Z.-R. ZHOU
KODAI MATH. J.
34 (2011), 272–290

STABILITIES OF F-STATIONARY MAPS

Zhen-Rong Zhou*

Abstract

An F-stationary map is a critical point of the F-energy with respect to variations in the domain. It is a generalization of F-harmonic maps. In [11, 10], we discuss the theorems of Liouville type and the monotonicity of F-stationary maps. In this paper, we discuss the stabilities of F-stationary maps from submanifolds of spheres and the Euclidean spaces.

1. Introduction

Let $F : [0, +\infty) \to [0, +\infty)$ be a C^2-function. For a smooth map $u : M \to N$ between two Riemannian manifolds (M, g) and (N, h), M. Ara ([1]) introduced the following F-energy functional

$$E_F(u) = \int_M F\left(\frac{|\mathrm{d}u|^2}{2}\right), \tag{1}$$

and discussed the geometry of the critical points. Let $u_t : M \to N$ $(-\epsilon < t < \epsilon)$ be a variation of u, i.e. $u_t = \Phi(t, \cdot)$ with $u_0 = u$, where $\Phi : (-\varepsilon, \varepsilon) \times M \to N$ is a smooth map. Let $\psi = \left.\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} \in \Gamma(u^{-1}TN)$ be the variational field, where $u^{-1}TN$ is the pullback vector bundle on M by u, and $\Gamma(u^{-1}TN)$ is the set of all smooth cross sections of the bundle. Let $\Gamma_0(u^{-1}TN)$ be a subset of $\Gamma(u^{-1}TN)$ consisting of all elements with compact supports contained in the interior of M. If M is compact and without boundary, then $\Gamma_0(u^{-1}TN) = \Gamma(u^{-1}TN)$. For each $\psi \in \Gamma_0(u^{-1}TN)$, there exists a variation $u_t(x) = \exp_{u(x)} t\psi$ (for t small enough) of u, which has the variational field ψ. Such a variation is called to have a compact support. Let $D_\psi E_F(u) \equiv \left.\frac{\mathrm{d}E_F(u_t)}{\mathrm{d}t}\right|_{t=0}$. An F-harmonic map u is a critical point of the F-energy functional, i.e., for any $\psi \in \Gamma_0(u^{-1}TN)$, one has $D_\psi E_F(u) = 0$, where $\psi = \left.\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} \in \Gamma_0(u^{-1}TN)$ is the variational field. When

*Research supported by National Science Fundation of China No.10871149.
Received October 8, 2010; revised December 17, 2010.

$F(t) = t, \dfrac{(2t)^{p/2}}{p}, e^t$, the F-harmonic maps are harmonic maps, p-harmonic maps and exponential harmonic maps respectively.

It is known that $\mathrm{d}uX \in \Gamma(u^{-1}TN)$ for any vector field X of M. If X has a compact support which is contained in the interior of M, then $\mathrm{d}uX \in \Gamma_0(u^{-1}TN)$.

If $D_{\mathrm{d}uX}E_F(u) = 0$ for any vector field X on M with compact support contained in the interior of M, we call u an F-stationary map. Because $\mathrm{d}uX \in \Gamma_0(u^{-1}TN)$, F-harmonic maps must be F-stationary ones.

In [11, 10], we discussed the theorems of Liouville type and the monotonicity of F-stationary maps.

Y. L. Xin in [8] proved that any stable harmonic map from S^m $(m > 2)$ must be constant and P. F. Leung in [5] proved that any stable harmonic map from M^m $(m > 2)$ to some hypersurfaces of Euclidean space is constant. Q. Chen in [2] generalized them to harmonic maps with potential. Ohnita in [7] verified that stable harmonic maps from or into minimal submanifolds of the sphere are constant if the Ricci curvatures of the submanifolds are bigger than half the dimensions. In this paper, we investigate the stabilities of F-stationary maps from more general submanifolds of the sphere and the Euclidean space.

2. F-Stationary maps and the F-conservation law

Let $u : M \to N$ be a smooth map. In the following, we will denote the Riemannian connection of any Riemannian manifold M by ∇^M. The connection of the pullback vector bundle $u^{-1}TN$ is denoted by ∇. Taking a local field of orthonormal frame $\left\{\dfrac{\partial}{\partial t}, e_1, \dots, e_m\right\}$ on $\overline{M} = (-\epsilon, \epsilon) \times M$, then we have $\nabla^{\overline{M}}_{\partial/\partial t} e_i = \nabla^{\overline{M}}_{e_i} \dfrac{\partial}{\partial t} = 0$. For any fixed point $p \in M$, we can require $\nabla^M_{e_i} e_j(p) = 0$. By a straightforward calculation, we get the first variational formula:

$$D_\psi E_F(u) = -\int_M \langle \tau_F(u), \psi \rangle, \tag{2}$$

where $\psi \in \Gamma_0(u^{-1}TN)$, and

$$\tau_F(u) = \sum \nabla_{e_i}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\mathrm{d}ue_i\right] \tag{3}$$

is the F-tension of u. Then u is F-harmonic if and only if

$$\tau_F(u) = 0. \tag{4}$$

Let $u : M \to N$ be an F-stationary map, $X \in \Gamma_0(TM)$. Then by (2) and the definition of F-stationary maps, we have

$$D_{\mathrm{d}uX}E_F(u) = -\int_M \langle \tau_F(u), \mathrm{d}uX \rangle = 0. \tag{5}$$

Ara in [1] defined an F-stress-energy tensor of u by

$$S_u^F = F\left(\frac{|du|^2}{2}\right)g - F'\left(\frac{|du|^2}{2}\right)u^*h, \tag{6}$$

and the divergence of the F-stress-energy tensor by

$$\begin{aligned}(\text{div } S_u^F)(X) &= \sum (\nabla_{e_i}^M S_u^F)(e_i, X) \\ &= \sum \nabla_{e_i}^M [S_u^F(e_i, X)] - \sum S_u^F(e_i, \nabla_{e_i}^M X),\end{aligned} \tag{7}$$

where, X is any smooth vector field of M.

Lemma 1. *For any smooth vector field X of M, we have (see [1])*

$$(\text{div } S_u^F)(X) = -\langle \tau_F(u), duX \rangle. \tag{8}$$

By a straightforward calculation similar to [8], we have

Lemma 2. *Let $D \subseteq M$ be a compact smooth domain of M. If $u : (M, g) \to (N, h)$ is a smooth map and X is a smooth vector field of M, then*

$$\begin{aligned}\int_{\partial D} F\left(\frac{|du|^2}{2}\right)\langle X, \boldsymbol{n}\rangle &= \int_{\partial D} F'\left(\frac{|du|^2}{2}\right)\langle duX, du\boldsymbol{n}\rangle \\ &\quad + \int_D (\text{div } S_u^F)(X) + \int_D \langle S_u^F, \nabla X\rangle,\end{aligned} \tag{9}$$

where, ∂D is the boundary of D, $\boldsymbol{n}$ is the outward unit normal vector field of ∂D, and ∇X is defined by $\nabla X(V, W) := \langle \nabla_V X, W\rangle$.

Proof. It is not difficult to check that

$$\begin{aligned}\text{div}\left[F\left(\frac{|du|^2}{2}\right)X\right] &= \sum \left\langle \nabla_{e_i}^M \left[F\left(\frac{|du|^2}{2}\right)X\right], e_i\right\rangle \\ &= XF\left(\frac{|du|^2}{2}\right) + F\left(\frac{|du|^2}{2}\right)\sum \langle \nabla_{e_i}^M X, e_i\rangle,\end{aligned} \tag{10}$$

and by the symmetry of $\nabla\, du := \nabla^{T^*M \otimes u^{-1}TN}\, du$ (the second fundamental form of u), we have

$$\begin{aligned}XF\left(\frac{|du|^2}{2}\right) &= F'\left(\frac{|du|^2}{2}\right)\sum \langle (\nabla_X\, du)e_i, due_i\rangle \\ &= F'\left(\frac{|du|^2}{2}\right)\sum \langle (\nabla_{e_i}\, du)X, due_i\rangle\end{aligned} \tag{11}$$

$$= \sum \left\langle \nabla_{e_i}(\mathrm{d}uX), F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \mathrm{d}ue_i \right\rangle$$
$$- F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}u\nabla^M_{e_i}X, \mathrm{d}ue_i\rangle$$
$$= \sum e_i \left\langle \mathrm{d}uX, F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \mathrm{d}ue_i \right\rangle$$
$$- \sum \left\langle \mathrm{d}uX, \nabla_{e_i}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \mathrm{d}ue_i\right]\right\rangle$$
$$- F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}u\nabla^M_{e_i}X, \mathrm{d}ue_i\rangle$$
$$= \mathrm{div}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}uX, \mathrm{d}ue_i\rangle e_i\right]$$
$$- \langle \mathrm{d}uX, \tau_F(u)\rangle - F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla X, u^*h\rangle.$$

Here, we have used that

$$\sum \langle \mathrm{d}u\nabla^M_{e_i}X, \mathrm{d}ue_i\rangle = \sum u^*h(\nabla^M_{e_i}X, e_i)$$
$$= \sum g(\nabla^M_{e_i}X, e_j)u^*h(e_j, e_i) = \langle \nabla X, u^*h\rangle.$$

Inserting (11) into (10), we obtain

$$\mathrm{div}\left[F\left(\frac{|\mathrm{d}u|^2}{2}\right)X\right] = \mathrm{div}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}uX, \mathrm{d}ue_i\rangle e_i\right] - \langle \mathrm{d}uX, \tau_F(u)\rangle \tag{12}$$
$$- F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla X, u^*h\rangle + F\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \nabla^M_{e_i}X, e_i\rangle$$
$$= \mathrm{div}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}uX, \mathrm{d}ue_i\rangle e_i\right] - \langle \mathrm{d}uX, \tau_F(u)\rangle$$
$$- F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla X, u^*h\rangle + F\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \nabla X, g\rangle$$
$$= \mathrm{div}\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum \langle \mathrm{d}uX, \mathrm{d}ue_i\rangle e_i\right]$$
$$- \langle \mathrm{d}uX, \tau_F(u)\rangle + \langle S^F_u, \nabla X\rangle.$$

Integrating both sides of (12) on D, and taking use of (8) and Green's formula, we have

$$\int_{\partial D} F\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle X, \boldsymbol{n}\rangle = \int_{\partial D} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \mathrm{d}uX, \mathrm{d}u\boldsymbol{n}\rangle + \int_D (\mathrm{div}\, S_u^F)(X) + \int_D \langle S_u^F, \nabla X\rangle. \tag{13}$$ □

Corollary 3. *If X is a smooth vector field with a compact support contained in the interior of M, then*

$$\int_M (\mathrm{div}\, S_u^F)(X) + \int_M \langle S_u^F, \nabla X\rangle = 0. \tag{14}$$

From Lemma 1 and (5), we get

$$D_{\mathrm{d}uX} E_F(u) = -\int_M \langle \tau_F(u), \mathrm{d}uX\rangle = \int_M (\mathrm{div}\, S_u^F)(X), \tag{15}$$

if $X \in \Gamma_0(TM)$.

If $\mathrm{div}\, S_u^F = 0$, we call u to satisfy the F-conservation law; if $\int_M \mathrm{div}\, S_u^F(X) = 0$ for all $X \in \Gamma(TM)$, we call u to satisfy the integral F-conservation law.

From (15), we have

Theorem 4. *A smooth map $u: M \to N$ is an F-stationary map if and only if $\int_M (\mathrm{div}\, S_u^F)(X) = 0$ for all $X \in \Gamma_0(TM)$. Especially, if M is compact and without boundary, then u is an F-stationary map if an only if u satisfies the integral F-conservation law.*

Apparently, if u satisfies the F-conservation law or the integral F-conservation law, then u must be an F-stationary map.

3. Stabilities

Let M be an $m+k_0$-submanifold of $\mathbf{R}^{m+k_0}$, Ric^M and R_{ij}^M the Ricci operator and the Ricci curvature tensor of M respectively, h_{ij}^μ the second fundamental tensor, b a function. In this section, we suppose that M satisfied the following condition (with respected to any local orthonormal frame field):

$$-2R_{lj}^M + \sum h_{ss}^\mu h_{jl}^\mu \le b\delta_{lj}. \tag{16}$$

By Gauss equations, (16) is equivalent to

$$-R_{lj}^M + \sum h_{sj}^\mu h_{sl}^\mu \le b\delta_{lj} \tag{17}$$

or to

$$-\sum h_{ss}^\mu h_{jl}^\mu + 2\sum h_{sj}^\mu h_{sl}^\mu \le b\delta_{lj}. \tag{18}$$

Note: $a_{ij} \le 0$ means that $\sum a_{ij}\xi_i\xi_j \le 0$ for any $(\xi_1, \dots, \xi_m) \in \mathbf{R}^m$, and $a_{ij} \le b_{ij}$ means that $a_{ij} - b_{ij} \le 0$.

For example, if $M = S^m$, the unit sphere of m-dimension, then, we can choose a local orthonormal frame such that $R_{ij}^M = (m-1)\delta_{ij}$, $h_{ij}^{m+1} = \delta_{ij}$, $h_{ij}^{\mu} = 0$ for $\mu \ge m+2$. Hence (16) is satisfied for $b = -(m-2)$. If M^m is a minimal submanifold of $S^{m+k_0} \subseteq \mathbf{R}^{m+k_0+1}$, then it is also a submanifold of $\mathbf{R}^{m+k_0+1}$. We can choose a local orthonormal frame such that h_{ij}^{μ} is the second fundamental tensor of M in S^{m+k_0} for $\mu = m+1, \dots, m+k_0$, and that $h_{ij}^{m+k_0+1} = \delta_{ij}$. Hence (16) is satisfied for $b \le 0$ if $R_{ij}^M \ge \frac{m}{2}\delta_{ij}$.

3.1. The second variation

Let M be a compact Riemannian manifold without boundary, $u : M \to N$, u_t a variation of u, i.e. $u_t = \Phi(t, \cdot)$ with $u_0 = u$, where $\Phi : (-\varepsilon, \varepsilon) \times M \to N$ is a smooth map. The first variational formula is given by

$$\frac{d}{dt} E_F(u_t) = -\int_M \left\langle \frac{d\Phi}{dt}, \tau_F(u_t) \right\rangle.$$

Please note that $\frac{d\Phi}{dt} = \Phi_*\left(\frac{\partial}{\partial t}\right) \in \Gamma(\Phi^{-1}TN)$ and that $\tau_F(u_t)$ is the F-tension of $u_t : M \to N$, not of $\Phi : (-\varepsilon, \varepsilon) \times M \to N$. Let $a(x,t) = \tau_F(u_t)(x)$, then a is a cross section of $\Phi^{-1}TN$, although $\tau_F(u_t) \in \Gamma(u_t^{-1}TN)$ for any fixed t. In the following, we don't distinguish $\tau_F(u_t)$ from a.

By a straight-forward calculation, we have (We denote the connection of $\Phi^{-1}TN$ also by ∇)

$$\begin{aligned} \frac{d^2}{dt^2} E_F(u_t) &= -\frac{d}{dt}\int_M \left\langle \frac{d\Phi}{dt}, \tau_F(u_t) \right\rangle \\ &= -\int_M \left\langle \frac{d^2\Phi}{dt^2}, \tau_F(u_t) \right\rangle - \int_M \left\langle \frac{d\Phi}{dt}, \nabla_{\partial/\partial t}\tau_F(u_t) \right\rangle, \end{aligned} \tag{19}$$

where $\frac{d^2\Phi}{dt^2} = \nabla_{\partial/\partial t}\frac{d\Phi}{dt}$. Now, we calculate $\nabla_{\partial/\partial t}\tau_F(u_t)$.

$$\begin{aligned} \nabla_{\partial/\partial t}\tau_F(u_t) &= \sum \nabla_{\partial/\partial t}\nabla_{e_i}\left(F'\left(\frac{|du_t|^2}{2}\right) du_t e_i\right) \\ &= -\sum R\left(e_i, \frac{\partial}{\partial t}\right)\left(F'\left(\frac{|du_t|^2}{2}\right) du_t e_i\right) \\ &\quad + \sum \nabla_{e_i}\nabla_{\partial/\partial t}\left(F'\left(\frac{|du_t|^2}{2}\right) du_t e_i\right) \end{aligned} \tag{20}$$

$$
\begin{aligned}
&= -\sum R^N\left(\mathrm{d}u_te_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\left(F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\mathrm{d}u_te_i\right) \\
&\quad + \sum \nabla_{e_i}\left[\nabla_{\partial/\partial t}\left(F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\mathrm{d}u_te_i\right)\right] \\
&= -\sum F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)R^N\left(\mathrm{d}u_te_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\mathrm{d}u_te_i \\
&\quad + \sum \nabla_{e_i}\left[\nabla_{\partial/\partial t}\left(F'\left(\frac{|\mathrm{d}\Phi|^2}{2}\right)\mathrm{d}u_te_i\right)\right],
\end{aligned}
$$

where, R is the curvature tensor of $\Phi^{-1}TN$ and R^N is the Riemannian curvature tensor of N. Because N is torsion-free, we have

$$
\begin{aligned}
\nabla_{\partial/\partial t}(\mathrm{d}u_te_j) &= \nabla^N_{\mathrm{d}\Phi(\partial/\partial t)}(\mathrm{d}\Phi(e_j)) \\
&= \nabla^N_{\mathrm{d}\Phi(e_j)}\left(\mathrm{d}\Phi\left(\frac{\partial}{\partial t}\right)\right) + \mathrm{d}\Phi\left[\frac{\partial}{\partial t}, e_j\right] \\
&= \nabla^N_{\mathrm{d}\Phi(e_j)}\left(\mathrm{d}\Phi\left(\frac{\partial}{\partial t}\right)\right) = \nabla_{e_j}\frac{\mathrm{d}\Phi}{\mathrm{d}t}.
\end{aligned}
$$

Hence, the second term of the last line of (20) becomes as

$$
\begin{aligned}
(21)\quad &\sum \nabla_{e_i}\left[\nabla_{\partial/\partial t}\left(F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\mathrm{d}u_te_i\right)\right] \\
&= \sum \nabla_{e_i}\left[F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\langle\nabla_{\partial/\partial t}\,\mathrm{d}u_te_j, \mathrm{d}u_te_j\rangle\,\mathrm{d}u_te_i + F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\nabla_{\partial/\partial t}\,\mathrm{d}u_te_i\right] \\
&= \sum \nabla_{e_i}\left[F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\nabla_{e_j}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_te_j\right\rangle\mathrm{d}u_te_i + F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right].
\end{aligned}
$$

Substituting (21) into (20), and then making an inner product with $\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$ yield:

$$
\begin{aligned}
(22)\quad \left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \nabla_{\partial/\partial t}\tau_F(u_t)\right\rangle &= -F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \sum R^N\left(\mathrm{d}u_te_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\mathrm{d}u_te_i\right\rangle \\
&\quad + \left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \sum \nabla_{e_i}\left[F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\nabla_{e_j}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_te_j\right\rangle\mathrm{d}u_te_i\right]\right\rangle \\
&\quad + \left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \sum \nabla_{e_i}\left[F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right]\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \sum R^N\left(\mathrm{d}u_t e_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\, \mathrm{d}u_t e_i\right\rangle \\
&\quad + \sum \nabla_{e_i}\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \left[F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\nabla_{e_j}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_t e_j\right\rangle\, \mathrm{d}u_t e_i\right]\right\rangle \\
&\quad - \sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\nabla_{e_j}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_t e_j\right\rangle\, \mathrm{d}u_t e_i\right\rangle \\
&\quad + \sum \nabla_{e_i}\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right\rangle \\
&\quad - \sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right\rangle \\
&= -\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum R^N\left(\mathrm{d}u_t e_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\, \mathrm{d}u_t e_i\right\rangle \\
&\quad - F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_t e_i\right\rangle^2 \\
&\quad - F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right\rangle + \cdots,
\end{aligned}
$$

where, "$\cdots$" is a divergence. Inserting (22) into (19), and taking use of the divergence theorem, we obtain a second variational formula of the F-energy:

$$
\begin{aligned}
(23)\qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}E_F(u_t) &= -\int_M\left\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}, \tau_F(u_t)\right\rangle \\
&\quad + \int_M\left[\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum R^N\left(\mathrm{d}u_t e_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)\, \mathrm{d}u_t e_i\right\rangle\right. \\
&\qquad + F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_t e_i\right\rangle^2 \\
&\qquad \left. + F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right\rangle\right] \\
&= \int_M\left[-\left\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}, \tau_F(u_t)\right\rangle\right.
\end{aligned}
$$

$$+ F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \sum R^N\left(\mathrm{d}u_t e_i, \frac{\mathrm{d}\Phi}{\mathrm{d}t}\right) \mathrm{d}u_t e_i\right\rangle$$
$$+ F''\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \mathrm{d}u_t e_i\right\rangle^2$$
$$+ F'\left(\frac{|\mathrm{d}u_t|^2}{2}\right)\sum\left\langle\nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \nabla_{e_i}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right\rangle\Bigg].$$

For any fixed $X \in \Gamma(TM)$, we take a variation $u_t = \Phi(t,\cdot)$ of u as follows:

$$\begin{cases} \dfrac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} = \dfrac{\mathrm{d}\Phi}{\mathrm{d}t}, \\ \left.\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} = \mathrm{d}u(X), \\ u_0 = u. \end{cases}$$

Because

$$\int_M \left\langle\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}, \tau_F(u_t)\right\rangle\Bigg|_{t=0} = \int_M \left\langle\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tau_F(u_t)\right\rangle\Bigg|_{t=0} = \int_M \langle\, \mathrm{d}u(X), \tau_F(u)\rangle = 0$$

for an F-stationary map u, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}E_F(u_t)|_{t=0} \tag{24}$$
$$= \sum\int_M \left[F''\left(\frac{|\mathrm{d}u_0|^2}{2}\right)\langle\nabla_{e_i}\psi, \mathrm{d}u_0 e_i\rangle^2 + F'\left(\frac{|\mathrm{d}u_0|^2}{2}\right)(\langle\nabla_{e_i}\psi, \nabla_{e_i}\psi\rangle + \langle\psi, R^N(\mathrm{d}u_0 e_i, \psi)\,\mathrm{d}u_0 e_i\rangle)\right],$$

where $\psi = \left.\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} = \mathrm{d}u(X)$. Let

$$I_F(V,V) = \int_M F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle\nabla V, \mathrm{d}u\rangle^2 + \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)[|\nabla V|^2 + \sum\langle R^N(\mathrm{d}ue_i, V)\,\mathrm{d}ue_i, V\rangle], \tag{25}$$

where $V \in \Gamma_0(u^{-1}TN)$.

Definition 5. If for any $X \in \Gamma_0(TM)$ we have $I_F(duX, duX) \geq 0$, then the F-stationary map u is called to be stable; Otherwise, it is called to be unstable.

Then we have

THEOREM 6. *Let M be a compact m-submanifold of $\mathbf{R}^{m+k_0}$, which satisfies the condition* (16, 17 *or* 18) *for some function b. If $u : M \to N$ is a stable F-stationary map with $F'' \leq 0$ and $F' \geq 0$, then*

$$\int_M bF'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle \geq 0.$$

Especially, if u is a stable F-harmonic map with $F'' \leq 0$, $F'(t) > 0$ for $t > 0$ and $b < 0$, then it must be constant.

In Theorem 6, we assume that $F'' \leq 0$. In the following theorem, we suppose that F satisfies another condition.

THEOREM 7. *Let M be a compact m-submanifold of $\mathbf{R}^{m+k_0}$, which satisfies the condition* (16, 17 *or* 18) *for some function b. If $u : M \to N$ is a stable F-stationary map with $2tF''(t) \leq pF'(t)$ for a number p and $F' \geq 0$, then*

$$\int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(p|B|^2 + b)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle \geq 0,$$

where B is the second fundamental form. Especially, if $M = S^m$, and u is a stable F-harmonic map with $2tF''(t) \leq pF'(t)$ with $p < \dfrac{m-2}{m}$ and $F'(t) > 0$ for $t > 0$, then it must be constant.

For expnential stationary maps, we have

THEOREM 8. *If $u : S^m \to N$ is a stable expnential stationary map, then*

$$\int_M \exp\left(\frac{|\mathrm{d}u|^2}{2}\right)\sum(|\mathrm{d}u|^2 + 2 - m)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_e(u)\rangle \geq 0,$$

where, $\tau_e(u)$ is the exponential tension of u. Especially, when $|\mathrm{d}u|^2 < m-2$, and u is an exponential harmonic map, then u must be constant.

3.2. Lemmas

Let M^m and N^n be Riemannian manifolds, $u : M \to N$ a smooth map. Denote ∇^M, ∇^N and $\nabla^{\mathbf{R}^{m+k_0}}$ are Riemannian connections of M, N and $\mathbf{R}^{m+k_0}$ respectively. Assume that M is a submanifold of $\mathbf{R}^{m+k_0}$. Let $\nabla^\perp$ be the connection of the normal bundle of M in $\mathbf{R}^{m+k_0}$, and ∇ the induced connection of $u^{-1}TN$ by ∇^N. In the following, the ranges of indices are given by

$$1 \leq A, B, C, \ldots \leq m + k_0;$$

$$1 \leq i, j, k, \ldots \leq m;$$

$$m + 1 \leq \mu, p, s \leq m;$$

$$1 \leq \alpha, \beta \leq n.$$

Let $\{X_A\}$ be the canonical orthonormal base of $\mathbf{R}^{m+k_0}$, $\{e_i; e_\mu\}$ a local field of orthonormal frames of M, such that $\{e_i\}$ are tangent to M and that $\{e_\mu\}$ are normal to M. At any fixed point we considered, we can suppose that $\nabla^M_{e_i} e_j = 0$. Denote the tangent part of X_A by X_A^T and the normal part by X_A^N. Then we have

$$X_A^T = \sum \langle X_A, e_i \rangle e_i =: \sum v_A^i e_i,$$

$$X_A^N = \sum \langle X_A, e_\mu \rangle e_\mu =: \sum v_A^\mu e_\mu,$$

$$X_A = X_A^T + X_A^N = \sum v_A^B e_B = \sum \langle X_A, e_B \rangle e_B,$$

$$e_A = \sum \langle e_A, X_B \rangle X_B.$$

Lemma 9. *We have*

$$\sum v_A^B v_A^C = \delta_{BC}.$$

$$\nabla^M_{e_i} X_A^T = \sum h_{ij}^\mu v_A^\mu e_j.$$

$$\nabla^M_{e_k} \nabla^M_{e_j} X_A^T = \sum (v_A^\mu h_{ijk}^\mu - v_A^l h_{kl}^\mu h_{ij}^\mu) e_i.$$

Proof. It is not difficult to see

$$\begin{aligned} \sum v_A^B v_A^C &= \sum \langle e_B, X_A \rangle \langle e_C, X_A \rangle \\ &= \sum \langle e_B, X_A \rangle \langle e_D, X_A \rangle \langle e_D, e_C \rangle \\ &= \sum \langle e_B, X_A \rangle \langle X_A, e_C \rangle = \langle e_B, e_C \rangle = \delta_{BC}. \end{aligned} \tag{26}$$

By Weingarten's equations, we have

$$\begin{aligned} \nabla^M_{e_i} X_A^T &= (\nabla^{\mathbf{R}^{m+k_0}}_{e_i} X_A^T)^T = (\nabla^{\mathbf{R}^{m+k_0}}_{e_i} (X_A - X_A^N))^T \\ &= -(\nabla^{\mathbf{R}^{m+k_0}}_{e_i} X_A^N)^T = A^{X_A^N}(e_i) \\ &= \sum v_A^\mu A^{e_\mu}(e_i) = \sum h_{ij}^\mu v_A^\mu e_j. \end{aligned} \tag{27}$$

where, $A^{e_\mu}(e_i) = \sum h_{ij}^\mu e_j$. By Gaussian equations and Weigarten's equations, we have

$$\begin{aligned} \nabla^{\mathbf{R}^{m+k_0}}_{e_k} X_A^N &= \sum (e_k v_A^\mu) e_\mu + \sum v_A^\mu \nabla^{\mathbf{R}^{m+k_0}}_{e_k} e_\mu \\ &= \sum (e_k v_A^\mu) e_\mu + \sum v_A^\mu (-A^{e_\mu}(e_k) + \nabla^\perp_{e_k} e_\mu) \\ &= \sum (e_k v_A^\mu) e_\mu - \sum v_A^\mu h_{kj}^\mu e_j + \sum v_A^\mu \nabla^\perp_{e_k} e_\mu \end{aligned}$$

and

$$\begin{aligned}\nabla^{\mathbf{R}^{m+k_0}}_{e_k} X_A^T &= \sum (e_k v_A^j) e_j + \sum v_A^j \nabla^{\mathbf{R}^{m+k_0}}_{e_k} e_j \\ &= \sum (e_k v_A^j) e_j + \sum v_A^j (\nabla^M_{e_k} e_j + B(e_k, e_j)) \\ &= \sum (e_k v_A^j) e_j + \sum v_A^j \nabla^M_{e_k} e_j + \sum v_A^j h^\mu_{kj} e_\mu.\end{aligned}$$

Therefore, we have

$$\begin{aligned}0 = \nabla^{\mathbf{R}^{m+k_0}}_{e_k} X_A &= \nabla^{\mathbf{R}^{m+k_0}}_{e_k} X_A^T + \nabla^{\mathbf{R}^{m+k_0}}_{e_k} X_A^N \\ &= \sum (e_k v_A^\mu) e_\mu - \sum v_A^\mu h^\mu_{kj} e_j + \sum v_A^\mu \nabla^\perp_{e_k} e_\mu \\ &\quad + \sum (e_k v_A^j) e_j + \sum v_A^j \nabla^M_{e_k} e_j + \sum v_A^j h^\mu_{kj} e_\mu.\end{aligned}$$

This implies that

$$\sum (e_k v_A^\mu) e_\mu = -\sum v_A^j h^\mu_{kj} e_\mu - \sum v_A^\mu \nabla^\perp_{e_k} e_\mu.$$

Taking dot product of both sides of this equation by e_μ yields

$$e_k v_A^\mu = -\sum v_A^j h^\mu_{kj} - \sum v_A^\nu \langle \nabla^\perp_{e_k} e_\nu, e_\mu \rangle.$$

From $h^\mu_{ij} = \langle B(e_i, e_j), e_\mu \rangle$ we have

$$\begin{aligned}e_k h^\mu_{ij} &= \langle \nabla^\perp_{e_k}(B(e_i, e_j)), e_\mu \rangle + \langle B(e_i, e_j), \nabla^\perp_{e_k} e_\mu \rangle \\ &= \langle (\nabla^\perp_{e_k} B)(e_i, e_j), e_\mu \rangle + h^\nu_{ij} \langle e_\nu, \nabla^\perp_{e_k} e_\mu \rangle \\ &= h^\mu_{ijk} + h^\nu_{ij} \langle e_\nu, \nabla^\perp_{e_k} e_\mu \rangle.\end{aligned}$$

Hence

$$\begin{aligned}\nabla^M_{e_k} \nabla^M_{e_j} X_A^T &= \nabla^M_{e_k} \left(\sum h^\mu_{ij} v_A^\mu e_i\right) \\ &= \sum (e_k h^\mu_{ij}) v_A^\mu e_i + \sum h^\mu_{ij} (e_k v_A^\mu) e_i + \sum h^\mu_{ij} v_A^\mu \nabla^M_{e_k} e_i \\ &= \sum h^\mu_{ijk} v_A^\mu e_i - \sum v_A^l h^\mu_{ij} h^\mu_{kl} e_i + \sum h^\mu_{ij} v_A^\mu \nabla^M_{e_k} e_i.\end{aligned}$$

At any fixed point, we have

$$\nabla^M_{e_k} \nabla^M_{e_j} X_A^T = \sum (v_A^\mu h^\mu_{ijk} - v_A^l h^\mu_{kl} h^\mu_{ij}) e_i, \tag{28}$$

since we have required $\nabla_{e_k} e_i = 0$. □

LEMMA 10. *Under the assumptions of Theorem* 6, *then for any stable* F*-stationary map* $u : M \to N$, *we have*

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) = \int_M \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right) h_{ij}^\mu h_{kl}^\mu a_{\alpha i} a_{\alpha j} a_{\beta k} a_{\beta l} + \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^\mu h_{ik}^\mu - R_{jk}^M) a_{\alpha j} a_{\alpha k} + \int_M \langle \tau(u), \tau_F(u)\rangle,$$

where $a_{\alpha j}$ *is determined by* $\mathrm{d}ue_j = a_{\alpha j}(e'_\alpha \circ u)$, $\{e'_\alpha\}$ *is a local field of orthonormal frame of* N, R_{ij}^M *is the Ricci tensor of* M.

Proof. By the second variational formula of F-stationary map u, we have

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) = \sum \int_M F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla(\mathrm{d}uX_A^T), \mathrm{d}u\rangle^2 + \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\nabla(\mathrm{d}uX_A^T)|^2 + \langle \sum R^N(\mathrm{d}ue_i, \mathrm{d}uX_A^T)\, \mathrm{d}ue_i, \mathrm{d}uX_A^T\rangle\}. \tag{29}$$

By Weitzenbock formula (See [4] 1.34, but the curvature operators there are different from those here by a signature.) we have (∇ is the connection of $T^*M \otimes u^{-1}TN$ and Δ is the Laplacian acting on $T^*M \otimes u^{-1}TN$)

$$(\Delta\, \mathrm{d}u)X_A^T = -(\nabla^2\, \mathrm{d}u)X_A^T + \sum R^N(\mathrm{d}ue_i, \mathrm{d}uX_A^T)\, \mathrm{d}ue_i - \sum \mathrm{d}uR^M(e_i, X_A^T)e_i. \tag{30}$$

On the other hand, because $\mathrm{dd}u = 0$, we have

$$\begin{aligned} &\sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle(\Delta\, \mathrm{d}u)X_A^T, \mathrm{d}uX_A^T\rangle \\ &= \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) v_A^i v_A^j \langle(\Delta\, \mathrm{d}u)e_i, \mathrm{d}ue_j\rangle \\ &= \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle(\Delta\, \mathrm{d}u)e_i, \mathrm{d}ue_i\rangle \\ &= \int_M \left\langle \mathrm{d}^*\mathrm{d}u, \mathrm{d}^*\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\mathrm{d}u\right)\right\rangle = \int_M \langle \tau(u), \tau_F(u)\rangle. \end{aligned} \tag{31}$$

By (29), (30) and (31) we get

$$(32)\qquad \sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T)$$
$$= \sum \int_M F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla(\mathrm{d}uX_A^T), \mathrm{d}u\rangle^2 + \sum \int_M \langle \tau(u), \tau_F(u)\rangle$$
$$+ \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\nabla(\mathrm{d}uX_A^T)|^2 + \langle(\nabla^2\,\mathrm{d}u)X_A^T, \mathrm{d}uX_A^T\rangle$$
$$+ \sum \langle \mathrm{d}uR^M(e_i, X_A^T)e_i, \mathrm{d}uX_A^T\rangle\}.$$

On the other hand, we have

$$(33)\qquad \begin{aligned}(\nabla^2\,\mathrm{d}u)X_A^T &= \sum(\nabla_{e_i}\nabla_{e_i}\,\mathrm{d}u)X_A^T\\ &= \sum \nabla_{e_i}((\nabla_{e_i}\,\mathrm{d}u)X_A^T) - \sum(\nabla_{e_i}\,\mathrm{d}u)(\nabla^M_{e_i}X_A^T)\\ &= \sum \nabla_{e_i}[\nabla_{e_i}(\mathrm{d}uX_A^T) - \mathrm{d}u(\nabla^M_{e_i}X_A^T)]\\ &\quad - \sum \nabla_{e_i}[\mathrm{d}u(\nabla^M_{e_i}X_A^T)] + \sum \mathrm{d}u(\nabla^M_{e_i}\nabla^M_{e_i}X_A^T)\\ &= \sum \nabla_{e_i}\nabla_{e_i}(\mathrm{d}uX_A^T) - 2\sum \nabla_{e_i}[\mathrm{d}u(\nabla^M_{e_i}X_A^T)]\\ &\quad + \sum \mathrm{d}u(\nabla^M_{e_i}\nabla^M_{e_i}X_A^T)\end{aligned}$$

and

$$(34)\qquad \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \nabla_{e_i}\nabla_{e_i}(\mathrm{d}uX_A^T), \mathrm{d}uX_A^T\rangle$$
$$= -\sum \int_M \left\langle \nabla_{e_i}(\mathrm{d}uX_A^T), e_i\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right]\,\mathrm{d}uX_A^T\right\rangle$$
$$-\sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\nabla(\mathrm{d}uX_A^T)|^2.$$

From (33) and (34) we have

$$(35)\qquad \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\nabla(\mathrm{d}uX_A^T)|^2 + \langle(\nabla^2\,\mathrm{d}u)(X_A^T), \mathrm{d}uX_A^T\rangle\}$$
$$= \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\nabla(\mathrm{d}uX_A^T)|^2 + \langle \nabla_{e_i}\nabla_{e_i}(\mathrm{d}uX_A^T), \mathrm{d}uX_A^T\rangle$$
$$- 2\langle \nabla_{e_i}(\mathrm{d}u(\nabla^M_{e_i}X_A^T)), \mathrm{d}uX_A^T\rangle + \langle \mathrm{d}u(\nabla_{e_i}\nabla_{e_i}X_A^T), \mathrm{d}uX_A^T\rangle\}$$

$$= \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle -2\nabla_{e_i}(\mathrm{d}u(\nabla^M_{e_i} X^T_A)) + \mathrm{d}u\nabla^M_{e_i}\nabla^M_{e_i} X^T_A, \mathrm{d}uX^T_A\rangle$$
$$- \sum \int_M \left\langle \nabla_{e_i}(\mathrm{d}uX^T_A), e_i\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right] \mathrm{d}uX^T_A\right\rangle.$$

Let $\mathrm{d}ue_j = a_{\alpha j}(e'_\alpha \circ u)$ and $(\nabla_{e_i}\, \mathrm{d}u)e_j = a_{\alpha ji}(e'_\alpha \circ u)$. Then we have

$$\begin{aligned}(36)\quad & -2\sum \nabla_{e_i}(\mathrm{d}u(\nabla^M_{e_i} X^T_A)) \\ &= -2\sum \mathrm{d}u\nabla^M_{e_i}\nabla^M_{e_i} X^T_A - 2\sum(\nabla_{e_i}\, \mathrm{d}u)\nabla^M_{e_i} X^T_A \\ &= -2\sum \langle\nabla^M_{e_i}\nabla^M_{e_i} X^T_A, e_j\rangle\, \mathrm{d}ue_j - 2\sum \langle\nabla^M_{e_i} X^T_A, e_j\rangle(\nabla_{e_i}\, \mathrm{d}u)e_j \\ &= -2\sum \langle\nabla^M_{e_i}\nabla^M_{e_i} X^T_A, e_j\rangle a_{\alpha j}(e'_\alpha \circ u) - 2\sum \langle\nabla^M_{e_i} X^T_A, e_j\rangle a_{\alpha ji}(e'_\alpha \circ u).\end{aligned}$$

Applying (26), (27) and (28) to (36) we can get

$$\begin{aligned}(37)\quad & \sum F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle -2\nabla_{e_i}(\mathrm{d}u(\nabla^M_{e_i} X^T_A)), \mathrm{d}uX^T_A\rangle \\ &= \sum F'\left(\frac{|\mathrm{d}u|^2}{2}\right)[-2\langle\nabla^M_{e_i}\nabla^M_{e_i} X^T_A, e_j\rangle a_{\alpha j}a_{\alpha k}v^k_A - 2\langle\nabla^M_{e_i} X^T_A, e_j\rangle a_{\alpha ji}a_{\alpha k}v^k_A] \\ &= \sum F'\left(\frac{|\mathrm{d}u|^2}{2}\right)[-2a_{\alpha j}a_{\alpha k}h^\mu_{iij}v^\mu_A v^k_A + 2h^\mu_{il}h^\mu_{ij}a_{\alpha j}a_{\alpha k}v^l_A v^k_A - 2h^\mu_{ij}a_{\alpha ji}a_{\alpha k}v^\mu_A v^k_A] \\ &= 2\sum h^\mu_{ij}h^\mu_{ik}a_{\alpha j}a_{\alpha k}F'\left(\frac{|\mathrm{d}u|^2}{2}\right).\end{aligned}$$

Again by (26), (27) and (28) we get

$$(38)\quad \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \mathrm{d}u(\nabla^M_{e_i}\nabla^M_{e_i} X^T_A, \mathrm{d}uX^T_A)\rangle = -\sum h^\mu_{ki}h^\mu_{ij}a_{\alpha j}a_{\alpha k}F'\left(\frac{|\mathrm{d}u|^2}{2}\right)$$

and

$$\begin{aligned}(39)\quad & -\sum\left\langle \nabla_{e_i}(\mathrm{d}uX^T_A), e_i\left[F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right] \mathrm{d}uX^T_A\right\rangle \\ &= -\sum\left\langle (\nabla_{e_i}\, \mathrm{d}u)X^T_A + \mathrm{d}u\nabla_{e_i}X^T_A, F''\left(\frac{|\mathrm{d}u|^2}{2}\right)a_{\beta li}a_{\beta l}\, \mathrm{d}uX^T_A\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\sum\left\langle v_A^j a_{\alpha ji}(e'_\alpha \circ u), F''\left(\frac{|\mathrm{d}u|^2}{2}\right) a_{\beta li} a_{\beta l} v_A^k a_{\gamma k}(e'_\gamma \circ u)\right\rangle \\
&\quad -\sum\left\langle h_{ij}^\mu v_A^\mu a_{\alpha j}(e'_\alpha \circ u), F''\left(\frac{|\mathrm{d}u|^2}{2}\right) a_{\beta li} a_{\beta l} v_A^k a_{\gamma k}(e'_\gamma \circ u)\right\rangle \\
&= -\sum a_{\alpha ji} a_{\alpha j} a_{\beta li} a_{\beta l} F''\left(\frac{|\mathrm{d}u|^2}{2}\right).
\end{aligned}
$$

Substituting (37), (38) and (39) into (35), we have

$$
\begin{aligned}
(40)\quad &\sum\int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\nabla(\mathrm{d}uX_A^T)|^2 + \langle(\nabla^2\,\mathrm{d}u)(X_A^T), \mathrm{d}uX_A^T\rangle\} \\
&= \sum\int_M h_{ij}^\mu h_{ik}^\mu a_{\alpha j} a_{\alpha k} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) - \sum\int_M a_{\alpha ji} a_{\alpha j} a_{\beta li} a_{\beta l} F''\left(\frac{|\mathrm{d}u|^2}{2}\right).
\end{aligned}
$$

By a calculation, we have

$$
\begin{aligned}
(41)\quad &\sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle\nabla(\mathrm{d}uX_A^T), \mathrm{d}u\rangle^2 \\
&= \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle(\nabla_{e_i}\,\mathrm{d}u)X_A^T + \mathrm{d}u\nabla_{e_i}X_A^T, \mathrm{d}ue_i\rangle^2 \\
&= \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle a_{\alpha ji} v_A^j (e'_\alpha \circ u) + v_A^\mu h_{ij}^\mu a_{\alpha j}(e'_\alpha \circ u), a_{\beta i}(e'_\beta \circ u)\rangle^2 \\
&= \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right)[a_{\alpha ji} a_{\alpha i} a_{\beta jl} a_{\beta l} + h_{ij}^\mu h_{kl}^\mu a_{\alpha j} a_{\alpha i} a_{\beta l} a_{\beta k}]
\end{aligned}
$$

and

$$
\begin{aligned}
(42)\quad \sum\langle \mathrm{d}uR^M(e_i, X_A^T)e_i, \mathrm{d}uX_A^T\rangle &= \sum v_A^k v_A^j \langle \mathrm{d}uR^M(e_i, e_k)e_i, \mathrm{d}ue_j\rangle \\
&= \sum\langle \mathrm{d}uR^M(e_i, e_j)e_i, \mathrm{d}ue_j\rangle \\
&= -\sum R_{jk}^M \langle \mathrm{d}ue_k, \mathrm{d}ue_j\rangle \\
&= -\sum R_{jk}^M a_{\alpha k} a_{\alpha j}.
\end{aligned}
$$

Taking use of (40), (41), (42) and (32), we have

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) = \int_M \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right) h_{ij}^{\mu} h_{kl}^{\mu} a_{\alpha i} a_{\alpha j} a_{\beta k} a_{\beta l} + \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^{\mu} h_{ik}^{\mu} a_{\alpha j} a_{\alpha k} - R_{ij}^{M} a_{\alpha i} a_{\alpha j}) + \int_M \langle \tau(u), \tau_F(u) \rangle \tag{43}$$

which is desired. □

3.3. Proof of Theorem 6

By Lemma 10, we get

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) = \int_M \sum F''\left(\frac{|\mathrm{d}u|^2}{2}\right) h_{ij}^{\mu} h_{kl}^{\mu} a_{\alpha i} a_{\alpha j} a_{\beta k} a_{\beta l} + \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^{\mu} h_{ik}^{\mu} - R_{jk}^{M}) a_{\alpha j} a_{\alpha k} + \int_M \langle \tau(u), \tau_F(u) \rangle. \tag{44}$$

Because $F'' \le 0$, we have

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) \le \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^{\mu} h_{ik}^{\mu} - R_{jk}^{M}) a_{\alpha j} a_{\alpha k} + \int_M \langle \tau(u), \tau_F(u) \rangle. \tag{45}$$

By the assumption on the curvatures of M, we get

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) \le \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) b|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u) \rangle.$$

By the stabilities of u, we have

$$0 \le \sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) \le \int_M bF'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u) \rangle.$$

3.4. Proof of Theorem 7

If $2tF''(t) \le pF'(t)$, then by Lemma 10 we have

$$\begin{aligned}\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) &= \int_M F''\left(\frac{|\mathrm{d}u|^2}{2}\right)|B|^2|\mathrm{d}u|^4 + \int_M \langle \tau(u), \tau_F(u)\rangle \\ &\quad + \sum \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^\mu h_{ik}^\mu - R_{jk}^M)a_{\alpha j}a_{\alpha k} \\ &\le p\int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|B|^2|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle \\ &\quad + \int_M bF'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2 \\ &= \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(p|B|^2 + b)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle.\end{aligned}$$

Therefore, from the stability of u, we have

$$\int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(p|B|^2 + b)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle \ge 0.$$

Especially, if $M = S^m$, then $b = 2 - m$ and $|B|^2 = m$. Hence, when $p < \frac{m-2}{m}$, and u is a stable F-harmonic map, we have $\int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2 = 0$, and hence $\mathrm{d}u = 0$.

3.5. Proof of Theorem 8

If $F(t) = e^t$, then by Lemma 10 we have

$$\begin{aligned}\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) &= \int_M \sum \exp\left(\frac{|\mathrm{d}u|^2}{2}\right) h_{ij}^\mu h_{kl}^\mu a_{\alpha i}a_{\alpha j}a_{\beta k}a_{\beta l} \\ &\quad + \sum \int_M \exp\left(\frac{|\mathrm{d}u|^2}{2}\right)(h_{ij}^\mu h_{ik}^\mu - R_{jk}^M)a_{\alpha j}a_{\alpha k} + \int_M \langle \tau(u), \tau_F(u)\rangle \\ &= \int_M \exp\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum (h_{jk}^\mu h_{il}^\mu a_{\beta i}a_{\beta l} + h_{ij}^\mu h_{ik}^\mu - R_{jk}^M)a_{\alpha j}a_{\alpha k} \\ &\quad + \int_M \langle \tau(u), \tau_F(u)\rangle\end{aligned}$$

If $M = S^m \subseteq \mathbf{R}^{m+1}$, then $h_{ij}^\mu =: h_{ij} = \delta_{ij}$. Hence

$$\sum I(\mathrm{d}uX_A^T, \mathrm{d}uX_A^T) = \int_M \exp\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum (|\mathrm{d}u|^2 + 2 - m)|\mathrm{d}u|^2 + \int_M \langle \tau(u), \tau_F(u)\rangle.$$

If u is a stable expnential stationary map, then

$$\int_M \exp\left(\frac{|du|^2}{2}\right) \sum (|du|^2 + 2 - m)|du|^2 + \int_M \langle \tau(u), \tau_F(u) \rangle \geq 0.$$

When $|du|^2 < m - 2$, and u is an exponential harmonic map, then u must be constant.

References

[1] M. Ara, Geometry of F-harmonic maps, Kodai Math. J. **22** (1999), 243–263.

[2] Q. Chen, Stability and constant boundary-value problems of harmonic maps with potential, J. Aust. Math. Soc. Ser. A **68** (2000), 145–154.

[3] Q. Chen and Z. R. Zhou, On gap properties and stabilities of p-Yang-Mills fields, Canadian J. Math. **59** (2007), 1245–1259.

[4] J. Eells and L. Lemaire, Selected topics on harmonic maps, Expository Lectures from the CBMS Regional Conf. held at Tulane Univ., Dec. 15–19, 1980.

[5] P. F. Leung, On the stability of harmonic maps, Lecture notes in math. **949**, 1982, 122–129.

[6] J. T. Li, F-harmonic maps for minimal submanifolds with positive Ricci curvature, Acta. Math. Sci. Ser. A. **24** (2004), 152–156.

[7] Y. Ohnita, Stability of harmonic maps and standard minimal immersions, Tohoku Math. J. **38** (1986), 259–267.

[8] Y. L. Xin, Some results on stable harmonic Maps, Duke Math. J. **47** (1980), 600–613.

[9] Z. R. Zhou, Stability and Quantum phenomenen and Liouville theorems of p-harmonic maps with potential, Kodai Math. J. **26** (2003), 101–118.

[10] Z. R. Zhou, Monotonicity of F-stationary maps, to appear in J. Math. (PRC) (2011).

[11] Z. R. Zhou, Liouville theorems of F-stationary maps, to appear in Acta Math. Sci. Ser. B (2010).

Zhen-Rong Zhou
Department of Mathematics
Central China Normal University
430079, Wuhan
P.R. China
E-mail: zrzhoumath@sina.cn