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NORMAL FAMILIES AND UNIQUENESS THEOREM OF HOLOMORPHIC FUNCTIONS

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Abstract

In the paper, we have two purposes. Firstly, we prove two theorems and two corollaries of normal families which improve and generalize some results of Pang and Zalcman [9], Zhang, Sun and Pang [13], Chang and Fang [2]. Secondly, we use the theory of normal families and differential equations to obtain a uniqueness theorem of entire function which is an improvement of Chang and Fang [1].

1. Introduction and main results

Let f and g denote some non-constant meromorphic functions. We say f and g share a value b IM(CM) if $f(z) - b = 0 \Leftrightarrow g(z) - b = 0$ ($f(z) - b = 0 \Leftrightarrow g(z) - b = 0$ counting multiplicities) (see [12]).

In 2000, X. Pang and L. Zalcman [9] proved the following famous theorem.

Theorem A. *Let $\mathscr{F}$ be a family of meromorphic functions on domain D, all of whose zeros are of multiplicity* (*at least*) *k. Suppose that there exist $a, b, c \in \mathbf{C}$ such that $b, c \neq 0$ and, for every $f \in \mathscr{F}$,*

$$\overline{E}_f(a) = \overline{E}_{f^{(k)}}(b) \subset \overline{E}_{f^{(k+1)}}(c).$$

Then $\mathscr{F}$ is normal in D.

In 2005, G. Zhang, W. Sun and X. Pang [13] obtained a related result.

Theorem B. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, and let $h(z)$ be a function holomorphic in D such that $h(z)$ has only simple zeros. If, for every function $f \in \mathscr{F}$, we have*

(a) *$f(z) = 0 \Leftrightarrow f'(z) = h(z)$ and $f'(z) = h(z) \Rightarrow |f''(z)| \leq M$, where M is a positive number*;

(b) *$f(z)$ and $h(z)$ don't have common zeros,*

then $\mathscr{F}$ is normal in D.

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It's naturally to ask whether the conditions (a) and (b) can be weakened or not? We study the problem and obtain the following result.

THEOREM 1. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let $h(z)(\not\equiv 0)$ be a function holomorphic in D, and let $k \geq 2$ be a positive integer. If for every function $f \in \mathscr{F}$, we have*

(a) $f(z) = 0 \Rightarrow f'(z) = h(z)$, $f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq M$, *where $M > 0$ is a constant*;

(b) $\dfrac{f_n' - h(z)}{f_n}$ *is holomorphic in D*,

then $\mathscr{F}$ is normal in D.

Remark 1. If in addition $f(z)$ and $h(z)$ don't have common zeros, it is easy to deduce that $\dfrac{f_n' - h(z)}{f_n}$ is holomorphic in D. Thus, we immediately have the following corollary.

COROLLARY 1. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let $h(z)(\not\equiv 0)$ be a function holomorphic in D, and let $k \geq 2$ be a positive integer. If for every function $f \in \mathscr{F}$, we have*

(a) $f(z) = 0 \Rightarrow f'(z) = h(z)$, $f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq M$, *where $M > 0$ is a constant*;

(b) *$f(z)$ and $h(z)$ don't have common zeros,*

then $\mathscr{F}$ is normal in D.

Clearly, Corollary 1 is an improvement of Theorem B.

Remark 2. The following example shows that there exists normal family that does not satisfy the conditions of Theorem B yet does satisfy the conditions of Theorem 1.

Example 1. Let $\mathscr{F} = \left\{f_n : f_n = \dfrac{1}{n}z^3 + z^2, n = 2, 3, \dots\right\}$, let $D = \{z : |z| < 1\}$, and let $k \geq 4$ and $h(z) = 2z$. Then $\mathscr{F}$ is normal in D. We have

$$f_n(z) = 0 \Leftrightarrow f_n'(z) = 2z, \quad f_n'(z) = 2z \Rightarrow f_n^{(k)}(z) = 0$$

and $\dfrac{f_n' - h(z)}{f_n} = \dfrac{3}{z+n}$ is holomorphic in D. Thus, the family satisfies the conditions of Theorem 1. But f_n and $h(z)$ have common zeros at $z = 0$, so it does not satisfies the conditions of Theorem B.

The following example shows that condition (b) of Theorem 1 is necessary.

Example 2. Let $\mathscr{F} = \{f_n : f_n = nz^2, n \in N\}$ and $h(z) = z$. Then $f_n(z) = 0 \Rightarrow f_n'(z) = z$, $f_n'(z) = z \Rightarrow f_n'''(z) = 0$. But $\dfrac{f_n' - h(z)}{f_n} = \dfrac{2n-1}{nz}$ has a pole at $z = 0$, and indeed $\mathscr{F}$ is not normal at $z = 0$.

In 2005, J. Chang and M. Fang [2] derived a theorem of normal family.

THEOREM C. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, and let $a(z)$ be an analytic function in D such that $a(z) \not\equiv a'(z)$. If for every function $f \in \mathscr{F}$, $f(z) = a(z) \Leftrightarrow f'(z) = a(z)$, $f'(z) = a(z) \Leftrightarrow f''(z) = a(z)$ and $f(z) - a(z) = 0 \to f'(z) - a(z) = 0$ in D, then $\mathscr{F}$ is normal in D.*

Here $f(z) - a(z) = 0 \to f'(z) - a(z) = 0$ means: if z_0 is a zero of $f(z) - a(z)$ with multiplicity n, then z_0 is a zero of $f'(z) - a(z)$ with multiplicity at least n.

From the Theorem 1, it is not difficult to deduce the following corollary.

COROLLARY 2. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, and let $a(z)$ be an analytic function in D such that $a(z) \not\equiv a'(z)$, and let $k \geq 2$ be an integer. If, for every function $f \in \mathscr{F}$,*

$$f(z) - a(z) = 0 \to f'(z) - a(z) = 0, \quad f'(z) - a(z) = 0 \Rightarrow |f^{(k)}(z)| \leq M$$

in D, where $M > 0$ is a constant. Then $\mathscr{F}$ is normal in D.

Remark 3. Let $\mathscr{G} = \{F : F = f - a, f \in \mathscr{F}\}$ and $h = a - a'$. Then, for every $z_0 \in D$, there exist a disc $D(z_0, r) = \{z : |z - z_0| < r\}$ such that for $z \in D(z_0, r)$,

$$F(z) = 0 \to F'(z) = h(z), \quad F'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq \tilde{M},$$

where $\tilde{M} = \tilde{M}(z_0) = M + \max_{z \in D(z_0, r)} |a^{(k)}(z)|$. Since normality is a local property, with Theorem 1, it is easy to deduce that $\mathscr{G}$ is normal in D. Hence, the family $\mathscr{F}$ is normal as well. In fact, Corollary 2 improves the Theorem C.

In the same paper [2], they also obtained a corollary.

THEOREM D. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D. If, for every function $f \in \mathscr{F}$, f, f' and f'' have the same fixed points in D, then $\mathscr{F}$ is normal in D.*

From Theorem 1, we deduce the following result which is an improvement of Theorem D.

THEOREM 2. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D. If, for every function $f \in \mathscr{F}$, we have*

$$f(z) = z \Rightarrow f'(z) = z, \quad f'(z) = z \Rightarrow f''(z) = z,$$

then $\mathscr{F}$ is normal in D.

In 2002, J. Chang and M. Fang [1] proved a uniqueness theorem.

Theorem E. *Let $f(z)$ be a nonconstant entire function. If*

$$f(z) = z \Leftrightarrow f'(z) = z, \quad f'(z) = z \Rightarrow f''(z) = z,$$

then $f(z) = f'(z)$.

Naturally, we will ask what will happen if we replace the assumption $f(z) = z \Leftrightarrow f'(z) = z$ by $f(z) = z \Rightarrow f'(z) = z$. With the theory of normal family, we study the problem and find the conclusion of Theorem D still holds. In fact, we deduce the following result.

Theorem 3. *Let $f(z)$ be a nonconstant entire function. If*

$$f(z) = z \Rightarrow f'(z) = z, \quad f'(z) = z \Rightarrow f''(z) = z,$$

then $f(z) = f'(z)$.

Remark 4. Some ideas of the paper are based on [7].

2. Some lemmas

Lemma 2.1 [9]. *Let $\mathscr{F}$ be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}| \leq A$ whenever $f = 0$, then if $\mathscr{F}$ is not normal, there exist, for each* $0 \leq \alpha \leq k$,

(a) *a number* $0 < r < 1$;

(b) *points* z_n, $z_n < 1$;

(c) *functions* $f_n \in \mathscr{F}$, *and*

(d) *positive number $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant entire function on* $\mathbf{C}$, *all of whose zeros have multiplicity at least k, such that* $g^{\sharp}(\xi) \leq g^{\sharp}(0) = kA + 1$.

Here, as usual, $g^{\sharp}(\xi) = \dfrac{|g'(\xi)|}{1 + |g(\xi)|^2}$ is the spherical derivative.

Lemma 2.2 [3]. *Let g be a nonconstant entire function with $\rho(g) \leq 1$, let $k \geq 2$ be an integer, and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

Lemma 2.3 [14]. *If g is a meromorphic function with bounded spherical derivative, then the order of g is at most two.*

Lemma 2.4 [6, Corollary 1]. *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $0 \leq j_i < k_i$, for $i = 1, \ldots, q$. And let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero,* *such that if* $\psi \in [0, 2\pi] \backslash E$, *then there is a constant* $R_0 = R_0(\psi) > 1$ *such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geq R_0$ *and for all* $(k, j) \in H$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Relying on Markushevich's book [8, see p. 253–255], we can deduce the following lemma. It also can be seen in [4].

Lemma 2.5. *Let*

$$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0$$

where n *is a positive integer and* $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. *For any given* $0 < \varepsilon < \dfrac{\pi}{4n}$, *we introduce* $2n$ $(j = 0, 1, \dots, 2n-1)$ *open angles*

$$S_j = \left\{re^{i\theta} : r > 0, -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon\right\}$$

Then there exists a positive number $R = R(\varepsilon)$ *such that for* $|z| = r > R$,

$$Re\{Q(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon) r^n \tag{2.2}$$

if $z \in S_j$ *where* j *is even; while*

$$Re\{Q(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon) r^n \tag{2.3}$$

if $z \in S_j$ *where* j *is odd.*

Proof. Suppose $z = re^{i\theta}$, $b_k = \alpha_k e^{i\theta_k}$ and $\alpha_k > 0$, $k = 0, 1 \dots, n-1$. Then

$$\begin{aligned} Re\, Q(z) &= \alpha_n r^n \cos(\theta_n + n\theta) + \sum_{k=0}^{n-1} \alpha_k r^k \cos(\theta_k + k\theta) \\ &= \alpha_n r^n \left[\cos(\theta_n + n\theta) + \sum_{k=0}^{n-1} \frac{\alpha_k \cos(\theta_k + k\theta)}{\alpha_n r^{n-k}}\right]. \end{aligned} \tag{2.4}$$

For any $0 < \varepsilon < \dfrac{\pi}{4n}$, we introduce $2n$ open angles

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon \quad (j = 0, 1, \dots, 2n-1).$$

Thus, we have

$$(2j-1)\frac{\pi}{2} + n\varepsilon < \theta_n + n\theta < (2j+1)\frac{\pi}{2} - n\varepsilon \quad (j = 0, 1, \dots, 2n-1).$$

Furthermore,

$$(2.5)\quad (2j-1)\frac{\pi}{2} < (2j-1)\frac{\pi}{2} + n\varepsilon < \theta_n + n\theta < (2j+1)\frac{\pi}{2} - n\varepsilon < (2j+1)\frac{\pi}{2}.$$

Now, we consider into two cases.

CASE 1. j is even.
Then, it is not difficult to deduce that

$$(2.6)\quad \cos(\theta_n + n\theta) > \cos\left((2j-1)\frac{\pi}{2} + n\varepsilon\right) = \sin(n\varepsilon) > \cos\left((2j-1)\frac{\pi}{2}\right) = 0.$$

Noting that

$$\left|\sum_{k=0}^{n-1} \frac{\alpha_k \cos(\theta_k + k\theta)}{\alpha_n r^{n-k}}\right| \to 0, \quad \text{as } r \to \infty,$$

we deduce that there exists a positive number $R = R(\varepsilon)$ satisfying

$$(2.7)\qquad \left|\sum_{k=0}^{n-1} \frac{\alpha_k \cos(\theta_k + k\theta)}{\alpha_n r^{n-k}}\right| < \varepsilon \sin(n\varepsilon), \quad \text{if } r > R.$$

Combining (2.4), (2.6) and (2.7) yields that there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$Re\{Q(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon) r^n.$$

CASE 2. j is odd.
With the similar way, we can obtain that there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$Re\{Q(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon) r^n.$$

Thus, we finish the proof of this lemma.

With the idea in [4], we deduce the following result.

LEMMA 2.6. *Let $P(z)(\not\equiv 0)$, $H(z)(\not\equiv 0)$ and $Q(z)$ be three polynomials with that $Q(z)$ is nonconstant. Then, every entire solution $F(z)$ of the following differential equation*

$$(2.8)\qquad F'(z) - P(z)e^{Q(z)}F(z) = H(z)$$

has infinite order.

Proof. Obviously, $F(z)$ is transcendental. Now, we suppose that $F(z)$ is of finite order, we will deduce that $F(z)$ is a polynomial. By Lemma 2.4, we see that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta \in [0, 2\pi] \backslash E$ and any given $0 < \varepsilon < 1$, there is a $R(> 0)$, as $r > R$,

$$\left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| \le r^{\sigma(F)-1+\varepsilon}. \tag{2.9}$$

Set $\deg H(z) = h$ and $Q(z) = b_n z^n + \cdots + b_0$, where n is a positive integer and $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, \pi)$. By Lemma 2.5, we know that if $\theta \ne -\dfrac{\theta_n}{n} + (2j-1)\dfrac{\pi}{2n}$ $(j = 0, \dots, 2n-1)$, as r sufficiency large, we have

$$Re\{Q(z)\} > \alpha_{n\theta} r^n \quad or \quad Re\{Q(z)\} < -\alpha_{n\theta} r^n,$$

where $\alpha_{n\theta} > 0$ is a constant.

Now, we take

$$\arg z = \theta \in [0, 2\pi) \Bigg\backslash \left(E \cup \left[\bigcup_{j=0}^{2n-1} \left\{ \frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} \right\} \right] \right).$$

By (2.8), we get

$$\frac{F'(re^{i\theta})}{F(re^{i\theta})} - P(re^{i\theta})e^{Q(re^{i\theta})} = \frac{H(re^{i\theta})}{F(re^{i\theta})}. \tag{2.10}$$

If $Re\{Q(re^{i\theta})\} > \alpha_{n\theta} r^n$, from (2.9), we see that as $r \to \infty$,

$$\left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| \frac{1}{r^{\sigma(F)+h+1}} \to 0, \quad \left| \frac{H(re^{i\theta})}{r^{\sigma(F)+h+1}} \right| \to 0, \quad \left| \frac{P(z)e^{Q(re^{i\theta})}}{r^{\sigma(F)+h+1}} \right| \to \infty. \tag{2.11}$$

From (2.10) and (2.11), we see that as $r \to \infty$,

$$|F(re^{i\theta})| \to 0. \tag{2.12}$$

If $Re\{Q(re^{i\theta})\} < -\alpha_{n\theta} r^n$, by (2.8) we get

$$1 - \frac{F(re^{i\theta})}{F'(re^{i\theta})} P(re^{i\theta})e^{Q(re^{i\theta})} = \frac{H(re^{i\theta})}{F'(re^{i\theta})}. \tag{2.13}$$

Let

$$M(r, F', \theta) = \max\{|F'(z)| : 0 \le |z| \le r, \arg z = \theta\}.$$

We claim that

$$|F'(z)| = o(|z|^{h+1})$$

as $r \to \infty$ for all $z = re^{i\theta}$.

Otherwise, there exists a positive number M_1 and an infinite sequence of points $z_n = r_n e^{i\theta}$ satisfying $r_n \to \infty$ and

$$|F'(r_n e^{i\theta})| = M(r_n, F', \theta) > M_1 |z_n|^{h+1}.$$

Thus,

$$\left|\frac{H(z_n)}{F'(z_n)}\right| \to 0 \quad as\ r_n \to \infty. \tag{2.14}$$

Since

$$F(z_n) = F(z_1) + \int_{z_1}^{z_n} F'(\omega)\, d\omega,$$

it is easy to deduce

$$|F(z_n)| \le |F(z_1)| + |F'(z_n)|\,|z_n|.$$

Dividing $|F'(z_n)|$ on both sides of the above inequality yields

$$\left|\frac{F(z_n)}{F'(z_n)}\right| \le (1 + o(1))|z_n| \quad as\ r_n \to \infty. \tag{2.15}$$

By (2.15) and the fact $Re\{Q(re^{i\theta})\} < -\alpha_{n\theta} r^n$, we deduce

$$\left|\frac{F(z_n)}{F'(z_n)} P(z_n) e^{Q(z_n)}\right| \to 0, \tag{2.16}$$

which, together with (2.13) and (2.14), implies a contradiction. Thus, the claim is proved.

From the claim, we have

$$|F(z)| = o(|z|^{h+2}) \tag{2.17}$$

as $r \to \infty$ for all $z = re^{i\theta}$, where M_2 is a positive number.

In view of (2.12) and (2.17), it is obvious that

$$|F(re^{i\theta})| = o(r^{h+2}) \tag{2.18}$$

as $r \to \infty$ for each $\theta \in [0, 2\pi) \Big\backslash \left(E \cup \left[\bigcup_{j=0}^{2n-1} \left\{\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}\right\}\right]\right)$, where M is a positive integer.

The facts that the linear measure of $E \cup \left[\bigcup_{j=0}^{2n-1} \left\{\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}\right\}\right]$ equal to 0 and F is of finite order, together with (2.18) and Phragmén-Lindelöf theorem yield F is a polynomial. It is a contradiction.

3. Proof of Theorem 1

In the following, we prove Theorem 1 with the method of J. Grahl and Meng C. in [7].

Since normality is a local property, it is enough to show that $\mathscr{F}$ is normal at each $z_0 \in D$. We distinguish two cases.

CASE 1. $h(z_0) \ne 0$.

Then, there exists a disc (which we may assume to be Δ) contained in D, on which $\{f_n\}$ is not normal, $h(z) \neq 0$ and $|h(z)| \leq M > 1$, where M is a positive number. Thus, $f_n = 0$ implies that $|f_n'| = |h| \leq M$.

Taking an appropriate subsequence of f_n and renumbering, we have, by Lemma 2.1 (with $\alpha = k = 1$ and $A = M$), points $z_n \to z_0$ $(|z_n| < r < 1)$ and numbers $\rho_n \to 0$ such that

$$\frac{f_n(z_n + \rho_n \zeta)}{\rho_n} = g_n(\zeta) \to g(\zeta) \tag{3.1}$$

locally uniformly, where g is a nonconstant entire function on $\mathbf{C}$ satisfying $\rho(g) \leq 1$ and

$$g^\sharp(\zeta) \leq g^\sharp(0) = M + 1.$$

We claim:

$$g(\zeta) = 0 \Rightarrow g'(\zeta) = h(z_0), \quad g'(\zeta) = h(z_0) \Rightarrow g^{(k)}(\zeta) = 0.$$

From (3.1), it is easy to derive that

$$g_n'(\zeta) = f_n'(z_n + \rho_n \zeta) \to g'(\zeta) \tag{3.2}$$

and

$$g_n^{(k)}(\zeta) = \rho_n^{k-1} f_n^{(k)}(z_n + \rho_n \zeta) \to g^{(k)}(\zeta). \tag{3.3}$$

The (3.2) leads to

$$f_n'(z_n + \rho_n \zeta) - h(z_n + \rho_n \zeta) \to g'(\zeta) - h(z_0). \tag{3.4}$$

Suppose that $g(a_0) = 0$, then by Hurwitz's theorem, there exists a sequence $\{a_n\}$ such that $a_n \to a_0$ and (for n sufficiently large) $f_n(z_n + \rho_n a_n) = 0$. With the assumption, we have $f_n'(z_n + \rho_n a_n) = h(z_n + \rho_n a_n)$. Thus

$$g'(a_0) = \lim_{n\to\infty} f_n'(z_n + \rho_n a_n) = \lim_{n\to\infty} h(z_n + \rho_n a_n) = h(z_0),$$

which implies that $g(\zeta) = 0 \Rightarrow g'(\zeta) = h(z_0)$.

Now suppose that $g'(b_0) = h(z_0)$. We assume that $g'(z) \not\equiv h(z_0)$. Otherwise, $g(z) = h(z_0)(z - b)$, b is a constant. Therefore, $g^\sharp(z) \leq g^\sharp(0) \leq |h(z_0)| < M + 1$, a contradiction. Since $g'(b_0) = h(z_0)$ and $g' \not\equiv h(z_0)$, by Hurwitz's theorem and (3.4), there exist a sequence $\{b_n\}$ such that $b_n \to b_0$ and (for n sufficiently large)

$$f_n'(z_n + \rho_n b_n) - h(z_n + \rho_n b_n) = 0.$$

Furthermore, with (3.3) we deduce that

$$g^{(k)}(b_0) = \lim_{n\to\infty} \rho_n^{k-1} f_n^{(k)}(z_n + \rho_n b_n) = 0.$$

Thus, we have shown that $g'(z) = h(z_0) \Rightarrow g^{(k)}(z) = 0$. This completes the proof of the claim.

By Lemma 2.2 and the claim, we obtain $g(z) = h(z_0)(z - b_1)$, where b_1 is a constant. But, we have $g^\sharp(0) \le |h(z_0)| < M + 1$, a contradiction.

Case 2. $h(z_0) = 0$.

Since $h(z) \not\equiv 0$, there exists a r such that $h(z) \ne 0$ in $D'(z_0, r) = \{z : 0 < |z - z_0| < r\}$. Then, Case 1 implies that $\mathscr{F}$ is normal in $D'(z_0, r)$. Then for any sequence $\{f_n\} \subset \mathscr{F}$, there exist a subsequence $\{f_{n,j}\}$ such that $\{f_{n,j}\}$ converges locally uniformly to a function H in $D'(z_0, r)$, where H is either holomorphic or identically infinite in $D'(z_0, r)$.

Case 2.1. H is holomorphic in $D'(z_0, r)$.

Then there exists a positive number M_1 such that $|H(z)| \le M_1$ on $|z - z_0| = r/2$. It follows that $|f_{n,j}(z)| \le 2M_1$ on $|z - z_0| = r/2$ for large j. By the maximum principle, we have $|f_{n,j}(z)| \le 2M_1$ in $D(z_0, r/2) = \{z : |z - z_0| \le r/2\}$. Then H is bounded in $D(z_0, r/2)$, and H extends to be holomorphic in $D(z_0, r/2)$. Again by the maximum principle, we get $f_{n,j}(z) \to H(z)$ in $D(z_0, r/2)$.

Case 2.2. $H \equiv \infty$.

Note that $f_{n,j}(z) \to \infty$ on $\Gamma := \{z : |z - z_0| = r/2\}$. Thus we have (for sufficiently large n)

$$\left| \int_\Gamma \frac{h(z)}{f_{n,j}(z)}\, dz \right| \le \pi. \tag{3.5}$$

We know

$$\frac{f'_{n,j} - h(z)}{f_{n,j}}$$

is holomorphic in $D(z_0, r)$. Thus by Cauchy's Theorem, we have

$$\int_\Gamma \frac{f'_{n,j}(z) - h(z)}{f_{n,j}(z)}\, dz = 0 \tag{3.6}$$

By $n(\Gamma, f_{n,j})$ we denote the number of zeros of $f_{n,j}$ in $D_2 = \{z : |z - z_0| < r/2\}$ counting multiplicities. By the argument principle (3.5) and (3.6) (for sufficiently large n), we get

$$n(\Gamma, f_{n,j}) = \frac{1}{2\pi i} \int_\Gamma \frac{f'_{n,j}(z)}{f_{n,j}(z)}\, dz = \frac{1}{2\pi} \left| \int_\Gamma \frac{h(z)}{f_{n,j}(z)}\, dz \right| \le \frac{1}{2},$$

hence

$$n(\Gamma, f_{n,j}) = 0.$$

So $f_{n,j}$ has no zeros in $D(z_0, r/2)$. Thus, $\dfrac{1}{f_{n,j}}$ is holomorphic and $\dfrac{1}{f_{n,j}} \to 0$ on $D'(z_0, r/2)$. Similarly as Case 2.1, we can get $f_{n,j}(z) \to \infty$ in $D(z_0, r/2)$.

From the above discussion, we get $\mathscr{F}$ is normal at z_0. Hence, we complete the proof of the Theorem 1.

4. Proof of Theorem 2

From the assumption of Theorem 2, for each $f \in \mathscr{F}$ we have

$$f(z) = z \Rightarrow f'(z) = z, \quad f'(z) = z \Rightarrow f''(z) = z.$$

Let $F = f(z) - z$, then

$$F(z) = 0 \Rightarrow F'(z) = z - 1, \quad F'(z) = z - 1 \Rightarrow F''(z) = z.$$

Suppose that a_0 is a zero of $F(z)$.

If $a_0 \neq 1$, then a_0 is a simple zero of $F(z)$. Suppose that $G = F' - (z - 1)$, then a_0 is also a zero of $G(z)$.

If $a_0 = 1$, then $F'(a_0) = a_0 - 1 = 0$ and $F''(a_0) = a_0 = 1$, which indicates that a_0 is a zero of $F(z)$ with multiplicity 2. Note that $G(a_0) = 0$ and $G'(a_0) = F''(a_0) - 1 = 0$, we know that a_0 is a zero of $G(z)$ with multiplicity at least 2.

By the above discussion, we obtain

$$\frac{G(z)}{F(z)} = \frac{F' - (z - 1)}{F(z)}$$

is holomorphic in D. Thus, the family $\mathscr{G} = \{F : F = f - z, f \in \mathscr{F}\}$ satisfies the conditions of Theorem 1. By Theorem 1, we get $\mathscr{G}$ is normal in D. Hence $\mathscr{F}$ is normal in D. This completes the proof of Theorem 2.

5. Proof of Theorem 3

We consider the function $F = \dfrac{f}{z}$.

Case 1. F has bounded spherical derivative.

Then by Lemma 2.3, F has finite order. Hence $f = Fz$ has finite order as well.

Let $h = f - z$, then h has finite order and

$$h = 0 \Rightarrow h' = z - 1, \quad h' = z - 1 \Rightarrow h'' = z. \tag{5.1}$$

Set

$$\mu = \frac{zh' - (z - 1)h''}{h}. \tag{5.2}$$

Suppose that $\mu \equiv 0$, then $zh' = (z - 1)h''$. Integrating the differential equation yields

$$h' = A(z - 1)e^z, \tag{5.3}$$

and

$$h = A(z-2)e^z + B, \tag{5.4}$$

where $A \neq 0$ and B are two constants. With (5.1), (5.3) and (5.4), it is not different to obtain a contradiction. Thus, $\mu \not\equiv 0$.

Now, we consider the equation (5.2). It is easy to see that

$$\begin{aligned} m(r,\mu) &= m\left(r, \frac{zh' - (z-1)h''}{h}\right) \\ &\leq m\left(r, \frac{zh'}{h}\right) + m\left(r, \frac{(z-1)h''}{h}\right) + O(1) \leq O(\log r). \end{aligned} \tag{5.5}$$

Next we discuss the poles of μ. From (5.1) we obtain h has at most one zero which is multiple, at $z = 1$. And the points which are the simple zeros of h are not poles of μ. Then we derive that

$$N(r,\mu) = N\left(r, \frac{zh' - (z-1)h''}{h}\right) \leq O(\log r). \tag{5.6}$$

Combining (5.5) and (5.6) yields

$$T(r,\mu) = m(r,\mu) + N(r,\mu) = O(\log r),$$

which implies that μ is a rational function.

We denote by $N(r, h' - (z-1); h \neq 0)$ the counting function of those 0-points of $h' - (z-1)$, counted according to multiplicity, which are not the 0-points of h. Because of μ is a rational function we get $N\left(r, \frac{1}{\mu}\right) = O(\log r)$. Furthermore, we have

$$N(r, h' - (z-1); h \neq 0) \leq N\left(r, \frac{1}{\mu}\right) + O(\log r) = O(\log r). \tag{5.7}$$

Put

$$\phi = \frac{h' - (z-1)}{h}. \tag{5.8}$$

Suppose that $\phi \equiv 0$, then $h'(z) = z - 1$. But from (5.1) we know that $h'(z) = z - 1$ implies $h'' = z$, and this is a contradiction. Thus, $\phi \not\equiv 0$. In the following, we discuss the zeros and poles of ϕ.

We know h has at most one multiple zero.

If $z = 1$ is not a zero of h, then h has only simple zeros. Thus, ϕ does not has poles and ϕ is an entire function.

If $z = 1$ is a zero of h, then $h'(1) = z - 1 = 0$ and $h''(1) = 1$. Thus, $z = 1$ is a zero of h with multiplicity 2. Meanwhile, $z = 1$ is a zero of $h' - (z-1)$. So, $h''(1) - 1 = 0$, which implies that $z = 1$ is a zero of $h' - (z-1)$ with multiplicity at least 2. It also yields that ϕ is an entire function.

Thus, we deduce that ϕ is an entire function. From (5.1) we obtain $h' - (z-1)$ has at most one multiple zero at $z = 1$. It follows from (5.7) and (5.8) that $N\left(r, \frac{1}{\phi}\right) = O(\log r)$ and ϕ has only finitely many zeros. Hence we can assume that

$$\phi = P(z)e^{Q(z)},$$

where $P(z) \not\equiv 0$ and $Q(z)$ are two polynomials. From (5.8), we have

$$h' - P(z)e^{Q(z)}h = z - 1. \tag{5.9}$$

Noting that h is of finite order, by Lemma 2.6, we can easily deduce that $Q(z) = C$, a constant. Let $P_1(z) = e^C P(z)$. Rewriting (5.9) as

$$h' - P_1(z)h = z - 1. \tag{5.10}$$

Now, we discuss the equation (5.10) by considering two subcases.

Case 1.1. h has infinite many zeros.

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $h(z_n) = 0$ and $|z_n| \to \infty$ as $n \to \infty$. It is clear from (5.1) that

$$h'(z_n) = z_n - 1 \quad \text{and} \quad h''(z_n) = z_n.$$

By differentiating both sides of Eq. (5.10), we have

$$h'' - P_1'(z)h - P_1(z)h' = 1. \tag{5.11}$$

Substitute z_n into Eq. (5.11) yields

$$z_n - P_1(z_n)(z_n - 1) \equiv 1. \tag{5.12}$$

If $\deg(P_1(z)) \geq 1$, the left side of Eq. (5.12) $z_n - P_1(z_n)(z_n - 1) \to \infty$ as $n \to \infty$, this is a contradiction. Thus, $P_1(z)$ is a constant. Again by (5.12), we obtain $P_1 = 1$. Then we have

$$h' - h = z - 1, \tag{5.13}$$

which implies that $f \equiv f'$.

Case 1.2. h has finitely many zeros.

Then we can set $h(z) = P_2(z)e^{Q_2(z)}$, where $P_2(z)$ and $Q_2(z)$ are two polynomials. Substituting h into Eq. (5.10) yields that

$$[P_2' + P_2Q_2' - P_1P_2]e^{Q_2(z)} = z - 1. \tag{5.14}$$

From the above equation, it is obvious that $Q_2(z)$ is a constant and h is a polynomial. Let $Q_2 = C_1$. Rewriting (5.14) as $e^{C_1}(P_2' - P_1P_2) = z - 1$. Thus,

$$\deg(P_2' - P_1P_2) = 1.$$

Suppose $\deg(P_1) \geq 1$, then P_2 is a constant and h is a constant, which is a contradiction. Thus, $\deg(P_1) = 0$ and P_1 is a constant. Again by $\deg(P_2' - P_1P_2)$

$= 1$, we derive that $\deg P_2 = 1$. Thus, $\deg h = 1$. Furthermore, we can assume that $h(z) = A_2(z - B_2)$, where $A_2 \neq 0$, B_2 are two constants. By (5.1), it is not difficult that $A_2 = -1$ and $B_2 = 0$. Thus, $h(z) = -z$ and $f \equiv 0$, which is a contradiction. Hence, we finish the proof of Case 1.

CASE 2. F has unbounded spherical derivative.

Next, with a similar way in [7], we will prove this case cannot occur.

From the assumption of Case 2, there exists a sequence $(w_n)_n$ such that $\lim_{n\to\infty} F^\sharp(w_n) = \infty$. Since $F^\sharp$ is continuous and bounded in every compact set, so $w_n \to \infty$ as $n \to \infty$. Let $D = \{z : |z| \geq 1\}$, then F is analytic in D. We may assume $|w_n| \geq 2$ for all n. We define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z).$$

Then all $F_n(z)$ are analytic in D_1 and $F_n^\sharp(0) = F^\sharp(w_n) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Assume that $F_n(z_0) = 1$ for some $z_0 \in D_1$. Then for n large enough, we have

$$|F_n'(z_0)| = \left|\frac{f'(w_n + z_0)}{w_n + z_0} - \frac{f(w_n + z_0)}{(w_n + z_0)^2}\right| = \left|1 - \frac{1}{w_n + z_0}\right| \leq 2.$$

Therefore, we can apply Lemma 2.1 with $\alpha = 1$. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequence $(z_n)_n \in D_1$ and $(\rho_n)_n$ such that $z_n \to 0$, $\rho_n \to 0$ and

$$g_n(\zeta) = \rho_n^{-1}(F_n(z_n + \rho_n\zeta) - 1) = \rho_n^{-1}\left(\frac{f(w_n + z_n + \rho_n\zeta)}{w_n + z_n + \rho_n\zeta} - 1\right) \to g(\zeta) \tag{5.15}$$

locally uniformly in $\mathbf{C}$ with g is a nonconstant entire function. We also have $g^\sharp(\zeta) \leq g^\sharp(0) = 3$ for all $\zeta \in \mathbf{C}$ and $\rho(g) \leq 1$. We claim that

$$g = 0 \Rightarrow g' = 1, \quad g' = 1 \Rightarrow g'' = 0.$$

From (5.15), we deduce that

$$G_n(\zeta) = \frac{f'(w_n + z_n + \rho_n\zeta)}{w_n + z_n + \rho_n\zeta} = g_n'(\xi) + \frac{\rho_n g_n(\zeta) + 1}{w_n + z_n + \rho_n\zeta} \to g'(\zeta) \tag{5.16}$$

locally uniformly in $\mathbf{C}$.

Suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem, there exist a sequence $\{\zeta_n\}$ such that $\zeta_n \to \zeta_0$ and (for n sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-1}(F_n(z_n + \rho_n\zeta_n) - 1) = 0.$$

Thus $F_n(z_n + \rho_n\zeta_n) = 1$ and $f(w_n + z_n + \rho_n\zeta_n) = w_n + z_n + \rho_n\zeta_n$. It follows from the assumption that

$$\frac{f'(w_n + z_n + \rho_n\zeta_n)}{w_n + z_n + \rho_n\zeta_n} = 1.$$

Thus, by (5.16) we derive that

$$g'(\zeta_0) = \lim_{n\to\infty} \frac{f'(w_n + z_n + \rho_n\zeta_n)}{w_n + z_n + \rho_n\zeta_n} = 1,$$

which implies that $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$. Next we prove $g'(\zeta) = 1 \Rightarrow g''(\zeta) = 0$. Again by (5.16), we obtain

$$\rho_n \frac{f''(w_n + z_n + \rho_n\zeta)}{w_n + z_n + \rho_n\zeta} = G_n'(\zeta) + \rho_n \frac{G_n(\zeta)}{w_n + z_n + \rho_n\zeta} \to g''(\zeta). \tag{5.17}$$

Suppose that $g'(\eta_0) = 1$. Obviously $g' \not\equiv 1$, for otherwise $g^\sharp(0) \le g'(0) = 1 < 3$, which is a contradiction. Again by Hurwitz's theorem, there exist a sequence $\{\eta_n\}$, $\eta_n \to \eta_0$ and (for n sufficiently large)

$$\frac{f'(w_n + z_n + \rho_n\eta_n)}{w_n + z_n + \rho_n\eta_n} = 1.$$

Thus $f'(w_n + z_n + \rho_n\eta_n) = w_n + z_n + \rho_n\eta_n$. By the assumption, we have $f''(w_n + z_n + \rho_n\eta_n) = w_n + z_n + \rho_n\eta_n$. Then

$$g''(\eta_0) = \lim_{n\to\infty} \rho_n \frac{f''(w_n + z_n + \rho_n\eta_n)}{w_n + z_n + \rho_n\eta_n} = \lim_{n\to\infty} \rho_n = 0.$$

Thus we prove the claim. By Lemma 2.2 an the claim, we get $g = \zeta - b$, where b is a constant. Thus we have $g^\sharp(0) \le 1 < 3$, a contradiction. So the case cannot occur.

Hence, we complete the proof of Theorem 3.

For further study, we propose the following questions.

Question 1. Let $f(z)$ be a nonconstant entire function and $k \ge 2$ be a positive integer. If

$$f(z) = z \Rightarrow f'(z) = z, \quad f'(z) = z \Rightarrow f^{(k)}(z) = z,$$

what will happen?

Question 2. Let $f(z)$ be a nonconstant entire function and $Q(z)$ be a nonzero polynomial. If

$$f(z) = Q(z) \Rightarrow f'(z) = Q(z), \quad f'(z) = Q(z) \Rightarrow f''(z) = Q(z),$$

what will happen?

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