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LOCAL PROPERTIES ON THE REMAINDERS OF THE TOPOLOGICAL GROUPS

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Abstract

When does a topological group G have a Hausdorff compactification bG with a remainder belonging to a given class of spaces? In this paper, we mainly improve some results of A. V. Arhangel'skiĭ and C. Liu's. Let G be a non-locally compact topological group and bG be a compactification of G. The following facts are established: (1) If $bG\setminus G$ has locally a k-space with a point-countable k-network and π-character of $bG\setminus G$ is countable, then G and bG are separable and metrizable; (2) If $bG\setminus G$ has locally a $\delta\theta$-base, then G and bG are separable and metrizable; (3) If $bG\setminus G$ has locally a quasi-G_δ-diagonal, then G and bG are separable and metrizable. Finally, we give a partial answer for a question, which was posed by C. Liu in [16].

1. Introduction

By a remainder of a space X we understand the subspace $bX\setminus X$ of a Hausdorff compactification bX of X. In [3, 4, 5, 13, 16], many topologists studied the following question of a Hausdorff compactification: When does a Tychonoff space X have a Hausdorff compactification bX with a remainder belonging to a given class of spaces? A famous classical result in this study is the following theorem of M. Henriksen and J. Isbell [13]:

(M. Henriksen and J. Isbell) A space X is of countable type if and only if the remiander in any (in some) compactification of X is Lindelöf

Recall that a space X is of *countable type* [10] if every compact subspace F of X is contained in a compact subspace $K \subset X$ with a countable base of open neighborhoods in X. Suppose that X is a non-locally compact topological

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group, and that bX is a compactification of X. In [4], A. V. Arhangel'skiĭ showed that if the remainder $Y = bX \setminus X$ has a G_δ-diagonal or a point-countable base, then both X and Y are separable and metrizable. In [16], C. Liu improved the results of A. V. Arhangel'skiĭ, and proved that if Y satisfies one of the following conditions (i) and (ii), then X and bX are separable and metrizable.

(i) $Y = bX \setminus X$ is a quotient s-image of a metrizable space, and π-character of Y is countable;

(ii) $Y = bX \setminus X$ has locally a G_δ-diagonal.

In this paper, we mainly concerned with the following statement, and under what condition Φ it is true.

Statement Suppose that G is a non-locally compact topological group, and that $Y = bG \setminus G$ has locally a property-Φ. Then G and bG are separable and metrizable.

Recall that a space X has *locally a property*-Φ if for each point $x \in X$ there exists an open set U with $x \in U$ such that U has a property-Φ.

In Section 2 we mainly study some local properties on the remainders of the topological group G such that G and bG are separable and metrizable if the π-character of $bG \setminus G$ is countable. Therefore, we extend some results of A. V. Arhangel'skiĭ and C. Liu.

In Section 3 we prove that if the remainders of a topological group G has locally a quasi-G_δ-diagonal, then G and bG are separable and metrizable. Therefore, we improve a result of C. Liu in [16]. Also, we study the remainders that are the unions of G_δ-diagonals.

In Section 4 we mainly give a partial answer for a question, which was posed by C. Liu in [16]. Finally, we also study the remainders that are locally hereditarily D-spaces.

Recall that a family $\mathscr{U}$ of non-empty open sets of a space X is called a *π-base* if for each non-empty open set V of X, there exists an $U \in \mathscr{U}$ such that $V \subset U$. The *π-character* of x in X is defined by $\pi\chi(x, X) = \min\{|\mathscr{U}| : \mathscr{U}$ is a local π-base at x in $X\}$. The *π-character of* X is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.

The p-spaces are a class of generalized metric spaces [1]. It is well-known that every metrizable space is a p-space, and every p-space is of countable type.

Throughout this paper, all spaces are assumed to be Hausdorff. The positively natural numbers is denoted by $\mathbf{N}$. We refer the readers to [10, 11] for notations and terminology not explicitly given here.

2. Remainders with the countable π-characters

Let $\mathscr{A}$ be a collection of subsets of X. $\mathscr{A}$ is a *p-network* [7] for X if for distinct points $x, y \in X$, there exists an $A \in \mathscr{A}$ such that $x \in A \subset X - \{y\}$. The collection $\mathscr{A}$ is called a *p-base* (i.e., *T_1-point-separating open cover*) [7] for X if $\mathscr{A}$

is a p-network and each element of $\mathscr{A}$ is an open subset of X. The collection $\mathscr{A}$ is a *p-metabase* [15] (in [7], p-metabase is denoted by the condition (1.5)) for X if for distinct points $x, y \in X$, there exists an $\mathscr{F} \in \mathscr{A}^{<\omega}$ such that $x \in (\bigcup \mathscr{F})^\circ \subset \bigcup \mathscr{F} \subset X - \{y\}$. The collection $\mathscr{A}$ is a *p-k-network* [15] (in [12], p-k-network is denoted by the condition $(1.4)_p$) for X if, whenever $K \subset X \backslash \{y\}$ with K compact in X, then $K \subset \bigcup \mathscr{F} \subset X \backslash \{y\}$ for some $\mathscr{F} \in \mathscr{A}^{<\omega}$.

First, we give some technique lemmas.

LEMMA 2.1 [3]. *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

LEMMA 2.2 [16]. *Let G be a non-locally compact topological group. Then G is locally separable and metrizable if for each point $y \in Y = bG \backslash G$, there is an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is metrizable and π-character of Y is countable.*

LEMMA 2.3. *Suppose that X has a point-countable p-metabase. Then each countably compact subset of X is a compact, metrizable, G_δ-subset*[1] *of X.*

Proof. Suppose that $\mathscr{U}$ is a point-countable p-metabase of X, and that K is a countably compact subset of X. Then K is compact by [7]. According to a generalized Miščenko's Lemma in [22, Lemma 6], there are only countably many minimal neighborhood-covers[2] of K by finite elements of $\mathscr{U}$, say $\{\mathscr{V}(n) : n \in \mathbf{N}\}$. Let $V(n) = \bigcup \mathscr{V}(n)$. Then $K \subset \bigcap \{V(n) : n \in \mathbf{N}\}$. Suppose that $x \in X \backslash K$. For each point $y \in K$, there is an $\mathscr{F}_y \in \mathscr{U}^{<\omega}$ with $y \in (\bigcup \mathscr{F}_y)^\circ \subset \bigcup \mathscr{F}_y \subset X - \{x\}$. Then there is some sub-collection of $\bigcup \{\mathscr{F}_y : y \in K\}$ is a minimal finite neighborhood-covers of K since K is compact. Therefore, we obtain one of the collections $\mathscr{V}(n)$ with $K \subset V(n) = \bigcup \mathscr{V}(n) \subset X - \{x\}$. □

LEMMA 2.4. *Suppose that X is a Lindelöf space with locally a point-countable p-metabase. Then X has a point-countable p-metabase.*

Proof. For each point $x \in X$, there is an open neighborhood $U(x)$ with $x \in U(x)$ such that $U(x)$ has a point-countable p-metabase $\mathscr{F}_x$. Let $\mathscr{U} = \{U(x) : x \in X\}$. Since X is Lindelöf, it follows that there exists a countable subfamily $\mathscr{U}' \subset \mathscr{U}$ such that $X = \bigcup \mathscr{U}'$. Denoted $\mathscr{U}'$ by $\{U_{x_i} : i \in \mathbf{N}\}$. Obviously, $\mathscr{F} = \bigcup_i \mathscr{F}_{x_i}$ is a point-countable p-metabase for X. □

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[1] A subset K of X is called a *G_δ-subset* of X if K is the intersection of countably open subsets of X.

[2] Let $\mathscr{P}$ be a collection of subsets of X and $A \subset X$. The collection $\mathscr{P}$ is a *neighborhood-cover* of A if $A \subset (\bigcup \mathscr{P})^\circ$. A neighborhood-cover $\mathscr{P}$ of A is a *minimal neighborhood-cover* if for each $P \in \mathscr{P}$, $\mathscr{P} \backslash \{P\}$ is not a neighborhood-cover of A.

Theorem 2.5. *Suppose that G is a non-locally compact topological group, and that $Y = bG \setminus G$ has locally a point-countable p-metabase. Then G and bG are separable and metrizable if π-character of Y is countable.*

Proof. It is easy to see that G is locally separable and metrizable by Lemmas 2.2 and 2.3. Then G is a p-space. Hence Y is Lindelöf by Henriksen and Isbell's theorem. From Lemma 2.4 it follows that $Y = bG \setminus G$ has a point-countable p-metabase.

Claim: The space Y has a G_δ-diagonal.

Put $G = \bigoplus_{\alpha \in \wedge} G_\alpha$, where G_α is a separable and metrizable subset for each $\alpha \in \wedge$. Let $\zeta = \{G_\alpha : \alpha \in \wedge\}$, and let F be the set of all points of bG at which ζ is not locally finite. Since ζ is discrete in G, it follows that $F \subset bG \setminus G$. It is easy to see that F is compact. Therefore, it follows from Lemma 2.3 that F is separable and metrizable. Hence F has a countable network.

Let $M = Y \setminus F$. For each point $y \in M$, there is an open neighborhood O_y in bG such that $\overline{O_y} \cap F = \emptyset$. Since ζ is discrete, $\overline{O_y}$ meets at most finitely many G_α. Let $L = \bigcup\{G_\alpha : G_\alpha \cap \overline{O_y} \neq \emptyset\}$. Then L is separable and metrizable. By Lemma 2.1, $\overline{L} \setminus L$ is a Lindelöf p-space. Obviously, $\overline{L} \setminus L \subset Y$. Therefore, $\overline{L} \setminus L$ has a point-countable p-metabase. Hence $\overline{L} \setminus L$ is separable and metrizable by [12], which implies that $\overline{L}$ has a countable network. It follows that $\overline{L}$ is separable and metrizable. Clearly, $O_y \subset \overline{L}$ and $O_y \cap M$ is separable and metrizable. Therefore, M is locally separable and metrizable. From Lemma 2.3 it follows that each compact subset of Y is a G_δ-subset of Y. Since F is compact and Y is Lindelöf, it follows that M is Lindelöf. Therefore, M is separable. Then M has a countable network. So Y has a countable network, which implies that Y has a G_δ-diagonal. Thus, Claim is verified.

Therefore, G and bG are separable and metrizable by [4, Theorem 5]. □

Corollary 2.6. *Suppose that G is a non-locally compact topological group, and that $Y = bG \setminus G$ has locally a point-countable p-base. Then G and bG are separable and metrizable if π-character of Y is countable.*

Corollary 2.7. *Suppose that G is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a k-space with a point-countable p-k-network. Then G and bG are separable and metrizable if π-character of Y is countable.*

Proof. Note that if $\mathscr{P}$ is a point-countable p-k-network for a k-space X, then $\mathscr{P}$ is a point-countable p-metabase for X by [12]. □

A collection $\mathscr{P}$ of subsets of a space X is a *k-network* [11] for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathscr{F} \subset U$ for some $\mathscr{F} \in \mathscr{P}^{<\omega}$.

Obviously, if a space X has a point-countable k-network, then X has a point-countable p-k-network. So we have the following Theorem 2.8, which improves the result [16, Theorem 4] of C. Liu.

THEOREM 2.8. *Suppose that G is a non-locally compact topological group, and that $Y = bG \setminus G$ is locally a k-space with a point-countable k-network. Then G and bG are separable and metrizable if π-character of Y is countable.*

COROLLARY 2.9 [4]. *Suppose that G is a non-locally compact topological group. If $Y = bG \setminus G$ has a point-countable base, then G and bG are separable and metrizable.*

Next, we consider the remainders with locally a $\delta\theta$-base[3] of the topological groups.

LEMMA 2.10. *Let X be a Lindelöf space with locally a $\delta\theta$-base. Then X has a $\delta\theta$-base.*

Proof. For each point $x \in X$, there is an open neighborhood $U(x)$ with $x \in U(x)$ such that $U(x)$ has a $\delta\theta$-base $\mathscr{B}_x = \bigcup_n \mathscr{B}_{n,x}$. Let $\mathscr{U} = \{U(x) : x \in X\}$. Since X is Lindelöf, it follows that there exists a countable subfamily $\mathscr{U}' \subset \mathscr{U}$ such that $X = \bigcup \mathscr{U}'$. Denoted $\mathscr{U}'$ by $\{U_{x_i} : i \in \mathbf{N}\}$. Obviously, $\mathscr{B} = \bigcup_{i,n} \mathscr{B}_{n,x_i}$ is a $\delta\theta$-base for X. □

THEOREM 2.11. *Let G be a non-locally compact topological group. If $Y = bG \setminus G$ has locally a $\delta\theta$-base. Then G and bG are separable and metrizable.*

Proof. Obviously, Y is first countable. By [8, Proposition 2.1], each countably compact subset of Y is a compact, metrizable, G_δ-subset of Y. From Lemma 2.2 it follows that G is locally separable and metrizable. Then G is a p-space. Hence Y is Lindelöf by Henriksen and Isbell's theorem. From Lemma 2.10 it follows that $Y = bG \setminus G$ has a $\delta\theta$-base.

By the same notations in Theorem 2.5, it is easy to see from [8, Propostion 2.1] that $F \subset bG \setminus G$ is compact and metrizable in view of the proof of Theorem 2.5. By [11, Corollary 8.3] and Lemma 2.1, $\overline{L} \setminus L$ is separable and metrizable. In view of the proof of Theorem 2.5, G and bG are separable and metrizable by [8, Propostion 2.1]. □

COROLLARY 2.12 [16]. *Let G be a non-locally compact topological group. If $Y = bG \setminus G$ is locally a quasi-developable*[4]. *Then G and bG are separable and metrizable.*

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[3] Recall that a collection $\mathscr{B} = \bigcup_n \mathscr{B}_n$ of open subsets of a space X is a *$\delta\theta$-base* [11] if whenever $x \in U$ with U open, there exist an $n \in \mathbf{N}$ and a $B \in \mathscr{B}$ such that

(i) $1 \leq \operatorname{ord}(x, \mathscr{B}_n) \leq \omega$;

(ii) $x \in B \subset U$.

[4] A space X is *quasi-developable* if there exists a sequence $\{\mathscr{G}_n\}_n$ of families of open subsets of X such that for each point $x \in X$, $\{\operatorname{st}(x, \mathscr{G}_n) : n \in \mathbf{N}, \operatorname{st}(x, \mathscr{G}_n) \neq \emptyset\}$ is a base at x.

Finally, we consider the remainders with locally a c-semistratifiable space of the topological group.

Let X be a topological space. X is called a *c-semistratifiable space*(CSS) [17] if for each compact subset K of X and each $n \in \mathbf{N}$ there is an open set $G(n, K)$ in X such that:

(i) $\bigcap\{G(n, K) : n \in \mathbf{N}\} = K$;

(ii) $G(n+1, K) \subset G(n, K)$ for each $n \in \mathbf{N}$; and

(iii) if for any compact subsets K, L of X with $K \subset L$, then $G(n, K) \subset G(n, L)$ for each $n \in \mathbf{N}$.

THEOREM 2.13. *Suppose that G is a non-locally compact topological group, and that $Y = bG \backslash G$ is locally a CSS-space. Then G and bG are separable and metrizable if π-character of Y is countable.*

Proof. By [8, Proposition 3.8(c)] and the definition of *CSS*-spaces, it is easy to see that each countably compact subset of Y is a compact, metrizable, G_δ-subset of Y. From Lemma 2.2 it follows that G is locally separable and metrizable. Then G is a p-space. Hence Y is Lindelöf by Henriksen and Isbell's theorem. From Lemma 2.10 it follows that $Y = bG \backslash G$ is a *CSS*-space by [8, Proposition 3.5].

By the same notations in Theorem 2.5, it is easy to see from [8, Proposition 3.8] that $F \subset bG \backslash G$ is compact and metrizable in view of the proof of Theorem 2.5. By [8, Proposition 3.8], $\overline{L} \backslash L$ is separable and metrizable. In view of the proof of Theorem 2.5, it is easy to see that G and bG are separable and metrizable. □

COROLLARY 2.14. *Suppose that G is a non-locally compact topological group, and that $Y = bG \backslash G$ is locally a $\sigma^{\#}$-space*[5]. *Then G and bG are separable and metrizable if π-character of Y is countable.*

Proof. By [8, Lemma 3.1], it follows that every $\sigma^{\#}$-space is a *CSS*-space. Hence G and bG are separable and metrizable by Theorem 2.13. □

QUESTION 2.15. Let G be a non-locally compact topological group. If $Y = bG \backslash G$ satisfies the following conditions (1) and (2), are G and bG separable and metrizable?

(1) For each point $y \in Y$, there exists an open neighborhood $U(y)$ of y such that every countably compact subset of $U(y)$ is metirzable and G_δ-subset of $U(y)$;

(2) π-character of Y is countable.

[5] A space X is called a *$\sigma^{\#}$-space* [17] if X has a σ-closure-preserving closed p-network.

3. Remainders that are locally quasi-G_δ-diagonals, and that are unions

First, we study the remainders with locally a quasi-G_δ-diagonal[6] and improve a result of C. Liu.

We call a space X is *Ohio complete* [3] if in each compactification bX of X there is a G_δ-subset Z such that $X \subset Z$ and each point $y \in Z\backslash X$ is separated from X by a G_δ-subset of Z.

LEMMA 3.1. *Let X be a p-space and every compact subset of $bX\backslash X$ be metrizalbe. Then there exists a G_δ-subset Y of bX such that $X \subset Y$ and satisfies the following conditions*:

(1) *bX is first countable at every point $y \in Y\backslash X$*;

(2) *If X is a topological group and $\overline{Y\backslash X} \cap X \neq \emptyset$, then X is metrizable.*

Proof. Since X is a p-space, X is Ohio complete [3, Corollary 3.7]. It follows that there is a G_δ-subset Y of bX such that $X \subset Y$ and every point $y \in Y\backslash X$ can be separated from X by a G_δ-subset. We now prove that Y satisfies the conditions (1) and (2).

(1) From the choice of Y, it is easy to see that for every point $y \in Y\backslash X$ there exists a compact G_δ-subset C of bX such that $y \in C \subset Y\backslash X \subset bX\backslash X$. Since C is compact, the compact subset C is metrizable. Therefore, y is a G_δ-point in bX and hence, bX is first countable at y.

(2) We choose a point $a \in \overline{Y\backslash X} \cap X$. Since X is a p-space, there exists a compact subset F of X such that $a \in F$ and F has a countable base of neighborhoods in X. Since X is dense in bX, the set F has a countable base of open neighborhoods $\phi = \{U_n : n \in \omega\}$ in bX. Since $a \in \overline{Y\backslash X}$, we can fix a $b_n \in U_n \cap (Y\backslash X)$ for each $n \in \omega$. Obviously, there is a point $c \in F$ which is a limit point for the sequence $\{b_n\}$. By (1), we know that bX is first countable at b_n for every $n \in \omega$. We can fix a countable base η_n of bX at b_n. Then $\bigcup\{\eta_n : n \in \omega\}$ is a countable π-base of bX at c. Then the space X also has a countable π-base at c, since $c \in X$ and X is dense in bX. Since X is a topological group, the space X is metrizable. □

THEOREM 3.2. *Let G be a non-locally compact topological group. If $Y = bG\backslash G$ has a quasi-G_δ-diagonal. Then G and bG are separable and metrizable.*

Proof. Obviously, Y has a countable pseudocharacter. By [5, Theorem 5.1], G is a paracompact p-space or Y is first countable.

Case 1: The space Y is first countable.

From [8, Proposition 2.3] it follows that each countably compact subset of Y is a compact, metrizable, G_δ-subset of Y. Note that a Lindelöf p-space with a

[6] A space X has a *quasi-G_δ-diagonal* [14] if there exists a sequence $\{\mathscr{G}_n\}_n$ of families of open subsets of X such that for each point $x \in X$, $\{\mathrm{st}(x, \mathscr{G}_n) : n \in \mathbf{N}, \mathrm{st}(x, \mathscr{G}_n) \neq \emptyset\}$ is a p-network at point x.

quasi-G_δ-diagonal is metrizable by [14, Corollary 3.6]. In view of the proof of Theorem 2.5, it is easy to see that G and bG are separable and metrizable.

Case 2: The space G is a paracompact p-space.

By [3, Corollary 3.7], G is Ohio complete. Therefore, there exists a G_δ-subset X of bG such that $G \subset X$ and every point $x \in X \backslash G$ can be separated from G by a G_δ-set of X. Let $M = X \backslash G$. Then bG is first countable at every point $y \in M$ by Lemma 3.1.

Subcase 1: $\overline{M} \cap G = \emptyset$. Then $X \backslash \overline{M} = G$. Hence G is a G_δ-subset of bG. It follows that Y is σ-compact. Since Y has a quasi-G_δ-diagonal, every compact subspace of Y is separable and metrizable by [8, Proposition 2.3]. Hence Y is separable. Since both Y and G are dense in bG, it follows that the souslin number of G is countable. The space G is Lindelöf, since G is paracompact. Therefore, G is a Lindelöf p-space. Then Y is a Lindelöf p-space by Lemma 2.1. Since Y has a quasi-G_δ-diagonal, the space Y is metrizable by [14, Corollary 3.6]. It follows that Y has a G_δ-diagonal. Therefore, G and bG are separable and metrizable by [4, Theorem 5].

Subcase 2: $\overline{M} \cap G \neq \emptyset$. Then G is metrizable by Lemma 3.1.

Subcase 2(a): G is locally separable. By [8, Proposition 2.3], it is easy to see that G and bG are separable and metrizable by the proof of Theorem 2.5.

Subcase 2(b): G is nowhere locally separable. Fix a base $\mathscr{B} = \bigcup\{\mathscr{U}_n : n \in \mathbf{N}\}$ of G such that each $\mathscr{U}_n$ is discrete in G. Let F_n be the set of all accumulation points for $\mathscr{U}_n$ in bG for each $n \in \mathbf{N}$. Put $Z = \bigcup\{F_n : n \in \mathbf{N}\}$. Then Z is dense in Y and σ-compact by [4, Proposition 4]. Since every compact space with a quasi-G_δ-diagonal is separable and metrizable by [8, Proposition 2.3], the space Z has a countable network. Because G is nowhere locally compact, the space Y is dense in bG. It follows that Z is dense in bG. Hence bG is separable, which implies that the Souslin number of G is countable. Since G is metrizable, the space G is separable. Then Y is a Lindelöf p-space by Lemma 2.1. Hence Y is metrizable by [14, Corollary 3.6]. It follows that Y is separable and metrizable, which implies that G and bG are separable and metrizable. □

LEMMA 3.3. *Let X be a Lindelöf space with locally a quasi-G_δ-diagonal. Then X has a quasi-G_δ-diagonal.*

Proof. For each point $x \in X$, there exists an open neighborhood $U(x)$ such that $x \in U(x)$ and $U(x)$ has a quasi-G_δ-diagonal. Then $\mathscr{U} = \{U(x) : x \in X\}$ is an open cover of X. Since X is a Lindelöf space, there exists a countable subfamily $\mathscr{V} \subset \mathscr{U}$ such that $X = \bigcup \mathscr{V}$. Denoted $\mathscr{V}$ by $\{U_n : n \in \mathbf{N}\}$. For each $n \in \mathbf{N}$, let $\{\mathscr{U}_{nk}\}_{k \in \mathbf{N}}$ be a quasi-G_δ-diagonal sequence of U_n. Let $\mathscr{F} = \{\mathscr{U}_{nk}\}_{n,k \in \mathbf{N}}$. Then $\mathscr{F}$ is a quasi-G_δ-diagonal sequence of X.

Indeed, for distinct points $x, y \in X$, there exists an $n \in \mathbf{N}$ such that $x \in U_n$.

If $y \notin U_n$, then $x \in U_n \subset X - \{y\}$. Since $\{\mathscr{U}_{nk}\}_{k \in \mathbf{N}}$ is a quasi-G_δ-diagonal sequence of U_n, there exists a $k \in \mathbf{N}$ such that $x \in \bigcup \mathscr{U}_{nk}$. Hence $x \in \mathrm{st}(x, \mathscr{U}_{nk}) \subset \bigcup \mathscr{U}_{nk} \subset U_n \subset X - \{y\}$.

If $y \in U_n$, then $x \in U_n - \{y\} \subset X - \{y\}$. Since $\{\mathscr{U}_{nk}\}_{k \in \mathbf{N}}$ is a quasi-G_δ-diagonal sequence of U_n, there exists a $k \in \mathbf{N}$ such that $x \in \mathrm{st}(x, \mathscr{U}_{nk}) \subset U_n - \{y\} \subset X - \{y\}$.

Therefore, $\mathscr{F}$ is a quasi-G_δ-diagonal sequence of X. □

THEOREM 3.4. *Let G be a non-locally compact topological group. If $Y = bG \backslash G$ has locally a quasi-G_δ-diagonal, then G and bG are separable and metrizable.*

Proof. By [8, Proposition 2.1 and 2.5] and Lemma 2.2, it is easy to see that G is locally a separable and metrizable space. Then Y is a Lindelöf space by Henriksen and Isbell's theorem. From Lemma 3.3 it follows that Y has a quasi-G_δ-diagonal. Then G and bG are separable and metrizable by Theorem 3.2. □

QUESTION 3.5. *Is there a topological group G such that the $Y = bG \backslash G$ has a W_δ-diagonal*[7], *G is not reparable and metrizable*?

COROLLARY 3.6 [16]. *Let G be a non-locally compact topological group. If $Y = bG \backslash G$ has locally a G_δ-diagonal, then G and bG are separable and metrizable.*

Next, we study the remainder that are the unions of the G_δ-diagonals.

LEMMA 3.7. *Let G be a non-locally compact topological group. If there exists a point $a \in Y = bG \backslash G$ such that $\{a\}$ is a G_δ-set in Y, then at least one of the following conditions holds*:

(1) *G is a paracompact p-space*;

(2) *Y is first-countable at some point.*

Proof. Suppose that Y is not first-countable at point a. Since a is a G_δ-point in Y, there exists a compact subset $F \subset bG$ with a countable base at F in bG such that $\{a\} = F \cap (bG \backslash G)$. We have $F \backslash \{a\} \neq \emptyset$, since Y is not first-countable at point a. Therefore, there exists a non-empty compact subset $B \subset F$ with a countable base at B in bG. Obviously, $B \subset G$. It follows that G is a topological group of countable type [18]. Therefore, G is a paracompact p-space [18]. □

LEMMA 3.8. *Let G be a non-locally compact topological group, and $Y = bG \backslash G = Y_1 \cup Y_2$, where both Y_1 and Y_2 have a countable pseudocharacter. If at most one of the Y_1 and Y_2 is dense in bG, then at least one of the following conditions holds*:

(1) *G is a paracompact p-space*;

(2) *Y is first-countable at some point.*

[7] A space X is said to have a *W_δ-diagonal* if there is a sequence $(\mathscr{B}_n)$ of bases for X such that whenever $x \in B_n \in \mathscr{B}_n$, and (B_n) is decreasing (by set inclusion), then $\{x\} = \bigcap\{B_n : n \in \omega\}$.

Proof. Without loss of generality, we can assume that $\overline{Y_1} \neq bG$. Let $U = bG \backslash \overline{Y_1}$. Then $V = U \cap Y = U \cap Y_2 \neq \emptyset$. It follows that V is an open subset of Y and each point of V is a G_δ-point. By Lemma 3.7, we complete the proof. □

Theorem 3.9. *Let G be a non-locally compact topological group, and $Y = bG \backslash G = Y_1 \cup Y_2$, where both Y_1 and Y_2 have a countable pseudocharacter. If both Y_1 and Y_2 are Ohio complete, then at least one of the following conditions holds*:

(1) *G is a paracompact p-space*;

(2) *Y is first-countable at some point.*

Proof. Case 1: $\overline{Y_1} \neq bG$ or $\overline{Y_2} \neq bG$.

It is easy to see by Lemma 3.8.

Case 2: $\overline{Y_1} = bG$ and $\overline{Y_2} = bG$.

Then bG is the Hausdorff compactification of Y_1 and Y_2. Since Y_1 and Y_2 are Ohio complete, there exist G_δ-subsets X_1 and X_2 satisfy the definition of Ohio complete, respectively.

Case 2(a): $Y_1 = X_1$ and $Y_2 = X_2$.

Then Y has countable pseudocharacter. By [5, Theorem 5.1], we complete the proof.

Case 2(b): $Y_1 \neq X_1$ or $Y_2 \neq X_2$.

Without loss of generality, we can assume that $Y_1 \neq X_1$. If $(X_1 \backslash Y_1) \cap Y_2 \neq \emptyset$, then for each $y \in (X_1 \backslash Y_1) \cap Y_2$ there exists a compact subset C such that $y \in C$ and $C \cap Y_1 = \emptyset$. Obviously, y is a G_δ-point of Y. By Lemma 3.7, we also complete the proof. If $(X_1 \backslash Y_1) \cap Y_2 = \emptyset$, then there exists a compact subset $C \subset G$ with a countable base at C in bG. It follows that G is a topological group of countable type [18]. Therefore, G is a paracompact p-space [18]. □

A space with a G_δ-diagonal is Ohio complete [2]. Therefore, by Theorem 3.9, we have the following result.

Theorem 3.10. *Let G be a non-locally compact topological group, and $Y = bG \backslash G = Y_1 \cup Y_2$, where both Y_1 and Y_2 have a G_δ-diagonal. Then at least one of the following conditions holds*:

(1) *G is a paracompact p-space*;

(2) *Y is first-countable at some point.*

Question 3.11. Let G be a non-locally compact topological group, and $Y = bG \backslash G = \bigcup_{i=1}^{i=n} Y_i$, where Y_i has a G_δ-diagonal for every $1 \leq i \leq n$. Is G a paracompact p-space or is Y first-countable at some point?

Question 3.12. Let G be a non-locally compact topological group, and $Y = bG \backslash G = Y_1 \cup Y_2$, where both Y_1 and Y_2 have quasi-G_δ-diagonal. Is G a paracompact p-space or is Y first-countable at some point?

4. Remainders of locally BCO and locally hereditarily D-spaces

First, we study the following question, which was posed by C. Liu in [16].

Question 4.1. Let G be a non-locally compact topological group, and $Y = bG \setminus G$ have a BCO[8]. Are G and bG separable and metrizable?

Now we give a partial answer for Question 4.1.

Theorem 4.2. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ has a BCO. If Y is Ohio complete, then G and bG are separable and metrizable.*

Proof. Since Y is Ohio complete, G is a paracompact p-space or σ-compact space by [3, Theorem 4.3].

Case 1: The space G is a paracompact p-space.

Since G is a p-space, the space Y is Lindelöf by Henriksen and Isbell's theorem. Hence Y is developable by [11, Theorem 6.6]. Then G and bG are separable and metrizable by Theorem 3.4.

Case 2: The space G is a σ-compact space.

We claim that G is metrizable. Suppose that G is not metrizable. Then Y is ω-bounded[9] by [5, Theorem 3.12]. Since G is a σ-compact topological group, the Souslin number $c(G)$ of G is countable by a theorem of Tkachenko [21, Corollary 2]. Therefore, $c(bG) \leq \omega$. Y is dense in bG, since G is non-locally compact. It follows that $c(Y) \leq \omega$ as well. Since Y is Čech-complete, there exists a dense subspace $Z \subset Y$ such that Z is a paracompact and Čech-complete subspace of Y by [19]. Then Z is a paracompact space with a BCO. Therefore, Z is metrizable by [11, Theorem 1.2 and 6.6]. Since $c(Y) \leq \omega$ and Z is dense for Y, $c(Z) \leq \omega$ as well. It follows that Z is separable. Since Y is ω-bounded, it is compact. Therefore, G is locally compact, which is a contradiction. It follows that G is metrizable. Therefore, G and bG are separable and metrizable by Case 1. □

Theorem 4.3. *Let G be a non-locally compact topological group, and $Y = bG \setminus G$ have a BCO. If G is an Σ-space, then G and bG are separable and metrizable.*

Proof. From [6, Theorem 2.8] it follows that every compact subspace of Y has countable character in Y. Since G is non-locally compact, Y is also a dense subset of bG. Hence G is Lindelöf space by Henriksen and Isbell's theorem. If

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[8] A space X is said to have a *base of countable order*(BCO) [11] if there is a sequence $\{\mathscr{B}_n\}$ of base for X such that whenever $x \in b_n \in \mathscr{B}_n$ and (b_n) is decreasing (by set inclusion), then $\{b_n : n \in \mathbf{N}\}$ is a base at x.

[9] A space X is said to be *ω-bounded* if the clourse of every countable subset of X is compact.

G is a σ-compact space, then G and bG are separable and metrizable by Case 2 in Theorem 4.2. Hence we assume that G is non-σ-compact. Since G is a Lindelöf Σ-space, it is easy to see that G is a Lindelöf p-space by the proof of [5, Theorem 4.2]. It follows that G and bG are separable and metrizable by Case 1 in Theorem 4.2. □

Finally, we study the remainders of topological groups with locally a hereditarily D-space.

Theorem 4.4. *Let G be a topological group. If for each $y \in Y = bG \setminus G$ there exists an open neighborhood $U(y)$ of y such that every ω-bounded subset of $U(y)$ is compact, then at least one of the following conditions holds*:

(1) *G is metrizable*;

(2) *bG can be continuously mapped onto the Tychonoff cube I^{ω_1}*.

Proof. Case 1: The space G is locally compact.

If G is not metrizable, then G contains a topological copy of D^{ω_1}. Since the space G is normal, the space G can be continuously mapped onto the Tychonoff cube I^{ω_1}

Case 2: The space G is not locally compact.

Obviously, both G and Y are dense in bG. Suppose that the condition (2) doesn't hold. Then, by a theorem of Šapirovskiĭ in [20], the set A of all points $x \in bG$ such that the π-character of bG at x is countable is dense in bG. Since G is dense in bG, it can follow that the π-character of G is countable at each point of $A \cap G$.

Subcase 2(a): $A \cap G \neq \emptyset$.

Since G is a topological group, it follows that G is first countable, which implies that G is metrizable.

Subcase 2(b): $A \cap G = \emptyset$.

Obviously, $A \subset Y$. For each $y \in Y$, there exists an open neighborhood $U(y)$ in Y such that $y \in U(y)$ and every ω-bounded subset of $U(y)$ is compact. Obviously, $A \cap U(y)$ is dense of $U(y)$. Also, it is easy to see that $A \cap U(y)$ is ω-bounded subset for $U(y)$. Therefore, $A \cap U(y)$ is compact. Then $A \cap U(y) = U(y)$, since $A \cap U(y)$ is dense of $U(y)$. Hence Y is locally compact, a contradiction. □

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for each point $x \in X$. A space X is a *D-space* [9], if for any neighborhood assignment φ for X there is a closed discrete subset D of X such that $X = \bigcup_{d \in D} \varphi(d)$.

It is easy to see that every countably compact D-space is compact. Hence we have the following result by Theroem 4.4.

Theorem 4.5. *Let G be a topological group. If $Y = bG \setminus G$ is locally a hereditarily D-space, then at least one of the following conditions holds*:

(1) *G is metrizable*;
(2) *bG can be continuously mapped onto the Tychonoff cube* I^{ω_1}.

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