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ON THE EULER CHARACTERISTICS OF REAL MILNOR FIBRES OF PARTIALLY PARALLELIZABLE MAPS OF $(\mathbf{R}^n, 0)$ TO $(\mathbf{R}^2, 0)$

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Abstract

We consider a real analytic map-germ $(f, g) : (\mathbf{R}^n, 0) \to (\mathbf{R}^2, 0)$ such that the fibres of f are simultaneously parallelizable. We call such a map a partially parallelizable map. We establish degree formulas for the following quantities:

$$\chi(\{f = \alpha\} \cap \{g = \delta\} \cap B_\varepsilon^n),$$

$$\chi(\{f = \alpha\} \cap \{g \geq \delta\} \cap B_\varepsilon^n) - \chi(\{f = \alpha\} \cap \{g \leq \delta\} \cap B_\varepsilon^n),$$

where (α, δ) is a regular value of (f, g) and $0 < |(\alpha, \delta)| \ll \varepsilon \ll 1$.

1. Introduction

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be an analytic function-germ with an isolated critical point at 0. The real Milnor fibres of f are the sets $f^{-1}(\delta) \cap B_\varepsilon^n$, where B_ε^n is the closed ball centered at the origin of radius ε and δ is a regular value of f such that $0 < |\delta| \ll \varepsilon \ll 1$. We will denote these fibres by $W_{f-\delta}^\varepsilon$. The Khimshiashvili formula [Kh] states that:

$$\chi(W_{f-\delta}^\varepsilon) = 1 - \operatorname{sign}(-\delta)^n \deg_0 \nabla f,$$

where ∇f is the gradient of f and $\deg_0 \nabla f$ is the topological degree of the mapping $\dfrac{\nabla f}{|\nabla f|} : \partial B_\varepsilon \to S_1^{n-1}$.

In [Fu], Fukui proved a relative version of this formula. He considered the map-germ $H : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, $x \mapsto (f(x), f_{x_2}(x), \ldots, f_{x_n}(x))$ and showed that, if 0 is isolated in $H^{-1}(0)$, then:

$$\chi(W_{f-\delta}^\varepsilon \cap \{x_1 \geq 0\}) - \chi(W_{f-\delta}^\varepsilon \cap \{x_1 \leq 0\}) = -\operatorname{sign}(-\delta)^n \deg_0 H,$$

where as above $\deg_0 H$ is the topological degree of the map $\dfrac{H}{|H|} : \partial B_\varepsilon \to S_1^{n-1}$. Here $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbf{R}^n$ and f_{x_i} denotes the partial derivative of f by x_i, $i = 1, \ldots, n$.

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In [Du1], we also gave a relative version of Khimshiashvili's formula. We restricted ourselves to the cases $n = 2, 4$ or 8 and we considered a function-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. Then we defined a mapping $H : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ in terms of f and g and proved that if 0 is isolated in $H^{-1}(0)$ then:

$$\chi(W^{\varepsilon}_{f-\delta} \cap \{g \geq \alpha\}) - \chi(W^{\varepsilon}_{f-\delta} \cap \{g \leq \alpha\}) = -\deg_0 H,$$

where α is a regular value of g such that $0 < |\alpha| \ll |\delta|$. These results were generalized by Fukui and Khovanskii [FK]. In that paper, the authors consider an analytic function-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ that satisfies the following Condition (P): there exist C^∞-vector fields $v_2, \ldots, v_n$ which span the tangent space at x to $g^{-1}(g(x))$, whenever x is a regular point of g, and ∇g, $v_2, \ldots, v_n$ agree with the orientation of $\mathbf{R}^n$. They define a mapping $H : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ by $H = (f, v_2 f, \ldots, v_n f)$ and they prove that if 0 is isolated in $H^{-1}(0)$, if the set of critical points of g does not intersect $W^{\varepsilon}_{f-\delta}$ and if $(\delta, 0)$ is a regular value of (f, g) then:

$$\chi(W^{\varepsilon}_{f-\delta} \cap \{g \geq 0\}) - \chi(W^{\varepsilon}_{f-\delta} \cap \{g \leq 0\}) = \operatorname{sign}(-\delta)^n \deg_0 H.$$

In this paper, we continue this work of computing Euler-Poincaré characteristics of real Milnor fibres. We are especially interested in partially parallelizable mappings of $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^2, 0)$. A partially parallelizable map is defined as follows.

Definition 1.1. Let U be an open subset of $\mathbf{R}^n$ and let $F = (f_1, \ldots, f_k) : U \to \mathbf{R}^k$, $1 < k < n$, be a C^∞-map. We say that F is partially parallelizable in U if there exist an integer l with $1 \leq l < k$ and l integers $i_1, \ldots, i_l$ in $\{1, \ldots, k\}$ with $i_1 < \cdots < i_l$ such that there are C^∞-vector fields $V_{i_1}, \ldots, V_{i_l}$ defined in U such that $V_{i_1}(x), \ldots, V_{i_l}(x)$ span the tangent space at x to $\tilde{F}^{-1}(\tilde{F}(x))$, where $\tilde{F}$ is the mapping defined in U by $\tilde{F}(x) = (f_{i_1}(x), \ldots, f_{i_l}(x))$ and x is a regular point of $\tilde{F}$.

We remark that this notion depends on the choice of coordinates.

Before describing our results, we need a notation: if $F : \mathbf{R}^n \to \mathbf{R}^k$ is a mapping then W^{ε}_F denotes the set $F^{-1}(0) \cap B^n_{\varepsilon}$, where B^n_{ε} is the ball of radius ε centered at the origin and $\partial W^{\varepsilon}_F$ is $F^{-1}(0) \cap S^{n-1}_{\varepsilon}$. In Section 2, we start our study of Euler characteristics of Milnor fibres of partially parallelizable mappings of $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^2, 0)$. More precisely, we consider an analytic function-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$, with an isolated critical point at 0, that satisfies Condition (P) described above. Let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be another function-germ. The mapping (f, g) is clearly partially parallelizable in a neighborhood of the origin. We define a mapping $k(f, g) : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ in terms of f and g and we assume that it has an isolated zero at the origin. We prove (Theorem 2.1) that:

if n is even: $\chi(W^{\varepsilon}_{(f, g-\delta)}) = 1 - \deg_0 \nabla f + \operatorname{sign}(\delta) \deg_0 k(f, g)$,

if n is odd: $\chi(W^{\varepsilon}_{(f, g-\delta)}) = 1 - \deg_0 k(f, g)$.

We also show that if n is even:

$$\chi(W^{\varepsilon}_{f-\alpha} \cap \{g \geq \delta\}) - \chi(W^{\varepsilon}_{f-\alpha} \cap \{g \leq \delta\}) = \deg_0 k(f,g),$$

where (α, δ) is an appropriate regular value of (f,g). Then we assume that g has an isolated critical point at the origin as well and we define another mapping $l(f,g)$. If it has an isolated zero at the origin, then we have (Theorem 2.9):

$$\text{if } n \text{ is even: } \chi(W^{\varepsilon}_{(f-\delta,g)}) = 1 - \deg_0 \nabla g - \operatorname{sign}(\delta) \deg_0 l(f,g),$$

$$\text{if } n \text{ is odd: } \chi(W^{\varepsilon}_{f-\delta} \cap \{g \geq 0\}) - \chi(W^{\varepsilon}_{f-\delta} \cap \{g \leq 0\}) = \deg_0 \nabla g + \operatorname{sign}(\delta) \deg_0 l(f,g),$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

In Section 3, we study partially parallelizable maps from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^3, 0)$ of the type (F, G, x_0). More precisely, we work in $\mathbf{R}^{1+n}$ equipped with the coordinate system $(x_0, x_1, \ldots, x_n)$ and we consider a function-germ $F : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}, 0)$ with an isolated critical point at 0. We assume that F satisfies the following Condition (P_{x_0}): there exist C^{∞}-vector fields $V_2, \ldots, V_n$ on $\mathbf{R}^{1+n}$ such that $V_2(p), \ldots, V_n(p)$ span the tangent space at p to $F^{-1}(F(p)) \cap x_0^{-1}(x_0(p))$ whenever p is a regular point of (F, x_0) and such that $(e_0, \nabla F(p), V_2(p), \ldots, V_n(p))$ agrees with the orientation of $\mathbf{R}^{1+n}$. Here e_0 is the vector $(1, 0, \ldots, 0)$. Let $G : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}, 0)$ be another function-germ. The map (F, G, x_0) is partially parallelizable in a neighborhood of the origin. We define two mappings $H(F,G)$ and $J(F,G) : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}^{1+n}, 0)$ in terms of F and G. We prove that if 0 is isolated in $H(F,G)^{-1}(0)$ and $(0, \delta, 0)$ is a regular value of (F, G, x_0) then (Theorem 3.1):

$$\deg_0 H(F,G) = \operatorname{sign}(-\delta)^n [\chi(W^{\varepsilon}_{(F,\,G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F,\,G-\delta)} \cap \{x_0 \leq 0\})],$$

where $0 < |\delta| \ll \varepsilon \ll 1$. We also prove that if 0 is isolated in $J(F,G)^{-1}(0)$ and $(0, \delta, 0)$ is a regular value of (F, G, x_0) then (Theorem 3.8):

$$\deg_0 J(F,G) = \operatorname{sign}(-\delta)^n [\chi(W^{\varepsilon}_{(F,\,G-\delta)}) - \chi(W^{\varepsilon}_{(G-\delta, x_0)})].$$

In Section 4, we apply these formulas to the case where F and G are one-parameter deformations of two function-germs f and $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. Denoting by f_t and g_t the deformations given by $f_t(x) = F(t,x)$ and $g_t(x) = G(t,x)$, we prove degree formulas for $\chi(W^{\varepsilon}_{(f_t, g_t)})$ and

$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}),$$

where $0 < |t| \ll \varepsilon \ll 1$ (Theorem 4.3). Then we finish our study of partially parallelizable mappings of $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^2, 0)$. Namely, we consider a partially parallelizable map (f,g) from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^2, 0)$ as in Section 2 and we construct the following deformations:

$$F(t,x) = f(x) - \gamma_1(t) \quad \text{and} \quad G(t,x) = g(x) - \gamma_2(t),$$

where $\gamma = (\gamma_1, \gamma_2) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is an analytic arc such that $\gamma(t) \neq 0$ if $t \neq 0$, $\gamma_1'(t) \neq 0$ if $t \neq 0$, and the image of γ consists of regular values of (f, g) (except the origin). Applying Theorem 4.3, we get formulas for $\chi(W^{\varepsilon}_{(f-\gamma_1(t), g-\gamma_2(t))})$ and

$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}),$$

where $0 < |t| \ll \varepsilon \ll 1$ (Theorem 4.3).

In Section 5, we present different cases where we can apply the results of the previous sections. There are two cases: when $n = 2, 4$ or 8 and when $\dfrac{\partial f}{\partial x_1} \geq 0$ and $\dfrac{\partial F}{\partial x_1} \geq 0$. We end the paper with an example in Section 6.

We will use the following notations: if $F = (F_1, \ldots, F_k) : \mathbf{R}^n \to \mathbf{R}^k$, $0 < k \leq n$, is a smooth mapping then DF is its Jacobian matrix and $\dfrac{\partial(F_1, \ldots, F_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}$ is the determinant of the following $k \times k$ minor of DF:

$$\begin{pmatrix} F_{1x_{i_1}} & \cdots & F_{1x_{i_k}} \\ \vdots & \ddots & \vdots \\ F_{kx_{i_1}} & \cdots & F_{kx_{i_k}} \end{pmatrix}.$$

2. First results on partially parallelizable mappings from $(\mathbf{R}^n, 0)$ to $(\mathbf{R}^2, 0)$

Let $(x_1, \ldots, x_n)$ be a coordinate system in $\mathbf{R}^n$ and let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. We assume that f satisfies Condition (P) introduced in [FK]: there exist C^∞-vector fields $v_2, \ldots, v_n$ on $\mathbf{R}^n$ such that $v_2(x), \ldots, v_n(x)$ span the tangent space at x to $f^{-1}(f(x))$, whenever x is a regular point of f, and such that the orientation of $(\nabla f(x), v_2(x), \ldots, v_n(x))$ agrees with the orientation of $\mathbf{R}^n$. Let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be another analytic function-germ.

2.1. Restriction of g to the regular levels of f

We define a mapping $k(f, g) : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ in the following way:

$$k(f, g) = (f, v_2 g, \ldots, v_n g).$$

We will prove the following theorem:

THEOREM 2.1. *If 0 is an isolated critical point of f and is isolated in $k(f, g)^{-1}(0)$, then we have*:

if n is even: $\chi(W^{\varepsilon}_{(f, g-\delta)}) = 1 - \deg_0 \nabla f + \operatorname{sign}(\delta) \deg_0 k(f, g)$,

if n is odd: $\chi(W^{\varepsilon}_{(f, g-\delta)}) = 1 - \deg_0 k(f, g)$, *where* $0 < |\delta| \ll \varepsilon \ll 1$. *Furthermore, if* n *is even, we also have*:

$$\chi(W^{\varepsilon}_{f-\alpha} \cap \{g \geq \delta\}) - \chi(W^{\varepsilon}_{f-\alpha} \cap \{g \leq \delta\}) = \deg_0 k(f,g),$$

where $0 \leq |\delta| \ll |\alpha| \ll \varepsilon \ll 1$ *and* (α, δ) *is a regular value of* (f,g).

To prove this theorem, we study the critical points of $g_{|W^{\varepsilon}_{f-\alpha}}$, where α is a regular value of f such that $0 < |\alpha| \ll \varepsilon$. More precisely, after a small perturbation of g, we can assume that they are all non-degenerate. To each of these critical points, we assign a sign: $+1$ if its Morse index is even and -1 if it is odd. Then we consider the algebraic sum of these points. This sum has two interpretations. On the one hand, by Morse theory, it is possible to relate it to Euler characteristics. On the other hand, since the critical points of $g_{|W^{\varepsilon}_{f-\alpha}}$ are the points of $f^{-1}(\alpha)$ where ∇f and ∇g are colinear, Condition (P) implies that these critical points are exactly the zeros of $f - \alpha, v_2 g, \dots, v_n g$. This enables us to prove that this sum is equal to $\deg_0 k(f,g)$.

From now on, we will assume that the hypothesis of Theorem 2.1 are fullfilled. For all $(i,j) \in \{1, \dots, n\}^2$, we will set $m_{ij} = \dfrac{\partial(g,f)}{\partial(x_i, x_j)}$.

Lemma 2.2. *For* $\delta \neq 0$ *sufficiently small*, $(0, \delta)$ *is a regular value of* (f,g).

Proof. Since f has an isolated critical point, $f^{-1}(0) \setminus \{0\}$ is smooth (or empty). By the Curve Selection Lemma, the critical points of $g_{|f^{-1}(0)\setminus\{0\}}$ lie in $g^{-1}(0)$. □

Lemma 2.3. *Let* p *be a regular point of* f. *The function* $g_{|f^{-1}(f(p))}$ *has a critical point at* p *if and only if* $v_i g(p) = 0$ *for all* $i \in \{2, \dots, n\}$.

Proof. If p is a regular point of f then $v_2(p), \dots, v_n(p)$ span the tangent space at $f^{-1}(f(p))$. Therefore $g_{|f^{-1}(f(p))}$ has a critical point at p if and only if $\langle v_i(p), \nabla g(p) \rangle = 0$ for all $i \in \{2, \dots, n\}$. □

Lemma 2.4. *The origin is an isolated singularity of* $f^{-1}(0) \cap g^{-1}(0)$ *if and only if* 0 *is isolated in* $k(f,g)^{-1}(0)$.

Proof. A point p, distinct from the origin, is in $k(f,g)^{-1}(0)$ if and only if $g_{|f^{-1}(0)\setminus\{0\}}$ has a critical point at p. But, as noticed above, such a point lies in $g^{-1}(0)$. □

Lemma 2.5. *Let* $\alpha \neq 0$ *be a regular value of* f. *Let* p *be a point in* $f^{-1}(\alpha)$. *The function* $g_{|f^{-1}(\alpha)}$ *has a non-degenerate critical point at* p *if and only if* $k(f,g)(p) = (\alpha, 0, \dots, 0)$ *and* $\det Dk(f,g)(p) \neq 0$. *Furthermore if* $\lambda(p)$ *is the Morse index of* $g_{|f^{-1}(\alpha)}$ *at* p *then we have*:

$$(-1)^{\lambda(p)} = \operatorname{sign}[\det Dk(f,g)(p)].$$

Proof. Since α is a regular value of f, there exists j such that $f_{x_j}(p) \neq 0$. Assume that $j = 1$. From [Sz, p349–350], p is a non-degenerate critical point of $g_{|f^{-1}(\alpha)}$ if and only if:

$$\det\begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}_{2 \leq i \leq n} \neq 0.$$

Furthermore, we have:

$$(-1)^{\lambda(p)} = (-1)^{n-1} \operatorname{sign}(f_{x_1}(p))^n \det\begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}.$$

We have to relate $\det\begin{bmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{bmatrix}$ to $\det\begin{bmatrix} \nabla f(p) \\ \nabla v_i g(p) \end{bmatrix}$. For $i \in \{2, \dots, n\}$, let $u_i(p)$ be the vector $(f_{x_i}(p), 0, \dots, 0, -f_{x_1}(p), 0, \dots, 0)$, where $-f_{x_1}(p)$ is the i-th coordinate. Then $(u_2(p), \dots, u_n(p))$ is a basis of $T_p f^{-1}(\alpha)$ and it is not difficult to see that:

$$\det(\nabla f(p), u_2(p), \dots, u_n(p)) = (-1)^{n-1} f_{x_1}(p)^{n-2}\left(\sum_{i=1}^{n} f_{x_i}^2(p)\right).$$

Hence there exists an $(n-1) \times (n-1)$ matrix $B(p)$ such that:

$$\begin{pmatrix} \nabla f(p) \\ u_i(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B(p) \end{pmatrix}\begin{pmatrix} \nabla f(p) \\ v_i(p) \end{pmatrix},$$

with $\operatorname{sign}[\det B(p)] = (-1)^{n-1} \operatorname{sign}[f_{x_1}(p)^{n-2}]$. Hence, for $i \in \{2, \dots, n\}$:

$$u_i(p) = \sum_{j=2}^{n} B_{ij}(p) v_j(p),$$

and:

$$m_{1i}(p) = u_i g(p) = \sum_{j=2}^{n} B_{ij}(p) v_j g(p).$$

Since $v_j g(p) = 0$, we have:

$$\nabla m_{1i}(p) = \sum_{j=2}^{n} B_{ij}(p) \nabla v_j g(p),$$

and:

$$\begin{pmatrix} \nabla f(p) \\ \nabla m_{1i}(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B(p) \end{pmatrix}\begin{pmatrix} f(p) \\ \nabla v_i g(p) \end{pmatrix}.$$

With this equality, it is easy to conclude. □

To prove Theorem 2.1, we will use Morse theory for manifolds with corners. The reader may refer to [Du3, Section 2] for a brief description of this theory. The following lemma deals with the critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$.

Lemma 2.6. *For all α and ε such that $0 < |\alpha| \ll \varepsilon \ll 1$, we have:*
- *at all correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ with $g > 0$, $\nabla g_{|f^{-1}(\alpha)}$ points outwards,*
- *at all correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ with $g < 0$, $\nabla g_{|f^{-1}(\alpha)}$ points inwards,*
- *there are no correct critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ in $g^{-1}(0)$.*

Proof. The proof is the same as in [Du1], Lemma 4.1. □

Lemma 2.7. *We can choose α small enough and we can perturb g into $\tilde{g}$ in such a way that $\tilde{g}_{|W^{\varepsilon}_{f-\alpha}}$ has only Morse critical points.*

Proof. Let $(x;t) = (x_1, \dots, x_n; t_1, \dots, t_n)$ be a coordinate system of $\mathbf{R}^{2n}$ and let $\bar{g}(x,t) = g(x) + t_1 x_1 + \cdots t_n x_n$. For $(i,j) \in \{1,\dots,n\}^2$, we define $\bar{m}_{ij}(x,t)$ by $\bar{m}_{ij}(x,t) = \dfrac{\partial(f,\bar{g})}{\partial(x_i,x_j)}(x,t)$. Notice that:

$$\bar{m}_{ij}(x,t) = m_{ij}(x,t) + f_{x_i}(x)t_j - t_i f_{x_j}(x).$$

Let Γ be defined by:

$$\Gamma = \{(x,t) \in \mathbf{R}^{2n} \mid \bar{m}_{ij}(x,t) = 0 \text{ for } (i,j) \in \{1,\dots,n\}^2\}.$$

At a point p, if f does not vanish then there exists $i \in \{1,\dots,n\}$ such that $f_{x_i}(p) \neq 0$. This implies that $\Gamma \backslash \{f = 0\}$ is a smooth manifold (or empty) of dimension $n+1$. Actually if p belongs to $\Gamma \backslash \{f = 0\}$, then one can assume that $f_{x_1}(p) \neq 0$. In this case, around p, Γ is defined by the vanishing of $\bar{m}_{12}, \dots, \bar{m}_{1n}$ and the gradient vector fields of these functions are linearly independent. Let π be the following mapping:

$$\begin{aligned} \pi : \Gamma \backslash \{f = 0\} &\to \mathbf{R}^{1+n} \\ (x,t) &\mapsto (f(x), t). \end{aligned}$$

By the Bertini-Sard theorem, we can choose (α, s) close to 0 in $\mathbf{R}^{1+n}$ such that π is regular at each point in $\pi^{-1}(\alpha, s)$ close to the origin. If we denote by $\tilde{g}$ the function defined by $\tilde{g}(x) = \bar{g}(x,s)$, this means that $\tilde{g}_{|f^{-1}(\alpha)}$ admits only Morse critical points in a neighborhood of the origin. □

Proof of Theorem 2.1. Let $\omega : \mathbf{R}^n \to \mathbf{R}$ be the distance function to the origin. Let $\varepsilon > 0$ be sufficiently small so that $g_{|f^{-1}(0)\backslash\{0\}}$ has no critical point in $f^{-1}(0)\backslash\{0\} \cap \{\omega < \varepsilon\}$. Let δ be such that $0 < |\delta| \ll \varepsilon \ll 1$. We want to express $\chi(W^{\varepsilon}_{(f,g-\delta)})$ in terms of $\deg_0 k(f,g)$. Let α be a regular value of f such that $0 < |\alpha| \ll |\delta|$ and the following properties are satisfied:

(1) $W^{\varepsilon}_{(f-\alpha,g-\delta)}$ is diffeomorphic to $W^{\varepsilon}_{(f,g-\delta)}$,

(2) the critical points of $g_{|f^{-1}(\alpha)\cap\{\omega<\varepsilon\}}$ lie in $\{|g| < \delta\} \cap \left\{\omega < \dfrac{\varepsilon}{2}\right\}$.

Hence the critical points of $g_{|\partial W^{\varepsilon}_{f-\alpha}}$ are correct. Furthermore by the previous lemmas, we can assume that $g_{|f^{-1}(\alpha)\cap\{\omega<\varepsilon\}}$ has only Morse critical points, that at the correct critical points of $g_{|W^{\varepsilon}_{f-\alpha}}$ lying in $\{g>0\}$ (resp. $\{g<0\}$), $\nabla g_{|W^{\varepsilon}_{f-\alpha}}$ points outwards (resp. inwards) and that there are no correct critical points of $g_{|W^{\varepsilon}_{f-\alpha}}$ in $g^{-1}(0)$.

We assume that $\delta>0$ and we apply Morse theory for manifolds with boundary to obtain:

$$\chi(W^{\varepsilon}_{f-\alpha}\cap\{g\geq-\delta\}, W^{\varepsilon}_{(f-\alpha,g+\delta)})=\sum_i(-1)^{\lambda(p_i)},$$

where $\{p_i\}$ is the set of critical points of $g_{|f^{-1}(\alpha)\cap\{\omega<\varepsilon\}}$, and:

$$\chi(W^{\varepsilon}_{f-\alpha}\cap\{g\leq-\delta\}, W^{\varepsilon}_{(f-\alpha,g+\delta)})=0.$$

Summing these equalities and using the Mayer-Vietoris sequence, we obtain:

$$\chi(W^{\varepsilon}_{f-\alpha})-\chi(W^{\varepsilon}_{(f-\alpha,g+\delta)})=\sum_i(-1)^{\lambda(p_i)}.$$

By Lemma 2.5, $\sum_i(-1)^{\lambda(p_i)}$ is equal to $\deg_0 k(f,g)$. By Khimshiashvili's formula, $\chi(W^{\varepsilon}_{f-\alpha})=1-\operatorname{sign}(-\alpha)^n\deg_0\nabla f$. Now by Proposition 1.1 in [FK], we know that $\deg_0\nabla f=0$ if n is odd. This gives the result for the fibre $W^{\varepsilon}_{(f,g-\delta)}$ with $\delta<0$. The formula for the fibre $W^{\varepsilon}_{(f,g-\delta)}$ with $\delta>0$ is obtained replacing g with $-g$. It remains to prove the third formula. Let δ be such that (α,δ) is a regular value of (f,g) and $0\leq|\delta|\ll|\alpha|\ll\varepsilon$. Since n is even, we have:

$$\chi(W^{\varepsilon}_{f-\alpha}\cap\{g\geq\delta\})-\chi(W^{\varepsilon}_{(f-\alpha,g-\delta)})=\sum_{i|g(p_i)>\delta}(-1)^{\lambda(p_i)},$$

$$\chi(W^{\varepsilon}_{f-\alpha}\cap\{g\leq\delta\})-\chi(W^{\varepsilon}_{(f-\alpha,g-\delta)})=-\sum_{i|g(p_i)<\delta}(-1)^{\lambda(p_i)}.$$

Making the difference and using Lemma 2.5, we obtain the result. □

COROLLARY 2.8. *If* 0 *is an isolated critical point of* f *and is isolated in* $k(f,g)^{-1}(0)$, *then one has*:

if n *is odd*: $\chi(\partial W^{\varepsilon}_{(f,g)})=2-2\deg_0 k(f,g)$,

if n *is even*: $\chi(\partial W^{\varepsilon}_f\cap\{g\geq0\})-\chi(\partial W^{\varepsilon}_f\cap\{g\leq0\})=2\deg_0 k(f,g)$.

Proof. The first point is easy. For the second assertion, see [Du1], Theorem 5.2. □

2.2. Restriction of f to the regular levels of g

Now we suppose that g also has an isolated critical point at the origin and we consider the mapping $l(f,g):(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$ defined by:

$$l(f,g) = (g, v_2 g, \dots, v_n g).$$

In [FK], Theorem 4.1, Fukui and Khovanskii prove that if 0 is isolated in $l(f,g)^{-1}(0)$ and if the set of critical points of f does not intersect $W^{\varepsilon}_{g-\delta}$ then:

$$\deg_0 l(f,g) = -\text{sign}(-\delta)^n \{\chi(W^{\varepsilon}_{g-\delta} \cap \{f \geq 0\}) - \chi(W^{\varepsilon}_{g-\delta} \cap \{f \leq 0\})\}.$$

In our situation the second condition is fullfilled because f has an isolated critical point. In the following theorem, we give another interpretation of this degree.

Theorem 2.9. *If f and g have an isolated critical point at the origin and 0 is isolated in $l(f,g)^{-1}(0)$ then*:

if n is even: $\chi(W^{\varepsilon}_{(f-\delta,g)}) = 1 - \deg_0 \nabla g - \text{sign}(\delta) \deg_0 l(f,g)$,

if n is odd: $\chi(W^{\varepsilon}_{f-\delta} \cap \{g \geq 0\}) - \chi(W^{\varepsilon}_{f-\delta} \cap \{g \leq 0\})$

$$= \deg_0 \nabla g + \text{sign}(\delta) \deg_0 l(f,g),$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

As in the previous theorem, Theorem 2.9 is proved giving two different interpretations of the algebraic sum of the critical points of a Morse perturbation of $f_{|W^{\varepsilon}_{g-\alpha}}$, where $0 < |\alpha| \ll \varepsilon$. We need some lemmas.

Lemma 2.10. *For $\delta \neq 0$ sufficiently small, $(\delta, 0)$ is a regular value of (f,g).*

Lemma 2.11. *Let p be a regular point of g. The function $f_{|g^{-1}(g(p))}$ has a critical point at p if and only if $v_i g(p) = 0$ for all $i \in \{2, \dots, n\}$.*

Proof. The function $f_{|g^{-1}(g(p))}$ has a critical point at p if and only if $\nabla f(p)$ and $\nabla g(p)$ are colinear. Since these two vectors are non zero, this is equivalent to the fact that $g_{|f^{-1}(f(p))}$ has a critical point at p. It is enough to use Lemma 2.3. □

Lemma 2.12. *The origin is an isolated singularity of $f^{-1}(0) \cap g^{-1}(0)$ if and only if 0 is isolated in $l(f,g)^{-1}(0)$.*

Lemma 2.13. *Let $\alpha \neq 0$ be a regular value of g. Let p be a point in $g^{-1}(\alpha)$. The function $f_{|g^{-1}(\alpha)}$ has a non-degenerate critical point at p if and only if $l(f,g)(p) = (\alpha, 0, \dots, 0)$ and $\det Dl(f,g)(p) \neq 0$. Furthermore if $\lambda(p)$ is the Morse index of $f_{|g^{-1}(\alpha)}$ at p and if $\mu(p)$ is the real number such that $\nabla f(p) = \mu(p) \nabla g(p)$ then we have*:

$$(-1)^{\lambda(p)} = (-1)^{n-1} \text{sign}[\mu(p)^n \det Dl(f,g)(p)].$$

Proof. Since α is a regular value of g, there exists j such that $g_{x_j}(p) \neq 0$. Assume that $j = 1$. From [Sz, p349–350], p is a non-degenerate critical point of $f_{|g^{-1}(\alpha)}$ if and only if:

$$\det\begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix}_{2\le i\le n} \neq 0.$$

Furthermore, we have:

$$(-1)^{\lambda(p)} = (-1)^{n-1}\operatorname{sign}\left(g_{x_1}(p)^n \det\begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix}\right).$$

Since $g_{x_1}(p) \neq 0$, $f_{x_1}(p)$ does not vanish for otherwise $\mu(p)$ and $\nabla f(p)$ would vanish as well. Then the computations done in Lemma 2.5 show that:

$$\operatorname{sign}\left(\det\begin{bmatrix} \nabla g(p) \\ -\nabla v_i g(p) \end{bmatrix}\right) = \operatorname{sign}\left(f_{x_1}(p)^{n-2} \det\begin{bmatrix} \nabla g(p) \\ -\nabla m_{1i}(p) \end{bmatrix}\right),$$

and it is easy to finish the proof. □

The following lemma deals with the critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$.

Lemma 2.14. *For all α and ε such that $0 < |\alpha| \ll \varepsilon \ll 1$, we have*:
- *at all correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ with $f > 0$, $\nabla f_{|g^{-1}(\alpha)}$ points outwards,*
- *at all correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ with $f < 0$, $\nabla f_{|g^{-1}(\alpha)}$ points inwards,*
- *there are no correct critical points of $f_{|\partial W^{\varepsilon}_{g-\alpha}}$ in $f^{-1}(0)$.*

Similarly, we have:

Lemma 2.15. *For ε sufficiently small, we have*:
- *at all correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ with $f > 0$, ∇f points outwards,*
- *at all correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ with $f < 0$, ∇f points inwards,*
- *there are no correct critical points of $f_{|S^{n-1}_{\varepsilon}}$ in $f^{-1}(0)$.*

Lemma 2.16. *We can choose α small enough and we can perturb g into $\tilde{g}$ in such a way that $f_{|W^{\varepsilon}_{\tilde{g}-\alpha}}$ has only Morse critical points.*

Proof. With the method of Lemma 2.7, we can prove that there exists a small perturbation $\tilde{g}$ of g such that $f_{|W^{\varepsilon}_{\tilde{g}-\alpha}}$ has only Morse critical points outside $\{f = 0\}$. But Lemma 2.2 states that $(0, \alpha)$ is a regular value of $(f, \tilde{g})$ for α small enough. □

Proof of Theorem 2.9. When n is even, the theorem is proved as in Theorem 2.1. So let us assume that n is odd. Let $\omega : \mathbf{R}^n \to \mathbf{R}$ be the distance function to the origin. Let $\varepsilon > 0$ be sufficiently small so that $f_{|g^{-1}(0)\setminus\{0\}}$ has no critical point in $g^{-1}(0)\setminus\{0\} \cap \{\omega < \varepsilon\}$. Let (δ, α) be a regular value of (f, g) such that:

(1) $0 < |\alpha| \ll |\delta| \ll \varepsilon$,
(2) the critical points of $f_{|g^{-1}(\alpha)}$ lie in $\{|f| < \delta\} \cap \left\{\omega < \frac{\varepsilon}{2}\right\}$,
(3) $\{g?0\} \cap W^{\varepsilon}_{f-\delta}$ is diffeomorphic to $\{g?\alpha\} \cap W^{\varepsilon}_{f-\delta}$, where $? \in \{\le, =, \ge\}$.

Thanks to the three previous lemmas, we can assume as in Theorem 2.1 that we are in a good situation to apply Morse theory for manifolds with corners. Let us assume that $\delta > 0$. By Morse Theory, we obtain:

$$(1) \qquad \chi(\{g \geq \alpha\} \cap \{f \geq \delta\} \cap B_\varepsilon^n) - \chi(\{g \geq \alpha\} \cap W_{f-\delta}^\varepsilon) = 0,$$

$$(2) \qquad \chi(\{g \geq \alpha\} \cap \{f \leq \delta\} \cap B_\varepsilon^n) - \chi(\{g \geq \alpha\} \cap W_{f-\delta}^\varepsilon) = \sum_{i \mid \mu(p_i) < 0} (-1)^{\lambda(p_i)},$$

$$(3) \qquad \chi(\{g \leq \alpha\} \cap \{f \geq \delta\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap W_{f-\delta}^\varepsilon) = 0,$$

$$(4) \qquad \chi(\{g \leq \alpha\} \cap \{f \leq \delta\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap W_{f-\delta}^\varepsilon) = -\deg_0 \nabla f + \sum_{i \mid \mu(p_i) > 0} (-1)^{\lambda(p_i)}.$$

In the equality (4), the terms $-\deg_0 \nabla f$ appears because we can perturb f in such a way that its critical points lie in $\{|g| \leq \alpha\} \cap \{|f| \leq \delta\}$. The combination $(1) + (2) - (3) - (4)$ together with the Mayer-Vietoris sequence gives:

$$\chi(\{g \geq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \geq \alpha\} \cap W_{f-\delta}^\varepsilon) + \chi(\{g \leq \alpha\} \cap W_{f-\delta}^\varepsilon) = -\sum_i \operatorname{sign} \mu(p_i)(-1)^{\lambda(p_i)} + \deg_0 \nabla f.$$

We have already seen that $\deg_0 \nabla f = 0$. Moreover, by the remark after Theorem 3.2 in [Du3], we have:

$$\chi(\{g \geq \alpha\} \cap B_\varepsilon^n) - \chi(\{g \leq \alpha\} \cap B_\varepsilon^n) = \deg_0 \nabla g.$$

Using Lemma 2.13, we find that:

$$\chi(\{g \geq 0\} \cap W_{f-\delta}^\varepsilon) - \chi(\{g \leq 0\} \cap W_{f-\delta}^\varepsilon) = \deg_0 \nabla g + \deg_0 l(f, g).$$

The proof for δ negative is obtained replacing f with $-f$. □

3. On partially parallelizable maps of $(\mathbf{R}^{n+1})$ to $(\mathbf{R}^3, 0)$ of the type (F, G, x_0)

Let $(x_0, x_1, \ldots, x_n)$ be a coordinate system in $\mathbf{R}^{1+n}$ and $F : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}, 0)$ be an analytic function-germ with an isolated critical point at the origin. We assume that F satisfies the following Condition (P_{x_0}): there exist C^∞-vector fields $V_2, \ldots, V_n$ on $\mathbf{R}^{1+n}$ such that $V_2(p), \ldots, V_n(p)$ span the tangent space at p to $F^{-1}(F(p)) \cap x_0^{-1}(x_0(p))$ whenever p is a regular point of (F, x_0) and such that $(e_0, \nabla F(p), V_2(p), \ldots, V_n(p))$ agrees with the orientation of $\mathbf{R}^{1+n}$. Here e_0 is the vector $(1, 0, \ldots, 0)$. Let $G : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}, 0)$ be another analytic function-germ. In our study, we will investigate the critical points of the restriction of x_0 to regular levels of (F, G) and the critical points of the restriction of F to regular levels of (G, x_0). These critical points are the points where ∇F, ∇G and e_0 are linearly dependent. This last condition is realized when the vectors $(F_{x_1}, \ldots, F_{x_n})$

and $(G_{x_1}, \dots, G_{x_n})$ are colinear. Since F satisfies Condition (P_{x_0}), its restriction to the levels of x_0 satisfies Condition (P), with the vectors $V_2, \dots, V_n$. Hence the critical points that we will study will be points where $V_2 G, \dots, V_n G$ vanish.

3.1. Restriction of x_0 to the levels of (F, G)

We define a mapping $H(F,G) : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}^{1+n}, 0)$ by: $H(F,G) = (F, G, V_2 G, \dots, V_n G)$. Our aim is to prove the following theorem:

THEOREM 3.1. *If F has an isolated critical point at the origin, 0 is isolated in $H(F,G)^{-1}(0)$ and $(0, \delta, 0)$ is a regular value of (F, G, x_0), then we have*:

$$\deg_0 H(F,G) = \operatorname{sign}(-\delta)^n [\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\})],$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

To establish this theorem, we use the same strategy as in Theorem 2.1 and 2.9: we count in two different ways the critical points of a perturbation of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$. Note that thanks to Condition (P_{x_0}) these critical points are exactly the roots of $F, G-\delta, V_2 G, \dots, V_n G$.

From now on, we will assume that the three assumptions of Theorem 3.1 are fullfilled. For all $(i,j) \in \{1, \dots, n\}^2$, we will set $M_{ij} = \dfrac{\partial(F,G)}{\partial(x_i, x_j)}$.

LEMMA 3.2. *For $\delta \neq 0$ sufficiently small, $(0, \delta)$ is a regular value of (F, G).*

LEMMA 3.3. *The origin is an isolated singularity of $F^{-1}(0) \cap G^{-1}(0)$.*

LEMMA 3.4. *Let $\delta \neq 0$ be sufficiently small so that $F^{-1}(0) \cap G^{-1}(\delta)$ is a smooth submanifold* (*or empty*) *of codimension 2 near the origin. Let p be a point in $F^{-1}(0) \cap G^{-1}(\delta)$. The function $x_{0|F^{-1}(0) \cap G^{-1}(\delta)}$ has a critical point at p if and only if $H(F,G)(p) = (0, \delta, 0, \dots, 0)$.*

Proof. The function $x_{0|F^{-1}(0) \cap G^{-1}(\delta)}$ has a critical point at p if and only if $F(p) = 0$, $G(p) = \delta$ and

$$\operatorname{rank} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ F_{x_0}(p) & F_{x_1}(p) & \cdots & F_{x_n}(p) \\ G_{x_0}(p) & G_{x_1}(p) & \cdots & G_{x_n}(p) \end{bmatrix} < 3.$$

First let us suppose that p is a critical point of $x_{0|F^{-1}(0) \cap G^{-1}(\delta)}$ and remark that necessarily $x_0(p) \neq 0$ because $(0, \delta, 0)$ is a regular value of (F, G, x_0). This implies that p is a regular point of (F, x_0) for the critical points of $x_{0|F^{-1}(0) \setminus \{0\}}$ lie in $\{x_0 = 0\}$ by the Curve Selection Lemma and, so, $(V_2(p), \dots, V_n(p))$ is a basis of $T_p[F^{-1}(0) \cap x_0^{-1}(x_0(p))]$. Since $\nabla G(p)$ belongs to the normal space at p to $F^{-1}(0) \cap x_0^{-1}(x_0(p))$, we find that for each $i \in \{2, \dots, n\}$, $\langle V_i(p), \nabla G(p) \rangle = 0$.

Let us show the inverse implication. Let p be such that $H(F,G)(p) = (0,\delta,0,\dots,0)$. If (F,x_0) is not regular at p then $x_0(p)=0$ and $(0,\delta,0)$ is not a regular value of (F,G,x_0), which is impossible. Hence $(V_2(p),\dots,V_n(p))$ is a basis of $T_p[F^{-1}(0)\cap x_0^{-1}(x_0(p))]$ and $\nabla G(p)$ is normal to this last tangent space. □

LEMMA 3.5. *Under the assumptions of Lemma* 3.4, $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ *has a non-degenerate critical point at* p *if and only if* $H(F,G)(p) = (0,\delta,0,\dots,0)$ *and* $\det DH(F,G)(p)\neq 0$. *Furthermore if* $\lambda(p)$ *is the Morse index of this function at* p *then*:

$$(-1)^{\lambda(p)} = (-1)^n \operatorname{sign}\left[\left(\frac{G(p)}{x_0(p)}\right)^n \det DH(F,G)(p)\right].$$

Proof. First observe that, since $(0,\delta)$ is a regular value of (F,G) and the M_{ij}'s, $i,j\in\{1,\dots,n\}$, vanish at p, there exists $k\in\{1,\dots,n\}$ such that $\frac{\partial(F,G)}{\partial(x_0,x_k)}(p)\neq 0$. Assume that $k=1$. This implies that $F_{x_1}(p)\neq 0$ for otherwise $G_{x_1}(p)\neq 0$ and $F_{x_j}(p)=0$ for $j\in\{2,\dots,n\}$, which means that p is not a regular point of (F,x_0) and $x_0(p)=0$.

From [Sz, p349–350], p is a Morse critical point of $x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ if and only if

$$\det\begin{bmatrix}\nabla F(p)\\ \nabla G(p)\\ \nabla N_i(p)\end{bmatrix}_{2\le i\le n}\neq 0,$$

where $N_i = \dfrac{\partial(x_0,F,G)}{\partial(x_0,x_1,x_i)} = M_{1i}$. Moreover, we have:

$$(-1)^{\lambda(p)} = \operatorname{sign}\left(\det\begin{bmatrix}\nabla F(p)\\ \nabla G(p)\\ \nabla M_{1i}(p)\end{bmatrix}\frac{\partial(F,G)}{\partial(x_0,x_1)}(p)^n\right).$$

Let us relate $\det(\nabla F(p),\nabla G(p),\nabla M_{1i}(p))$ to $\det(\nabla F(p),\nabla G(p),\nabla V_i G(p))$. For $i\in\{2,\dots,n\}$, let $U_i(p)$ be the vector:

$$(0,F_{x_i}(p),0,\dots,0,-F_{x_1}(p),0,\dots,0),$$

where $-F_{x_1}(p)$ is the $(i+1)$-th coordinate. Then $(U_2(p),\dots,U_n(p))$ is a basis $T_p[F^{-1}(0)\cap x_0^{-1}(x_0(p))]$ and

$$\det(e_0,\nabla F(p),U_2(p),\dots,U_n(p)) = (-1)^{n-1}F_{x_1}(p)^{n-2}\cdot\left(\sum_{i=1}^n F_{x_i}(p)^2\right).$$

Hence there exists an $(n-1)\times(n-1)$-matrix $B(p)$ such that:

$$\begin{pmatrix} e_0 \\ \nabla F(p) \\ U_i(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & B(p) \end{pmatrix} \begin{pmatrix} e_0 \\ \nabla F(p) \\ V_i(p) \end{pmatrix},$$

with $\mathrm{sign}[\det B(p)] = (-1)^{n-1}\,\mathrm{sign}[F_{x_1}(p)^{n-2}]$. As we proceeded in Lemma 2.5, we have:

$$\mathrm{sign}[\det(e_0, \nabla F(p), \nabla U_i G(p))] = (-1)^{n-1}\,\mathrm{sign}[F_{x_1}(p)^{n-2}\det(e_0, \nabla F(p), \nabla V_i G(p))].$$

Since e_0 is a linear combination of $\nabla F(p)$ and $\nabla G(p)$, it is easy to see that:

$$\begin{aligned} &\mathrm{sign}[\det(\nabla F(p), \nabla G(p), \nabla U_i G(p))] \\ &\quad = (-1)^{n-1}\,\mathrm{sign}[F_{x_1}(p)^{n-2}\det(\nabla F(p), \nabla G(p), \nabla V_i G(p))]. \end{aligned}$$

Using the fact that $U_i G(p) = -M_{1i}(p)$, we find that:

$$(-1)^{\lambda(p)} = \mathrm{sign}\left(\det\begin{bmatrix} \nabla F(p) \\ \nabla G(p) \\ \nabla V_i G(p) \end{bmatrix} \frac{\partial(F,G)}{\partial(x_0,x_1)}(p)^n F_{x_1}(p)^{n-2}\right).$$

It remains to study the sign of $\frac{\partial(F,G)}{\partial(x_0,x_1)}(p)$. By the Curve Selection Lemma, we can assume that p is on the image of an analytic arc $\gamma :]0,\nu[\to F^{-1}(0)$ such that $M_{ij}(\gamma(t)) = 0$ for $t \in]0,\nu[$ and $(i,j) \in \{1,\dots,n\}^2$. We have $\sum_{i=1}^n F_{x_i}(\gamma)\gamma_i' = 0$ since $F\circ\gamma = 0$ and $(G\circ\gamma)' = \sum_{i=1}^n G_{x_i}(\gamma)\gamma_i'$. Multiplying the first equality by G_{x_1}, the second by F_{x_1} and making the difference, we obtain:

$$F_{x_0}G_{x_1} - G_{x_0}F_{x_1} = -\frac{(G\circ\gamma)'}{\gamma_0'}F_{x_1}.$$

Hence if $\delta \neq 0$ is small enough, $\mathrm{sign}\left(\frac{\partial(F,G)}{\partial(x_0,x_1)}^n F_{x_1}^{n-2}\right) = -\mathrm{sign}\left(\frac{G}{x_0}\right)$ at p. □

The following lemma deals with the critical points of $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$.

LEMMA 3.6. *Assume that* $(0,\delta,0)$ *is a regular value of* (F,G,x_0) *for* δ *sufficiently small. Then, for* ε *and* δ *such that* $0 < |\delta| \ll \varepsilon \ll 1$:

- *the vector* $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ *points outwards at all correct critical points of* $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$ *with* $x_0 > 0$,
- *the vector* $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ *points inwards at all correct critical points of* $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$ *with* $x_0 < 0$,
- *there are no correct critical points of* $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$ *in* $\{x_0 = 0\}$.

Proof. The proof is the same as in [Du1], Lemma 4.1. □

LEMMA 3.7. *If for δ small enough, $(0,\delta,0)$ is a regular value of (F,G,x_0), then we can perturb G into $\tilde{G}$ in such a way that $x_{0|W^{\varepsilon}_{(F,\tilde{G}-\delta)}}$ has only Morse critical points in $W^{\varepsilon}_{(F,\tilde{G}-\delta)}\backslash\{x_0=0\}$.*

Proof. The proof is similar to the proofs of Lemma 2.7 and Lemma 4.2 in [Du1]. Let us describe it briefly. Let $(x_0,\dots,x_n;t_1,\dots,t_n)=(x;t)$ be a coordinate system of $\mathbf{R}^{2n+1}$ and let

$$\bar{G}(x,t)=G(x)+t_1x_1+\cdots+t_nx_n.$$

For $(i,j)\in\{1,\dots,n\}^2$, we define $\overline{M}_{ij}(x,t)$ by $M_{ij}(x,t)=\dfrac{\partial(F,\bar{G})}{\partial(x_i,x_j)}$. Note that $\overline{M}_{ij}(x,t)=M_{ij}(x)+F_{x_i}t_j-F_{x_j}t_i$. Let Γ be defined by:

$$\Gamma=\{(x,t)\in\mathbf{R}^{2n+1}\mid F(x)=0 \text{ and } \overline{M}_{ij}(x,t)=0 \text{ for } (i,j)\in\{1,\dots,n\}^2\}.$$

In the same way as in Lemma 2.7 and Lemma 4.2, we can prove that $\Gamma\backslash\{x_0=0\}$ is a smooth manifold (or empty) of dimension $n+1$. Then we conclude with the following mapping:

$$\begin{aligned}\pi:\Gamma\backslash\{x_0=0\}&\to\mathbf{R}^{1+n}\\(x,t)&\mapsto(\bar{G}(x,t),t).\end{aligned}$$

□

Proof of Theorem 3.1. Let $\omega:\mathbf{R}^{1+n}\to\mathbf{R}$ be the distance function to the origin. Because 0 is isolated in $H(F,G)^{-1}(0)$, $x_{0|F^{-1}(0)\cap G^{-1}(0)\backslash\{0\}}$ has no critical point and then, choosing δ sufficiently small, we can assume that $x_{0|F^{-1}(0)\cap G^{-1}(\delta)\cap\{\omega<\varepsilon\}}$ admits its critical points in $W^{\varepsilon/4}_{(F,G-\delta)}$. Thus the critical points of $x_{0|\partial W^{\varepsilon}_{(F,G-\delta)}}$ are correct. By Lemmas 3.6 and 3.7, we can suppose that $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ is a correct Morse function, that its critical points lie in $B_{\varepsilon/2}$, that at the correct critical points of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ lying in $\{x_0>0\}$ (resp. in $\{x_0<0\}$), $\nabla x_{0|F^{-1}(0)\cap G^{-1}(\delta)}$ points outwards (resp. inwards) and that there are no correct critical points of $x_{0|F_\delta}$ in $\{x_0=0\}$. Applying Morse Theory for manifolds with boundary, we find:

$$\chi(W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\geq 0\},W^{\varepsilon}_{(F,G-\delta,x_0)})=\sum_{i|x_0(p_i)>0}(-1)^{\lambda(p_i)},$$

where $\{p_i\}$ is the set of Morse critical points of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$. Similarly, we have:

$$\chi(W^{\varepsilon}_{(F,G-\delta)}\cap\{x_0\leq 0\},W^{\varepsilon}_{(F,G-\delta,x_0)})=(-1)^{n-1}\sum_{i|x_0(p_i)<0}(-1)^{\lambda(p_i)}.$$

By Lemma 3.4, p is a critical point of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ if and only if

$$H(F,G)(p)=(0,\delta,0,\dots,0).$$

Hence $H(F,G)^{-1}(0,\delta,0,\dots,0)$ is the set of critical points of $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$. Since $x_{0|W^{\varepsilon}_{(F,G-\delta)}}$ is a Morse function, $\det DH(F,G)(p)\neq 0$ for each p in

$H^{-1}(0,\delta,0,\dots,0)$ by Lemma 3.5. Hence $(0,\delta,0,\dots,0)$ is a regular value of $H(F,G)$ and

$$\deg_0 H = \sum_{p \in H^{-1}(0,\delta,0,\dots,0)} \operatorname{sign}[\det DH(F,G)(p)].$$

Combining this with the above equalities and Lemma 3.5, we obtain the required equality. □

3.2. Restriction of F to the levels of (G, x_0)

We define two maps $I(F,G)$ and $J(F,G) : (\mathbf{R}^{1+n},0) \to (\mathbf{R}^{1+n},0)$ by: $I(F,G) = (x_0, G, V_2G, \dots, V_nG)$ and $J(F,G) = (x_0F, G, V_2G, \dots, V_nG)$. Our aim is to prove the following theorem:

THEOREM 3.8. *If F has an isolated critical point at the origin, 0 is isolated in $J(F,G)^{-1}(0)$ and $(0,\delta,0)$ is a regular value then we have*:

$$\deg_0 J(F,G) = \operatorname{sign}(-\delta)^n \cdot [\chi(W^{\varepsilon}_{(F,G-\delta)}) - \chi(W^{\varepsilon}_{(G-\delta,x_0)})],$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

From now on, we will assume that the three assumptions of Theorem 3.8 are fullfilled. We keep the notations of the previous subsection: $M_{ij} = \dfrac{\partial(F,G)}{\partial(x_i,x_j)}$.

LEMMA 3.9. *The function $G_{|\{x_0=0\}}$ has an isolated critical point at the origin.*

Proof. Since $J(F,G)$ has an isolated zero at 0, the point $(0,0,0)$ is isolated in $I(F,G)^{-1}(0)$. This would not be the case if 0 in $\mathbf{R}^n$ was not an isolated critical point of $G_{|\{x_0=0\}}$. □

LEMMA 3.10. *Let $\delta \neq 0$ be sufficiently small so that $\{x_0 = 0\} \cap G^{-1}(\delta)$ is a smooth submanifold of codimension 2 (or empty) near the origin. Let s be a point in $\{x_0 = 0\} \cap G^{-1}(\delta)$. The function $F_{|\{x_0=0\}\cap G^{-1}(\delta)}$ has a critical point at s if and only if $I(F,G)(s) = (0,\delta,0,\dots,0)$.*

Proof. Since $(0,\delta,0)$ is a regular value of (F,G,x_0), we can apply the proof of Lemma 2.11. □

LEMMA 3.11. *Under the assumptions of Lemma 3.10, $F_{|G^{-1}(\delta)\cap x_0^{-1}(0)}$ has a non-degenerate critical point at s if and only if $I(F,G)(s) = (0,\delta,0,\dots,0)$ and* $\det DI(F,G)(s) \neq 0$. *Furthermore if $\mu(s)$ is the Morse index of this function at s then*:

$$(-1)^{\mu(s)} = (-1)^{n-1} \operatorname{sign}\left[\left(\frac{G(s)}{F(s)}\right)^n \det DI(F,G)(s)\right].$$

Proof. The proof is the same as in Lemmas 2.5, 2.13 and 3.5. We leave it to the reader. □

The following lemma deals with the critical points of $F_{|\partial W^{\varepsilon}_{(G-\delta,x_0)}}$.

LEMMA 3.12. *Assume that* $(0,\delta,0)$ *is a regular value of* (F,G,x_0) *for* δ *sufficiently small. Then, for* ε *such that* $0<|\delta|\ll\varepsilon\ll 1$:

- *the vector* $\nabla F_{|x_0^{-1}(0)\cap G^{-1}(\delta)}$ *points outwards at all correct critical points of* $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ *with* $F>0$,
- *the vector* $\nabla F_{|x_0^{-1}(0)\cap G^{-1}(\delta)}$ *points inwards at all correct critical points of* $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ *with* $F<0$,
- *there are no correct critical points of* $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$ *in* $F^{-1}(0)$.

LEMMA 3.13. *We can perturb* G *into* $\tilde{G}$ *in such a way that* $F_{|W^{\varepsilon}_{(\tilde{G}-\delta,x_0)}}$ *has only Morse critical point.*

Proof. The same method as in Lemma 2.16 can be applied, because we have assumed that $(0,\delta,0)$ is a regular value of (F,G,x_0). □

Proof of Theorem 3.8. It is easy to see that 0 is isolated in $I(F,G)^{-1}(0)$. Let us study the critical points of $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$. Thanks to Lemmas 3.12 and 3.13, we can assume that we are in a good situation to apply Morse theory. We have:

$$\chi(W^{\varepsilon}_{(G-\delta,x_0)}\cap\{F\geq 0\})-\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})=\sum_{j|F(s_j)>0}(-1)^{\mu(s_j)},$$

where $\{s_j\}$ is the set of Morse critical points of $F_{|W^{\varepsilon}_{(G-\delta,x_0)}}$. Similarly, we have:

$$\chi(W^{\varepsilon}_{(G-\delta,x_0)}\cap\{F\leq 0\})-\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})=(-1)^{n-1}\sum_{j|F(s_j)>0}(-1)^{\mu(s_j)}.$$

Hence, we get:

$$\chi(W^{\varepsilon}_{(G-\delta,x_0)})-\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})=\sum_{j|F(s_j)>0}(-1)^{\mu(s_j)}+(-1)^{n-1}\sum_{j|F(s_j)<0}(-1)^{\mu(s_j)}.$$

An application of Lemma 3.11 gives:

$$\chi(W^{\varepsilon}_{(G-\delta,x_0)})-\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})=-\text{sign}(-\delta)^n\sum_j\text{sign}[F(s_j)\det DI(F,G)(s_j)].$$

Similarly, we have:

$$\chi(W^{\varepsilon}_{(F,G-\delta)})-\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})=\text{sign}(-\delta)^n\sum_i\text{sign}[x_0(p_i)\det DH(F,G)(p_i)].$$

But the sets $\{p_i\}$ and $\{s_j\}$ are exactly the preimages of $(0,\delta,0,\dots,0)$ by $J(F,G)$. Furthermore, each p_i is a regular point of $J(F,G)$ and:

$$\text{sign}[\det DJ(F,G)(p_i)] = \text{sign}[x_0(p_i) \det DH(F,G)(p_i)].$$

Each s_j is a regular value of $J(F,G)$ as well and:

$$\text{sign}\big[\det DJ(F,G)(s_j)\big] = \text{sign}\big[F(s_j) \det DI(F,G)(s_j)\big].$$

With all these informations, it is easy to finish the proof. □

4. Application to 1-parameter families and to partially parallelizable mappings of $(\mathbf{R}^n,0)$ to $(\mathbf{R}^2,0)$

In this section, we apply the formulas of Section 3 when F and G are one-parameter deformations of two germs f and g. We use the same strategy as Fukui does in [Fu].

We consider two function-germs $F,G : (\mathbf{R}^{1+n},0) \to (\mathbf{R},0)$ such that F satisfies Condition (P_{x_0}). We also assume that the conditions of Theorem 3.8 are satisfied. Note that the fact that $J(F,G)$ has an isolated zero at the origin implies that $H(F,G)$ has an isolated zero at the origin as well. Let us define the function-germ $F_0 : (\mathbf{R}^n,0) \to (\mathbf{R},0)$ by $F_0(x_1,\dots,x_n) = F(0,x_1,\dots,x_n)$.

Lemma 4.1. *Assume that the function-germ F_0 has an isolated critical point at the origin. Then for δ sufficiently small, $(0,\delta,0)$ is a regular of (F,G,x_0). Let us suppose that $\delta > 0$, then for $0 < \delta \ll \varepsilon \ll 1$, we have*:

$$W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\} \simeq \partial W^{\varepsilon}_F \cap \{G \geq 0\} \cap \{x_0 \geq 0\} \simeq W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\},$$
$$W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\} \simeq \partial W^{\varepsilon}_F \cap \{G \geq 0\} \cap \{x_0 \leq 0\} \simeq W^{\varepsilon}_{(F,x_0+\delta)} \cap \{G \geq 0\},$$
$$W^{\varepsilon}_{(F,G+\delta)} \cap \{x_0 \geq 0\} \simeq \partial W^{\varepsilon}_F \cap \{G \leq 0\} \cap \{x_0 \geq 0\} \simeq W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \leq 0\},$$
$$W^{\varepsilon}_{(F,G+\delta)} \cap \{x_0 \leq 0\} \simeq \partial W^{\varepsilon}_F \cap \{G \leq 0\} \cap \{x_0 \leq 0\} \simeq W^{\varepsilon}_{(F,x_0+\delta)} \cap \{G \leq 0\},$$

where $\simeq$ means diffeomorphic to.

Proof. Let us prove the first line. It is an adaptation to our case of the deformation argument given by Milnor [Mi, Lemma 11.3]. We can construct a vector field v_1 on $W^{\varepsilon}_F \backslash \{G=0\}$ such that $\langle v_1(x), \nabla G(x)\rangle$ and $\langle v_1(x), x\rangle$ are both positive. Similarly there exists a vector field v_2 on $W^{\varepsilon}_{(F,x_0)} \backslash \{G=0\}$ such that $\langle v_2(x), \nabla G(x)\rangle$ and $\langle v_2(x), x\rangle$ are both positive. Using a collar, we can extend v_2 to a vector field $\tilde{v}_2$ defined in a neighborhood of $W^{\varepsilon}_{(F,x_0)} \backslash \{G=0\}$ in $W^{\varepsilon}_F \cap \{x_0 \geq 0\} \backslash \{G=0\}$ such that $\langle \tilde{v}_2(x), \nabla G(x)\rangle$ and $\langle \tilde{v}_2(x), x\rangle$ are positive. Gluing v_1 and $\tilde{v}_2$, we construct a new vector field w on $W^{\varepsilon}_F \cap \{x_0 \geq 0\} \backslash \{G=0\}$. The diffeomorphism between $W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}$ and $\partial W^{\varepsilon}_F \cap \{G \geq 0\} \cap \{x_0 \geq 0\}$ is obtained integrating the trajectories of w. Similarly $W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\}$ is diffeomorphic to $\partial W^{\varepsilon}_F \cap \{G \geq 0\} \cap \{x_0 \geq 0\}$ because, by Lemma 3.3, $F^{-1}(0) \cap G^{-1}(0)$ is smooth outside the origin. □

We want to compute $\chi(W^{\varepsilon}_{(F,G,x_0-\delta)})$. By the Mayer-Vietoris sequence, we know that:

$$\chi(W^{\varepsilon}_{(F,x_0-\delta)}) = \chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\}) + \chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \leq 0\}) - \chi(W^{\varepsilon}_{(F,G,x_0-\delta)}).$$

Hence, by Lemma 4.1, we find that if $\delta > 0$, then:

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F,G+\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F,x_0-\delta)}),$$

$$\chi(W^{\varepsilon}_{(F,G,x_0+\delta)}) = \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) + \chi(W^{\varepsilon}_{(F,G+\delta)} \cap \{x_0 \leq 0\}) - \chi(W^{\varepsilon}_{(F,x_0+\delta)}).$$

The Euler-Poincaré characteristic of $W^{\varepsilon}_{(F,x_0\pm\delta)}$ can be computed thanks to formulas established in [Fu], as explained in [Du2, Theorem 3.2]. More precisely, let $L(F) : (\mathbf{R}^{1+n}, 0) \to (\mathbf{R}^{1+n}, 0)$ be the mapping defined by $L(F) = (F, F_{x_1}, \dots, F_{x_n})$. If $L(F)$ and ∇F_0 have an isolated zero at the origin, then ∇F has an isolated zero at the origin and the following theorem explains how to compute $\chi(W^{\varepsilon}_{(F,x_0\pm\delta)})$.

Theorem 4.2. *Let δ and ε be such that $0 < |\delta| \ll \varepsilon \ll 1$. If n is even then*:

$$\chi(W^{\varepsilon}_{(F,x_0-\delta)}) = 1 - \deg_0 \nabla F_0.$$

If n is odd then:

$$\chi(W^{\varepsilon}_{(F,x_0-\delta)}) = 1 - \deg_0 \nabla F - \operatorname{sign}(\delta) \deg_0 L(F).$$

Proof. See [Fu] and [Du2]. □

At this point, we have assumed that:
(1) F has an isolated critical point at the origin,
(2) $J(F,G)$ has an isolated zero at the origin,
(3) $(0,\delta,0)$ is a regular value of (F,G,x_0),
(4) F_0 has an isolated critical point at the origin.

By the Curve Selection Lemma, Assumption (4) implies Assumption (3). Moreover, it means that 0 is isolated in $\{F = F_{x_1} = \cdots = F_{x_n} = x_0 = 0\}$. Since this last set is equal to $\{F = F_{x_1} = \cdots = F_{x_n} = 0\}$ near the origin thanks to the Curve Selection Lemma and the fact that $F^{-1}(0)$ has an isolated singularity, we have that (4) implies that 0 is isolated in $L(F)^{-1}(0)$. So, under Assumption (4), we can apply the above theorem.

It remains to compute $\chi(W^{\varepsilon}_{(F,G\pm\delta)} \cap \{x_0 ? 0\})$, $? \in \{\leq, \geq\}$. By the Mayer-Vietoris sequence, we have:

$$\chi(W^{\varepsilon}_{(F,G-\delta)}) = \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) - \chi(W^{\varepsilon}_{(F,G-\delta,x_0)}).$$

But Theorem 2.1 enables us to compute $\chi(W^{\varepsilon}_{(F,G-\delta)})$ and $\chi(W^{\varepsilon}_{(F,G-\delta,x_0)})$. Let $G_0 : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}, 0)$ be defined by $G_0(x_1, \dots, x_n) = G(0, x_1, \dots, x_n)$ and let us assume that it has an isolated critical point at the origin. Then using Theorem 3.8 and Khimshiashvili's formula, we find that:

$$\chi(W^{\varepsilon}_{(F,G-\delta)}) = 1 + \operatorname{sign}(-\delta)^n [\deg_0 J(F,G) - \deg_0 \nabla G_0].$$

Now observe that, since F satisfies Condition (P_{x_0}), F_0 satisfies Condition (P) of Section 2 with the vector fields $V_2^0, \dots, V_n^0$ given by:

$$V_i^0(x_1, \dots, x_n) = V_i(0, x_1, \dots, x_n).$$

Let $k(F_0, G_0) : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ be defined by:

$$k(F_0, G_0) = (F_0, V_2^0 G_0, \dots, V_n^0 G_0).$$

By Theorem 2.1, we have:

$$\text{if } n \text{ is even: } \chi(W^{\varepsilon}_{(F, G-\delta, x_0)}) = 1 - \deg_0 \nabla F_0 + \operatorname{sign}(\delta) \deg_0 k(F_0, G_0),$$

$$\text{if } n \text{ is odd: } \chi(W^{\varepsilon}_{(F, G-\delta, x_0)}) = 1 - \deg_0 k(F_0, G_0).$$

Let us focus first on the case n even. By Theorem 3.1, we have:

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\}) = \deg_0 H(F, G).$$

As explained above, we also have:

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\})$$
$$= 2 + \deg_0 J(F, G) - \deg_0 \nabla G_0 - \deg_0 \nabla F_0 + \operatorname{sign}(\delta) \deg_0 k(F_0, G_0).$$

This gives:

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) = 1 + \frac{1}{2}[\deg_0 J(F, G) - \deg_0 \nabla G_0 - \deg_0 \nabla F_0$$
$$+ \operatorname{sign}(\delta) \deg_0 k(F_0, G_0) + \deg_0 H(F, G)],$$

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\}) = 1 + \frac{1}{2}[\deg_0 J(F, G) - \deg_0 \nabla G_0 - \deg_0 \nabla F_0$$
$$+ \operatorname{sign}(\delta) \deg_0 k(F_0, G_0) - \deg_0 H(F, G)].$$

Collecting all these informations, we obtain:

$$\chi(W^{\varepsilon}_{(F, G, x_0-\delta)}) = 1 + \deg_0 J(F, G) - \deg_0 \nabla G_0 + \operatorname{sign}(\delta) \deg_0 H(F, G),$$
$$\chi(W^{\varepsilon}_{(F, x_0-\delta)} \cap \{G \geq 0\}) - \chi(W^{\varepsilon}_{(F, x_0-\delta)} \cap \{G \leq 0\}) = \deg_0 k(F_0, G_0).$$

If n is odd, we have:

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) - \chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\}) = -\operatorname{sign}(\delta) \deg_0 H(F, G),$$
$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) + \chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \leq 0\})$$
$$= 2 - \operatorname{sign}(\delta)[\deg_0 J(F, G) - \deg_0 \nabla G_0] - \deg_0 k(F_0, G_0).$$

This gives:

$$\chi(W^{\varepsilon}_{(F, G-\delta)} \cap \{x_0 \geq 0\}) = 1 - \frac{1}{2}[\operatorname{sign}(\delta)(\deg_0 J(F, G) - \deg_0 \nabla G_0$$
$$+ \deg_0 H(F, G)) + \deg_0 k(F_0, G_0)],$$

$$\chi(W^{\varepsilon}_{(F,G-\delta)} \cap \{x_0 \leq 0\}) = 1 - \frac{1}{2}[\text{sign}(\delta)(\deg_0 J(F,G) - \deg_0 \nabla G_0 - \deg_0 H(F,G)) + \deg_0 k(F_0,G_0)].$$

Finally we find:

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = 1 - \deg_0 k(F_0,G_0) - \deg_0 \nabla F - \text{sign}(\delta)\deg_0 L(F),$$
$$\chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \geq 0\}) - \chi(W^{\varepsilon}_{(F,x_0-\delta)} \cap \{G \leq 0\}) = -\deg_0 J(F,G) + \deg_0 \nabla G_0 - \text{sign}(\delta)\deg_0 H(F,G).$$

Here, we have to remark that:

$$\chi(W^{\varepsilon}_{(F,G,x_0-\delta)}) = \frac{1}{2}\chi(\partial W^{\varepsilon}_{(F,G,x_0-\delta)}) = \frac{1}{2}\chi(\partial W^{\varepsilon}_{(F,G,x_0)}) = 1 - \deg_0 k(F_0,G_0),$$

by Corollary 2.8. Hence, we get that $\deg_0 \nabla F = \deg_0 L(F) = 0$.

Let us reformulate these results in terms of one-parameter deformations of function-germs. Let $(x_1,\ldots,x_n)$ be a coordinate system of $\mathbf{R}^n$. Let $f:(\mathbf{R}^n,0) \to (\mathbf{R},0)$ be a function-germ with an isolated critical point at the origin. Let $g:(\mathbf{R}^n,0) \to (\mathbf{R},0)$ be a function-germ with an isolated critical point at the origin such that the mapping $k(f,g):(\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ has an isolated zero where $k(f,g)$ is defined as in Section 2. Let $(\lambda,x_1,\ldots,x_n)$ be a coordinate system in $\mathbf{R}^{1+n}$ and let $F:(\mathbf{R}^{1+n},0) \to (\mathbf{R},0)$ (resp. $G:(\mathbf{R}^{1+n},0) \to (\mathbf{R},0)$) be a one-parameter deformation of f (resp g), i.e. $F(0,x) = f(x)$ (resp. $G(0,x) = g(x)$). We will use the notations $f_t(x) = F(t,x)$ and $g_t(x) = G(t,x)$. We assume that:

(1) F has an isolated critical point at the origin,

(2) the mapping $J(F,G)$ has an isolated zero at the origin,

(3) F satisfies Condition (P_λ) (which implies that f satisfies Condition (P)).

We note that F_0 and G_0 have an isolated critical point because $F_0 = f$ and $G_0 = g$. So we are in situation to apply the above process.

THEOREM 4.3. *For t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have*:
– *if n is odd*:

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 - \deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = -\deg_0 J(F,G) + \deg_0 \nabla g - \text{sign}(t)\deg_0 H(F,G),$$

– *if n is even*:

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 + \deg_0 J(F,G) - \deg_0 \nabla g + \text{sign}(t)\deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = \deg_0 k(f,g).$$

Let us apply Theorem 4.3 when (f, g) is a partially parallelizable map. More precisely, let us assume that f satisfies Condition (P) and let us consider the following deformations of f and g:

$$F(\lambda, x) = f(x) - \gamma_1(\lambda) \quad \text{and} \quad G(\lambda, x) = g(x) - \gamma_2(\lambda),$$

where $\gamma = (\gamma_1, \gamma_2) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is an analytic arc such that $\gamma(t) \neq 0$ if $t \neq 0$ and $\gamma_1'(t) \neq 0$ if $t \neq 0$. With this last condition, the function F has an isolated critical point at the origin. Furthermore F satisfies Condition (P_λ) with $V_i(\lambda, x) = v_i(x)$ for $i = 2, \ldots, n$. Let us denote by $\mathrm{Disc}(f, g)$ the discriminant of the mapping (f, g). The following lemma tells us when the points in the image of γ are regular value of (f, g) near the origin.

Lemma 4.4. *The origin* $(0, 0)$ *is isolated in* $H(F, G)^{-1}(0)$ *if and only if* 0 *is isolated in* $\mathrm{Disc}(f, g) \cap \gamma(I)$, *where* I *is a small open interval in* $\mathbf{R}$ *containing* 0.

Proof. The point $(0, 0)$ is isolated in $H(F, G)^{-1}(0)$ if and only if for all $(t, x) \neq (0, 0)$ such that $F(t, x) = G(t, x) = 0$, there exists $i \in \{2, \ldots, n\}$ such that $v_i G(t, x) \neq 0$. Let us remark that if $x \neq 0$ is such that $F(0, x) = G(0, x) = 0$ then $v_i G(0, x) \neq 0$ for some i in $\{2, \ldots, n\}$ because $f^{-1}(0) \cap g^{-1}(0)$ has an isolated singularity. Therefore the point $(0, 0)$ is isolated in $H(F, G)^{-1}(0)$ if and only if for all (t, x) with $t \neq 0$ such that $F(t, x) = G(t, x) = 0$ there exists $i \in \{2, \ldots, n\}$ such that $v_i G(t, x) \neq 0$. This is equivalent to the fact that for all $t \neq 0$ and for all x such that $f(x) = \gamma_1(t)$ and $g(x) = \gamma_2(t)$, $\nabla f(x)$ and $\nabla g(x)$ are not colinear. □

Theorem 4.3 can be restated in this situation.

Theorem 4.5. *Assume that* f *and* g *have an isolated singularity and that* $\gamma_1'(t) \neq 0$ *if* $t \neq 0$. *Assume that* $J(F, G)$ *and* $k(f, g)$ *have an isolated zero at the origin then for* t *and* ε *with* $0 < |t| \ll \varepsilon \ll 1$, *we have*:

– *if* n *is odd*:

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t), g-\gamma_2(t))}) = 1 - \deg_0 k(f, g),$$

$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\})$$
$$= -\deg_0 J(F, G) + \deg_0 \nabla g - \mathrm{sign}(t) \deg_0 H(F, G),$$

– *if* n *is even*:

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t), g-\gamma_2(t))}) = 1 + \deg_0 J(F, G) - \deg_0 \nabla g + \mathrm{sign}(t) \deg_0 H(F, G),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = \deg_0 k(f, g).$$

Let us examine the situation when $\lambda_1(t) = t$ and $\lambda_2(t) = 0$. In this case, we can check that $\deg_0 J(F, G) = 0$ and that $\deg_0 H = -\deg_0 l(f, g)$, where $l(f, g)$ is defined in Section 2. Hence, we recover the results of Theorem 2.9.

5. Explicit formulas

In this section, we present some situations where Conditions (P) and (P_{x_0}) are satisfied.

5.1. Case $n = 2$, 4 or 8

As explained in [FK], when $n = 2$, 4 or 8, Condition (P) is satisfied for any function-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. If ∂_{x_i} denotes the vector $e_i = (0, \dots, 0, 1, 0 \dots, 0)$ where 1 is the i-th coordinate, then the vectors $v_2, \dots, v_n$ are given by, if $n = 2$:

$$v_2 = -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2},$$

if $n = 4$:

$$\begin{aligned}
v_2 &= -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2} - f_{x_4}\partial_{x_3} + f_{x_3}\partial_{x_4},\\
v_3 &= -f_{x_3}\partial_{x_1} + f_{x_4}\partial_{x_2} + f_{x_1}\partial_{x_3} - f_{x_2}\partial_{x_4},\\
v_4 &= -f_{x_4}\partial_{x_1} - f_{x_3}\partial_{x_2} + f_{x_2}\partial_{x_3} + f_{x_1}\partial_{x_4},
\end{aligned}$$

if $n = 8$:

$$\begin{aligned}
v_2 &= -f_{x_2}\partial_{x_1} + f_{x_1}\partial_{x_2} - f_{x_4}\partial_{x_3} + f_{x_3}\partial_{x_4} - f_{x_6}\partial_{x_5} + f_{x_5}\partial_{x_6} + f_{x_8}\partial_{x_7} - f_{x_7}\partial_{x_8},\\
v_3 &= -f_{x_3}\partial_{x_1} + f_{x_4}\partial_{x_2} + f_{x_1}\partial_{x_3} - f_{x_2}\partial_{x_4} - f_{x_7}\partial_{x_5} - f_{x_8}\partial_{x_6} + f_{x_5}\partial_{x_7} + f_{x_6}\partial_{x_8},\\
v_4 &= -f_{x_4}\partial_{x_1} - f_{x_3}\partial_{x_2} + f_{x_2}\partial_{x_3} + f_{x_1}\partial_{x_4} - f_{x_8}\partial_{x_5} + f_{x_7}\partial_{x_6} - f_{x_6}\partial_{x_7} - f_{x_5}\partial_{x_8},\\
v_5 &= -f_{x_5}\partial_{x_1} + f_{x_6}\partial_{x_2} + f_{x_7}\partial_{x_3} + f_{x_8}\partial_{x_4} + f_{x_1}\partial_{x_5} - f_{x_2}\partial_{x_6} - f_{x_3}\partial_{x_7} - f_{x_4}\partial_{x_8},\\
v_6 &= -f_{x_6}\partial_{x_1} - f_{x_5}\partial_{x_2} + f_{x_8}\partial_{x_3} - f_{x_7}\partial_{x_4} + f_{x_2}\partial_{x_5} + f_{x_1}\partial_{x_6} + f_{x_4}\partial_{x_7} - f_{x_3}\partial_{x_8},\\
v_7 &= -f_{x_7}\partial_{x_1} - f_{x_8}\partial_{x_2} - f_{x_5}\partial_{x_3} + f_{x_6}\partial_{x_4} + f_{x_3}\partial_{x_5} - f_{x_4}\partial_{x_6} + f_{x_1}\partial_{x_7} + f_{x_2}\partial_{x_8},\\
v_8 &= -f_{x_8}\partial_{x_1} + f_{x_7}\partial_{x_2} - f_{x_6}\partial_{x_3} - f_{x_5}\partial_{x_4} + f_{x_4}\partial_{x_5} + f_{x_3}\partial_{x_6} - f_{x_2}\partial_{x_7} + f_{x_1}\partial_{x_8}.
\end{aligned}$$

Condition (P_{x_0}) is also fullfilled, the vectors V_i being given by, if $n = 2$:

$$V_2 = -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2},$$

if $n = 4$:

$$\begin{aligned}
V_2 &= -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2} - F_{x_4}\partial_{x_3} + F_{x_3}\partial_{x_4},\\
V_3 &= -F_{x_3}\partial_{x_1} + F_{x_4}\partial_{x_2} + F_{x_1}\partial_{x_3} - F_{x_2}\partial_{x_4},\\
V_4 &= -F_{x_4}\partial_{x_1} - F_{x_3}\partial_{x_2} + F_{x_2}\partial_{x_3} + F_{x_1}\partial_{x_4},
\end{aligned}$$

if $n = 8$:

$$\begin{aligned}
V_2 &= -F_{x_2}\partial_{x_1} + F_{x_1}\partial_{x_2} - F_{x_4}\partial_{x_3} + F_{x_3}\partial_{x_4} - F_{x_6}\partial_{x_5} + F_{x_5}\partial_{x_6} + F_{x_8}\partial_{x_7} - F_{x_7}\partial_{x_8},\\
V_3 &= -F_{x_3}\partial_{x_1} + F_{x_4}\partial_{x_2} + F_{x_1}\partial_{x_3} - F_{x_2}\partial_{x_4} - F_{x_7}\partial_{x_5} - F_{x_8}\partial_{x_6} + F_{x_5}\partial_{x_7} + F_{x_6}\partial_{x_8},\\
V_4 &= -F_{x_4}\partial_{x_1} - F_{x_3}\partial_{x_2} + F_{x_2}\partial_{x_3} + F_{x_1}\partial_{x_4} - F_{x_8}\partial_{x_5} + F_{x_7}\partial_{x_6} - F_{x_6}\partial_{x_7} - F_{x_5}\partial_{x_8},\\
V_5 &= -F_{x_5}\partial_{x_1} + F_{x_6}\partial_{x_2} + F_{x_7}\partial_{x_3} + F_{x_8}\partial_{x_4} + F_{x_1}\partial_{x_5} - F_{x_2}\partial_{x_6} - F_{x_3}\partial_{x_7} - F_{x_4}\partial_{x_8},
\end{aligned}$$

$$V_6 = -F_{x_6}\partial_{x_1} - F_{x_5}\partial_{x_2} + F_{x_8}\partial_{x_3} - F_{x_7}\partial_{x_4} + F_{x_2}\partial_{x_5} + F_{x_1}\partial_{x_6} + F_{x_4}\partial_{x_7} - F_{x_3}\partial_{x_8},$$
$$V_7 = -F_{x_7}\partial_{x_1} - F_{x_8}\partial_{x_2} - F_{x_5}\partial_{x_3} + F_{x_6}\partial_{x_4} + F_{x_3}\partial_{x_5} - F_{x_4}\partial_{x_6} + F_{x_1}\partial_{x_7} + F_{x_2}\partial_{x_8},$$
$$V_8 = -F_{x_8}\partial_{x_1} + F_{x_7}\partial_{x_2} - F_{x_6}\partial_{x_3} - F_{x_5}\partial_{x_4} + F_{x_4}\partial_{x_5} + F_{x_3}\partial_{x_6} - F_{x_2}\partial_{x_7} + F_{x_1}\partial_{x_8}.$$

So all the results of Section 2, Section 3 and Section 4 can be applied. Note also that the vector fields v_i and V_i are analytic.

5.2. Case $f_{x_1} \geq 0$ and $F_{x_1} \geq 0$

Condition (P) is satisfied for a function-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ such that $f_{x_1} \geq 0$ (see [FK, p151]). The vectors $v_2, \ldots, v_n$ are defined by:

$$v_i = -f_{x_i}\partial_{x_1} - \sum_{j=2}^{n}(f_{x_i}f_{x_j} - \delta_{i,j}T)\partial_{x_j},$$

where $T = f_{x_1} + \sum_{j=2}^{n} f_{x_j}^2$ and $\delta_{i,j}$ is the Kronecker symbol. Here we notice that there is a mistake in the computation of the determinant of the matrix M defined p. 151 in [FK]. This determinant is $(-1)^n T^{n-1} \sum_{i=0}^{n} g_{x_i}^2$. That is why our v_i's are the opposite of the v_i's defined by Fukui and Khovanskii.

If $F_{x_1} \geq 0$, Condition (P_{x_0}) is satisfied with the vectors V_i's defined by:

$$V_i = -F_{x_1}\partial_{x_1} - \sum_{j=2}^{n}(F_{x_i}F_{x_j} - \delta_{i,j}T')\partial_{x_j},$$

where $T' = F_{x_1} + \sum_{j=2}^{n} F_{x_j}^2$. Let us remark that in this situation the computation of $\chi(W^{\varepsilon}_{(F, G-\delta)})$ can be simplified thanks to Theorem 2.1. Actually, the function F satisfies Condition (P) with the following vectors:

$$Z_0 = F_{x_0}\partial_{x_1} + \sum_{j=0 \,|\, j \neq 1}^{n} (F_{x_i}F_{x_j} - \delta_{i,j}S)\partial_{x_j},$$

$$Z_i = -F_{x_i}\partial_{x_1} - \sum_{j=0 \,|\, j \neq 1}^{n} (F_{x_i}F_{x_j} - \delta_{i,j}S)\partial_{x_j}, \quad i = 2, \ldots, n,$$

where $S = F_{x_1} + F_{x_0}^2 + \sum_{j=2}^{n} F_{x_j}^2$. Let $K(F, G) : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1}, 0)$ be defined by:

$$K(F, G) = (F, Z_0 G, Z_2 G, \ldots, Z_n G).$$

Since $F^{-1}(0) \cap G^{-1}(0)$ has an isolated singularity at the origin (Lemma 3.3) then $K(F, G)$ has an isolated zero at the origin (Lemma 2.4). Hence, by Theorem 2.1 and since $\deg_0 \nabla F = 0$ for $F_{x_1} \geq 0$, we have:

if n is odd: $\chi(W^{\varepsilon}_{(F, G-\delta)}) = 1 + \mathrm{sign}(\delta) \deg_0 K(F, G)$,

if n is even: $\chi(W^{\varepsilon}_{(F, G-\delta)}) = 1 - \deg_0 K(F, G)$.

So Theorem 4.3 can be rewritten without the assumption that g has an isolated critical point at the origin. Namely, with the obvious assumptions, we obtain:

THEOREM 5.1. *For t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have*:
– *if n is odd*:

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 - \deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = +\deg_0 K(F,G) - \operatorname{sign}(t) \deg_0 H(F,G),$$

– *if n is even*:

$$\chi(W^{\varepsilon}_{(f_t,g_t)}) = 1 - \deg_0 K(F,G) + \operatorname{sign}(t) \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f_t} \cap \{g_t \geq 0\}) - \chi(W^{\varepsilon}_{f_t} \cap \{g_t \leq 0\}) = \deg_0 k(f,g).$$

If the deformation (F,G) of (f,g) is of the form $F(\lambda,x) = f(x) - \gamma_1(\lambda)$, $G(\lambda,x) = f(x) - \gamma_2(\lambda)$, then we just need to suppose that $f_{x_1} \geq 0$. Therefore Theorem 4.5 becomes:

THEOREM 5.2. *Assume that f has an isolated singularity and that $\gamma_1'(t) \neq 0$ if $t \neq 0$. Assume that $J(F,G)$ and $k(f,g)$ have an isolated zero at the origin then for t and ε with $0 < |t| \ll \varepsilon \ll 1$, we have*:
– *if n is odd*:

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 - \deg_0 k(f,g),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\})$$
$$= +\deg_0 K(F,G) - \operatorname{sign}(t) \deg_0 H(F,G),$$

– *if n is even*:

$$\chi(W^{\varepsilon}_{(f-\gamma_1(t),g-\gamma_2(t))}) = 1 - \deg_0 K(F,G) + \operatorname{sign}(t) \deg_0 H(F,G),$$
$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = \deg_0 k(f,g).$$

6. An example

Let $f(x_1,x_2,x_3,x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$ and $g(x_1,x_2,x_3,x_4) = x_1x_2 + x_3x_4$. These functions have an isolated critical point at the origin and $\deg_0 \nabla f = -1$ and $\deg_0 \nabla g = 1$. The mappings $k(f,g)$ and $l(f,g)$ of Section 2 are:

$$k(f,g)(x) = (x_1^2 + x_2^2 + x_3^2 - x_4^2, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4),$$
$$l(f,g)(x) = (x_1x_2 + x_3x_4, 2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4).$$

It is not difficult to see that 0 is an isolated root of $k(f,g)$ and $l(f,g)$. Furthermore, $\deg_0 k(f,g) = 0$ because $k(f,g)^{-1}(0,\beta,0,0) = \emptyset$ if $\beta < 0$. If $\beta < 0$

then $l(f,g)^{-1}(0,\beta,0,0)$ consists of the points $p_1 = \left(0, \sqrt{-\frac{\beta}{2}}, 0, 0\right)$ and $p_2 = \left(0, -\sqrt{-\frac{\beta}{2}}, 0, 0\right)$. Since $\det[Dl(f,g)(p_i)] > 0$, $\deg_0 l(f,g)$ is equal to 2. By Theorem 2.1 and Theorem 2.9, we get that $\chi(W^{\varepsilon}_{(f,g-\delta)}) = 2$, $\chi(W^{\varepsilon}_{(f-\delta,g)}) = -2$ if $\delta > 0$ and $\chi(W^{\varepsilon}_{(f-\delta,g)}) = 2$ if $\delta < 0$. By Theorem 4.5, we have:

$$\chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g_t \geq \gamma_2(t)\}) - \chi(W^{\varepsilon}_{f-\gamma_1(t)} \cap \{g \leq \gamma_2(t)\}) = 0,$$

for an appropriate analytic arc (γ_1, γ_2).

Let us compute $\chi(W^{\varepsilon}_{(f-t,g-t)})$ using Theorem 4.5. The mappings H and J of Section 3 are given by:

$$\begin{aligned} H(t,x) = (&x_1^2 + x_2^2 + x_3^2 - x_4^2 - t, x_1x_2 + x_3x_4 - t, \\ &2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4), \\ J(t,x) = (&t \cdot (x_1^2 + x_2^2 + x_3^2 - x_4^2 - t), x_1x_2 + x_3x_4 - t, \\ &2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2x_3, 4x_2x_4). \end{aligned}$$

Let us search the points (t,x) such that $H(t,x) = 0$. If $x_2 = 0$ then clearly $x_1 = x_3 = x_4 = t = 0$. If $x_2 \neq 0$ then $x_3 = x_4 = 0$ and:

$$\begin{cases} x_1^2 + x_2^2 - t = 0 \\ x_1x_2 - t = 0 \\ 2x_1^2 - 2x_2^2 = 0 \end{cases}$$

This implies that $t^2 = 4x_2^4 = x_2^4$, which is a contradiction. Hence H admits an isolated zero at the origin. Furthermore $\deg_0 H = 0$. To see this, let (t,x) be such that $H(t,x) = (0,0,\beta,0,0)$ where $\beta < 0$. Necessary $x_2 \neq 0$ and $x_3 = x_4 = 0$. Hence x_1, x_2 and t satisfy the system:

$$\begin{cases} x_1^2 + x_2^2 - t = 0 \\ x_1x_2 - t = 0 \\ 2x_1^2 - 2x_2^2 = \beta \end{cases}$$

Putting $\gamma = \frac{\beta}{2}$, we find that $x_1^2 = \frac{t+\gamma}{2}$, $x_2^2 = \frac{t-\gamma}{2}$ and $t^2 = \frac{t^2-\gamma^2}{4}$. This last equality is equivalent to $3t^2 = -\gamma^2$, which is impossible.

Let us search the points (t,x) such that $J(t,x) = 0$. As above, if $x_2 = 0$ then $x_1 = x_3 = x_4 = t = 0$. If $x_2 \neq 0$ then $x_3 = x_4 = 0$ and

$$\begin{cases} t(x_1^2 + x_2^2 - t) = 0 \\ x_1x_2 - t = 0 \\ 2x_1^2 - 2x_2^2 = 0 \end{cases}$$

If $t = 0$ then $x_1 = x_2 = 0$, which is a contradiction. The case $x_1^2 + x_2^2 - t = 0$ is also impossible as we have already explained. Hence J admits an isolated zero at the origin. Let $\beta < 0$ and let us search the points (t, x) such that $J(t, x) = \left(\frac{\beta^2}{8}, 0, \beta, 0, 0\right)$. Necessarily $x_2 \neq 0$ and $x_3 = x_4 = 0$. Hence x_1, x_2 and t satisfy the system:

$$\begin{cases} t(x_1^2 + x_2^2 - t) = \dfrac{\beta^2}{8} \\ x_1 x_2 - t = 0 \\ 2x_1^2 - 2x_2^2 = \beta \end{cases}$$

Furthermore, $t > 0$ because $t(x_1^2 + x_2^2) = t^2 + \frac{\beta^2}{8}$ and x_1 and x_2 have the same sign. Putting $\gamma = \frac{\beta}{2}$ and $\lambda = t + \frac{\beta^2}{8t}$, we find that $x_1^2 = \frac{\lambda + \gamma}{2}$, $x_2^2 = \frac{\lambda - \gamma}{2}$ and $t^2 = \frac{\lambda^2 - \gamma^2}{4}$. Hence, we get that $3t^4 = \frac{\beta^4}{64}$. Thus $\left(\frac{\beta^2}{8}, 0, \beta, 0, 0\right)$ has two preimages $q_1 = (t_0, a_1, b_1, 0, 0)$ and $q_2 = (t_0, a_2, b_2, 0, 0)$, where $t_0 > 0$, $a_1, b_1 > 0$ and $a_2, b_2 < 0$. An easy computation shows that $DJ(q_i) = -128 b_i^2 t_0 (a_i - b_i)^2$. Finally we find that $\deg_0 J = -2$. Theorem 4.5 gives that $\chi(W^{\varepsilon}_{(f-t, g-t)}) = -2$.

Let us now compute $\chi(W^{\varepsilon}_{(f-t, g-(1/4)t)})$. The mappings H and J are:

$$H(t, x) = \Big(x_1^2 + x_2^2 + x_3^2 - x_4^2 - t, x_1 x_2 + x_3 x_4 - \frac{1}{4} t,$$
$$2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2 x_3, 4x_2 x_4\Big),$$
$$J(t, x) = \Big(t \cdot (x_1^2 + x_2^2 + x_3^2 - x_4^2 - t), x_1 x_2 + x_3 x_4 - \frac{1}{4} t,$$
$$2x_1^2 - 2x_2^2 + 2x_3^2 + 2x_4^2, -4x_2 x_3, 4x_2 x_4\Big).$$

We use the same technics as in the previous example. We find that H and J have an isolated root at the origin. If $\beta < 0$ then $(0, 0, \beta, 0, 0)$ has two preimages by $H : p_1 = (t_0, a_1, b_1, 0, 0)$ and $p_2 = (t_0, a_2, b_2, 0, 0)$ where $t_0 > 0$, $a_1, b_1 > 0$ and $a_2, b_2 < 0$. A computation gives that $DH(p_i) = -48 b_i^2 t_0$, which implies that $\deg_0 H = -2$. Let us search the preimages of $\left(\frac{\beta^2}{8}, 0, \beta, 0, 0\right)$, $\beta < 0$, by J. If (t, x) is such a preimage then necessarily $x_2 \neq 0$, $x_3 = x_4 = 0$ and $t > 0$. Moreover x_1, x_2 and t satisfy the system:

$$\begin{cases} t(x_1^2 + x_2^2 - t) = \dfrac{\beta^2}{8} \\ x_1 x_2 - \dfrac{1}{4} t = 0 \\ 2x_1^2 - 2x_2^2 = \beta \end{cases}$$

This gives that $-\frac{3}{4}t^2 = \frac{\beta^4}{64t^2}$, a contradiction. We have proved that $\deg_0 J = 0$. Applying Theorem 4.5, we obtain that $\chi(W^{\varepsilon}_{(f-t,g-(1/4)t)}) = -2$ if $t > 0$ and $\chi(W^{\varepsilon}_{(f-t,g-(1/4)t)}) = 2$ if $t < 0$.

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