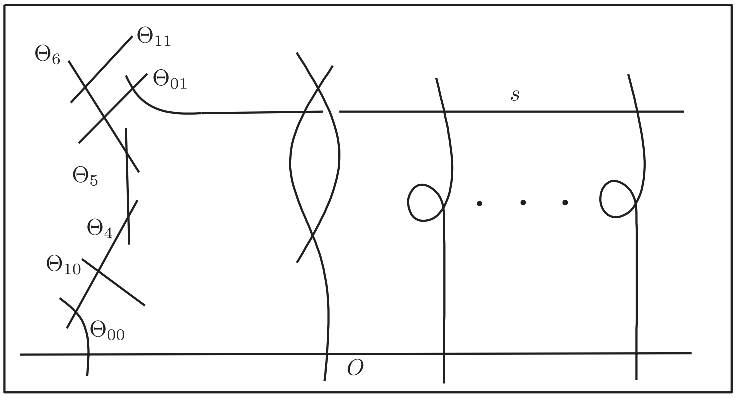
Then we have $\gamma_{38}(s_3) = (1,0,1,0)$ and $\gamma_{38}(s_4) = (1,0,0,1)$. Since $\langle s_3, s_3\rangle = \langle s_4, s_4\rangle = 3/4$, $\langle s_3, s_4\rangle = 0$, s_3 and s_4 are integral and $s_3 s_4 = 0$ by Corollaries 1.3 and 1.4. Therefore we have the following figure for No. 38:



No. 38

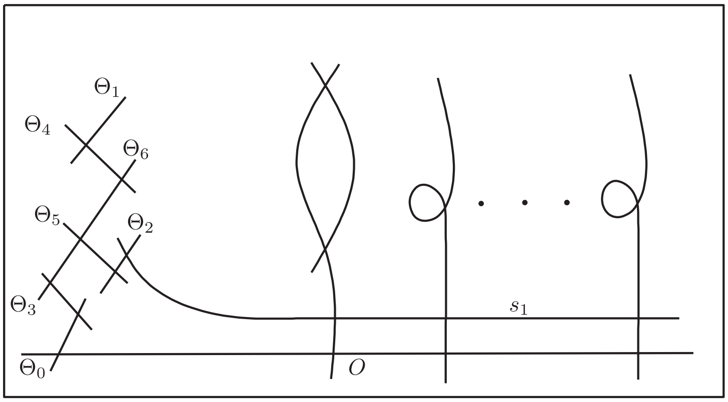
No. 43 and 44: For each case, we label irreducible components of its unique reducible singular fiber as in §1.

No. 48: $\mathrm{MW}(X_{48}) \cong A_1^* \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi_{48})} \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z}$. By Theorem 1.1, there exist a 2-torsion τ and an integral section s with $\langle s, s\rangle = 1/2$. As $\langle \tau, \tau\rangle = 0$, we may assume $\gamma_{48}(\tau) = (1,1,1)$. Since $\langle s, s\rangle = \langle s \dotplus \tau, s \dotplus \tau\rangle = 1/2$, we may assume that $\gamma_{48}(s) = (0,1,0)$, after exchange s and $s \dotplus \tau$, if necessary. Thus we have the following figure for No. 48:



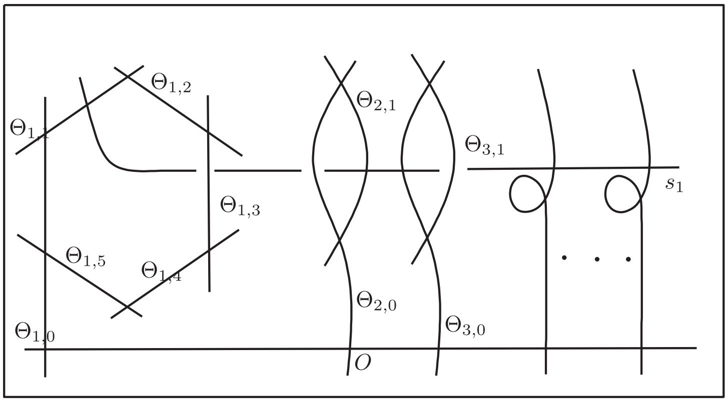
No. 48

No. 49: $\mathrm{MW}(X_{49}) \cong \langle 1/6 \rangle$ and $G_{\mathrm{Sing}(\varphi)} \cong \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. By Theorem 1.1, there exists an integral section s with $\langle s, s \rangle = 1/6$. By possible values of the explicit formula for $\langle\, , \rangle$, we may assume that $\gamma_{49}(s) = (1, 1)$. Put $s_1 := 2s$, then we have $\gamma_{49}(s_1) = (2, 0)$. Since $\langle s_1, s_1 \rangle = 2/3$, s_1 is integral by Corollary 1.3. Thus we obtain the following figure for No. 49:



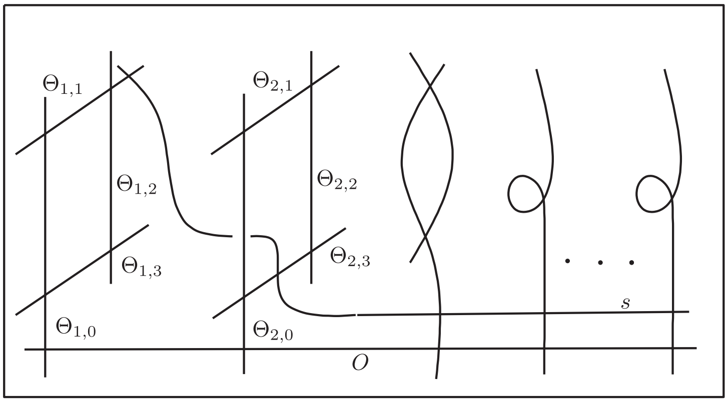
No. 49

No. 53: $\mathrm{MW}(X_{53}) \cong \langle 1/6 \rangle \oplus \mathbf{Z}/2\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi)} \cong \mathbf{Z}/6\mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. By Theorem 1.1, there exist a 2-torsion τ and an integral section s with $\langle s, s \rangle = 1/6$. Since $\langle \tau, \tau \rangle = 0$, we may assume that $\gamma_{53}(\tau) = (3, 1, 0)$. As $\langle s, s \rangle = 1/6$, we may assume that $\gamma_{53}(s)$ is either $(1, 1, 1)$ or $(\pm 2, 0, 1)$. If $\gamma_{53}(s) = (1, 1, 1)$, $\gamma_{53}(s \dotplus \tau) = (-2, 0, 1)$ and $\gamma_{53}(-(s \dotplus \tau)) = (2, 0, 1)$. Hence we may assume that $\gamma_{53}(s) = (2, 0, 1)$. Now put $s_1 := 2s \dotplus \tau$. Then $\gamma_{53}(s_1) = (1, 1, 0)$ and s_1 is integral. Thus we have the following figure:



No. 53

No. 58: $\mathrm{MW}(X_{58}) \cong A_1^* \oplus \mathbf{Z}/4\mathbf{Z}$ and $G_{\mathrm{Sing}(\varphi)} \cong (\mathbf{Z}/4\mathbf{Z})^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z}$. Let τ be a 4-torsion. Since any torsion section is integral (see [8]), by Corollary 1.1, we may assume that $\gamma_{58}(\tau) = (1,1,1)$. Let s be a generator of A_1^*. By Theorem 1.1, we may assume that s is integral. As $\langle s, s\rangle = \langle s \dot{+} \tau, s \dot{+} \tau\rangle = 1/2$, we infer that $\gamma_{58}(s)$ is either $(2,0,1)$ or $(1,3,0)$. If $\gamma_{58}(s) = (2,0,1)$, then $\gamma_{58}(s \dot{+} (-\tau)) = (1,3,0)$. Hence we may assume that $\gamma_{58}(s) = (1,3,0)$l. Thus we have the following figure for No. 58:



No. 58

No. 65 and 70: For each case, we label irreducible components of a III^* (resp. I_8) singular fiber for No. 65 (resp. No. 70) as in §1.

4. Construction of CQ configurations of types I and II via rational elliptic surfaces

We keep our notation in the previous sections. As we have seen in the last section, given a CQ-configuration of type I or II, we canonically obtain a rational elliptic surface $\varphi_{C+Q} : X_{C+Q} \to \mathbf{P}^1$. In order to obtain a CQ configuration with prescribed $I(C, Q)$, we consider the converse:

(i) Take an appropriate rational elliptic surface $\varphi : X \to \mathbf{P}^1$.

(ii) Blow down X to $\mathbf{P}^1 \times \mathbf{P}^1$ (resp. Σ_2) for the case of type I (resp. type II) in a suitable way.

(iii) Choose a singular fiber F_o of type I_1 and a smooth fiber F. Let C_{F_o} and Q_F be their proper images under the birational map Φ_{I}^{-1} (resp. Φ_{II}^{-1}).

We then infer that $C_{F_o} + Q_F$ is the desired CQ configuration. More precisely, we have the following proposition:

PROPOSITION 4.1. *Let* $\varphi_n : X_n \to \mathbf{P}^1$ *be the rational elliptic surface as in* §2. *Let* F_o *and* F *be as above. After the procedure* (i)–(iii), *we obtain a CQ configuration of Type I or II as in the table below*:

No. of X_n	Type	$I(C_{F_o}, Q_F)$	No. of X_n	Type	$I(C_{F_o}, Q_F)$
9	I	$(2,2,2,2)$	35	I	$(4,4)$
13	I	$(2,2,2,2)$	38	II	$(4,2,2)$
16	I	$(4,2,2)$	43	I	(8)
18	II	$(2,2,2,2)$	44	I	(8)
21	I	$(4,2,2)$	48	II	$(4,4)$
24	II	$(2,2,2,2)$	49	II	$(6,2)$
26	I	$(4,4)$	53	II	$(6,2)$
27	I	$(6,2)$	58	II	$(4,4)$
28	I	$(6,2)$	65	II	(8)
30	II	$(4,2,2)$	70	II	(8)

Proof. For each X_n, let $\nu_n : X_n \to \overline{X}_n$ be a birational morphism obtained by blowing down the curves in the middle column of the table below from the left to the right. By Corollary 1.2, we infer that $\overline{X}_n$ is either $\mathbf{P}^1 \times \mathbf{P}^1$, Σ_2 or Σ_1. We show that

- $\overline{X}_n$ is as in the right column in the table below, and
- $C_{F_o} + Q_F$ gives the desired CQ-configulration, if we choose F_o and F suitably.

This will be done by case-by-case.

No of X_n	Exceptional curves of v	$\bar{X}_n$
9	$O, \Theta_{00}, s_1, \Theta_{11}, s_2, \Theta_{01}, s_3, \Theta_{10}$	$\mathbf{P}^1 \times \mathbf{P}^1$
13	$O, \Theta_{1,0}, s_1, \Theta_{2,1}, s_2, \Theta_{3,1}, s_3, \Theta_{4,1}$	$\mathbf{P}^1 \times \mathbf{P}^1$
16	$O, \Theta_0, s_2, \Theta_2, s_1, \Theta_1, \Theta_5, \Theta_3$	$\mathbf{P}^1 \times \mathbf{P}^1$
18	$O, \Theta_{00}, s_4, \Theta_{11}, s_5, \Theta_{01}, s_6, \Theta_{10}$	Σ_2
21	$O, \Theta_{1,0}, \Theta_{1,3}, \Theta_{1,2}, s_1, \Theta_{2,1}, s_2, \Theta_{3,1}$	$\mathbf{P}^1 \times \mathbf{P}^1$
24	$O, \Theta_{5,0}, s_4, \Theta_{2,1}, s_5, \Theta_{4,1}, s_6, \Theta_{3,1}$	Σ_2
26	$O, \Theta_{00}, \Theta_4, \Theta_{10}, s, \Theta_{01}, \Theta_6, \Theta_{11}$	$\mathbf{P}^1 \times \mathbf{P}^1$
27	$O, \Theta_0, s, \Theta_1, \Theta_4, \Theta_6, \Theta_5, \Theta_2$	$\mathbf{P}^1 \times \mathbf{P}^1$
28	$O, \Theta_{1,0}, \Theta_{1,5}, \Theta_{1,4}, \Theta_{1,3}, \Theta_{1,2}, s_1, \Theta_{2,1}$	$\mathbf{P}^1 \times \mathbf{P}^1$
30	$O, \Theta_0, \Theta_4, \Theta_2, s_3, \Theta_1, s_4, \Theta_3$	Σ_2
35	$O, \Theta_{1,0}, \Theta_{1,3}, \Theta_{1,2}, s, \Theta_{2,1}, \Theta_{2,2}, \Theta_{2,3}$	$\mathbf{P}^1 \times \mathbf{P}^1$
38	$O, \Theta_{1,0}, \Theta_{1,3}, \Theta_{1,2}, s_3, \Theta_{3,1}, s_4, \Theta_{4,1}$	Σ_2
43	$O, \Theta_0, \Theta_2, \Theta_3, \Theta_4, \Theta_6, \Theta_7, \Theta_1$	$\mathbf{P}^1 \times \mathbf{P}^1$
44	$O, \Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6$	$\mathbf{P}^1 \times \mathbf{P}^1$
48	$O, \Theta_{00}, \Theta_4, \Theta_{10}, s, \Theta_{01}, \Theta_6, \Theta_{11}$	Σ_2
49	$O, \Theta_0, s_1, \Theta_2, \Theta_5, \Theta_6, \Theta_4, \Theta_1$	Σ_2
53	$O, \Theta_{1,0}, \Theta_{1,5}, \Theta_{1,4}, \Theta_{1,3}, \Theta_{1,2}, s_1, \Theta_{2,1}$	Σ_2
58	$O, \Theta_{1,0}, \Theta_{1,3}, \Theta_{1,2}, s, \Theta_{2,3}, \Theta_{2,2}, \Theta_{2,1}$	Σ_2
65	$O, \Theta_0, \Theta_2, \Theta_3, \Theta_4, \Theta_6, \Theta_7, \Theta_1$	Σ_2
70	$O, \Theta_0, \Theta_7, \Theta_6, \Theta_5, \Theta_4, \Theta_3, \Theta_2$	Σ_2

No. 9: By its definition of v_9, and we easily see that $v_9(s)^2 = 0$ for $\forall s \in \mathrm{MW}(X_9) \backslash \{O, s_1, s_2, s_3\}$ and $v_9(\Theta_4)^2 = 2$. Hence by Lemma 1.2, there is no curve with negative self-intersection number. Hence we infer that $\bar{X}_9 \cong \mathbf{P}^1 \times \mathbf{P}^1$. Let l_1 and l_2 be two lines on $\mathbf{P}^1 \times \mathbf{P}^1$ such that $l_i^2 = 0$ $(i = 1, 2)$, $l_1 l_2 = 1$ and $l_1 \cap l_2$ is the node of $v_9(F_o)$. Since $v_9^{-1}(l_i)$ $(i = 1, 2)$ are double sections of $\varphi_9 : X_9 \to \mathbf{P}^1$, we may assume that F meets both l_1 and l_2 transversely. Now by considering the proper images of $v_9(F_o)$ and $v_9(F)$ under Φ_{I}^{-1}, we have the desired *CQ*-configuration.

For the cases $n = 16, 26, 27, 43$, we similarly obtain the desired CQ-configurations, so we omit their proof.

No. 13. By Lemma 1.2, we infer that there is no curve whose self-intersection number is -2. Hence $\overline{X}_{13} \neq \Sigma_2$. We show that $\overline{X}_{13} \neq \Sigma_1$. Suppose that $\overline{X}_{13} \cong \Sigma_1$. Then the unique (-1) curve gives rise to an integral section $\tilde{s}$ with $\gamma_{13}(\tilde{s}) = (1,0,0,0)$, and we have $\langle \tilde{s}, \tilde{s} \rangle = 3/2$. On the other hand, as $\mathrm{MW}(X_{13}) \cong D_4^* \oplus \mathbf{Z}/2\mathbf{Z}$, $\langle s, s \rangle$ is an integer for $\forall s \in \mathrm{MW}(X_{13})$. This leads us to a contradiction. Hence $\overline{X}_{13} \cong \mathbf{P}^1 \times \mathbf{P}^1$. Now similar argument to the case of No. 9 shows the existence of the desired CQ-configuration.

No. 21. We only need to show that $\overline{X}_{21} \cong \mathbf{P}^1 \times \mathbf{P}^1$ as the remaining statement can be proved in a similar manner to the previous cases. By Lemma 1.2, $\overline{X}_{21} \not\cong \Sigma_2$. Suppose that $\overline{X}_{21} \cong \Sigma_1$. Then the (-1) curve gives rise to an integral section $\tilde{s}$ with $\gamma_{21}(\tilde{s}) = (1,0,0)$. Then $\gamma_{21}(\tilde{s} \dot{+} \tau) = (3,1,1)$, where τ denotes a 2-torsion. Thus we have $\langle \tilde{s}, \tilde{s} \rangle = 5/4$, and $\langle \tilde{s} \dot{+} \tau, \tilde{s} \dot{+} \tau \rangle = 1/4 + 2(\tilde{s} \dot{+} \tau)O$. On the other hand, by the property of the height pairing, we have $\langle \tilde{s} \dot{+} \tau, \tilde{s} \dot{+} \tau \rangle = \langle \tilde{s}, \tilde{s} \rangle$. This leads us to a contradiction. Hence $\overline{X}_{21} \cong \mathbf{P}^1 \times \mathbf{P}^1$.

No. 28. It is enough to show that $\overline{X}_{28} \cong \mathbf{P}^1 \times \mathbf{P}^1$. By Lemma 1.2, $\overline{X}_{28} \not\cong \Sigma_2$. Suppose that $\overline{X}_{28} \cong \Sigma_1$. Then the (-1) curve gives rise to an integral section $\tilde{s}$ with $\gamma_{28}(\tilde{s}) = (1,0)$. Hence we have $\langle \tilde{s}, \tilde{s} \rangle = 7/6$, but this is impossible as $\mathrm{MW}(X_{28}) \cong A_2^* \oplus \mathbf{Z}/2\mathbf{Z}$.

No. 35. It is enough to show that $\overline{X}_{35} \cong \mathbf{P}^1 \times \mathbf{P}^1$. By Lemma 1.2, $\overline{X}_{28} \not\cong \Sigma_2$. Suppose that $\overline{X}_{35} \cong \Sigma_1$. Then the (-1) curve gives rise to an integral section $\tilde{s}$ with $\gamma_{35}(\tilde{s}) = (1,0)$. Hence we have $\langle \tilde{s}, \tilde{s} \rangle = 5/4$, but this is impossible as $\mathrm{MW}(X_{35}) \cong (A_1^*)^{\oplus 2} \oplus \mathbf{Z}/2\mathbf{Z}$.

No. 44. It is enough to show that $\overline{X}_{44} \cong \mathbf{P}^1 \times \mathbf{P}^1$. By Lemma 1.2, $\overline{X}_{44} \not\cong \Sigma_2$. Suppose that $\overline{X}_{44} \cong \Sigma_1$. Then the (-1) curve gives rise to an integral section $\tilde{s}$ with $\gamma_{44}(\tilde{s}) = (1)$. Hence we have $\langle \tilde{s}, \tilde{s} \rangle = 9/8$, but this is impossible as $\mathrm{MW}(X_{44}) \cong A_1^* \oplus \mathbf{Z}/2\mathbf{Z}$.

No. 18, 24, 30, 38, 48, 49, 53, 58, 65, 70. For these cases, one can easily see that there exists a (-2)-curve on $\overline{X}_n$. Hence by Corollary 1.2, $\overline{X}_n \cong \Sigma_2$. Choose a fiber, $\mathfrak{f}_o$, of $\overline{X}_n$ passing through the node of $v_n(F_o)$. Since $v_n^{-1}(\mathfrak{f}_o)$ is a double section of φ_n, we may assume $v_n(F)$ meets $\mathfrak{f}_o$ transversely. Now by considering the proper images of $v_n(F_o)$ and $v_n(F)$ under Φ_{II}^{-1}, we have the desired CQ-configuration. □

In the following, we denote the CQ-configuration obtained from $\varphi_n : X_n \to \mathbf{P}^1$ as above by $C_n + Q_n$. In the next section, we determine the value (C_n/Q_n).

5. Proof of Theorem 0.2

We keep our notation as before. In this section, we consider the value of (C_n/Q_n) for the CQ-configurations given in §4. By combining Proposition 4.1, we obtain Theorem 0.2. Let us start with the following lemma.

LEMMA 5.1. *Let $C_n + Q_n$ be the CQ-configuration in the previous section. Let L_n be the line passing through two nodes (resp. the maximal tangent line) for type I (resp. type II). If $(C_n/Q_n) = 1$, then Q_n is given by an equation of the form*

$$g_2^2 + l_{L_n}^2 g_{C_n},$$

where g_2 is a homogeneous polynomial of degree 2 and g_{C_n} and l_{L_n} are defining equations of C_n and L_n, respectively.

Conversely, if there exists an irreducible conic $C_{o,n}$ such that $I_x(C_{o,n}, C_n) = 1/2 I_x(Q_n, C_n)$, $I_x(C_{o,n}, L_n) = 1/2 I_x(Q_n, L_n)$ for $\forall x \in (C_n \cap Q_n) \cup (L_n \cap Q_n)$, then $(C_n/Q_n) = 1$.

Proof. Let $q_n : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ be the double cover with $\Delta_{q_n} = C_n$. We choose affine coordinates (x, y) and (u, v) of $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ so that

$$q_n : \mathbf{P}^1 \times \mathbf{P}^1 \ni (x, y) \mapsto (u, v) = (x + y, xy) \in \mathbf{P}^2.$$

If $(C_n/Q_n) = 1$, i.e., $q_n^* Q_n = Q_n^+ + Q_n^-$, $Q_n^+ \neq Q_n^-$, we may assume that $Q_n^{\pm}$ are given by

$$Q_n^{\pm} : g_2(u, v) \pm g_1(u, v)(x - y) = 0, \quad g_i(u, v) \in \mathbf{C}[u, v], \quad \deg g_i = i.$$

Hence Q_n is given by $g_2^2 - g_1^2(u^2 - 4v) = 0$. Since any point satifying $g_1 = g_2 = 0$ is a singular point of Q_n, we infer that L_n is given by $g_1 = 0$.

Conversely, if such an irreducible conic $C_{o,n}$ as above exists, we infer that $2C_{o,n}$ is a member of the pencil generated by Q_n and $2L_n + C_n$. This means that Q_n is given by an equation of the form

$$g_{C_{o,n}}^2 + l_{L_n}^2 g_{C_n},$$

where $g_{C_{o,n}}$ is a defining equation of $C_{o,n}, L_n$ and C_n. Hence $(C_n/Q_n) = 1$. □

Now we determine the value of (C_n/Q_n).

PROPOSITION 5.1. *We have the following table*:

n	$9, 16, 18, 26, 27, 30, 43, 48, 49, 65$	$13, 21, 24, 28, 35, 38, 44, 53, 58, 70$
(C_n/Q_n)	1	-1

Proof. Suppose that $(C_n/Q_n) = 1$. Since $\varphi_n : X_n \to \mathbf{P}^1$ is determined by the pencil generated by Q_n and $C_n + 2L_n$, it has a singular fiber which is not of type

I_n. In fact, let $C'_{o,n}$ be a conic given by $g_2 = 0$ in Lemma 5.1 and let $\widetilde{C'}_{o,n}$ be the image of $C'_{o,n}$ under the birational maps Φ_{I} and Φ_{II} as in §3. Then we see that $2\widetilde{C'}_{o,n}$ is contained in a member of the pencil generated by $\tilde{Q}_n$ and $\widetilde{C'}_n$. Hence any irreducible component of $\widetilde{C'}_{o,n}$ gives rise to a non-reduced irreducible component of a singular fiber. Hence $C_n + Q_n$ for $n = 13, 21, 24, 28, 35, 38, 44, 43, 58$ and 70, we have $(C_n/Q_n) = -1$.

For $n = 9, 16, 18, 26, 27, 30, 43, 48, 49$ and 65, the irreducible component in the table below gives rise to an irreducible conic satisfying the conditions in Lemma 5.1. Hence our statement follows:

Singular fiber	the irreducible component
I_0^*	Θ_4 $(n = 9, 18)$
I_1^*	Θ_4 $(n = 16)$, Θ_5 $(n = 30)$
I_2^*	Θ_5 $(n = 26, 48)$
IV^*	Θ_4 $(n = 24, 49)$
III^*	Θ_4 $(n = 43, 65)$

□

Let $q_n : Z_n(\cong \mathbf{P}^1 \times \mathbf{P}^1) \to \mathbf{P}^2$ be the double cover with branch locus C_n. For n such that $(C_n/Q_n) = 1$, we see that $q_n^* Q_n = Q_n^+ + Q_n^-$ and $Q_n^+ \sim Q_n^- \sim (2, 2)$. Hence by [17, Theorem 0.2, Corollary 0.2], we have

Proposition 5.2. *Let k be an integer ≥ 2 and let $\mathscr{D}_{2k}$ denote the dihedral group of order $2k$.*

- *If $(C_n/Q_n) = 1$, there exists an epimorphism from the fundamental group $\pi_1(\mathbf{P}^2 \backslash (C_n + Q_n), *) \to \mathscr{D}_{2k}$ for any k.*
- *If $(C_n/Q_n) = -1$, there exists no epimorphism from the fundamental group $\pi_1(\mathbf{P}^2 \backslash (C_n + Q_n), *) \to \mathscr{D}_{2k}$ for any odd k.*

Now the following corollary is immediate:

Corollary 5.1. *The pairs of sextic curves*

$$(C_9 + Q_9, C_{13} + Q_{13}), \quad (C_{16} + Q_{16}, C_{21} + Q_{21}), \quad (C_{26} + Q_{26}, C_{35} + Q_{35}),$$
$$(C_{27} + Q_{27}, C_{28} + Q_{28}), \quad (C_{43} + Q_{43}, C_{44} + Q_{44}), \quad (C_{18} + Q_{18}, C_{24} + Q_{24}),$$
$$(C_{30} + Q_{30}, C_{38} + Q_{38}), \quad (C_{48} + Q_{48}, C_{58} + Q_{58}), \quad (C_{49} + Q_{49}, C_{53} + Q_{53}),$$
$$(C_{65} + Q_{65}, C_{70} + Q_{70})$$

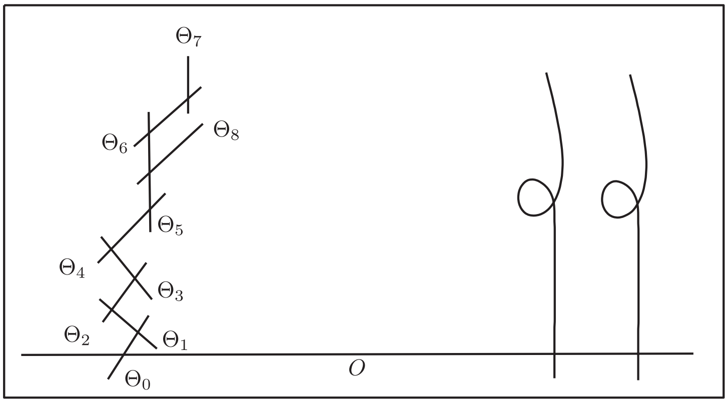
are Zariski pairs.

Remark 5.1. Zariski pairs in Corollary 5.1 can be found in [12]. Our justification is that their construction is different from that in [12].

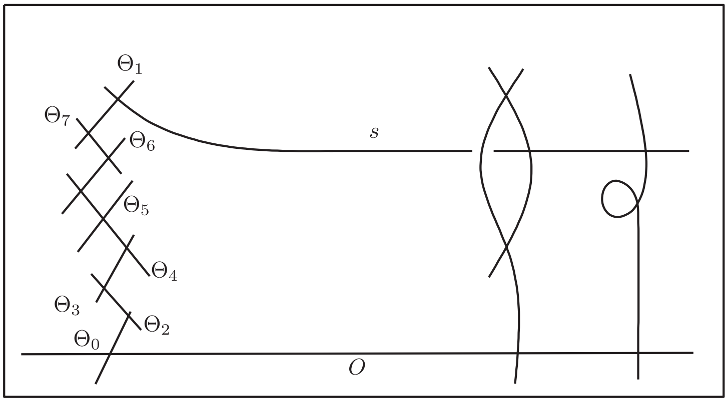
6. Further examples

We consider two more examples of *CQ*-configurations related to Zariski triples given in [12].

We label irreducible components of singular fibers and sections on rational elliptic surfaces X_{62} and X_{65} as follows:



No. 62



No. 65

We blow down

$$O, \Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_8 \quad \text{for } X_{62}$$

and

$$O, \Theta_0, \Theta_2, \Theta_3, s, \Theta_1, \Theta_7, \Theta_6 \quad \text{for } X_{65}$$

in this order. Then for both cases we have rational surfaces Σ with Picard number 2 and the images of Θ_7 for X_{62} and Θ_5 for X_{65} are (-2) curves. Hence by Corollary 1.2, $\Sigma = \Sigma_2$. Now for both cases, let F_o be one I_1-fiber and let F_1 be a general smooth fiber. Let $\tilde{C}_n$ and $\tilde{Q}_n$ $(n = 62, 65)$ be the proper transform of the birational map $p \circ q^{-1}$ of type II. Then by our construction, the following statement is immediate.

PROPOSITION 6.1. (i) *For* $n = 62$, $I(\tilde{C}_{62}, \tilde{Q}_{62}) = (8)$ *and the tangent line at* $\tilde{C}_{62} \cap \tilde{Q}_{62}$ *passes through the tacnode of* $\tilde{Q}_{62}$.

(ii) *For* $n = 65$, $I(\tilde{C}_{65}, \tilde{Q}_{65}) = (4, 4)$ *and the two points in* $\tilde{C}_{65} \cap \tilde{Q}_{65}$ *and the tacnode of* $\tilde{Q}_{65}$ *are collinear.*

Remark 6.1. (i) Note that $(C_{48} + Q_{48}, C_{58} + Q_{58}, \tilde{C}_{65} + \tilde{Q}_{65})$ and $(\tilde{C}_{62} + \tilde{Q}_{62}, C_{65} + Q_{65}, C_{70} + Q_{70})$ are Zariski triples given in [12].

(ii) By taking the affine coordinate as in the proof of Lemma 5.1, we can choose defining equations of $\tilde{Q}_{62}$ and $\tilde{Q}_{65}$ of the form $l^4 + g_1(u, v)(u^2 - 4v)$, where l is a defining equation of the tangent line at $\tilde{C}_{62} \cap \tilde{Q}_{62}$ for No. 62 and the line connecting $\tilde{C}_{65} \cap \tilde{Q}_{65}$ and the tacnode of $\tilde{Q}_{65}$.

REFERENCES

[1] E. ARTAL BARTOLO, J.-I. CODGOLLUDO AND H. TOKUNAGA, A survey on Zariski pairs, Adv. Stud. Pure Math. **50** (2008), 1–100.

[2] E. ARTAL BARTOLO AND H. TOKUNAGA, Zariski k-plets of rational curve arrangements and dihedral covers, Topology Appl. **142** (2004), 227–233.

[3] W. BARTH, K. HULEK, C. A. M. PETERS AND A. VAN DE VEN, Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete **4**, 2nd Enlarged Edition, Springer-Verlag, 2004.

[4] W. FULTON, Algebraic curves, Mathematics lecture note series, Benjamin, 1969.

[5] E. HORIKAWA, On deformation of quintic surfaces, Invent. Math. **31** (1975), 4–85.

[6] K. KODAIRA, On compact analytic surfaces II, Ann. of Math. **77** (1963), 563–626.

[7] R. MIRANDA, The moduli of Weierstrass fibrations over $\mathbf{P}^1$, Math. Ann. **255** (1981), 379–394.

[8] R. MIRANDA, Basic theory of elliptic surfaces, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.

[9] R. MIRANDA AND U. PERSSON, On extremal rational elliptic surfaces, Math. Z. **193** (1986), 537–558.

[10] K. OGUISO AND T. SHIODA, The Mordell-Weil lattice of rational elliptic surface, Comment. Math. Univ. St. Pauli **40** (1991), 83–99.

[11] U. PERSSON, Configurations of Kodaira fibers on rational elliptic surfaces, Math. Z. **205** (1990), 1–49.

[12] I. SHIMADA, Lattice Zariski k-ples of plane sextic curves and Z-splitting curves for double plane sextics, Michigan Math. J. **59** (2010), 621–665.

[13] T. SHIODA, On the Mordell-Weil lattices, Comment. Math. Univ. St. Pauli **39** (1990), 211–240.

[14] T. SHIODA AND H. USUI, Fundamental invariants of Weyl groups and excellent families of elliptic curves, Comment. Math. Univ. St. Pauli **41** (1992), 169–217.

[15] H. TOKUNAGA, On dihedral Galois coverings, Canadian J. of Math. **46** (1994), 1299–1317.

[16] H. TOKUNAGA, Dihedral covers and an elemetary arithmetic on elliptic surfaces, J. Math. Kyoto Univ. **44** (2004), 55–270.

[17] H. Tokunaga, Geometry of irreducible plane quartics and their quadratic residue conics, J. of Singul. **2** (2010), 170–190.

[18] O. Zariski, On the purity of the branch locus of algebraic functions, Proc. Nat. Acad. USA **44** (1958), 791–796.

Hiro-o Tokunaga
Department of Mathematics and Information Sciences
Graduate School of Science and Engineering
Tokyo Metropolitan University
1-1 Minami-Ohsawa, Hachiohji 192-0397
Japan
E-mail: tokunaga@tmu.ac.jp