I. HOTTA
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RUSCHEWEYH'S UNIVALENCE CRITERION AND QUASICONFORMAL EXTENSIONS

Ikkei Hotta

Abstract

Ruscheweyh extended the work of Becker and Ahlfors on sufficient conditions for a normalized analytic function on the unit disk to be univalent there. In this paper we refine the result to a quasiconformal extension criterion with the help of Becker's method. As an application, a positive answer is given to an open problem proposed by Ruscheweyh.

1. Introduction

Throughout the paper, $\mathbf{D}$ denotes the unit disk $\{|z| < 1\}$ in the complex plane $\mathbf{C}$ and $\mathbf{D}^*$ the exterior domain of $\mathbf{D}$ in the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

Let $\mathscr{A}$ be a family of normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on $\mathbf{D}$. We say that a sense-preserving homeomorphism f of a plane domain $G \subset \mathbf{C}$ is k-quasiconformal if f is absolutely continuous on almost all lines parallel to the coordinate axes and $|f_{\bar{z}}| \le k|f_z|$, almost everywhere G, where $f_{\bar{z}} = \partial f/\partial \bar{z}$, $f_z = \partial f/\partial z$ and k is a constant with $0 \le k < 1$.

Ahlfors [1] has shown that the following condition is sufficient for quasiconformal extensibility of univalent functions as an extension of Becker's univalence condition [2] (see also [7], p. 175);

Theorem A ([1], [3]). *Let $f \in \mathscr{A}$. If there exists a k, $0 \le k < 1$, such that for a constant $c \in \mathbf{C}$ satisfying $|c| \le k$ and all $z \in \mathbf{D}$*

$$\left| c|z|^2 + (1 - |z|^2) \frac{zf''(z)}{f'(z)} \right| \le k \tag{1}$$

then f has a k-quasiconformal extension to $\mathbf{C}$.

The limiting case $k \to 1$ in the above theorem ensures univalence of f in $\mathbf{D}$. Ruscheweyh [8] extended this univalence condition in the following way;

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Theorem B ([8]). *Let $s = a + ib$, $a > 0$, $b \in \mathbf{R}$ and $f \in \mathscr{A}$. Assume that for a constant $c \in \mathbf{C}$ and all $z \in \mathbf{D}$*

$$\left| c|z|^2 + s - a(1-|z|^2)\left\{ s\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)} \right\} \right| \le M \tag{2}$$

with

$$M = \begin{cases} a|s| + (a-1)|s+c|, & \text{if } 0 < a \le 1, \\ |s|, & \text{if } 1 < a, \end{cases}$$

then f is univalent in $\mathbf{D}$.

The case $s = 1$ with c replaced by $-1 - c$ is the special case of Theorem A. The purpose of this paper is to refine Ruscheweyh's univalence condition to a quasiconformal extension criterion which includes Theorem A;

Theorem 1. *Let $s = a + ib$, $a > 0$, $b \in \mathbf{R}$, $k \in [0,1)$ and $f \in \mathscr{A}$. Assume that for a constant $c \in \mathbf{C}$ and all $z \in \mathbf{D}$*

$$\left| c|z|^2 + s - a(1-|z|^2)\left\{ s\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)} \right\} \right| \le M \tag{3}$$

with

$$M = \begin{cases} ak|s| + (a-1)|s+c|, & \text{if } 0 < a \le 1, \\ k|s|, & \text{if } 1 < a, \end{cases}$$

then f has an l-quasiconformal extension to $\mathbf{C}$, where

$$l = \frac{2ka + (1-k^2)|b|}{(1+k^2)a + (1-k^2)|s|} < 1. \tag{4}$$

Remark 1.1. If $f \in \mathscr{A}$, then it is easy to verify that there exists a sequence $\{z_n\} \subset \mathbf{D}$ with $|z_n| \to 1$ such that for each $s \in \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$

$$\sup_n \left| s\left(1 + \frac{z_n f''(z_n)}{f'(z_n)}\right) + (1-s)\frac{z_n f'(z_n)}{f(z_n)} \right| < \infty$$

which shows that (3) implies the inequality

$$|c + s| \le M. \tag{5}$$

This inequality is needed for proving that $f(z)$ has no zeros in $0 < |z| < 1$ (see Lemma 7). In [8], it is mentioned that (3) implies $f(z) \neq 0$, $0 < |z| < 1$, without proof. The part of (5) can be found in [8].

Remark 1.2. A similar argument to Remark 1.1 is also valid for Theorem A. It follows that the assumption $|c| \le k$ is embedded in the inequality (1).

The next application follows from Theorem 1. Let $\alpha > 0$ and $\beta \in \mathbf{R}$. It follows from a result of Sheil-Small [9, Theorem 2] that

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbf{D}) \tag{6}$$

is sufficient for $f \in \mathcal{A}$ to be a Bazilevič function of type (α, β)[1] (see also [5]). Here, a function $f \in \mathcal{A}$ is called *Bazilevič of type* (α, β) if

$$f(z) = \left[(\alpha + i\beta)\int_0^z g(\zeta)^\alpha h(\zeta)\zeta^{i\beta - 1}\, d\zeta\right]^{1/(\alpha + i\beta)}$$

for a starlike univalent function $g \in \mathcal{A}$ and an analytic function h with $h(0) = 1$ satisfying $Re(e^{i\lambda}h) > 0$ in $\mathbf{D}$ for some $\lambda \in \mathbf{R}$. Together with this fact, the next theorem follows;

Theorem 2. *Let $\alpha > 0$, $\beta \in \mathbf{R}$ and $k \in [0, 1)$. If $f \in \mathcal{A}$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} + (\alpha + i\beta - 1)\frac{zf'(z)}{f(z)} - \frac{\alpha^2 + \beta^2}{\alpha}\right| \le M \tag{7}$$

for all $z \in \mathbf{D}$ with

$$M = \begin{cases} k & \text{if } \alpha < \alpha^2 + \beta^2, \\ k(\alpha^2 + \beta^2)/\alpha & \text{if } \alpha^2 + \beta^2 \le \alpha, \end{cases}$$

then f is a Bazilevič function of type (α, β) and can be extended to a $\tilde{k}$-quasiconformal automorphism of $\mathbf{C}$, where

$$\tilde{k} = \frac{2k\alpha + (1 - k^2)|\beta|}{(1 + k^2)\alpha + (1 - k^2)\sqrt{\alpha^2 + \beta^2}}.$$

Next, we shall discuss quasiconformal extensibility of functions $g(z) = z + \dfrac{d}{z} + \cdots$ analytic in $\mathbf{D}^*$.

Theorem 3. *Let $s = a + ib$, $a \ge 1$, $b \in \mathbf{R}$ and $k \in [0, 1)$ which satisfies $|b/s| \le k$. Let $g(\zeta) = \zeta + \dfrac{d}{\zeta} + \cdots$ be analytic in $\mathbf{D}^*$ and fulfill*

$$\left|ib + (1 - |\zeta|^2)a\left\{(1 - s)\left(1 - \frac{\zeta g'(\zeta)}{g(\zeta)}\right) - s\frac{\zeta g''(\zeta)}{g'(\zeta)}\right\}\right| \le ak|s| - |b|(a - 1) \tag{8}$$

[1] The author would like to thank Professor Yong Chan Kim for this remark.

for all $\zeta \in \mathbf{D}^*$. *Then* g *can be extended to an* l*-quasiconformal automorphism of* $\hat{\mathbf{C}}$, *where*

$$l = \frac{2ka + (1-k^2)|b|}{(1+k^2)a + (1-k^2)|s|}.$$

The case $k \to 1$ corresponds to a univalence criterion which is due to Ruscheweyh [8].

Theorem 3 yields the following corollary which gives a positive answer to an open problem proposed by Ruscheweyh [8], i.e., whether a function $g(\zeta) = \zeta + d/\zeta + \cdots$ with $(|\zeta|^2 - 1)|1 + (\zeta f''(\zeta)/f'(\zeta)) - (\zeta f'(\zeta)/f(\zeta))| \le k$ for all $\zeta \in \mathbf{D}^*$ admits a quasiconformal extension to $\mathbf{C}$;

Corollary 4. *Let* $g(\zeta) = \zeta + \dfrac{d}{\zeta} + \cdots$ *be analytic in* $\mathbf{D}^*$. *If there exists* $k \in [0,1)$ *such that*

$$(|\zeta|^2 - 1)\left|1 + \frac{\zeta g''(\zeta)}{g'(\zeta)} - \frac{\zeta g'(\zeta)}{g(\zeta)}\right| \le k$$

for all $\zeta \in \mathbf{D}^*$, *then* g *can be extended to a* k*-quasiconformal automorphism of* $\hat{\mathbf{C}} - \{0\}$.

From the above corollary we have another extension criterion for analytic functions on $\mathbf{D}$;

Corollary 5. *Let* $f \in \mathscr{A}$ *with* $f''(0) = 0$. *If there exists* $k \in [0,1)$ *such that*

$$(1 - |z|^2)\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \le k$$

for all $z \in \mathbf{D}$, *then* f *can be extended to a* k*-quasiconformal automorphism of* $\mathbf{C}$.

2. Preliminaries

Our investigations are based on the theory of Löwner chains. A function $f_t(z) = f(z,t) = a_1(t)z + \sum_{n=2}^{\infty} a_n(t)z^n$, $a_1(t) \ne 0$, defined on $\mathbf{D} \times [0,\infty)$ is called a *Löwner chain* if $f_t(z)$ is holomorphic and univalent in $\mathbf{D}$ for each $t \in [0,\infty)$ and satisfies $f_s(\mathbf{D}) \subsetneq f_t(\mathbf{D})$ and $f(0,s) = f(0,t)$ for $0 \le s \le t < \infty$, and if $a_1(t)$ is locally absolutely continuous in $t \in [0,\infty)$ with $\lim_{t\to\infty}|a_1(t)| = \infty$. Then $f(z,t)$ is absolutely continuous in $t \in [0,\infty)$ for each $z \in \mathbf{D}$ and satisfies the *Löwner differential equation*

$$\dot{f}(z,t) = h(z,t)zf'(z,t) \tag{9}$$

for $z \in \mathbf{D}$ and almost every $t \in [0,\infty)$. Here, $\dot{f}(z,t) = \partial f(z,t)/\partial t$, $f'(z,t) = \partial f(z,t)/\partial z$ and $h(z,t)$ is a function measurable on $t \in [0,\infty)$, holomorphic in $|z| < 1$ and $\operatorname{Re} h(z,t) > 0$ ([6]).

An interesting method connecting the theory of quasiconformal extensions with Löwner chains was obtained by Becker;

Theorem C ([2], see also [4]). *Suppose that $f(z,t)$ is a Löwner chain for which $h(z,t)$ of* (9) *satisfies the condition*

$$\left|\frac{h(z,t)-1}{h(z,t)+1}\right| \le k$$

for all $z \in \mathbf{D}$ and almost all $t \in [0,\infty)$. Then $f_t(z)$ admits a continuous extension to $\overline{\mathbf{D}}$ for each $t \ge 0$ and the map defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta},0) & \text{if } r<1,\\ f(e^{i\theta},\log r) & \text{if } r \ge 1, \end{cases}$$

is a k-quasiconformal extension of f_0 to $\mathbf{C}$.

3. Proof of Theorem 1

The proof is divided into two parts. The first part of the proof is based on [8].

(i) First we assume that $f(z)/z \neq 0$ for all $z \in \mathbf{D}$. Then we can define

$$f(z,t) = f(e^{-st}z)\left\{1 - \frac{a}{c}(e^{2t}-1)\frac{e^{-st}zf'(e^{-st}z)}{f(e^{-st}z)}\right\}^s$$

and let

$$F(z,t) = f(z,t/|s|). \tag{10}$$

A straightforward calculation shows

$$h(z,t) = \frac{\dot{F}(z,t)}{zF'(z,t)} = \frac{s}{|s|}\cdot\frac{1+P(e^{-st/|s|}z,t/|s|)}{1-P(e^{-st/|s|}z,t/|s|)}, \tag{11}$$

where

$$P(z,t) = \frac{c}{a}e^{-2t} + 1 + (e^{-2t}-1)H_s(z)$$

and

$$H_s(z) = s\left(1+\frac{zf''(z)}{f'(z)}\right) + (1-s)\frac{zf'(z)}{f(z)}.$$

Since $h(z,t)$ is holomorphic in $z \in \mathbf{D}$ and measurable on $t \in [0,\infty)$, applying Theorem C to (11), we see that the condition

$$\left|\frac{s(1+P(e^{-st/|s|}z,t/|s|)) - |s|(1-P(e^{-st/|s|}z,t/|s|))}{s(1+P(e^{-st/|s|}z,t/|s|)) + |s|(1-P(e^{-st/|s|}z,t/|s|))}\right| \le l$$

implies l-quasiconformal extensibility of $f(z)$. This is equivalent to

$$\left|P+\frac{(1+l^2)b}{(1+l^2)a+(1-l^2)|s|}i\right| \le \frac{2l|s|}{(1+l^2)a+(1-l^2)|s|}. \tag{12}$$

Here, we shall prove the following Lemma;

LEMMA 6. *Under the assumption of Theorem* 1, *we have*

$$|aP(e^{-st/|s|}z, t/|s|)+ib| < k|s| \tag{13}$$

for $z \in \mathbf{D}$ *and* $t \in [0,\infty)$.

Proof. We have

$$|aP+ib| \le m_1+m_2$$

by triangle inequality, where

$$m_1 = (1-e^{-2t/|s|})\left|\frac{ce^{-2at/|s|}+s}{1-e^{-2at/|s|}} - aH_s(e^{-st/|s|}z)\right|$$

and

$$m_2 = \left|(ce^{-2at/|s|}+s)\frac{1-e^{-2t/|s|}}{1-e^{-2at/|s|}} - (ce^{-2t/|s|}+s)\right|.$$

Then it is enough to show that $m_1+m_2<k|s|$. (3) implies

$$\left|\frac{c|e^{st/|s|}z|^2+s}{1-|e^{st/|s|}z|^2} - aH_s(e^{-st/|s|}z)\right| \le \frac{M}{1-|e^{st/|s|}z|^2} \le \frac{M}{1-e^{-2at/|s|}}$$

for $z \in \mathbf{D}$. Let $q(t) = (1-e^{-2t/|s|})/(1-e^{-2at/|s|})$. Applying the maximum modulus principle to the function

$$\frac{ce^{-2at/|s|}+s}{1-e^{-2at/|s|}} - aH_s(e^{-st/|s|}z)$$

we have

$$m_1 \le q(t)M.$$

On the other hand

$$m_2 \le |c+s|\,|1-q(t)|.$$

Since $1 \le q(t) < 1/a$ if $0<a\le 1$ and $1/a<q(t)\le 1$ if $1<a$ for all $t\in[0,\infty)$, we conclude that $m_1+m_2<k|s|$ which is our desired inequality. □

We now let Δ and Δ' be disks which are defined by replacing P in (12) and (13) to a complex variable w. It remains to find the smallest l so that Δ' is contained by Δ. Note that if $k = l = 1$ then these two disks coincide. The following condition is necessary and sufficient for $\Delta' \subset \Delta$;

$$\left| \frac{(1+l^2)b}{(1+l^2)a + (1-l^2)|s|} - \frac{b}{a} \right| \le \frac{2l|s|}{(1+l^2)a + (1-l^2)|s|} - \frac{k|s|}{a}. \tag{14}$$

Then we conclude

$$l \le \frac{2ka + (1-k^2)|b|}{(1+k^2)a + (1-k^2)\sqrt{a^2+b^2}}.$$

which is suitable for our purpose.

(ii) In order to eliminate the additional assumption that $f(z)/z \neq 0$ in $\mathbf{D}$, we need a sort of stability of the condition (3);

Lemma 7. *If $f \in \mathscr{A}$ satisfies the assumption of Theorem* 1, *then so does* $f_r(z) = \dfrac{1}{r} f(rz)$, $r \in (0,1)$.

Proof. It follows from the assumption that $aH_s(rz)$ is contained in the disk

$$\Delta = \left\{ w \in \mathbf{C} : \left| w - \frac{cr^2|z|^2 + s}{1 - r^2|z|^2} \right| \le \frac{M}{1 - r^2|z|^2} \right\}.$$

We want to deduce that $aH_s(rz)$ lies in the disk

$$\Delta' = \left\{ w \in \mathbf{C} : \left| w - \frac{c|z|^2 + s}{1 - |z|^2} \right| \le \frac{M}{1 - |z|^2} \right\}.$$

Therefore it is enough to see that $\Delta \subset \Delta'$, that is,

$$\left| \frac{c|z|^2 + s}{1 - |z|^2} - \frac{cr^2|z|^2 + s}{1 - r^2|z|^2} \right| \le \frac{M}{1 - |z|^2} - \frac{M}{1 - r^2|z|^2}. \tag{15}$$

In view of the identity

$$\frac{|z|^2}{1 - |z|^2} - \frac{r^2|z|^2}{1 - |z|^2} = \frac{1}{1 - |z|^2} - \frac{1}{1 - r^2|z|^2},$$

the inequality (15) is equivalent to (5). □

Now we shall show that the condition $f(z)/z \neq 0$ in $\mathbf{D}$ follows from the assumption of Theorem 1. Suppose, to the contrary, that $f(z_0) = 0$ for some $0 < |z_0| < 1$. We may assume that $f(z) \neq 0$ for $0 < |z| < |z_0|$. Then by Lemma 7 we can apply Theorem 1 to the function $f_{r_0}(z) = f(r_0 z)/r_0$, $r_0 = |z_0|$ to conclude that f_{r_0} has a quasiconformal extension to $\mathbf{C}$. In particular, f_{r_0} is injective on $\overline{\mathbf{D}}$. This, however, contradicts the relation $f_{r_0}(z_0/r_0) = f_{r_0}(0) = 0$. □

Remark 3.1. We can replace $|s|$ in (10) to any positive real value and continue our argument. However, it will be found that $|s|$ gives the smallest l by calculations.

Remark 3.2. We have $l \geq k$, where $l = k$ if and only if $b = 0$. Indeed, let $l = l(k)$. Then we have $l'(k) > 0$ and $l''(k) \leq 0$ which imply $l \geq k$. If we suppose $l = k \neq 0$, then the right-hand side of (14) is greater than or equal to 0 only if $b = 0$. In the case $l = k = 0$ we also have $b = 0$ by (14). It easily follows from (4) that $l = k$ if $b = 0$.

4. Proof of Theorem 2

It is easy to see from (6) that f is a Bazilevič function of type (α, β) under our assumption since M is always less than or equal to $(\alpha^2 + \beta^2)/\alpha$.

Let us now prove quasiconformal extensibility of f. Setting $1/s = \alpha + i\beta$ which implies $a = \operatorname{Re} s = \alpha/(\alpha^2 + \beta^2)$ and $b = \operatorname{Im} s = -\beta/(\alpha^2 + \beta^2)$, (7) turns to

$$\left| 1 + \frac{zf''(z)}{f'(z)} + \left(\frac{1}{s} - 1\right)\frac{zf'(z)}{f(z)} - \frac{1}{a}\right| \leq \begin{cases} k, & 0 < a < 1, \\ k/a, & 1 \leq a. \end{cases}$$

Therefore, Theorem 2 follows from Theorem 1 with $c = -s$. □

5. Proof of Theorem 3

First let $s \neq 1$. In that case we may assume $g(\zeta) \neq 0$ for all $\zeta \in \mathbf{D}^*$ because of a similar discussion of the proof of Theorem 1;

LEMMA 8. *Let* $g(\zeta) = \zeta + \dfrac{d}{\zeta} + \cdots$ *be analytic in* $\mathbf{D}^*$. *If* g *satisfies the same assumption of Theorem* 3, *then so does* $g_R(\zeta) = \dfrac{1}{R} f(R\zeta)$, $R > 1$.

Proof. We need to prove

$$\left| \frac{ib}{|\zeta|^2 - 1} - aG_s(R\zeta)\right| \leq \frac{ak|s| - |b|(a-1)}{|\zeta|^2 - 1}$$

by using

$$\left| \frac{ib}{R^2|\zeta|^2 - 1} - aG_s(R\zeta)\right| \leq \frac{ak|s| - |b|(a-1)}{R^2|\zeta|^2 - 1},$$

where

$$G_s(\zeta) = (1-s)\left(\frac{\zeta g'(\zeta)}{g(\zeta)} - 1\right) + s\frac{\zeta g''(\zeta)}{g'(\zeta)}.$$

In a similar way to the proof of Lemma 7, it suffices to see that

$$\left|\frac{ib}{|\zeta|^2-1}-\frac{ib}{R^2|\zeta|^2-1}\right| \le \frac{ak|s|-|b|(a-1)}{|\zeta|^2-1}-\frac{ak|s|-|b|(a-1)}{R^2|\zeta|^2-1}.$$

This is equivalent to $|b| \le k|s|$. □

Then we let

$$f(1/\zeta,t)=\frac{1}{g(e^{st}\zeta)}\left\{1-(1-e^{-2t})e^{st}\zeta\frac{g'(e^{st}\zeta)}{g(e^{st}\zeta)}\right\}^{-s}$$

and

$$F(1/\zeta,t)=f(1/\zeta,t/|s|).$$

Since

$$h(1/\zeta,t)=\frac{\dot{F}(1/\zeta,t)}{(1/\zeta)F'(1/\zeta,t)}=\frac{s}{|s|}\cdot\frac{1+P(e^{st/|s|}\zeta,t/|s|)}{1-P(e^{st/|s|}\zeta,t/|s|)}$$

where

$$P(\zeta,t)=(e^{2t/|s|}-1)G_s(\zeta),$$

it is sufficient to see that

$$|aP(e^{st/|s|}\zeta,t/|s|)+ib|<k|s| \tag{16}$$

for all $\zeta\in\mathbf{D}^*$ and $t\in[0,\infty)$ under the assumption of the theorem. By triangle inequality we have

$$|aP+ib| \le \left|\frac{1-e^{2t/|s|}}{1-e^{2at/|s|}}(ib+(1-e^{2at/|s|})aG_s(e^{st/|s|}\zeta))\right|+\left|ib\left(1-\frac{1-e^{2t/|s|}}{1-e^{2at/|s|}}\right)\right|$$

for $\zeta\in\mathbf{D}^*$ and $t\in[0,\infty)$. Following the lines of the proof of Lemma 6, one can obtain that (8) implies (16). Therefore, a similar argument of the proof of Theorem 1 implies our assertion. The case $s=1$ follows from a theorem of Becker [2]. □

6. Proof of Corollary 4 and 5

Proof of Corollary 4. Let $R>1$ be an arbitrary but fixed number. We would like to show that $g_R(\zeta)=g(R\zeta)/R$ can be extended to a k-quasiconformal mapping of $\hat{\mathbf{C}}-\{0\}$. Since $g(\zeta)\neq 0$ in $\zeta\in\mathbf{D}^*$ from the assumption, there exists a certain constant A such that

$$(|\zeta|^2-1)\left|1-\frac{\zeta g_R'(\zeta)}{g_R(\zeta)}\right| \le A<\infty$$

for all $\zeta \in \overline{\mathbf{D}^*}$. We also have

$$\left|1 - \frac{\zeta g_R'(\zeta)}{g_R(\zeta)} + \frac{\zeta g_R''(\zeta)}{g_R'(\zeta)}\right| \le \frac{k}{|\zeta R|^2 - 1}$$

for $\zeta \in \mathbf{D}^*$. Thus we obtain with $s = R^2 A/k(R^2 - 1)$

$$(|\zeta|^2 - 1)\left|\frac{1}{s}\left(1 - \frac{\zeta g_R'(\zeta)}{g_R(\zeta)}\right) - 1 - \frac{\zeta g_R''(\zeta)}{g_R'(\zeta)} + \frac{\zeta g_R'(\zeta)}{g_R(\zeta)}\right| \le \frac{A}{s} + k\frac{|\zeta|^2 - 1}{|\zeta R|^2 - 1} \le k$$

which implies quasiconformal extensibility of g_R by Theorem 3. A limiting procedure proves Corollary 4. □

Proof of Corollary 5. Note that the function $1 + (zf''(z)/f'(z)) - (zf'(z)/f(z))$ is analytic in $\mathbf{D}$ and has a zero of order 2 at the origin by the condition $f''(0) = 0$. Thus, we obtain from the assumption that

$$\frac{1}{|z|^2}(1 - |z|^2)\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \le k$$

by the maximum modulus principle. Let $g(\zeta)$ be a function defined by

$$g(\zeta) = \frac{1}{f(z)}$$

where $\zeta = 1/z$. Then g is analytic in $\mathbf{D}^*$ and has the form $g(\zeta) = \zeta + d/\zeta + \cdots$. From the relations

$$\frac{zf'(z)}{f(z)} = \frac{\zeta g'(\zeta)}{g(\zeta)}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = -1 - \frac{\zeta g''(\zeta)}{g'(\zeta)} + 2\frac{\zeta g'(\zeta)}{g(\zeta)},$$

we can deduce our assertion by applying Corollary 4. □

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Ikkei Hotta
Division of Mathematics
Graduate School of Information Sciences
Tohoku University
6-3-09 Aramaki-Aza-Aoba
Aoba-ku, Sendai 980-8579
Japan
E-mail: ikkeihotta@gmail.com