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# ON THE TRUNCATED DEFECT RELATION FOR HOLOMORPHIC CURVES

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**Abstract**

For a transcendental holomorphic curve and a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in subgeneral position, we consider the truncated defect relation by using a generalization of Nochka weight function introduced in [12] and its supplement in Section 3. When it is not extremal, we estimate the sum of defects and when it is extremal, we investigate the number of vectors each defect of which is equal to 1 or the structure of vectors each defect of which is positive.

## 1. Introduction

Let $f = [f_1, \dots, f_{n+1}]$ be a holomorphic curve from $\boldsymbol{C}$ into the $n$-dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\},$$

where $n$ is a positive integer. We use the following notations:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$

$$\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2}, \quad (\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$

$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z).$$

The characteristic function of $f$ is defined as follows (see [13]):

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| \, d\theta - \log\|f(0)\|.$$

We suppose throughout the paper that $f$ is transcendental; that is to say,

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty$$

and that $f$ is linearly non-degenerate over $\boldsymbol{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$.

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It is well-known that $f$ is linearly non-degenerate over $\boldsymbol{C}$ if and only if the Wronskian $W = W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero.

For meromorphic functions in the complex plane we use the standard notation of the Nevanlinna theory of meromorphic functions ([4, 5]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$, we write

$$m(r, \boldsymbol{a}, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|\boldsymbol{a}\| \, \|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|} \, d\theta, \quad N(r, \boldsymbol{a}, f) = N\left(r, \frac{1}{(\boldsymbol{a}, f)}\right).$$

We then have the First Fundamental Theorem ([13, p. 76]):

$$T(r, f) = m(r, \boldsymbol{a}, f) + N(r, \boldsymbol{a}, f) + O(1).$$

We call the quantity

$$\delta(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \boldsymbol{a}, f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

the defect of $\boldsymbol{a}$ with respect to $f$.

Let $\nu(c, (\boldsymbol{a}, f))$ be the order of zero of $(\boldsymbol{a}, f(z))$ at $z = c$ and

$$n_n(r, \boldsymbol{a}, f) = \sum_{|c| \le r} \min\{\nu(c, (\boldsymbol{a}, f)), n\}.$$

We put for $r > 0$

$$N_n(r, \boldsymbol{a}, f) = \int_0^r \frac{n_n(t, \boldsymbol{a}, f) - n_n(0, \boldsymbol{a}, f)}{t} \, dt + n_n(0, \boldsymbol{a}, f) \log r$$

and put

$$\delta_n(\boldsymbol{a}, f) = 1 - \limsup_{r \to \infty} \frac{N_n(r, \boldsymbol{a}, f)}{T(r, f)},$$

which is called the truncated defect of $\boldsymbol{a}$ with respect to $f$. It is easy to see that

$$0 \le \delta(\boldsymbol{a}, f) \le \delta_n(\boldsymbol{a}, f) \le 1 \tag{1.1}$$

since $0 \le N_n(r, \boldsymbol{a}, f) \le N(r, \boldsymbol{a}, f) \le T(r, f) + O(1)$ $(r \ge 1)$.

We denote by $S(r, f)$ the quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \to \infty, r \notin E),$$

where $E$ is a subset of $(0, \infty)$ of finite linear measure.

Let $X$ be a subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in $N$-subgeneral position satisfying $\#X \ge 2N - n + 1$, where $N$ is an integer satisfying $N \ge n$.

Let $q$ be an integer satisfying $2N - n + 1 \le q < \infty$ and $Q$ a subset of $X$ such that $\#Q = q$. For a non-empty subset $P$ of $Q$, we denote by $V(P)$ the vector space spanned by elements of $P$ and by $d(P)$ the dimension of $V(P)$. We put

$$\mathcal{O}_Q = \{P \subset Q \mid 0 < \#P \le N + 1\}.$$

Lemma 1.A (see [3, Theorem 2.4.11], [2], [7]). *There is a function* $\omega : Q \to (0,1]$ *and a constant* $\theta$ *satisfying the following properties*:

(1.a) *For any* $\boldsymbol{a} \in Q$, $0 < \theta\omega(\boldsymbol{a}) \le 1$;

(1.b) $q - (2N - n + 1) = \theta(\sum_{\boldsymbol{a} \in Q} \omega(\boldsymbol{a}) - n - 1)$;

(1.c) $(N+1)/(n+1) \le \theta \le (2N - n + 1)/(n+1)$;

(1.d) *For any* $P \in \mathcal{O}_Q$, $\sum_{\boldsymbol{a} \in P} \omega(\boldsymbol{a}) \le d(P)$.

We call $\omega$ the Nochka weight function and $\theta$ the Nochka constant. This lemma was used to prove the Cartan conjecture. The result is as follows. Cartan ([1], $N = n$) and Nochka ([6], $N > n$) gave the following

Theorem 1.A (see [3, Theorem 3.3.8 and Corollary 3.3.9]). *For any* $q$ *elements* $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ *of* $X$ $(2N - n + 1 \le q < \infty)$, *we have the following inequalities*:

(I) $\displaystyle\sum_{j=1}^{q} \omega(\boldsymbol{a}_j)\delta_n(\boldsymbol{a}_j, f) \le n + 1;$

(II) $\displaystyle\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \le 2N - n + 1.$

The Nochka weight function is defined for a *finite* subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in $N$-subgeneral position, so that we can not let $q$ tend to $\infty$ in Theorem 1.A(I). This is incovenient to apply it to holomorphic curves with an infinite number of positive truncated defects. To avoid this inconvenience we generalized it to *any* subset of $\boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$ in $N$-subgeneral position in [12]. A proposition similar to [3, Proposition 3.4.4], a generalizaton of the Nochka weight function, which has properties similar to Lemma 1.A, are given in Section 2. In Section 3, a supplement to Proposition 2.3 in Section 2 is given and in Section 4 a truncated defect relation with a new weight is given, which will be used later.

Let

$$D^+ = D_n^+(X, f) = \{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) > 0\}$$

and

$$D^1 = D_n^1(X, f) = \{\boldsymbol{a} \in D^+ \mid \delta_n(\boldsymbol{a}, f) = 1\}.$$

Then, we obtain that the set $D^+$ is at most countable as in the case of meromorphic functions (see [5, p. 79]) and the truncated defect relation

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) = \sum_{\boldsymbol{a} \in D^+} \delta_n(\boldsymbol{a}, f) \le 2N - n + 1. \tag{1.2}$$

In Section 5, we shall give an upper bound smaller than $2N - n + 1$ for

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f)$$

in several cases when $N > n \ge 2$.

We are also interested in a holomorphic curve $f$ such that the equality holds in the truncated defect relation (1.2). It is said that $f$ is extremal for the truncated defect relation when

$$\sum_{\boldsymbol{a} \in D^+} \delta_n(\boldsymbol{a}, f) = 2N - n + 1. \tag{1.3}$$

In [9, Theorems 3.2 and 3.3], we obtained the following results.

Theorem 1.B. *Suppose that $N > n \geq 2$ and that* (1.3) *is satisfied. Then,*
[I] *If $D^1$ contains $n+1$ linearly independent vectors, then $\#D^1 = 2N - n + 1$.*
[II] *If $D^1$ contains $n$ linearly independent vectors and if $\#D^1 < 2N - n + 1$, then $\#D^1 = N$.*

One of main purposes of this paper is to extend Theorem 1.B to the case when $D^1$ contains at most $n$ linearly independent vectors by using the generalization of the Nochka weight function given in Section 2 and the results in Sections 3 and 4. The result is given in Section 6. Further we unify Theorems 3.1(II) and 4.1(II) in [10] into one theorem, Theorem 6.2 in the section.

## 2. Generalization of Nochka weight function

Let $N$, $n$ and $X$ be as in Section 1 such that $2N - n + 1 \leq \#X \leq \infty$. We note that $X$ is in $N$-subgeneral position and that $\#X$ is not always finite. For a non-empty *finite* subset $S$ of $X$, we denote by $V(S)$ the vector space spanned by elements of $S$ and by $d(S)$ the dimension of $V(S)$. We put

$$\mathcal{O} = \{S \subset X \mid 0 < \#S \leq N + 1\}.$$

Lemma 2.1 ([3, p. 68]). *For $S_1, S_2 \in \mathcal{O}$,*

$$d(S_1 \cup S_2) + d(S_1 \cap S_2) \leq d(S_1) + d(S_2).$$

Lemma 2.2 ([3, p. 68]). *For $R \subset S$ $(R, S \in \mathcal{O})$,*

$$\#R - d(R) \leq \#S - d(S) \leq N - n.$$

For $R \subsetneq S (R, S \in \mathcal{O})$, we put

$$\Lambda(R; S) = \frac{d(S) - d(R)}{\#S - \#R}.$$

Then, by Lemma 2.2 we have the following

Proposition 2.1 ([3, p. 67]). $0 \leq \Lambda(R; S) \leq 1$.

Lemma 2.3 ([12, Lemma 2.3]). *$\#\{d(S)/\#S \mid S \in \mathcal{O}\}$ is finite.*

Definition 2.1 ([12, Definition 2.1]). $\lambda = \min_{S \in \mathcal{O}} \dfrac{d(S)}{\#S}$.

Proposition 2.2 ([12, Proposition 2.2]).

$$1/(N - n + 1) \leq \lambda \leq (n + 1)/(N + 1).$$

Lemma 2.4 ([12, Lemma 2.4]). *For a fixed $R \in \mathcal{O}$, $\#\{\Lambda(R; S) \mid R \subsetneqq S \in \mathcal{O}\} < \infty$.*

Proposition 2.3 ([12, Proposition 2.3]). (I) *When $\lambda \geq (n + 1)/(2N - n + 1)$, for any $S \in \mathcal{O}$ it holds that*

$$\frac{n+1}{2N - n + 1} \leq \frac{d(S)}{\#S}.$$

(II) *When $\lambda < (n + 1)/(2N - n + 1)$, there exist an integer $p$ $(1 \leq p < (n + 1)/2)$ and a subfamily $\{T_i \mid 1 \leq i \leq p\}$ of $\mathcal{O}$ satisfying the following conditions*:

(i) $\phi = T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_p$, $d(T_p) < (n + 1)/2$;

(ii) $\Lambda(T_0; T_1) < \Lambda(T_1; T_2) < \cdots < \Lambda(T_{p-1}; T_p) < \dfrac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p}$;

(iii) *When $1 \leq i \leq p$, for any $U \in \mathcal{O}$ such that $T_{i-1} \subsetneqq U$; if $d(T_{i-1}) < d(U)$, then*

(a) $\Lambda(T_{i-1}; T_i) \leq \Lambda(T_{i-1}; U)$ *and*

(b) $\Lambda(T_{i-1}; T_i) = \Lambda(T_{i-1}; U)$ *only if* $U \subseteqq T_i$;

(iv) *For any $U \in \mathcal{O}$ such that $T_p \subsetneqq U$, if $d(T_p) < d(U)$, then*

$$\frac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p} \leq \Lambda(T_p; U).$$

*Note* 2.1. (a) The case "$\lambda < (n + 1)/(2N - n + 1)$" occurs only when $N > n \geq 2$.

In fact, if $N = n$, then $\lambda = 1$ or if $n = 1$ then $1/N \leq \lambda$ from Proposition 2.2. They contradict the fact "$\lambda < (n + 1)/(2N - n + 1)$".

(b) From Proposition 2.3(II)(ii), we have the inequalities:

$$\lambda = \frac{d(T_1)}{\#T_1} < \frac{d(T_2)}{\#T_2} < \cdots < \frac{d(T_p)}{\#T_p} < \frac{n+1}{2N - n + 1} < \frac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p} \tag{2.1}$$

(see the proof of [12, Proposition 2.3]) and

$$0 < d(T_1) < d(T_2) < \cdots < d(T_{p-1}) < d(T_p). \tag{2.2}$$

According to Proposition 2.3, we define a weight function $w$ and a constant $h$ for $X$ as follows:

Definition 2.2 ([12, Definition 3.1]). (I) When $\lambda \geq (n+1)/(2N-n+1)$. For any $\boldsymbol{a} \in X$

$$w(\boldsymbol{a}) = \frac{n+1}{2N-n+1} \quad \text{and} \quad h = \frac{2N-n+1}{n+1}.$$

(II) When $\lambda < (n+1)/(2N-n+1)$.

$$w(\boldsymbol{a}) = \begin{cases} \Lambda(T_{i-1}; T_i) & \text{if} \quad \boldsymbol{a} \in T_i - T_{i-1} \quad (i = 1, \dots, p) \\ \dfrac{n+1-d(T_p)}{2N-n+1-\#T_p} & \text{if} \quad \boldsymbol{a} \in X - T_p \end{cases}$$

and

$$h = \frac{2N-n+1-\#T_p}{n+1-d(T_p)},$$

where $T_0 = \phi$, $T_i$ and $\Lambda(T_{i-1}; T_i)$ $(i = 1, \dots, p)$ are those given in Proposition 2.3(II).

*Note* 2.2.
(a) $\begin{cases} h = (2N-n+1)/(n+1) & \text{if} \quad \lambda \geq (n+1)/(2N-n+1); \\ h < (2N-n+1)/(n+1) & \text{if} \quad \lambda < (n+1)/(2N-n+1). \end{cases}$
(b) $\{\boldsymbol{a} \in X \mid hw(\boldsymbol{a}) < 1\} = \begin{cases} \phi & \text{if} \quad \lambda \geq (n+1)/(2N-n+1); \\ T_p & \text{if} \quad \lambda < (n+1)/(2N-n+1). \end{cases}$

Proposition 2.4 ([12, Theorem 3.1]).
(a) *For any* $\boldsymbol{a} \in X$, $0 < hw(\boldsymbol{a}) \leq 1$;
(b-1) *For any* $Q \subset X$ *satisfying* (i) $Q \supset \{\boldsymbol{a} \in X \mid hw(\boldsymbol{a}) < 1\}$ *and* (ii) $2N - n + 1 \leq \#Q < \infty$,

$$\#Q - (2N-n+1) = h\left(\sum_{\boldsymbol{a} \in Q} w(\boldsymbol{a}) - n - 1\right);$$

(b-2) $\sum_{\boldsymbol{a} \in X}(1 - hw(\boldsymbol{a})) = 2N - n + 1 - h(n+1)$;
(c) $N/n \leq h \leq (2N-n+1)/(n+1)$;
(d) *For any* $S \in \mathcal{O}$, $\sum_{\boldsymbol{a} \in S} w(\boldsymbol{a}) \leq d(S)$.

*Remark* 2.1. (b-1) is given in the proof of [12, Theorem 3.1] and (b-2) is [12, Theorem 3.1(b)].

## 3. Supplement to Proposition 2.3

Let $N$, $n$, $X$ and $\mathcal{O}$ etc. be as in Section 2. By taking Lemma 2.2 into consideration, we say that an element $S$ of $\mathcal{O}$ is *maximal* if it satisfies the equality

$$\#S = d(S) + N - n.$$

Proposition 3.1. *Let $R, S \in \mathcal{O}$ such that $R \subsetneqq S$. If $R$ is maximal, so is $S$.*

This is trivial from Lemma 2.2. From now on throughout this section we suppose that

$$\lambda < \frac{n+1}{2N-n+1}.$$

Then, $N > n \geq 2$ (Note 2.1(a)) and there exist sets

$$\phi = T_0, T_1, \dots, T_p \quad \left(1 \leq p < \frac{n+1}{2}\right)$$

in $\mathcal{O}$ satisfying Proposition 2.3(II)(i), (ii), (iii) and (iv).

We put

$$\mathcal{O}_p = \{S \in \mathcal{O} \mid T_p \subsetneqq S, d(T_p) < d(S)\}.$$

Definition 3.1. We say that

(I) $X$ is of type I if for any $S \in \mathcal{O}_p$

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} < \Lambda(T_p; S).$$

(II) $X$ is of type II if for some $S \in \mathcal{O}_p$

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = \Lambda(T_p; S).$$

(A) We first treat the case that $X$ is of type I.

Lemma 3.1. *Suppose that $X$ is of type I. Then, $\#\{\Lambda(T_p; S) \mid S \in \mathcal{O}_p\} < \infty$.*

This is a direct consequence of Lemma 2.4.

Definition 3.2. We put

$$\lambda_p = \min_{S \in \mathcal{O}_p} \Lambda(T_p; S).$$

Proposition 3.2. *Suppose that $X$ is of type I. Then,*

(a) *$h^{-1} < \lambda_p$.*

(b) *Further, if $T_p$ is not maximal,*

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} < \frac{n+1-d(T_p)}{N+1-d(T_p)}. \tag{3.1}$$

*Proof.* (a) This is trivial from Definitions 3.1(I) and 3.2.

(b) As $\#T_p < d(T_p) + N - n$, we easily have (3.1). □

Definition 3.3. When $X$ is of type I and $T_p$ is not maximal, we put

$$\Lambda_1 = \min\left\{\lambda_p, \frac{n+1-d(T_p)}{N+1-d(T_p)}\right\}.$$

Corollary 3.1. *Suppose that $X$ is of type I and $T_p$ is not maximal. Then,*

$$\Lambda_1 - \frac{1}{h} \geq \frac{1}{N(2N-n)}.$$

*Proof.* (a) For any $S \in \mathcal{O}_p$,

$$\Lambda(T_p; S) - \frac{1}{h} = \frac{d(S) - d(T_p)}{\#S - \#T_p} - \frac{n+1-d(T_p)}{2N-n+1-\#T}$$
$$= \frac{(d(S)-d(T_p))(2N-n+1-\#T_p) - (\#S-\#T_p)(n+1-d(T_p))}{(\#S-\#T_p)(2N-n+1-\#T_p)}.$$

As this fraction is positive (Proposition 3.2(a)), the numerator is a positive integer, so that the numerator $\geq 1$. Further, the denominator is at most equal to $(N+1-1)(2N-n+1-1) = N(2N-n)$. This implies that

$$\lambda_p - \frac{1}{h} = \min_{S \in \mathcal{O}_p} \Lambda(T; S) - \frac{1}{h} \geq \frac{1}{N(2N-n)}.$$

(b) Next, we estimate the following.

$$\frac{n+1-d(T_p)}{N+1-d(T_p)} - \frac{1}{h} = \frac{(n+1-d(T_p))(N-n+d(T_p)-\#T_p)}{(N+1-d(T_p))(2N-n+1-\#T_p)}. \tag{3.2}$$

As this fraction is positive (Proposition 3.2(b)) and $N-n+d(T_p) > \#T_p$ since $T_p$ is not maximal, we have the following inequalities.

$$N-n+d(T_p)-\#T_p \geq 1, \tag{3.3}$$

$$\frac{n+1-d(T_p)}{N+1-d(T_p)} > \frac{n+1}{2N-n+1} > \frac{1}{N}, \tag{3.4}$$

$$\frac{1}{2N-n+1-\#T_p} \geq \frac{1}{2N-n} \tag{3.5}$$

since $d(T_p) < (n+1)/2$ and $\#T_p \geq 1$. From (3.2), (3.3), (3.4) and (3.5) we have the inequality

$$\frac{n+1-d(T_p)}{N+1-d(T_p)} - \frac{1}{h} \geq \frac{1}{N(2N-n)}.$$

From (a) and (b) we obtain this corollary. □

PROPOSITION 3.3. *Suppose that $X$ is of type I and that $T_p$ is not maximal. Let*

$$w_1(\boldsymbol{a}) = \begin{cases} w(\boldsymbol{a}) & \text{if} \quad \boldsymbol{a} \in T_p; \\ \Lambda_1 & \text{if} \quad \boldsymbol{a} \in X - T_p. \end{cases}$$

*Then, for any $S \in \mathcal{O}$,*

$$\sum_{\boldsymbol{a} \in S} w_1(\boldsymbol{a}) \leq d(S). \tag{3.6}$$

*Proof.* Let $S \in \mathcal{O}$. a) When $d(S \cup T_p) = n+1$. From Lemma 2.1, we have the inequality

$$n + 1 - d(T_p) = d(S \cup T_p) - d(T_p) \leq d(S). \tag{3.7}$$

As $w_1(\boldsymbol{a}) = w(\boldsymbol{a}) < h^{-1}$ for $\boldsymbol{a} \in T_p$ (Note 2.2(b)) and $h^{-1} < \Lambda_1$ by Proposition 3.2, from Lemma 2.2 and (3.7) we have the inequality

$$\begin{aligned} \sum_{\boldsymbol{a} \in S} w_1(\boldsymbol{a}) &\leq \Lambda_1 \# S \leq \Lambda_1(d(S) + N - n) \\ &= d(S)\Lambda_1\left(1 + \frac{N-n}{d(S)}\right) \leq d(S)\Lambda_1\left(1 + \frac{N-n}{n+1-d(T_p)}\right) \\ &= d(S)\Lambda_1 \frac{N+1-d(T_p)}{n+1-d(T_p)} \leq d(S) \end{aligned}$$

since $\Lambda_1(N+1-d(T_p))/(n+1-d(T_p)) \leq 1$ by the definition of $\Lambda_1$.

b) When $d(S \cup T_p) \leq n$ and $S \subset T_p$. From Proposition 2.4(d),

$$\sum_{\boldsymbol{a} \in S} w_1(\boldsymbol{a}) = \sum_{\boldsymbol{a} \in S} w(\boldsymbol{a}) \leq d(S).$$

c) When $d(S \cup T_p) \leq n$ and $S - T_p \neq \phi$. We have that $S \cup T_p \in \mathcal{O}$ since $\#(S \cup T_p) \leq N$. We prepare two inequalities.

(c.1) $d(T_p) < d(S \cup T_p)$.

(Proof.) Suppose to the contrary that

$$d(T_p) = d(S \cup T_p). \tag{3.8}$$

Then, we have from (2.2) that

$$d(S \cup T_p) - d(T_{p-1}) = d(T_p) - d(T_{p-1}) > 0, \tag{3.9}$$

and from Proposition 2.3(II)(iii) that

$$\Lambda(T_{p-1}; T_p) = \frac{d(T_p) - d(T_{p-1})}{\# T_p - \# T_{p-1}} < \frac{d(S \cup T_p) - d(T_{p-1})}{\#(S \cup T_p) - \# T_{p-1}} = (*)$$

since $T_{p-1} \subsetneqq T_p \subsetneqq S \cup T_p$ and $d(T_{p-1}) < d(T_p) = d(S \cup T_p)$ from (3.8) and (3.9).

On the other hand from (3.8)

$$(*) < \frac{d(T_p) - d(T_{p-1})}{\# T_p - \# T_{p-1}} = \Lambda(T_{p-1}; T_p)$$

since $\# T_p < \#(S \cup T_p)$. This is a contradiction. (c.1) must hold.

(c.2) $(\# S - \#(S \cap T_p))\lambda_p \leq d(S) - d(S \cap T_p)$.

(Proof.) Note that $\# S - \#(S \cap T_p) > 0$. From the facts that

(i) $S \cup T_p \in \mathcal{O}$,

(ii) (c.1): $d(T_p) < d(S \cup T_p)$ and

(iii) $T_p \subsetneqq S \cup T_p$,

we have that $S \cup T_p \in \mathcal{O}_p$. Then, by Definition 3.2 we have the inequality

$$\lambda_p \leq \frac{d(S \cup T_p) - d(T_p)}{\#(S \cup T_p) - \# T_p} = (**).$$

Here, we have the relations

$$\#(S \cup T_p) = \# T_p + \# S - \#(S \cap T_p)$$

and

$$d(S \cup T_p) \leq d(T_p) + d(S) - d(S \cap T_p)$$

from Lemma 2.1, so that we have

$$(**) \leq \frac{d(S) - d(S \cap T_p)}{\# S - \#(S \cap T_p)},$$

which reduces to (c.2). Note that (c.2) is valid when $S \cap T_p = \phi$.

Now, we prove (3.2) in case c). As $S \cap T_p \in \mathcal{O}$ if $S \cap T_p \neq \phi$ and $S \cap T_p \subset T_p$, by using (c.2) and Proposition 2.4(d) we have the inequality

$$\begin{aligned}
\sum_{\boldsymbol{a} \in S} w_1(\boldsymbol{a}) &= \sum_{\boldsymbol{a} \in S \cap T_p} w(\boldsymbol{a}) + \sum_{\boldsymbol{a} \in S - S \cap T_p} w_1(\boldsymbol{a}) \\
&\leq d(S \cap T_p) + \Lambda_1 \#(S - S \cap T_p) \\
&\leq d(S \cap T_p) + \lambda_p(\# S - \# S \cap T_p) \\
&\leq d(S \cap T_p) + (d(S) - d(S \cap T_p)) = d(S).
\end{aligned}$$

since $w_1(\boldsymbol{a}) = \Lambda_1$ $(\boldsymbol{a} \in X - T_p)$. We obtain this proposition. □

(B) From now on in this subsection we suppose that $X$ is of type II. We put

$$\mathcal{O}_p(1/h) = \{S \in \mathcal{O}_p \mid \Lambda(T_p; S) = 1/h\}.$$

As $X$ is of type II,

Proposition 3.4. *$\mathcal{O}_p(1/h)$ is not empty.*

Proposition 3.5. *For any* $S \in \mathcal{O}_p(1/h)$,

$$d(S) < (n+1)/2 \quad \text{and} \quad \#S < (2N-n+1)/2.$$

*Proof.* We first note that $d(S) \le n$. In fact, if $d(S) = n+1$, then from Definition 3.1(II), $\#S = 2N - n + 1$, which is contrary to the fact that $S \in \mathcal{O}$ as $N > n \ge 2$. We have $\#S \le N$. From the equality

$$\frac{d(S) - d(T_p)}{\#S - \#T_p} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = h^{-1}$$

and Note 2.2(a), we have the inequality

$$\frac{n+1}{2N-n+1} < h^{-1} = \frac{n+1-d(S)}{2N-n+1-\#S},$$

from which we obtain the inequality

$$d(S)\frac{2N-n+1}{n+1} < \#S \le d(S) + N - n$$

due to Lemma 2.2, so that $d(S) < (n+1)/2$ and $\#S < (2N-n+1)/2$. □

Proposition 3.6. *If* $S_1, S_2 \in \mathcal{O}_p(1/h)$, *then* $S_1 \cup S_2 \in \mathcal{O}_p(1/h)$.

*Proof.* (a) First, we prove that $S_1 \cup S_2 \in \mathcal{O}_p$. As

$$\frac{d(S_1) - d(T_p)}{\#S_1 - \#T_p} = \frac{d(S_2) - d(T_p)}{\#S_2 - \#T_p} = h^{-1},$$

from Lemma 2.2, we have the inequality

$$\begin{aligned}
&d(S_1) + d(S_2) - 2d(T_p) \\
&\quad = h^{-1}(\#S_1 + \#S_2 - 2\#T_p) \\
&\quad \le h^{-1}(d(S_1) + N - n + d(S_2) + N - n - 2\#T_p) \\
&\quad = h^{-1}(d(S_1) + d(S_2) - 2d(T_p)) + 2h^{-1}(N - n + d(T_p) - \#T_p),
\end{aligned}$$

so that

$$d(S_1) + d(S_2) - 2d(T_p) \le \frac{2h^{-1}}{1-h^{-1}}(N - n + d(T_p) - \#T_p) = (*)$$

since $h^{-1} \le (n/N) < 1$ from Proposition 2.4(c). Here, we have the equality

$$1 - h^{-1} = 1 - \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = \frac{2N-2n+d(T_p)-\#T_p}{2N-n+1-\#T_p},$$

so that we have

$$(*) = 2h^{-1}\frac{N-n+d(T_p)-\#T_p}{2N-2n+d(T_p)-\#T_p}(2N-n+1-\#T_p)$$
$$< h^{-1}(2N-n+1-\#T_p) = n+1-d(T_p)$$

since $d(T_p) < \#T_p$ from (2.1) and $h^{-1} < 1$. We obtain the inequality

$$d(S_1)+d(S_2)-d(T_p) < n+1,$$

so that by Lemma 2.1 we have the inequality

$$d(S_1\cup S_2) \le d(S_1)+d(S_2)-d(S_1\cap S_2) \le d(S_1)+d(S_2)-d(T_p) < n+1$$

since $S_1\cap S_2 \supset T_p$. That is, $d(S_1\cup S_2)\le n$ and so $\#(S_1\cup S_2)\le N$. We have that $S_1\cup S_2 \in \mathcal{O}$. In addition, as

$$d(T_p) < d(S_1) \le d(S_1\cup S_2)$$

we have that $S_1\cup S_2\in \mathcal{O}_p$.

(b) Next, we prove the inequality

$$h^{-1}(\#(S_1\cap S_2)-\#T_p) \le d(S_1\cap S_2)-d(T_p). \tag{3.10}$$

As this inequality is trivial when $\#(S_1\cap S_2)-\#T_p = 0$, we prove (3.10) when $\#(S_1\cap S_2)-\#T_p > 0$. We first prove that

$$d(T_p) < d(S_1\cap S_2). \tag{3.11}$$

In fact, suppose to the contrary that $d(T_p) = d(S_1\cap S_2)$. Then,

$$S_1\cap S_2 \in \mathcal{O}_{p-1} = \{S\in\mathcal{O} \mid T_{p-1}\subsetneqq S, d(T_{p-1}) < d(S)\}$$

since $T_{p-1}\subsetneqq T_p \subsetneqq S_1\cap S_2$ and $d(T_{p-1}) < d(T_p) = d(S_1\cap S_2)$, so that we have the inequality

$$\Lambda(T_{p-1}; S_1\cap S_2) \ge \min_{S\in\mathcal{O}_{p-1}} \Lambda(T_{p-1}; S) = \Lambda(T_{p-1}; T_p) = \frac{d(T_p)-d(T_{p-1})}{\#T_p-\#T_{p-1}}$$
$$> \frac{d(S_1\cap S_2)-d(T_{p-1})}{\#(S_1\cap S_2)-\#T_{p-1}} = \Lambda(T_{p-1}; S_1\cap S_2).$$

This is a contradiction. We obtain (3.11) and $S_1\cap S_2\in\mathcal{O}_p$. From Proposition 2.3(II)(iv), we have the inequality

$$h^{-1} \le \Lambda(T_p; S_1\cap S_2).$$

This means that (3.10) holds.

(c) Finally, we prove that $S_1\cup S_2 \in \mathcal{O}_p(1/h)$. From Lemma 2.1 and (3.10) we have

$$h^{-1} \le \Lambda(T_p; S_1\cup S_2) \le \frac{d(S_1)+d(S_2)-d(S_1\cap S_2)-d(T_p)}{\#S_1+\#S_2-\#(S_1\cap S_2)-\#T_p} \le h^{-1}$$

since $S_1, S_2 \in \mathscr{O}_p(1/h)$ and the following inequality holds from (3.10):

$$\begin{aligned} &d(S_1) + d(S_2) - d(S_1 \cap S_2) - d(T_p) \\ &\quad = d(S_1) - d(T_p) + d(S_2) - d(T_p) - (d(S_1 \cap S_2) - d(T_p)) \\ &\quad \leq h^{-1}(\#S_1 - \#T_p + \#S_2 - \#T_p - (\#(S_1 \cap S_2) - \#T_p)) \\ &\quad = h^{-1}(\#S_1 + \#S_2 - \#(S_1 \cap S_2) - \#T_p). \end{aligned}$$

Namely, we have that $\Lambda(T_p; S_1 \cup S_2) = h^{-1}$. This means that $S_1 \cup S_2 \in \mathscr{O}_p(1/h)$. □

PROPOSITION 3.7. *$\#\mathscr{O}_p(1/h)$ is finite.*

*Proof.* We have only to prove this proposition when $\#X$ is not finite. Suppose to the contrary that $\#\mathscr{O}_p(1/h) = \infty$. Then, there are sets $S_1, S_2, \dots$ satsfying

$$\mathscr{O}_p(1/h) \supset \{S_1, S_2, \dots, S_i, \dots\}, \quad S_i \neq S_j \text{ if } i \neq j$$

and

$$\#\left\{\bigcup_{i=1}^{\infty} S_i\right\} = \infty.$$

There exists an integer $v$ satisfying

$$N + 1 < \#\left\{\bigcup_{i=1}^{v} S_i\right\}.$$

On the other hand, $\bigcup_{i=1}^{v} S_i \in \mathscr{O}_p(1/h)$ from Proposition 3.6 and so from Proposition 3.5

$$\#\left\{\bigcup_{i=1}^{v} S_i\right\} < \frac{2N - n + 1}{2}.$$

From these two inequalities we obtain that $n + 1 < 0$, which is absurd. This implies that $\#\mathscr{O}_p(1/h)$ is finite. □

DEFINITION 3.4. We put $T_{p+1} = \bigcup_{S \in \mathscr{O}_p(1/h)} S$.

PROPOSITION 3.8. (a) *$T_{p+1} \in \mathscr{O}_p(1/h)$. If $S \in \mathscr{O}_p(1/h)$, then $S \subset T_{p+1}$.*

(b)

$$\Lambda(T_{p-1}; T_p) < h^{-1} = \frac{n + 1 - d(T_p)}{2N - n + 1 - \#T_p} = \Lambda(T_p; T_{p+1}) = \frac{n + 1 - d(T_{p+1})}{2N - n + 1 - \#T_{p+1}}.$$

*Proof.* We obtain (a) from Definition 3.4 and Proposition 3.6. We have (b) from (a) and Proposition 2.3(II)(ii). □

We put

$$\mathscr{F}_p = \{S \in \mathcal{O} \mid T_p \subsetneqq S, d(T_p) < d(S), S - T_{p+1} \neq \phi\}.$$

PROPOSITION 3.9. *$\mathscr{F}_p$ is not empty.*

*Proof.* We can choose an element $S$ from $\mathcal{O}$ such that $T_p \subsetneqq S$ and $\#S = N+1$ since $\#T_p < \#T_{p+1} < (2N-n+1)/2 < N+1$ from Proposition 3.5. This set $S$ belongs to $\mathscr{F}_p$ since $d(T_p) < d(T_{p+1}) < (n+1)/2 < n+1 = d(S)$ from Proposition 3.5, so that $S - T_{p+1} \neq \phi$. □

PROPOSITION 3.10. *$\#\{\Lambda(T_p; S) \mid S \in \mathscr{F}_p\}$ is finite.*
This is due to Lemma 2.4.

DEFINITION 3.5. We put $\eta_p = \min_{S \in \mathscr{F}_p} \Lambda(T_p; S)$.

PROPOSITION 3.11. *$h^{-1} < \eta_p$.*

*Proof.* First we note that

$$h^{-1} < \Lambda(T_p; S) \quad (S \in \mathscr{F}_p). \tag{3.12}$$

In fact, by Proposition 2.3(II)(iv), $h^{-1} \leq \Lambda(T_p; S)$. If there exists an element $S \in \mathscr{F}_p$ such that $h^{-1} = \Lambda(T_p; S)$, then by Proposition 3.8(a), $S \subset T_{p+1}$, which is a contradiction. We have (3.12). By Proposition 3.10, we have this proposition. □

PROPOSITION 3.12. *$T_p$ is not maximal and we have*

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} < \frac{n+1-d(T_p)}{N+1-d(T_p)}.$$

*Proof.* Suppose that $T_p$ is maximal. Then, from Lemma 2.2, $T_{p+1}$ is maximal and we have $\#T_{p+1} - \#T_p = d(T_{p+1}) - d(T_p)$ so that from Proposition 3.8(b)

$$1 > h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = \Lambda(T_p; T_{p+1}) = 1,$$

which is absurd. This means that $T_p$ is not maximal. As $\#T_p < d(T_p) + N - n$, we have the inequality

$$h^{-1} = \frac{n+1-d(T_p)}{2N-n+1-\#T_p} < \frac{n+1-d(T_p)}{2N-n+1-(N-n+d(T_p))} = \frac{n+1-d(T_p)}{N+1-d(T_p)}.$$

□

Definition 3.6. When $X$ is of type II, we put

$$\Lambda_2 = \min\left\{\eta_p, \frac{n+1-d(T_p)}{N+1-d(T_p)}\right\}.$$

Corollary 3.2. *Suppose that $X$ is of type II. Then,*

$$\Lambda_2 - \frac{1}{h} \geq \frac{1}{N(2N-n)}.$$

*Proof.* (a) For any $S \in \mathscr{F}_p$,

$$\Lambda(T_p; S) - \frac{1}{h} = \frac{d(S) - d(T_p)}{\#S - \#T_p} - \frac{n+1-d(T_p)}{2N-n+1-\#T_p} \geq \frac{1}{N(2N-n)}$$

as in the case of Proof (a) of Corollary 3.1.

(b) As in Proof (b) of Corollary 3.1

$$\frac{n+1-d(T_p)}{N+1-d(T_p)} - \frac{1}{h} \geq \frac{1}{N(2N-n)}.$$

From (a) and (b) we have

$$\Lambda_2 - \frac{1}{h} \geq \frac{1}{N(2N-n)}. \qquad \square$$

Proposition 3.13. *Suppose that $X$ is of type II. Let*

$$w_2(\boldsymbol{a}) = \begin{cases} w(\boldsymbol{a}) & \text{if} \quad \boldsymbol{a} \in T_{p+1}; \\ \Lambda_2 & \text{if} \quad \boldsymbol{a} \in X - T_{p+1}. \end{cases}$$

*Then, for any $S \in \mathcal{O}$,*

$$\sum_{\boldsymbol{a} \in S} w_2(\boldsymbol{a}) \leq d(S). \tag{3.13}$$

*Proof.* We proceed this proof as in that of Proposition 3.3. Let $S \in \mathcal{O}$.

a) When $d(S \cup T_p) = n+1$. From Lemma 2.1, we have the inequality

$$n+1-d(T_p) = d(S \cup T_p) - d(T_p) \leq d(S). \tag{3.14}$$

As $w_2(\boldsymbol{a}) = w(\boldsymbol{a}) \leq h^{-1}$ for $\boldsymbol{a} \in T_{p+1}$ and $h^{-1} < \Lambda_2$ by Propositions 3.11 and 3.12, from Lemma 2.2 and (3.14) we have the inequality

$$\sum_{\boldsymbol{a}\in S} w_2(\boldsymbol{a}) \le \Lambda_2 \# S \le \Lambda_2(d(S)+N-n)$$
$$= d(S)\Lambda_2\left(1+\frac{N-n}{d(S)}\right) \le d(S)\Lambda_2\left(1+\frac{N-n}{n+1-d(T_p)}\right)$$
$$= d(S)\Lambda_2\frac{N+1-d(T_p)}{n+1-d(T_p)} \le d(S)$$

since $\Lambda_2(N+1-d(T_p))/(n+1-d(T_p)) \le 1$ by the definition of $\Lambda_2$.

b) When $d(S\cup T_p)\le n$ and $S\subset T_{p+1}$. As $w_2(\boldsymbol{a})=w(\boldsymbol{a})$ $(\boldsymbol{a}\in S)$, from Proposition 2.4(d),

$$\sum_{\boldsymbol{a}\in S} w_2(\boldsymbol{a}) = \sum_{\boldsymbol{a}\in S} w(\boldsymbol{a}) \le d(S).$$

c) When $d(S\cup T_p)\le n$ and $S-T_{p+1}\ne\phi$. As $\#(S\cup T_p)\le N$, $S\cup T_p\in\mathcal{O}$. We prepare two inequalities.

(c.1) $d(T_p)<d(S\cup T_p)$.

(Proof.) We suppose to the contrary that

$$d(T_p)=d(S\cup T_p). \tag{3.15}$$

Then, we have from (2.2) that

$$d(S\cup T_p)-d(T_{p-1})=d(T_p)-d(T_{p-1})>0 \tag{3.16}$$

and from Proposition 2.3(II)(iii) that

$$\Lambda(T_{p-1};T_p)=\frac{d(T_p)-d(T_{p-1})}{\#T_p-\#T_{p-1}}<\frac{d(S\cup T_p)-d(T_{p-1})}{\#(S\cup T_p)-\#T_{p-1}}=(*)$$

since $T_{p-1}\subsetneqq T_p\subsetneqq S\cup T_p$ and $d(T_{p-1})<d(T_p)=d(S\cup T_p)$ from (3.15) and (3.16).

On the other hand from (3.15)

$$(*)<\frac{d(T_p)-d(T_{p-1})}{\#T_p-\#T_{p-1}}=\Lambda(T_{p-1};T_p)$$

since $\#T_p<\#(S\cup T_p)$ as $S-T_{p+1}\ne\phi$. This is a contradiction. (c.1) must hold.

(c.2) $(\#S-\#(S\cap T_p))\eta_p\le d(S)-d(S\cap T_p)$.

(Proof.) Note that $\#S-\#(S\cap T_p)>0$. From the facts that

(i) $S\cup T_p\in\mathcal{O}$;

(ii) (c.1): $d(T_p)<d(S\cup T_p)$;

(iii) $T_p\subsetneqq S\cup T_p$ and

(iv) $S-T_{p+1}\ne\phi$,

we have that $S\cup T_p\in\mathcal{F}_p$. Then, by Definition 3.5 we have the inequality

$$\eta_p\le\frac{d(S\cup T_p)-d(T_p)}{\#(S\cup T_p)-\#T_p}=(**).$$

Here, we have the relations

$$\#(S \cup T_p) = \#T_p + \#S - \#(S \cap T_p)$$

and

$$d(S \cup T_p) \le d(T_p) + d(S) - d(S \cap T_p)$$

from Lemma 2.1, so that we have

$$(**) \le \frac{d(S) - d(S \cap T_p)}{\#S - \#(S \cap T_p)},$$

which reduces to (c.2). Note that (c.2) holds if $S \cap T_p = \phi$.

Now, we prove (3.13) in case c). As $S \cap T_p \in \mathcal{O}$ if $S \cap T_p \ne \phi$ and $S \cap T_p \subset T_p$, by using (c.2) and Proposition 2.4(d) we have the inequality

$$\begin{aligned}\sum_{\boldsymbol{a} \in S} w_2(\boldsymbol{a}) &= \sum_{\boldsymbol{a} \in S \cap T_p} w(\boldsymbol{a}) + \sum_{\boldsymbol{a} \in S - S \cap T_p} w_2(\boldsymbol{a}) \\ &\le d(S \cap T_p) + \Lambda_2 \#(S - S \cap T_p) \\ &\le d(S \cap T_p) + \eta_p(\#S - \#(S \cap T_p)) \\ &\le d(S \cap T_p) + (d(S) - d(S \cap T_p)) = d(S).\end{aligned}$$

since $w_2(\boldsymbol{a}) = w(\boldsymbol{a}) = 1/h < \Lambda_2$ for $\boldsymbol{a} \in T_{p+1} - T_p$. □

## 4. A defect relation

Let $f$, $X$, $N$ and $n$ etc. be as in Section 1. Let us remember the definition of $D^+$:

$$D^+ = \{\boldsymbol{a} \in X \,|\, \delta_n(\boldsymbol{a}, f) > 0\},$$

which is at most countable. We use the same notations used in Sections 2 and 3, such as

$$\lambda,\ w,\ h,\ \mathcal{O},\ etc..$$

The purpose of this section is to generalize Theorem 1.A(I) for later use. To that end, we consider the following set of weight functions on $X$:

Definition 4.1. $\mathcal{W} = \{\tau : X \to (0,1] \,|\, \forall S \in \mathcal{O}, \sum_{\boldsymbol{a} \in S} \tau(\boldsymbol{a}) \le d(S)\}$.

*Example* 4.1. (a) $w$ (in Definition 2.2 and Proposition 2.4), $w_1$ (in Proposition 3.3) and $w_2$ (in Proposition 3.13) are in $\mathcal{W}$.

(b) Let $\tau_\lambda : X \to (0,1]$ such that $\tau_\lambda(\boldsymbol{a}) = \lambda$ for any $\boldsymbol{a} \in X$. Then $\tau_\lambda \in \mathcal{W}$. In fact, for any $S \in \mathcal{O}$,

$$\sum_{\boldsymbol{a} \in S} \tau_\lambda(\boldsymbol{a}) = \lambda \#S \le (d(S)/\#S)\#S = d(S).$$

First of all, we prepare some lemmas for later use.

Lemma 4.1 (see [8, Proposition 10]). *Let* $\tau \in \mathscr{W}$ *and* $Q = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\} \subset X$ $(N+1 \le q < \infty)$, *then the following inequalities hold.*

(I) $\displaystyle\sum_{j=1}^{q} \tau(\boldsymbol{a}_j) m(r, \boldsymbol{a}_j, f) \le (n+1)T(r,f) - N(r, 1/W) + S(r,f);$

(II) $\displaystyle\sum_{j=1}^{q} \tau(\boldsymbol{a}_j) \delta(\boldsymbol{a}_j, f) \le n+1.$

For an entire function $g(z)$, let $v(c,g)$ be the order of zero of $g(z)$ at $z = c$.

Lemma 4.2 (cf. [3, (3.2.14)]). *Let* $\tau \in \mathscr{W}$ *and* $Q = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\} \subset X$ $(N+1 \le q < \infty)$. *Then, for* $c \in \boldsymbol{C}$

$$\sum_{\boldsymbol{a} \in Q} \tau(\boldsymbol{a})(v(c, (\boldsymbol{a}, f)) - n)^+ \le v(c, W),$$

*where* $x^+ = \max(x, 0)$ *for a real number* $x$.

In fact, as is seen from the proof of the inequality [3, (3.2.14), p. 102], among the four properties of $\omega$ in Lemma 1.A, only the property (1.d) is necessary to prove it. Therefore, the proof is effective if we change $\omega$ for our weight function $\tau \in \mathscr{W}$ which has the same property as Lemma 1.A(1.d) and we have this lemma.

As in [11, Lemmas 2.5 and 2.6], we obtain the following Lemmas 4.3 and 4.4:

Lemma 4.3. *Let* $\tau \in \mathscr{W}$ *and* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(N+1 \le q < \infty)$. *Then, we have the inequalities for* $r \ge 0$

(I) $\displaystyle\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\{n(r, \boldsymbol{a}_j, f) - n_n(r, \boldsymbol{a}_j, f)\} \le n(r, 1/W).$

(II) $$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\{(n(r, \boldsymbol{a}_j, f) - n_n(r, \boldsymbol{a}_j, f)) - (n(0, \boldsymbol{a}_j, f) - n_n(0, \boldsymbol{a}_j, f))\} \le n(r, 1/W) - n(0, 1/W).$$

Lemma 4.4 (cf. [3, p. 105]). *Let* $\tau \in \mathscr{W}$ *and* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X$ $(N+1 \le q < \infty)$. *Then, we have the inequality*

$$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\{N(r, \boldsymbol{a}_j, f) - N_n(r, \boldsymbol{a}_j, f)\} \le N\left(r, \frac{1}{W}\right) \quad (r \ge 1).$$

Theorem 4.1. *Let* $f$ *be as in Section* 1. *For any* $\tau \in \mathscr{W}$ *we have the inequality*

$$\sum_{\boldsymbol{a} \in X} \tau(\boldsymbol{a}) \delta_n(\boldsymbol{a}, f) \le n+1.$$

*Proof.* As is cited in the beginning of this section, we know that the set $D^+$ is at most countable. If $\# D^+ \le N+1$, then

$$\sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) \le \sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a}) \le d(D^+) \le n+1$$

since $D^+ \in \mathcal{O}$.

We have only to prove this theorem when $\# D^+ \ge N+2$. Let $Q = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ $(N+1 \le q < \infty)$ be a subset of $D^+$. Then, by Lemma 4.1(I), the first fundamental theorem and Lemma 4.4 we have the inequality

$$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)(T(r,f) - N_n(r, \boldsymbol{a}_j, f)) \le (n+1)T(r,f) + S(r,f) \quad (r \ge 1),$$

from which we easily obtain the inequality

$$\sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\delta_n(\boldsymbol{a}_j, f) \le n+1. \tag{4.1}$$

1) When $\# D^+ < +\infty$. Let $Q = D^+$ and we have

$$\sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) \le n+1.$$

2) When $\# D^+ = +\infty$. Let $D^+ = \{\boldsymbol{a}_j \mid j \in \boldsymbol{N}\}$. Then, from (4.1) we have the inequality

$$\sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) = \lim_{q \to \infty} \sum_{j=1}^{q} \tau(\boldsymbol{a}_j)\delta_n(\boldsymbol{a}_j, f) \le n+1.$$

As $\delta_n(\boldsymbol{a}, f) = 0$ for $\boldsymbol{a} \in X - D^+$, we have the inequality

$$\sum_{\boldsymbol{a} \in X} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) = \sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) \le n+1.$$

From 1) and 2) we have our theorem. □

Corollary 4.1. (I) (cf. Theorem 1.A(I)) $\sum_{\boldsymbol{a} \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) \le n+1$.
(II) $\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) \le (n+1)/\lambda$.
We have this corollary from Theorem 4.1 since $w$ and $\tau_\lambda$ are in $\mathscr{W}$.

## 5. Estimate of the sum of truncated defects

Let $f$, $X$, $N$ and $n$ etc. be as in Section 1, 2 or 3. The purpose of this section is to estimate

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f)$$

in several cases. We suppose that $N > n$ throughout this section.

Lemma 5.1. *For $S \in \mathcal{O}$, if*

$$\frac{n+1}{2N-n+1} < \frac{d(S)}{\#S}, \tag{5.1}$$

*then*

$$\frac{\#S}{d(S)} \leq \frac{2N-n+1}{n+1} - \frac{1}{n(n+1)}.$$

*Proof.* From (5.1) we have the inequality

$$(n+1)\#S < (2N-n+1)d(S),$$

which reduces to

$$(n+1)\#S \leq (2N-n+1)d(S) - 1$$

since two numbers $(n+1)\#S$ and $(2N-n+1)d(S)$ are positive integers. From this inequality we have the inequality

$$\frac{\#S}{d(S)} \leq \frac{2N-n+1}{n+1} - \frac{1}{d(S)(n+1)}. \tag{5.2}$$

(a) When $d(S) \leq n$, we easily have that

$$\frac{\#S}{d(S)} \leq \frac{2N-n+1}{n+1} - \frac{1}{n(n+1)}.$$

(b) When $d(S) = n+1$, we have the inequality

$$\frac{\#S}{d(S)} \leq \frac{N+1}{n+1} \leq \frac{2N-n+1}{n+1} - \frac{1}{n(n+1)}$$

since $S \in \mathcal{O}$ and

$$\frac{2N-n+1}{n+1} - \frac{1}{n(n+1)} - \frac{N+1}{n+1} = \frac{1}{n+1}\left(N - n - \frac{1}{n}\right) \geq 0.$$

We have our lemma. □

Lemma 5.2. *We have the equality*

$$2N-n+1 - \sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) = \sum_{\boldsymbol{a}\in X}(1 - hw(\boldsymbol{a}))(1-\delta_n(\boldsymbol{a}, f)) + h\left(n+1 - \sum_{\boldsymbol{a}\in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f)\right).$$

*Proof.* (A) When $\lambda \geq (n+1)/(2N-n+1)$. From Definition 2.2(I) we have the relations

$$(1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f)) = 0 \quad (\boldsymbol{a} \in X)$$

and

$$h\left(n + 1 - \sum_{\boldsymbol{a} \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f)\right) = 2N - n + 1 - \sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f)$$

since $hw(\boldsymbol{a}) = 1$ $(\boldsymbol{a} \in X)$, so that we have this lemma in this case.

(B) When $\lambda < (n+1)/(2N - n + 1)$. We note that $\#X \geq 2N - n + 1$. Let $Q$ be any finite subset of $X$ satisfying $\#Q \geq 2N - n + 1$ and $Q \supset T_p$. Then, as the equality

$$\begin{aligned} h\sum_{\boldsymbol{a} \in Q} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) + \#Q - h\sum_{\boldsymbol{a} \in Q} w(\boldsymbol{a}) \\ = \sum_{\boldsymbol{a} \in Q} \{\delta_n(\boldsymbol{a}, f) + (1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f))\} \end{aligned}$$

holds, from Proposition 2.4(b.1) we have the equality

$$\begin{aligned} (5.3) \qquad & h\left(\sum_{\boldsymbol{a} \in Q} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) - n - 1\right) \\ & = \sum_{\boldsymbol{a} \in Q} \{\delta_n(\boldsymbol{a}, f) + (1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f))\} - (2N - n + 1). \end{aligned}$$

We note that

$$(5.4) \qquad hw(\boldsymbol{a}) = 1 \quad (\boldsymbol{a} \in X - T_p).$$

(a) When $\#(T_p \cup D^+) < +\infty$. In (5.3), let $Q \supset T_p \cup D^+$. Then, since $\delta_n(\boldsymbol{a}, f) = 0$ $(\boldsymbol{a} \in X - Q)$ and (5.4) holds, we obtain this lemma from (5.3) in this case.

(b) When $\#(T_p \cup D^+) = +\infty$. Let $D^+ = \{\boldsymbol{a}_j \mid j \in \boldsymbol{N}\}$ and in (5.3) we take $Q = T_p \cup \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ $(k \geq 2N - n + 1)$ and then let $k$ tend to infinity. We then have the equality

$$\begin{aligned} (5.5) \qquad & h\left(\sum_{\boldsymbol{a} \in T_p \cup D^+} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) - n - 1\right) \\ & = \sum_{\boldsymbol{a} \in T_p \cup D^+} \{\delta_n(\boldsymbol{a}, f) + (1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f))\} - (2N - n + 1). \end{aligned}$$

As $\delta_n(\boldsymbol{a}, f) = 0$ $(\boldsymbol{a} \in X - T_p \cup D^+)$ and (5.4) holds, we obtain this lemma from (5.5) in this case.

From (A) and (B) we obtain this lemma. □

(I) The case when $\lambda > (n+1)/(2N - n + 1)$.

Theorem 5.1. *If* $\lambda > (n+1)/(2N-n+1)$, *then*

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) \le 2N-n+1-\frac{1}{n}.$$

*Proof.* By the definition of $\lambda$, there exists a set $S_o \in \mathcal{O}$ such that

$$\frac{n+1}{2N-n+1} < \lambda = \frac{d(S_o)}{\#S_o}.$$

From Lemma 5.1, we have the inequality

$$\frac{\#S_o}{d(S_o)} \le \frac{2N-n+1}{n+1} - \frac{1}{n(n+1)}.$$

From this inequality and Corollary 4.1(II), we have the estimate

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) \le \frac{n+1}{\lambda} \le 2N-n+1-\frac{1}{n},$$

which is our theorem. □

When $n$ is even, we obtain a little better result than Theorem 5.1.

Theorem 5.2. *Suppose that* $N > n = 2m$ $(m \in \boldsymbol{N})$ *and we put*

$$\delta = \min\{1/m, (N-n)/(m+1)\}.$$

*If* $\lambda > (n+1)/(2N-n+1)$, *then*

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) \le 2N-n+1-\delta.$$

*Proof.* Suppose to the contrary that

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) > 2N-n+1-\delta.$$

Then, from this inequality and Corollary 4.1(II) we have that

$$\lambda < \frac{n+1}{2N-n+1-\delta}$$

and by the definition of $\lambda$, there exists a set $S_o \in \mathcal{O}$ such that $\lambda = d(S_o)/\#S_o$ so that

$$\frac{d(S_o)}{\#S_o} < \frac{n+1}{2N-n+1-\delta}. \tag{5.6}$$

From (5.6) and Lemma 2.2, we have the inequality

$$d(S_o) < \frac{n+1}{2N-n+1-\delta}\#S_o \le \frac{n+1}{2N-n+1-\delta}(N-n+d(S_o)),$$

so that

$$d(S_o) < \frac{(n+1)(N-n)}{2(N-n)-\delta} = \frac{n+1}{2-\delta/(N-n)}.$$

From this inequality we have the inequality

$$d(S_o) - m < \frac{n+1}{2-\delta/(N-n)} - m = \frac{1+m\delta/(N-n)}{2-\delta/(N-n)} \le 1$$

since $\delta \le (N-n)/(m+1)$, so that we have the inequality

$$d(S_o) \le m. \tag{5.7}$$

As $(n+1)/(2N-n+1) < \lambda = d(S_o)/\#S_o$, from (5.7) we obtain the inequality

$$\frac{\#S_o}{d(S_o)} \le \frac{2N-n+1}{n+1} - \frac{1}{d(S_o)(n+1)} \le \frac{2N-n+1}{n+1} - \frac{1}{m(n+1)} \tag{5.8}$$

as in the case of (5.2).

On the other hand, from (5.6) and (5.8)

$$\frac{2N-n+1-\delta}{n+1} < \frac{\#S_o}{d(S_o)} \le \frac{2N-n+1}{n+1} - \frac{1}{m(n+1)},$$

from which we have that $\delta > 1/m$, which is a contradiction to the choice of $\delta$. This implies that this theorem must hold. □

*Note* 5.1. $\delta = 1/(m+1)$ when $N-n=1$ and $\delta = 1/m$ otherwise.

(II) The case when $\lambda = (n+1)/(2N-n+1)$.

LEMMA 5.3. *Suppose that* $N > n = 2m$ $(m \in \mathbf{N})$ *and* $\lambda = (n+1)/(2N-n+1)$. *Let*

$$\mathcal{O}(\lambda) = \left\{ S \in \mathcal{O} \,\middle|\, \frac{d(S)}{\#S} = \lambda \right\},$$

*then, we have the followings.*

(a) $\mathcal{O}(\lambda)$ *is not empty.*

(b) *For* $S \in \mathcal{O}(\lambda)$, (i) $S$ *is not maximal*; (ii) $\#S \le N-m$ *and* (iii) $d(S) \le m$.

(c) *If* $S_1, S_2 \in \mathcal{O}(\lambda)$, *then* $S_1 \cup S_2 \in \mathcal{O}(\lambda)$.

(d) $\#\mathcal{O}(\lambda)$ *is finite.*

(e) *Put* $U_1 = \bigcup_{S \in \mathcal{O}(\lambda)} S$. *Then,* $U_1 \in \mathcal{O}(\lambda)$, *and if* $S \in \mathcal{O}(\lambda)$, *then* $S \subset U_1$.

(f) *Let*

$$\mathcal{O}_1(\lambda) = \{S \in \mathcal{O} \mid S - U_1 \ne \phi\}.$$

*Then,* $\mathcal{O}_1(\lambda)$ *is not empty and* $\#\{d(S)/\#S \mid S \in \mathcal{O}_1(\lambda)\} < \infty$.

(g) *Let*

$$\lambda_1 = \min_{S \in \mathcal{O}_1(\lambda)} d(S)/\#S.$$

*Then,* $\lambda < \lambda_1$.

(h) *Let*

$$\tau_1 = \begin{cases} \lambda & \text{if} \quad \boldsymbol{a} \in U_1; \\ \lambda_1 & \text{if} \quad \boldsymbol{a} \in X - U_1. \end{cases}$$

*Then,* $\tau_1 \in \mathcal{W}$.

*Proof.* (a) This is tivial from our assumption.

(b) (i) As $S \in \mathcal{O}(\lambda)$,

$$\#S - d(S) = \#S - \frac{n+1}{2N-n+1}\#S = \frac{2(N-n)}{2N-n+1}\#S. \tag{5.9}$$

If $S$ is maximal: $\#S = d(S) + N - n$,

$$\#S = (2N - n + 1)/2 = N - m + 1/2,$$

which is absurd. We have (i).

(ii) From (i) and (5.9), $\#S < N - m + 1/2$, so that $\#S \leq N - m$.

(iii) As $S \in \mathcal{O}(\lambda)$,

$$d(S) = \frac{n+1}{2N-n+1}\#S \leq \frac{2m+1}{2(N-m)+1}(N-m) < m + \frac{1}{2},$$

so that $d(S) \leq m$.

(c) From Lemma 2.1 and (b)(iii),

$$d(S_1 \cup S_2) \leq d(S_1) + d(S_2) - d(S_1 \cap S_2) \leq d(S_1) + d(S_2) \leq 2m = n,$$

so that $\#(S_1 \cup S_2) \leq N$ and $S_1 \cup S_2 \in \mathcal{O}$.

On the other hand, by the definition of $\lambda$

$$\lambda\#(S_1 \cap S_2) \leq d(S_1 \cap S_2).$$

From Lemma 2.1 and this inequality we have the inequality

$$\lambda \leq \frac{d(S_1 \cup S_2)}{\#(S_1 \cup S_2)} \leq \frac{d(S_1) + d(S_2) - d(S_1 \cap S_2)}{\#S_1 + \#S_2 - \#(S_1 \cap S_2)} \leq \lambda,$$

namely, $d(S_1 \cup S_2)/\#(S_1 \cup S_2) = \lambda$ and we have (c).

(d) We have only to prove this proposition when $\#X$ is not finite. Suppose to the contrary that $\#\mathcal{O}(\lambda)$ is not finite. Then, there are sets $S_1, S_2, \ldots$ such that

$$\mathcal{O}(\lambda) \supset \{S_1, S_2, \ldots, S_i, \ldots\}, \quad S_i \neq S_j \text{ if } i \neq j$$

and

$$\#\left\{\bigcup_{i=1}^{\infty} S_i\right\} = \infty.$$

There exists an integer $\nu$ satisfying

$$N + 1 < \#\left\{\bigcup_{i=1}^{\nu} S_i\right\}.$$

On the other hand, $\bigcup_{i=1}^{\nu} S_i \in \mathcal{O}(\lambda)$ by (c) and so by (b)(ii)

$$\#\left\{\bigcup_{i=1}^{\nu} S_i\right\} \le N - m.$$

From these two inequalities we obtain that $m + 1 < 0$, which is absurd. This implies that $\#\mathcal{O}(\lambda)$ is finite.

(e) From (c) and (d) we easily obtain this assertion.

(f) A subset $S$ of $X$ such that $\#S = N + 1$ belongs to $\mathcal{O}$ and $S - U_1 \neq \phi$ since $\#U_1 \le N - m$ by (b)(ii). From Lemma 2.3 we obtain that $\#\{d(S)/\#S \mid S \in \mathcal{O}_1(\lambda)\} < \infty$.

(g) By the definitions of $\lambda$ and $\lambda_1$, we have $\lambda \le \lambda_1$. Suppose that $\lambda = \lambda_1$. Then, there exists a set $S \in \mathcal{O}_1(\lambda)$ satisfying $d(S)/\#S = \lambda$, which means that $S \in \mathcal{O}(\lambda)$.

On the other hand, as $S \in \mathcal{O}_1(\lambda)$, $S - U_1 \neq \phi$ and $S \cup U_1 \in \mathcal{O}(\lambda)$ by (c). But, $S \cup U_1 \supsetneqq U_1$, which contradicts (e). This means that (g) must hold.

(h) The fact that $\tau_1 : X \to (0, 1]$ is trivial. For any $S \in \mathcal{O}$,

(i) When $S \subset U_1$, by the definition of $\lambda$,

$$\sum_{\boldsymbol{a} \in S} \tau_1(\boldsymbol{a}) = \lambda \# S \le (d(S)/\#S)\#S = d(S).$$

(ii) When $S - U_1 \neq \phi$, by the definition of $\lambda_1$ and (g)

$$\sum_{\boldsymbol{a} \in S} \tau_1(\boldsymbol{a}) \le \lambda_1 \# S \le (d(S)/\#S)\#S = d(S).$$

(i) and (ii) imply that $\tau_1 \in \mathcal{W}$. □

Theorem 5.3. *Suppose that $N > n = 2m$. If $\lambda = (n+1)/(2N - n + 1)$, then*

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) \le 2N - n + 1 - \frac{1}{2n}.$$

*Proof.* Suppose that

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) > 2N - n + 1 - \frac{1}{2n}. \tag{5.10}$$

From Lemma 5.3(h), Theorem 4.1 and (5.10), we have the inequality

$$\sum_{\boldsymbol{a}\in X}\tau_1(\boldsymbol{a})\delta_n(\boldsymbol{a},f) \le n+1 < \sum_{\boldsymbol{a}\in X}\frac{n+1}{2N-n+1}\delta_n(\boldsymbol{a},f) + \frac{n+1}{2N-n+1}\cdot\frac{1}{2n},$$

so that we have the inequality

$$\left(\lambda_1 - \frac{n+1}{2N-n+1}\right)\sum_{\boldsymbol{a}\in X-U_1}\delta_n(\boldsymbol{a},f) = \sum_{\boldsymbol{a}\in X-U_1}\left(\tau_1(\boldsymbol{a}) - \frac{n+1}{2N-n+1}\right)\delta_n(\boldsymbol{a},f) < \frac{n+1}{2N-n+1}\cdot\frac{1}{2n}. \tag{5.11}$$

On the other hand, as $\#U_1 \le N-m$ due to Lemma 5.3(b) and (e), we have the inequality

$$\sum_{\boldsymbol{a}\in X-U_1}\delta_n(\boldsymbol{a},f) > N-m+1-\frac{1}{2n} \tag{5.12}$$

from (5.10) and (1.1). Further, by the definition of $\lambda_1$, there is a set $S \in \mathcal{O}_1(\lambda)$ such that

$$\lambda_1 = d(S)/\#S > (n+1)/(2N-n+1)$$

from Lemma 5.3(g). From Lemma 5.1 we obtain the inequality

$$\frac{\#S}{d(S)} \le \frac{2N-n+1}{n+1} - \frac{1}{n(n+1)}$$

and we have the inequality

$$d(S)/\#S \ge (n+1)/(2N-n+1-1/n),$$

so that

$$\lambda_1 - \frac{n+1}{2N-n+1} \ge \frac{n+1}{2N-n+1-1/n} - \frac{n+1}{2N-n+1} = \frac{n+1}{n}\frac{1}{(2N-n+1-1/n)(2N-n+1)}. \tag{5.13}$$

From (5.11), (5.12) and (5.13), we have the inequality

$$\frac{n+1}{n}\frac{1}{(2N-n+1-1/n)(2N-n+1)}\left(N-m+1-\frac{1}{2n}\right) < \frac{n+1}{2N-n+1}\cdot\frac{1}{2n}$$

and so we have the inequality

$$N-m+1-\frac{1}{2n} < \frac{1}{2}\left(2N-n+1-\frac{1}{n}\right) = N-m+\frac{1}{2}-\frac{1}{2n},$$

which is absurd. This implies that (5.10) does not hold and we have this theorem. □

Corollary 5.1. *Suppose that* $N > n = 2m$. *If*

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) > 2N - n + 1 - \frac{1}{2n},$$

*then* $\lambda < (n+1)/(2N - n + 1)$.

*Proof.* As $1/(2n) \le \min\{1/m, (N-n)/(m+1)\}$, we have this corollary from Theorems 5.2 and 5.3 immediately. □

(III) The case when $\lambda < (n+1)/(2N - n + 1)$.

Theorem 5.4. *Suppose that* (i) *X is of type I and* $T_p$ *is not maximal or* (ii) *X is of type II. Then,*

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) \le 2N - n + 1 - \frac{1}{2n}. \tag{5.14}$$

*Proof.* When $T_p$ is not maximal, we have

$$\#T_p < d(T_p) + N - n < \frac{n+1}{2} + N - n = \frac{2N - n + 1}{2} \tag{5.15}$$

from Proposition 2.3(II)(i).

When $X$ is of type II, we have

$$\#T_{p+1} < \frac{2N - n + 1}{2} \tag{5.16}$$

from Propositions 3.5 and 3.8.

We have only to prove (5.14) when

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) \ge 2N - n + 1 - \frac{1}{2}.$$

Let $j = 1$ or 2. From (1.1) and (5.15) or (5.16), we have the inequality

$$\begin{aligned}\sum_{\boldsymbol{a}\in X - T_{p+j-1}}\delta_n(\boldsymbol{a}, f) &= \sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f) - \sum_{\boldsymbol{a}\in T_{p+j-1}}\delta_n(\boldsymbol{a}, f)\\ &\ge 2N - n + 1 - \frac{1}{2} - \#T_{p+j-1}\\ &> 2N - n + 1 - \frac{1}{2} - \frac{2N - n + 1}{2}\\ &= (2N - n)/2,\end{aligned}$$

so that we have the inequality

$$(5.17) \qquad \sum_{a \in X} w_j(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) - \sum_{a \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f)$$
$$= \sum_{\boldsymbol{a} \in X - T_{p+j-1}} (w_j(\boldsymbol{a}) - w(\boldsymbol{a}))\delta_n(\boldsymbol{a}, f)$$
$$= \left(\Lambda_j - \frac{1}{h}\right) \sum_{\boldsymbol{a} \in X - T_{p+j-1}} \delta_n(\boldsymbol{a}, f)$$
$$\geq \frac{1}{N(2N-n)} \cdot \frac{2N-n}{2} = \frac{1}{2N}$$

from Corollary 3.1 or Corollary 3.2.

On the other hand, we obtain the inequality

$$(5.18) \qquad \sum_{a \in X} w_j(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) - \sum_{a \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f)$$
$$\leq n + 1 - \sum_{\boldsymbol{a} \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f)$$
$$\leq \frac{1}{h}\left(2N - n + 1 - \sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f)\right)$$

from Theorem 4.1 and Lemma 5.2. As $N/n \leq h$ (Proposition 2.4(c)), from (5.17) and (5.18) we have the inequality

$$\frac{1}{2N} \cdot \frac{N}{n} \leq 2N - n + 1 - \sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f),$$

which reduces to (5.14). □

## 6. Extremal truncated defect relation

Let $f$, $X$, $N$ and $n$ etc. be as in Section 1 or 2. We use notations in Sections 1 through 4 freely. We consider holomorphic curves extremal for the truncated defect relation in this section. First of all, we give the following lemma, which plays a fundamental role in this section.

Lemma 6.1. *Suppose that $N > n$. The truncated defect relation for $f$ is extremal*:

$$(6.1) \qquad \sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) = 2N - n + 1$$

*if and only if the following two conditions* (a) *and* (b) *hold*:

(a) $(1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f)) = 0 \ (\boldsymbol{a} \in X)$;

(b) $\sum_{\boldsymbol{a} \in X} w(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) = n + 1$.

*Proof.* As $(1 - hw(\boldsymbol{a}))(1 - \delta_n(\boldsymbol{a}, f)) \geq 0$ for any $\boldsymbol{a} \in X$ by (1.1) and Proposition 2.4(a), from Corollary 4.1(I) and (1.2), we easily obtain this lemma from Lemma 5.2. □

From now on throughout this section we suppose that
(i) $N > n$ and that
(ii) (6.1) holds.
As is given in Section 1, let us remember the following set:

$$D^1 = \{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) = 1\}.$$

One of the main purposes of this section is to estimate $\# D^1$ under the conditions (i) and (ii).

First of all, we can rewrite Theorem 1.B as follows.

Proposition 6.1. (I) *If* $d(D^1) = n + 1$, *then*, $\# D^1 = 2N - n + 1$.
(II) *If* $d(D^1) = n$, *then* $\# D^1 = N$.

According to this proposition, we have only to estimate $\# D^1$ when $d(D^1) \leq n$. Then, $\# D^1 \leq N$. We have $D^1 \in \mathcal{O}$ if $D^1 \neq \phi$ and $\# D^+ \geq 2N - n + 2$.

Lemma 6.2. $\lambda \leq (n + 1)/(2N - n + 1)$.

*Proof.* From Corollary 4.1(II) and (6.1) we have the inequality $2N - n + 1 \leq (n + 1)/\lambda$, so that $\lambda \leq (n + 1)/(2N - n + 1)$. □

From this lemma we consider the extremal truncated defect relation in two cases.

(**I**) The case when $\lambda < (n + 1)/(2N - n + 1)$.

We note that $\lambda < (n + 1)/(2N - n + 1)$ when $n$ is even due to Corollary 5.1 under the conditions (i) and (ii).

Theorem 6.1. *Suppose that* (i) $N > n$, (ii) (6.1) *holds and* (iii) $d(D^1) \leq n$. *If* $\lambda < (n + 1)/(2N - n + 1)$, *in particular, if* $n$ *is even, then* $D^1 \neq \phi$ *and* $D^1$ *is maximal*:

$$\# D^1 = d(D^1) + N - n.$$

*Proof.* We apply Proposition 2.3(II). (a) We first note that

$$T_p \subset D^1. \tag{6.2}$$

In fact, from Note 2.2(b) $T_p = \{\boldsymbol{a} \in X \mid hw(\boldsymbol{a}) < 1\}$ and due to Lemma 6.1, $\delta_n(\boldsymbol{a}, f) = 1$ for $\boldsymbol{a} \in T_p$. This implies that $D^1 \neq \phi$.

(b) $X$ is of type I.

In fact, suppose to the contrary that $X$ is of type II. Then, the truncated defect relation for $f$ is not extremal from Theorem 5.4. This implies that $X$ must be of type I.

(c) $T_p$ is maximal.

In fact, suppose to the contrary that $T_p$ is not maximal. Then, the truncated defect relation for $f$ is not extremal from Theorem 5.4. This implies that $T_p$ must be maximal.

From (6.2) and Proposition 3.1, $D^1$ is maximal:

$$\#D^1 = d(D^1) + N - n.$$

We obtain our theorem. □

(**II**) The case when $\lambda = (n+1)/(2N-n+1)$.

Let

$$\mathcal{O}^+ = \{S \subset D^+ \mid 0 < \#S \leq N+1\}$$

and

$$\mathcal{W}^+ = \left\{\tau^+ : D^+ \to (0,1] \mid \forall S \in \mathcal{O}^+, \sum_{\boldsymbol{a} \in S} \tau^+(\boldsymbol{a}) \leq d(S)\right\}.$$

We apply the results in Sections 2, 3 and 4 to $D^+$ in place of $X$.

Proposition 6.2. (a) $\#\{d(S)/\#S \mid S \in \mathcal{O}^+\} < \infty$.

(b) *Let*

$$\lambda^+ = \min_{S \in \mathcal{O}^+} \frac{d(S)}{\#S}$$

*and let* $\tau^+ : D^+ \to (0,1]$ *such that* $\tau^+(\boldsymbol{a}) = \lambda^+$. *Then,* $\tau^+ \in \mathcal{W}^+$.

(c) $\lambda^+ = \lambda$.

*Proof.* (a) As $\mathcal{O}^+ \subset \mathcal{O}$, we have that

$$\{d(S)/\#S \mid S \in \mathcal{O}^+\} \subset \{d(S)/\#S \mid S \in \mathcal{O}\},$$

so that from Lemma 2.3 we have (a).

(b) As in Example 4.1(b), we obtain that $\tau^+ \in \mathcal{W}^+$.

(c) By the definitions of $\lambda$ and $\lambda^+$, we have that $\lambda \leq \lambda^+$. On the other hand, by applying Corollary 4.1(II) to $D^+$ and $\tau^+$ we obtain the inequality

$$2N - n + 1 = \sum_{\boldsymbol{a} \in D^+} \delta_n(\boldsymbol{a}, f) \leq \frac{n+1}{\lambda^+},$$

so that $\lambda^+ \leq (n+1)/(2N-n+1) = \lambda$. That is, we obtain (c). □

We note that from Corollary 5.1, $n$ is odd. Let $n = 2m-1$ for a positive integer $m$. Then $\lambda^+ = m/(N-m+1)$.

We put

$$\mathcal{F}_0 = \{S \in \mathcal{O}^+ \mid d(S)/\#S = m/(N-m+1)\}.$$

As $\lambda^+ = m/(N-m+1)$,

Proposition 6.3. *$\mathscr{F}_0$ is not empty.*

Proposition 6.4. *For any $S \in \mathscr{F}_0$,* (a) $d(S) \le m$; (b) $\#S \le N - m + 1$.

*Proof.* (a) As $d(S)/\#S = m/(N - m + 1)$, we have

$$d(S) = \frac{m}{N - m + 1}\#S \le \frac{m}{N - m + 1}(d(S) + N - n)$$

by Lemma 2.2, so that

$$(N - 2m + 1)d(S) \le m(N - 2m + 1),$$

which reduces to $d(S) \le m$.

(b) $\#S = \{(N - m + 1)/m\}d(S) \le N - m + 1$. □

Proposition 6.5. *For any element $S_0 \in \mathscr{F}_0$, $\{S \in \mathscr{F}_0 \mid S - S_0 \ne \phi\} \ne \phi$.*

*Proof.* We put

$$\mathscr{F}_1 = \{S \in \mathscr{O}^+ \mid S - S_0 \ne \phi\}.$$

(a) $\mathscr{F}_1$ is not empty.

(Proof.) Suppose to the contrary that for some $S_0 \in \mathscr{F}_0, \mathscr{F}_1$ is empty. Then, any $S \in \mathscr{O}^+$ is a subset of $S_0$, so that $\bigcup_{S \in \mathscr{O}^+} S = S_0$. Since

$$2N - n + 1 \le \#D^+ = \#\left(\bigcup_{S \in \mathscr{O}^+} S\right) = \#S_0 \le N - m + 1,$$

by Proposition 6.4(b), we have that $N + 1 \le m \le n$, which is absurd. Therefore, $\mathscr{F}_1$ is not empty.

(b) $\#\{d(S)/\#S \mid S \in \mathscr{F}_1\}$ is finite.

We have (b) from Lemma 2.3.

(c) We put $\lambda_1 = \min_{S \in \mathscr{F}_1} d(S)/\#S$. Then, $\lambda^+ = \lambda_1$.

(Proof.) By the definitions of $\lambda^+$ and $\lambda_1$, we have $\lambda^+ \le \lambda_1$. Suppose that $\lambda^+ < \lambda_1$. Let

$$\tau(\boldsymbol{a}) = \begin{cases} \lambda^+ & \text{if} \quad \boldsymbol{a} \in S_0; \\ \lambda_1 & \text{if} \quad \boldsymbol{a} \in D^+ - S_0. \end{cases}$$

Then, $\tau \in \mathscr{W}^+$.

This is because

1) The fact that $\tau : D^+ \to (0, 1]$ is trivial.

2) For any $S \in \mathscr{O}^+$,

(i) When $S \subset S_0$, by the definition of $\lambda^+$,

$$\sum_{\boldsymbol{a} \in S} \tau(\boldsymbol{a}) = \lambda^+ \#S \le (d(S)/\#S)\#S = d(S).$$

(ii) When $S - S_0 \neq \phi$, by the definition of $\lambda_1$

$$\sum_{\boldsymbol{a} \in S} \tau(\boldsymbol{a}) \leq \lambda_1 \# S \leq (d(S)/\# S)\# S = d(S).$$

1) and 2) imply that $\tau \in \mathscr{W}^+$. By Theorem 4.1 for $D^+$ and the assumption (6.1) we obtain the inequality

$$\sum_{\boldsymbol{a} \in D^+} \tau(\boldsymbol{a})\delta_n(\boldsymbol{a}, f) \leq n + 1 = \sum_{\boldsymbol{a} \in D^+} \lambda^+ \delta_n(\boldsymbol{a}, f),$$

from which we obtain the inequality

$$0 < (\lambda_1 - \lambda^+) \sum_{\boldsymbol{a} \in D^+ - S_0} \delta_n(\boldsymbol{a}, f) = \sum_{\boldsymbol{a} \in D^+} (\tau(\boldsymbol{a}) - \lambda^+)\delta_n(\boldsymbol{a}, f) \leq 0$$

since $D^+ \supsetneqq S_0$ and $\tau(\boldsymbol{a}) = \lambda_1 > \lambda^+$ $(\boldsymbol{a} \in D^+ - S_0)$. This is a contradiction. We have that $\lambda^+ = \lambda_1$.

Now, there exists an element $S_1 \in \mathscr{F}_1$ satisfying

$$d(S_1)/\# S_1 = \lambda_1 = \lambda^+.$$

This $S_1$ belongs to $\mathscr{F}_0$ and satisfies that $S_1 - S_0 \neq \phi$. □

Proposition 6.6. *Let $S_1$ and $S_2$ be in $\mathscr{F}_0$. If $S_1 \cap S_2 \neq \phi$, then $S_1 \cup S_2 \in \mathscr{F}_0$.*

*Proof.* As $S_1, S_2 \in \mathscr{F}_0$,

$$\frac{d(S_1)}{\# S_1} = \frac{d(S_2)}{\# S_2} = \lambda^+. \tag{6.3}$$

From Proposition 6.4(a) and Lemma 2.1 we obtain the inequality

$$d(S_1 \cup S_2) \leq d(S_1) + d(S_2) - d(S_1 \cap S_2) \leq 2m - 1 = n$$

as $d(S_1 \cap S_2) \geq 1$ by our assumption, which implies that $\#(S_1 \cup S_2) \leq N$. This implies that $S_1 \cup S_2 \in \mathscr{O}^+$. As $\#(S_1 \cap S_2) \leq \#(S_1 \cup S_2) \leq N$, $S_1 \cap S_2 \in \mathscr{O}^+$.

Next, by the definition of $\lambda^+$, we have the inequalities

$$\lambda^+ \leq \frac{d(S_1 \cup S_2)}{\#(S_1 \cup S_2)} \quad \text{and} \quad \lambda^+ \leq \frac{d(S_1 \cap S_2)}{\#(S_1 \cap S_2)}.$$

From (6.3), Lemma 2.1 and these inequalities we have the inequality

$$\lambda \leq \frac{d(S_1 \cup S_2)}{\#(S_1 \cup S_2)} \leq \frac{d(S_1) + d(S_2) - d(S_1 \cap S_2)}{\# S_1 + \# S_2 - \#(S_1 \cap S_2)} \leq \lambda,$$

which implies that $d(S_1 \cup S_2)/\#(S_1 \cup S_2) = \lambda^+$, so that $S_1 \cup S_2 \in \mathscr{F}_0$. □

Here we give a definition.

Definition 6.1 ([10, Definition 2.3]). Let $\mathscr{F}$ be a family of non-empty subsets of $D^+$.

We say that two sets $S_1, S_2 \in \mathscr{F}$ have a relation $S_1 \sim S_2$ if and only if either

(i) $S_1 \cap S_2 \neq \phi$ or

(ii) there exist sets $R_1, \dots, R_s \in \mathscr{F}$ such that

$$R_{j-1} \cap R_j \neq \phi \quad (j = 1, \dots, s+1), \quad R_0 = S_1, \quad R_{s+1} = S_2.$$

Lemma 6.2 ([10, Lemma 2.6]). *The relation "$\sim$" in $\mathscr{F}$ is an equivalence relation.*

We apply Definition 6.1 and Lemma 6.2 to $\mathscr{F} = \mathscr{F}_0$ and classify $\mathscr{F}_0$ by the equivalence relation "$\sim$". We put

$$\mathscr{F}_0/\sim = \{\mathscr{P}_1, \dots, \mathscr{P}_p\}; \quad M_k = \bigcup_{S \in \mathscr{P}_k} S \quad (k = 1, \dots, p),$$

where $p$ is a positive integer or $+\infty$.

Proposition 6.7. *For each $k$, $\#\mathscr{P}_k$ is finite.*

*Proof.* We have only to prove this proposition when $\#D^+$ is not finite.

(a) Let $S_0$ be any element of $\mathscr{P}_k$ and put

$$\mathscr{A} = \{S \in \mathscr{P}_k \mid S_0 \cap S \neq \phi\}.$$

Then, $\#\mathscr{A}$ is finite.

(Proof.) Suppose that $\#\mathscr{A}$ is infinite. Then, there are sets $S_1, S_2, \dots$ such that

$$\mathscr{A} \supset \{S_1, S_2, \dots, S_i, \dots\}, \quad S_i \neq S_j \text{ if } i \neq j$$

and

$$\#\left\{\bigcup_{i=1}^{\infty} S_i\right\} = \infty.$$

There exists an integer $v$ satisfying

$$N + 1 < \#\left\{\bigcup_{i=1}^{v} S_i\right\}. \tag{6.4}$$

On the other hand, $\bigcup_{i=0}^{v} S_i \in \mathscr{F}_0$ by Proposition 6.6 and so by Proposition 6.4(b)

$$\#\left\{\bigcup_{i=1}^{v} S_i\right\} \leq N - m + 1,$$

which is a contradiction to (6.4). $\#\mathscr{A}$ must be finite.

(b) Suppose that there exist $S_1, \dots, S_q \in \mathscr{P}_k$ such that $S_i \cap S_j = \phi$ if $1 \le i \ne j \le q$. Then, $q \le N - m + 1$.

(Proof.) As $S_1, \dots, S_q$ belong to the same class $\mathscr{P}_k$, from Definition 6.1 and Proposition 6.6, there exists a set $S$ in $\mathscr{F}_0$ such that $\bigcup_{i=1}^q S_i \subset S$, so that due to Proposition 6.4(b)

$$q \le \#\left(\bigcup_{i=1}^q S_i\right) \le \#S \le N - m + 1,$$

that is, $q \le N - m + 1$.

(c) Now, we prove our proposition. Suppose to the contrary that for some $k$, $\#\mathscr{P}_k$ is infinite. It is easy to see that there are an infinite number of elements

$$S_1, S_2, \dots, S_i, \dots; \quad S_i \cap S_j = \phi \quad (1 \le i \ne j)$$

of $\#\mathscr{P}_k$ from (a). This is a contradiction to (b). We have that $\#\mathscr{P}_k$ is finite. □

Proposition 6.8 (see [10, Lemma 3.2 and Proposition 4.5]). *The sets* $M_k$ $(k = 1, \dots, p)$ *have the following properties*:

(a) $M_k \in \mathscr{F}_0$ $(1 \le k \le p)$;

(b) $p \ge 2$;

(c) $M_k \cap M_\ell = \phi$ $(k \ne \ell)$ *and*

(d) $d(M_k) = m$, $\#M_k = N - m + 1$ $(1 \le k \le p)$.

*Proof.* (a) From Definition 6.1, Propositions 6.6 and 6.7 we have this assertion.

(b) As $M_1$ belongs to $\mathscr{F}_0$, we apply Proposition 6.5 to $M_1$. There exists an element $S \in \mathscr{F}_0$ such that $S - M_1 \ne \phi$. In this case, $S \cap M_1 = \phi$. In fact, if $S \cap M_1 \ne \phi$, then, by the definition of the relation "$\sim$", $S \sim M_1$. This means that $S \in \mathscr{P}_1$, and so $S \subset M_1$ by the definition of $M_1$, which implies that $S - M_1 = \phi$. This is a contradiction. We have that $p \ge 2$.

(c) This is trivial by the definition of $\{M_k \mid k = 1, \dots, p\}$.

(d) By Proposition 6.4(a), we have $d(M_k) \le m$. Suppose to the contrary that there exists at least one $k$ such that $d(M_k) \le m - 1$. For simplicity we may suppose without loss of generality that $k = 1$. Then, by Lemma 2.1

$$d(M_1 \cup M_2) \le d(M_1) + d(M_2) \le 2m - 1 = n,$$

so that $\#(M_1 \cup M_2) \le N$ and $M_1 \cup M_2 \in \mathscr{O}^+$. As $M_1, M_2 \in \mathscr{F}_0$,

$$\lambda^+ \le \frac{d(M_1 \cup M_2)}{\#(M_1 \cup M_2)} \le \frac{d(M_1) + d(M_2)}{\#M_1 + \#M_2} = \lambda^+,$$

and we have that $\lambda^+ = d(M_1 \cup M_2)/\#(M_1 \cup M_2)$, and so $M_1 \cup M_2 \in \mathscr{F}_0$. Then, as

$$M_1 \sim M_1 \cup M_2 \sim M_2,$$

which is a contradiction since $M_1 \in \mathscr{P}_1$ and $M_2 \in \mathscr{P}_2$. This implies that $d(M_k) = m$ $(k = 1, \dots, p)$ and we have $\# M_k = ((N - m + 1)/m) d(M_k) = N - m + 1$ $(k = 1, \dots, p)$. We have (d). □

We put

$$X_0 = \bigcup_{k=1}^{p} M_k.$$

PROPOSITION 6.9 (see [10, Lemma 3.3 and Proposition 4.6]). (a) $X_0 = D^+$; (b) *When* $\# D^+ < \infty$, $(N - m + 1) \mid \# D^+$ *and* $p = \# D^+/(N - m + 1)$ *and when* $\# D^+ = \infty$, *then* $p = \infty$.

*Proof.* (a) Suppose to the contrary that $X_0 \subsetneq D^+$. We put

$$\mathscr{F}_2 = \{S \in \mathcal{O}^+ \mid S - X_0 \neq \phi\}.$$

1) $\mathscr{F}_2$ is not empty.
(Proof.) For example, $S = \{\boldsymbol{a}\}$, where $\boldsymbol{a} \in D^+ - X_0$, belongs to $\mathscr{F}_2$.
2) We put $\lambda_2 = \min_{S \in \mathscr{F}_2} d(S)/\# S$. Then, $\lambda^+ < \lambda_2$.

(Proof.) First, we note that $\#\{d(S)/\# S \mid S \in \mathscr{F}_2\}$ is finite by Lemma 2.3. Now, by the definition of $\lambda^+$ and $\lambda_2$, we have $\lambda^+ \leq \lambda_2$. Suppose that $\lambda^+ = \lambda_2$. Then, there exists an element $S \in \mathscr{F}_2$ such that

$$d(S)/\# S = \lambda^+ = m/(N - m + 1),$$

which implies that $S \in \mathscr{F}_0$; that is to say, $S \subset X_0$, which is a contradiction. We have that $\lambda^+ < \lambda_2$.
3) We define

$$\tau_2(\boldsymbol{a}) = \begin{cases} \lambda^+ & \text{if} \quad \boldsymbol{a} \in X_0; \\ \lambda_2 & \text{if} \quad \boldsymbol{a} \in D^+ - X_0. \end{cases}$$

Then, $\tau_2 \in \mathscr{W}^+$. This is because
α) The fact that $\tau_2 : D^+ \to (0, 1]$ is trivial.
β) For any $S \in \mathcal{O}^+$,
(i) When $S \subset X_0$, by the definition of $\lambda^+$,

$$\sum_{\boldsymbol{a} \in S} \tau_2(\boldsymbol{a}) = \lambda^+ \# S \leq (d(S)/\# S) \# S = d(S).$$

(ii) When $S - X_0 \neq \phi : S \in \mathscr{F}_2$, by the definition of $\lambda_2$ and 2) of this proof,

$$\sum_{\boldsymbol{a} \in S} \tau_2(\boldsymbol{a}) \leq \lambda_2 \# S \leq (d(S)/\# S) \# S = d(S).$$

4) By Theorem 4.1 for $D^+$ and the assumption (6.1) we obtain the inequality

$$\sum_{\boldsymbol{a} \in D^+} \tau_2(\boldsymbol{a}) \delta_n(\boldsymbol{a}, f) \leq n + 1 = \sum_{\boldsymbol{a} \in D^+} \lambda^+ \delta_n(\boldsymbol{a}, f)$$

from which we obtain the inequality

$$0 < (\lambda_2 - \lambda^+) \sum_{\boldsymbol{a} \in D^+ - X_0} \delta_n(\boldsymbol{a}, f) = \sum_{\boldsymbol{a} \in D^+} (\tau_2(\boldsymbol{a}) - \lambda^+)\delta_n(\boldsymbol{a}, f) \le 0$$

since $D^+ \supsetneqq X_0$ and $\tau_2(\boldsymbol{a}) = \lambda_2 > \lambda^+$ $(\boldsymbol{a} \in D^+ - X_0)$. This is a contradiction. We have that $X_0 = D^+$.

(b) When $\# D^+ < +\infty$. As $(N - m + 1)p = \# D^+$ from Proposition 6.8(a) and (a) of this proposition, $(N - m + 1) \mid \# D^+$ and $p = \# D^+/(N - m + 1)$.

When $\# D^+ = +\infty$, we easily obtain that $p = +\infty$ from (a) of this proposition. □

Proposition 6.10 (see [10, Lemma 3.4 and Proposition 4.7]). *Any $m$ elements of $D^+$ are linearly independent.*

*Proof.* Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m$ be any $m$ elements of $D^+$.

Case 1. $M_k \cap \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\} = \phi$ for some $k$ $(1 \le k \le p)$.

We suppose without loss of generality that $k = 1$. As $d(M_1) = m$, there are $m$ linearly independent vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ in $M_1$ and as $\# M_1 = N - m + 1$,

$$\#(M_1 \cup \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\}) = N + 1.$$

In addition, $D^+$ is in $N$-subgeneral position, there are $n + 1 = 2m$ linearly independent vectors in $M_1 \cup \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\}$. This implies that $n + 1$ vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\, \boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ are linearly independent, and so $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m$ are linearly independent.

We note that if $\# D^+ = +\infty$, only this case occurs.

Case 2. $M_k \cap \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\} \ne \phi$ for any $k$ $(1 \le k \le p)$. (This case occurs only when $\# D^+ < +\infty$.)

($\alpha$) First we note that any $m$ elements $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$ of $M_k$ $(1 \le k \le p)$ are linearly independent.

In fact, there is an integer $\ell \ne k$ such that $M_\ell \cap M_k = \phi$, so that $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\} \cap M_\ell = \phi$. From Case 1, $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$ are linearly independent.

($\beta$) Now we suppose without loss of generality that

$$M_1 \ni \boldsymbol{b}_1, \dots, \boldsymbol{b}_\ell \quad \text{and} \quad M_1 \cap \{\boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m\} = \phi \quad (1 \le \ell \le m - 1).$$

Let $\{\boldsymbol{c}_{\ell+1}, \dots, \boldsymbol{c}_m\}$ be any $m - \ell$ vectors in $M_1 - \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_\ell\}$. Then the vectors $\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_\ell, \boldsymbol{c}_{\ell+1}, \dots, \boldsymbol{c}_m\}$ are linearly independent since any $m$ vectors in $M_1$ are linearly independent from ($\alpha$).

Let $\boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell$ be any $\ell$ vectors in $D^+ - (M_1 \cup \{\boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m\})$. Then,

$$\{\boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell, \boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m\} \cap M_1 = \phi, \tag{6.5}$$

and so from Case 1, $m$ vectors $\boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell$, $\boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m$ are linearly independent. As $\# M_1 = N - m + 1$, (6.5) implies that

$$\#(M_1 \cup \{\boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell, \boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m\}) = N + 1.$$

As $D^+$ is in $N$-subgeneral position, there are $n+1=2m$ linearly independent vectors in $M_1 \cup \{\boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell, \boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m\}$. By taking into consideration that $d(M_1)=m$, $2m$ vectors

$$\boldsymbol{b}_1, \dots, \boldsymbol{b}_\ell,\ \boldsymbol{c}_{\ell+1}, \dots, \boldsymbol{c}_m,\ \boldsymbol{d}_1, \dots, \boldsymbol{d}_\ell,\ \boldsymbol{b}_{\ell+1}, \dots, \boldsymbol{b}_m$$

are linearly independent, so that the vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m$ are linearly independent. □

Summarizing Propositions 6.3 through 6.10, we have the following theorem when $\lambda = (n+1)/(2N-n+1)$.

THEOREM 6.2 (see [10, Theorems 3.1(II) and 4.1(II)]). *Suppose that* (i) $N > n$ *and that* (ii) (6.1) *holds*:

$$\sum_{\boldsymbol{a} \in X} \delta_n(\boldsymbol{a}, f) = 2N - n + 1.$$

If $\lambda = (n+1)/(2N-n+1)$, then $n$ is odd (we put $n = 2m-1$) and the following properties of $D^+$ hold:

There are mutually disjoint subsets $M_1, \dots, M_p$ of $D^+$ satisfying

(a) $D^+ = \bigcup_{k=1}^{p} M_k$;

(b) $d(M_k) = m$, $\#M_k = N - m + 1$ $(1 \le k \le p)$;

(c) any $m$ elements of $D^+$ are linearly independent,

where if $\#D^+ < +\infty$, $(N-m+1) \mid \#D^+$ and $p = \#D_n^+/(N-m+1)$, and if $\#D^+ = +\infty$, $p = +\infty$.

*Remark* 6.1. By using the inequality (1.1), we are able to obtain the results for $\delta(\boldsymbol{a}, f)$ corresponding to those obtained for $\delta_n(\boldsymbol{a}, f)$ in Sections 4, 5 and 6.

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