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NEW CHARACTERIZATIONS OF COMPLETE SPACELIKE SUBMANIFOLDS IN SEMI-RIEMANNIAN SPACE FORMS

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Abstract

In this paper we study n-dimensional complete spacelike submanifolds with constant normalized scalar curvature immersed in semi-Riemannian space forms. By extending Cheng-Yau's technique to these ambients, we obtain results to such submanifolds satisfying certain conditions on both the squared norm of the second fundamental form and the mean curvature. We also characterize compact non-negatively curved submanifolds in De Sitter space of index p.

1. Introduction

In recent years, the study of spacelike submanifolds in semi-Riemannian ambients has got increasing interest motivated by theirs importance in problems related to Physics, more specifically in the theory of general relativity.

Concerning to the mathematical viewpoint, such submanifolds appear in several uniqueness problems, for instance, constant mean curvature spacelike hypersurfaces exhibit nice Bernstein's type properties.

Here, we are interested in characterizing complete spacelike submanifolds with constant scalar curvature immersed in semi-Riemannian space forms by annalysing the growth of the squared of the second fundamental form of the immersion or the behaviour of its mean curvature. We recall that a submanifold immersed is said to be *spacelike* if its induced metric is positive definite.

The complete connected semi-Riemannian manifolds of index p with constant curvature c, defined as below, will be denoted by $\mathbf{Q}_p^{n+p}(c)$. They may be considered, up to isometries, as the De Sitter space $\mathbf{S}_p^{n+p}(c)$, if $c > 0$, the semi-Euclidean space $\mathbf{R}_p^{n+p}$, if $c = 0$ and the semi-hyperbolic space $\mathbf{H}_p^{n+p}(c)$, if $c < 0$. Those manifolds will be defined in Section 2.

Before stating our main results, we shall give a brief summary of principal results already current in this theory.

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The initial step in this context is due to Goddard [18], that conjectured that complete spacelike hypersurfaces with constant mean curvature in $\mathbf{S}_1^{n+1}(1)$ are totally umbilical. The totally umbilical hypersurfaces of $\mathbf{S}_1^{n+1}(1)$ are obtained by intersecting $\mathbf{S}_1^{n+1}(1)$ with linear hyperplanes of $\mathbf{R}_1^{n+2}$.

J. Ramanathan [27] proved Goddard's conjecture for $\mathbf{S}_1^3(1)$ and $0 \le H \le 1$. Moreover, if $H > 1$, he showed that the conjecture is false, as can be seen from an example due to Dajczer-Nomizu [14]. Independently, K. Akutagawa proved in [2] that Goddard's conjecture is true when $n = 2$ and either $H^2 \le c$ or M^2 is compact or when $n \ge 3$ and $H^2 < \dfrac{4(n-1)}{n^2}c$. He also constructed complete spacelike rotational surfaces in $\mathbf{S}_1^3(1)$ with constant H satisfying $H > 1$ which are not totally umbilical.

In [24], S. Montiel proved that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbf{S}_1^{n+1}(1)$ with constant mean curvature H satisfying $H^2 \ge \dfrac{4(n-1)}{n^2}$ and being not totally umbilical, the so called hyperbolic cylinders, which are isometric to the Riemannian product $\mathbf{H}^1(\sinh r) \times \mathbf{S}^{n-1}(\cosh r)$ of a hyperbolic line and an $(n-1)$-dimensional sphere of constant sectional curvatures $1 - \coth^2 r$ and $1 - \tanh^2 r$, respectively. We point out that in [25] Montiel characterized the hyperbolic cylinders as the only complete non-compact spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$ with constant mean curvature H satisfying $H^2 = \dfrac{4(n-1)}{n^2}$ and having more than one topological end.

In higher codimension, the condition on the mean curvature is replaced by a condition on the mean curvature vector. Let M^n be a spacelike submanifold of $\mathbf{Q}_p^{n+p}(c)$ with parallel mean curvature vector h. When M^n is maximal, i.e., $h \equiv 0$, T. Ishihara [21] established the following inequality for the squared norm S of the second fundamental form B of M^n

$$\frac{1}{2}\Delta S \ge S\left(nc + \frac{S}{p}\right). \tag{1.1}$$

We recall that a submanifold M^n of $\mathbf{Q}_p^{n+p}(c)$ is totally geodesic if its second fundamental form B vanishes identically. As an important application of (1.1), Ishihara proved that maximal complete spacelike submanifolds in $\mathbf{Q}_p^{n+p}(c)$, $c \ge 0$, are totally umbilical and, if $c < 0$, then $0 \le S \le -npc$. Moreover, he determined all the complete spacelike maximal submanifolds M^n of $\mathbf{Q}_p^{n+p}(c)$, $c < 0$, satisfying $S = -npc$ (cf. [21], Theorem 1.3).

R. Aiyama [3] studied compact spacelike submanifolds in $\mathbf{S}_p^{n+p}(c)$ with parallel mean curvature vector and proved that if the normal connection of M^n is flat, then M^n is totally umbilical. In the same work [3], it was proved that compact spacelike submanifolds in $\mathbf{S}_p^{n+p}(c)$ with parallel mean curvature vector and non-negative sectional curvatures are also totally umbilical.

Q. M. Cheng [11] showed that Akutagawa's result [2] is valid for complete spacelike submanifolds in $\mathbf{S}_p^{n+p}(c)$ with parallel mean curvature vector.

In [8], [9], Chaves-Sousa obtained the following inequality for the squared norm of the traceless tensor $\Phi = B - Hg$, where g stands for the induced metric on a spacelike submanifold in $\mathbf{Q}_p^{n+p}(c)$ with parallel mean curvature vector

$$\frac{1}{2}\Delta|\Phi|^2 \geq |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(c - H^2)\right). \tag{1.2}$$

As an application of (1.2), Brasil-Chaves-Mariano [5] obtained an other limitation for the supremum of the mean curvature

$$\sup H^2 < \frac{4(n-1)c}{(n-2)^2 p + 4(n-1)}, \tag{1.3}$$

as an extension of Akutagawa's [2] and Cheng's [11] results.

Moreover, Chaves-Sousa [9] obtained a Lorentzian version of results obtained by Yano-Ishihara [31] and also by Yau [32] for Riemannian submanifolds. More precisely they proved that complete spacelike submanifolds in $\mathbf{Q}_p^{n+p}(c)$ with parallel mean curvature vector, non-negative sectional curvatures and constant scalar curvature are totally umbilical or a product $M_1 \times M_2 \times \cdots \times M_k$, where each M_i is a totally umbilical submanifold of $\mathbf{Q}_p^{n+p}(c)$ and the $M_i's$ are mutually perpendicular along their intersections.

In [4], L. Alías and A. Romero developed some integral formulas for compact spacelike hypersurfaces in de Sitter space $\mathbf{S}_1^{n+1}(1)$ and obtained characterizations for totally umbilical spacelike hypersurfaces with constant higher order mean curvature.

Motivated by this brief description, it would be natural to replace the assumption on the mean curvature vector by a suitable one on the scalar curvature and to characterize the complete spacelike submanifolds in $\mathbf{Q}_p^{n+p}(c)$ satisfying this new condition.

In order to study hypersurfaces with constant scalar curvature, Cheng and Yau [13] introduced a new self-adjoint differential operator $\square$ acting on C^2-functions defined on Riemannian manifolds. Using this approach, they were able to classify compact hypersurfaces M^n with constant normalized scalar curvature R satisfying $R \geq c$ and non-negative sectional curvatures immersed in Riemannian space forms of constant curvature c.

There are some interesting and recent results related to the study of spacelike hypersurfaces with constant scalar curvature in De Sitter space. Y. Zheng [33] proved that a compact spacelike hypersurface in $\mathbf{S}_1^{n+1}(c)$ with constant normalized scalar curvature R, $R < c$, and non-negative sectional curvatures is totally umbilical. Later, Q. M. Cheng and S. Ishikawa [12] showed that Zheng's result in [33] is also true without additional assumptions on the sectional curvatures of the hypersurface.

In [20], Z. Hu, M. Scherfner and S. Zhai classified spacelike hypersurfaces in De Sitter space $\mathbf{S}_1^{n+1}(c)$ with constant scalar curvature and two distinct principal curvatures.

Recently, Camargo-Chaves-Sousa [7] answered a question posed by H. Li in Section 4 of [23] with an additional hypothesis on the mean curvature. The authors proved that a complete spacelike hypersurface in $\mathbf{S}_1^{n+1}(1)$, $n \geq 3$, with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leq R \leq 1$ and bounded mean curvature is totally umbilical.

We recall that the immersion $x : M^n \to \mathbf{Q}_p^{n+p}(c)$ is substantial if its codimension can not be reduced. The smallest codimension for which an immersion x can be reduced is called the *substantial codimension* of x.

It should be pointed out that the *normalized mean curvature vector* is defined by $\frac{h}{|h|}$, where h is the mean curvature vector of M^n, and the *normalized scalar curvature* R satisfies $n(n-1)R = \mathrm{tr}(\mathrm{Ric})$, where Ric is the Ricci curvature tensor of M^n.

In this paper, we extend Cheng-Yau's technique to complete submanifolds in $\mathbf{Q}_p^{n+p}(c)$ in order to prove the following results

Theorem 1.1. *Let $x : M^n \to \mathbf{Q}_p^{n+p}(c)$, $n \geq 3$, be a substantial isometric immersion of a complete Riemannian manifold. Assume that the normalized mean curvature vector of M^n in $\mathbf{Q}_p^{n+p}(c)$ is parallel and that M^n has constant normalized scalar curvature R satisfying $R \leq c$. For $x \geq -n(c-R)$, set*

$$P_R(x) = \frac{n-1-p}{np}x + \frac{(n-1)(p+1)(R-c)}{p} + nc - \frac{n-2}{n}\sqrt{(x - n(n-1)(R-c))(x + n(R-c))}. \tag{1.4}$$

If $P_R(\sup S) \geq 0$, then $p = 1$. Moreover, if $R < \left(\frac{n-2}{n}\right)c$, when $c \geq 0$, then either $S = n(c-R)$ and M^n is totally umbilical or $P_R(\sup S) = 0$ and $\sup S = C_n(R)$, where

$$C_n(R) = \frac{n[(-nR + (n-2)c)(n-2)(n-1)(R-c) + n((n-1)(R-c)+c)^2]}{(-nR + (n-2)c)(n-2)}.$$

Corollary 1.1. *Let $x : M^n \to \mathbf{Q}_p^{n+p}(c)$, $n \geq 3$ and $c > 0$, be a substantial isometric immersion of a complete Riemannian manifold. Assume that the normalized mean curvature vector of M^n in $\mathbf{Q}_p^{n+p}(c)$ is parallel and that M^n has constant normalized scalar curvature R satisfying $R \leq c$. For $x \geq -n(c-R)$, set*

$$P_R(x) = \frac{n-1-p}{np}x + \frac{(n-1)(p+1)(R-c)}{p} + nc - \frac{n-2}{n}\sqrt{(x - n(n-1)(R-c))(x + n(R-c))}. \tag{1.5}$$

If $P_R(\sup S) \geq 0$ *and* $\sup S$ *is attained on* M^n, *then* $p = 1$ *and* M^n *is either totally umbilical or isometric to a hyperbolic cylinder* $\mathbf{H}^1(\sinh r) \times \mathbf{S}^{n-1}(\cosh r)$.

Corollary 1.2. *Let* M^n *be a complete spacelike submanifold in* $\mathbf{Q}_p^{n+p}(c)$, $n \geq 3$, *with parallel normalized mean curvature vector and constant normalized scalar curvature* R *satisfying* $R \leq c$. *If the mean curvature* H *of* M^n *satisfies*

$$\sup H^2 < \frac{4(n-1)c}{(n-2)^2 p + 4(n-1)}, \tag{1.6}$$

then M^n *is totally umbilical.*

We also extend Theorem 4.1 in [30], characterizing the compact spacelike submanifolds in $\mathbf{S}_p^{n+p}(c)$ with parallel normalized mean curvature vector, constant normalized scalar curvature R satisfying $R \leq c$ and non-negative sectional curvatures. More precisely, we prove the following

Theorem 1.2. *Let* M^n *be a compact spacelike submanifold in* $\mathbf{S}_p^{n+p}(c)$, *with parallel normalized mean curvature vector, constant normalized scalar curvature* R *satisfying* $R \leq c$. *If* M^n *has non-negative sectional curvatures, then* M^n *is isometric to a sphere* $\mathbf{S}^n(c_1)$, $c_1 > 0$.

Corollary 1.3. *Let* M^n *be a complete spacelike submanifold in* $\mathbf{Q}_p^{n+p}(c)$, *with parallel normalized mean curvature vector, constant normalized scalar curvature* R *satisfying* $R \leq c$. *If* M^n *has non-negative sectional curvatures, then either*

(i) $\inf K = 0$, *where* $\inf K$ *denotes the infimum of the sectional curvatures of* M^n; *or*
(ii) $c > 0$ *and* M^n *is totally umbilical.*

Remark 1.1. The assumption about parallel normalized mean curvature vector was introduced by Chen in [10]. Submanifolds with nonzero parallel mean curvature vector also have parallel normalized mean curvature vector. The condition to have parallel normalized mean curvature vector is much weaker than the condition to have parallel mean curvature vector. For instance, every hypersurface in a semi-Riemannian manifold always has parallel normalized mean curvature vector.

2. Preliminaries

In this section we will introduce some basic facts and notations that will appear on the paper.

Let $\mathbf{R}_p^{n+p}$ denotes an $(n+p)$-dimensional real vector space endowed with an inner product of index p given by $\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n+p} x_j y_j$, where $x = (x_1, x_2, \dots x_{n+p})$ is the natural coordinate of $\mathbf{R}_p^{n+p}$. The manifold $\mathbf{R}_p^{n+p}$ is called *semi-Euclidean space* and it has constant curvature $c = 0$. We also define the semi-Riemannian manifolds $\mathbf{S}_p^{n+p}(c)$, with $c > 0$, *called De Sitter space*, and $\mathbf{H}_p^{n+p}(c)$, with $c < 0$, *called semi-Hyperbolic space*, as follows:

$$\mathbf{S}_p^{n+p}(c) = \left\{ (x_1, x_2, \dots x_{n+p+1}) \in \mathbf{R}_p^{n+p+1} : -\sum_{i=1}^{p} x_i^2 + \sum_{j=p+1}^{n+p+1} x_j^2 = \frac{1}{c} \right\}$$

$$\mathbf{H}_p^{n+p}(c) = \left\{ (x_1, x_2, \dots x_{n+p+1}) \in \mathbf{R}_{p+1}^{n+p+1} : -\sum_{i=1}^{p+1} x_i^2 + \sum_{j=p+2}^{n+p+1} x_j^2 = \frac{1}{c} \right\}.$$

Let M^n be an n-dimensional Riemannian manifold immersed in $\mathbf{Q}_p^{n+p}(c)$. When the indefinite Riemannian metric of $\mathbf{Q}_p^{n+p}(c)$ induces a Riemannian metric of on M^n, the immersion is called spacelike. We choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+p}$ in $\mathbf{Q}_p^{n+p}(c)$ such that, at each point p of M^n, $e_1, \dots, e_n$ span the tangent space T_pM to M^n at p. We make the following standard convention of indices

$$1 \le A, B, C, \dots \le n + p, \quad 1 \le i, j, k, \dots \le n, \quad n + 1 \le \alpha, \beta, \gamma, \dots \le n + p.$$

Take the correspondent dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$ so that the semi-Riemannian metric of $\mathbf{Q}_p^{n+p}(c)$ is given by

$$d\bar{s}^2 = \sum_A \varepsilon_A \omega_A^2, \quad \varepsilon_i = 1,\ \varepsilon_\alpha = -1,\ 1 \le i \le n, n+1 \le \alpha \le n+p.$$

Then the structure equations of $\mathbf{Q}_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0. \tag{2.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D. \tag{2.2}$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Next, we restrict those forms to M^n. First of all, we get

$$\omega_\alpha = 0, \quad n + 1 \le \alpha \le n + p. \tag{2.4}$$

So the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$.
Since $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$, from *Cartan's lemma*, we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{2.5}$$

Let $B = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \omega_j e_\alpha$ be the *second fundamental form*. We will denote by $h = \frac{1}{n} \sum_\alpha (\sum_i h_{ii}^\alpha) e_\alpha$ and by $H = |h| = \frac{1}{n} \sqrt{\sum_\alpha (\sum_i h_{ii}^\alpha)^2}$ the *mean curvature vector* and the *mean curvature* of M^n, respectively.

The structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0. \tag{2.6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \tag{2.7}$$

Using the structure equations we obtain the *Gauss equation*

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.8}$$

The components of the *Ricci curvature tensor* Ric and the *normalized scalar curvature* R are given, respectively, by

$$R_{jk} = c(n-1)\delta_{jk} - \sum_\alpha \left(\sum_i h^\alpha_{ii}\right) h^\alpha_{jk} + \sum_{\alpha,i} h^\alpha_{ik}h^\alpha_{ji}. \tag{2.9}$$

$$n(n-1)(R-c) = S - n^2H^2, \tag{2.10}$$

where $S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ denotes the square of the length of the second fundamental form of M^n.

We also have the structure equations of the normal bundle of M^n

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \tag{2.11}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2}\sum_{i,j} R_{\alpha\beta ij}\omega_i \wedge \omega_j, \tag{2.12}$$

where

$$R_{\alpha\beta ij} = \sum_l (h^\alpha_{il}h^\beta_{lj} - h^\alpha_{jl}h^\beta_{li}). \tag{2.13}$$

The covariant derivatives h^α_{ijk} of h^α_{ij} satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_k h^\alpha_{jk}\omega_{ki} - \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.14}$$

Then, by exterior differentiation of (2.5), we obtain the *Codazzi equations*

$$h^\alpha_{ijk} = h^\alpha_{jik} = h^\alpha_{ikj}. \tag{2.15}$$

Similarly, we have the second covariant derivatives h^α_{ijkl} of h^α_{ij} so that

$$\begin{aligned} \sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} &+ \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} \\ &+ \sum_l h^\alpha_{ijl}\omega_{lk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \end{aligned} \tag{2.16}$$

By exterior differentiation of (2.14), we can get the following *Ricci formulas*

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{jm} R_{mikl} + \sum_\beta h^\beta_{ij} R_{\alpha\beta kl}. \tag{2.17}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.15) and (2.17), we have

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum_{m,k} h^\alpha_{km} R_{mijk} + \sum_{m,k} h^\alpha_{mi} R_{mkjk} + \sum_{k,\beta} h^\beta_{ik} R_{\alpha\beta jk}. \tag{2.18}$$

If $H \neq 0$, we choose $e_{n+1} = \dfrac{h}{H}$. Thus

$$H^{n+1} := \frac{1}{n}\,\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha := \frac{1}{n}\,\mathrm{tr}(h^\alpha) = 0, \quad \alpha \geq n+2, \tag{2.19}$$

where h^α denotes the matrix $[h^\alpha_{ij}]$.

Putting together (2.14) and (2.19) we get

$$\sum_k H^{n+1}_k \omega_k = dH, \quad \sum_k H^\alpha_k \omega_k = -H\omega_{n+1\alpha}, \quad \forall \alpha > n+1, \tag{2.20}$$

where $dH = \sum_i H_i \omega_i$.

The formulas (2.16), (2.19) and (2.20) yield

$$H^{n+1}_{kl} = H_{kl} - \frac{1}{H} \sum_{\beta > n+1} H^\beta_k H^\beta_l, \tag{2.21}$$

where $\nabla H_k = \sum_l H_{kl}\omega_l = dH_k + \sum_l H_l \omega_{lk}$.

It follows from (2.8), (2.13), (2.18) and (2.19) that

$$\begin{aligned} \triangle h^{n+1}_{ij} = {} & nH_{ij} + cnh^{n+1}_{ij} - cnH\delta_{ij} + \sum_{\beta,k,m} h^{n+1}_{km} h^\beta_{mk} h^\beta_{ij} \\ & - 2\sum_{\beta,k,m} h^{n+1}_{km} h^\beta_{mj} h^\beta_{ik} + \sum_{\beta,k,m} h^{n+1}_{mi} h^\beta_{mk} h^\beta_{kj} \\ & - nH \sum_m h^{n+1}_{mi} h^{n+1}_{mj} + \sum_{\beta,k,m} h^{n+1}_{jm} h^\beta_{mk} h^\beta_{ki}; \end{aligned} \tag{2.22}$$

$$\begin{aligned} \triangle h^\alpha_{ij} = {} & nH^\alpha_{ij} + nch^\alpha_{ij} + \sum_{\beta,k,m} h^\alpha_{km} h^\beta_{mk} h^\beta_{ij} - 2\sum_{\beta,k,m} h^\alpha_{km} h^\beta_{mj} h^\beta_{ik} \\ & + \sum_{\beta,k,m} h^\alpha_{mi} h^\beta_{mk} h^\beta_{kj} - nH \sum_m h^\alpha_{mi} h^{n+1}_{mj} \\ & + \sum_{\beta,k,m} h^\alpha_{jm} h^\beta_{mk} h^\beta_{ki}, \quad \forall \alpha \geq n+2. \end{aligned} \tag{2.23}$$

Since $\frac{1}{2}\triangle S = \frac{1}{2}\sum_{\alpha,i,j}\triangle(h_{ij}^{\alpha})^2 = \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j}h_{ij}^{\alpha}\triangle h_{ij}^{\alpha}$, by using (2.22) and (2.23), it is straightforward to verify that

$$\begin{aligned}\frac{1}{2}\triangle S = &\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + n\sum_{\alpha,i,j}h_{ij}^{\alpha}H_{ij}^{\alpha} + nc(S - nH^2)\\ &- nH\sum_{\alpha}\operatorname{tr}(h^{n+1}(h^{\alpha})^2) + \sum_{\alpha,\beta}[\operatorname{tr}(h^{\alpha}h^{\beta})]^2\\ &+ \sum_{\alpha,\beta}N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}),\end{aligned} \tag{2.24}$$

where $N(A) = \operatorname{tr}(AA^t)$, for all matrix $A = [a_{ij}]$.

Remark 2.1. *Recall that M^n is a submanifold with parallel normalized mean curvature vector if $\nabla^{\perp}\frac{h}{H} \equiv 0$, where $\nabla^{\perp}$ is the normal connection of M^n in $\mathbf{Q}_p^{n+p}(c)$. It implies that $\omega_{n+1\alpha} = 0$, for all α and by* (2.12) *and* (2.13) *it is possible to show that $h^{n+1}h^{\alpha} = h^{\alpha}h^{n+1}$, for all α. Furthermore,* (2.20) *and* (2.21) *yield*

$$H_k^{\alpha} = 0, \quad \forall k, \alpha > n+1, \quad H_{kl}^{n+1} = H_{kl}. \tag{2.25}$$

From (2.16) *and* (2.25) *we obtain*

$$\sum_l H_{kl}^{\alpha}\omega_l = -H_k^{n+1}\omega_{n+1\alpha} = 0 \quad \text{and so}$$

$$H_{kl}^{\alpha} = 0, \quad \alpha > n+1. \tag{2.26}$$

We will need the following algebraic lemma, whose proof can be found in [28].

Lemma 2.1. *Let $A, B : \mathbf{R}^n \to \mathbf{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. Then*

$$|\operatorname{tr} A^2B| \le \frac{n-2}{\sqrt{n(n-1)}}N(A)\sqrt{N(B)}. \tag{2.27}$$

Moreover, the equality holds if, and only if, $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$\begin{aligned}&|x_i| = \sqrt{\frac{N(A)}{n(n-1)}}, \quad x_ix_j \ge 0,\\ &y_i = \sqrt{\frac{N(B)}{n(n-1)}} \left(\text{resp. } y_i = -\sqrt{\frac{N(B)}{n(n-1)}}\right).\end{aligned} \tag{2.28}$$

Consider the following symmetric tensor

$$\Phi = \sum_{\alpha, i, j} \Phi_{ij}^{\alpha} \omega_i \omega_j e_\alpha, \tag{2.29}$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$.

It is easy to check that Φ is traceless and

$$\begin{aligned} &N(\Phi^\alpha) = N(h^\alpha) - n(H^\alpha)^2, \quad n+1 \le \alpha \le n+p \quad \text{and} \\ &|\Phi|^2 = \sum_\alpha N(\Phi^\alpha) = S - nH^2, \end{aligned} \tag{2.30}$$

where Φ^α denotes the matrix $[\Phi_{ij}^{\alpha}]$.

Let $T = \sum_{i,j} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}^{n+1}. \tag{2.31}$$

According to Cheng-Yau [13], we introduce the operator $\square$ associated to T acting on any C^2-function f by

$$\square(f) = \sum_{i,j} T_{ij} f_{ij}. \tag{2.32}$$

Since T_{ij} is divergence-free, if M^n is a compact orientable manifold, it follows, from Proposition 1 in [13], that $\square$ is self-adjoint relative to the L^2-inner product of M^n, i.e, $\int_M f\square(g) = \int_M g\square(f)$. In particular, it implies $\int_M \square(f) = 0$.

The proof of the next result follows essentially from the pattern of the proof of Theorem 2.1 in [19].

Lemma 2.2. *Let M^n be a spacelike submanifold in $\mathbf{Q}_p^{n+p}(c)$. Suppose that the normalized scalar curvature R is constant and $R \le c$. Then*

$$\sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 \ge 0 \tag{2.33}$$

and the symmetric tensor T defined by (2.31) *is positive semi-definite. Moreover*

i) *when $R - c < 0$, if the equality holds on M^n, then H is constant and T is positive definite*;

ii) *when $R - c = 0$, if the equality occurs on M^n, then either H is constant or M^n lies in a totally geodesic subspace $\mathbf{Q}_1^{n+1}(c)$ of $\mathbf{Q}_p^{n+p}(c)$ and, in the former case, the matrix h^{n+1} has rank* 1.

We also will need the well known generalized *Maximum Principle* due to H. Omori [26].

Lemma 2.3. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to \mathbf{R}$ be a smooth* *function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}$ in M^n such that*

$$\lim_{k\to\infty} f(p_k) = \sup f, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0 \quad \textit{and}$$
$$\limsup_{k\to\infty} \max\{(\nabla^2 f(p_k))(X,X) : |X| = 1\} \le 0.$$

The next proposition has an essential role in the proofs of our results.

PROPOSITION 2.1. *Let M^n be a complete spacelike submanifold in $\mathbf{Q}_p^{n+p}(c)$ with parallel normalized mean curvature vector and constant normalized scalar curvature R, $R \le c$. Then the following inequality holds*

$$\square(nH) \ge |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(c-H^2)\right). \tag{2.34}$$

Proof. Take a local field of orthonormal frames $\{e_1,\dots,e_{n+p}\}$ such that $e_{n+1} = \dfrac{h}{H}$. By (2.32), $\square(nH)$ takes the form

$$\square(nH) = nH\triangle(nH) - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij}. \tag{2.35}$$

Notice that

$$nH\triangle(nH) = \frac{1}{2}\triangle(nH)^2 - n^2|\nabla H|^2. \tag{2.36}$$

Combining (2.35) and (2.36), we get

$$\square(nH) = \frac{1}{2}\triangle(nH)^2 - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1}H_{ij}. \tag{2.37}$$

Moreover, as R is constant, from (2.10), we have $\triangle S = \triangle(nH)^2$. Therefore, from (2.24) and (2.37) we can write

$$\begin{aligned}\square(nH) = {} & \sum_{\alpha,i,j,k}(h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + n\sum_{\alpha,i,j} h_{ij}^\alpha H_{ij}^\alpha \\ & - n\sum_{i,j} h_{ij}^{n+1}H_{ij} + nc(S - nH^2) - nH\sum_\alpha \mathrm{tr}(h^{n+1}(h^\alpha)^2) \\ & + \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2 + \sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha).\end{aligned} \tag{2.38}$$

Since M^n has parallel normalized mean curvature vector, (2.25), (2.26) and (2.38) yield

$$\square(nH) = \sum_{\alpha,i,j,k} (h^{\alpha}_{ijk})^2 - n^2|\nabla H|^2 + nc(S - nH^2) - nH\sum_{\alpha} \operatorname{tr}(h^{n+1}(h^{\alpha})^2) + \sum_{\alpha,\beta}[\operatorname{tr}(h^{\alpha}h^{\beta})]^2 + \sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}). \tag{2.39}$$

From (2.19) and (2.30), we get

$$\begin{aligned} &\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \\ &N(\Phi^{n+1}) = \operatorname{tr}(\Phi^{n+1})^2 = \operatorname{tr}(h^{n+1})^2 - nH^2 = N(h^{n+1}) - nH^2, \\ &\operatorname{tr}(h^{n+1})^3 = \operatorname{tr}(\Phi^{n+1})^3 + 3HN(\Phi^{n+1}) + nH^3 \\ &\Phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \quad N(\Phi^{\alpha}) = N(h^{\alpha}), \quad \alpha \geq n+2. \end{aligned} \tag{2.40}$$

By (2.40), (2.39) and Lemma 2.2, we see that

$$\square(nH) \geq n|\Phi|^2(c - H^2) + \sum_{\alpha,\beta}[\operatorname{tr}(\Phi^{\alpha}\Phi^{\beta})]^2 - nH\sum_{\alpha} \operatorname{tr}(\Phi^{n+1}(\Phi^{\alpha})^2) + \sum_{\alpha,\beta} N(\Phi^{\alpha}\Phi^{\beta} - \Phi^{\beta}\Phi^{\alpha}). \tag{2.41}$$

As it was already seen in Remark 2.1, the matrix h^{n+1} commutes with every matrix h^{α}, for all α and, therefore, by definition, the traceless matrix Φ^{n+1} commutes with the traceless matrices Φ^{α}, for all α. Hence we can apply Lemma 2.1 in order to obtain

$$\sum_{\alpha} \operatorname{tr}(\Phi^{n+1}(\Phi^{\alpha})^2) \leq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{N(\Phi^{n+1})}|\Phi|^2 \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \tag{2.42}$$

Moreover, *Cauchy-Schwarz inequality* implies that

$$|\Phi|^4 \leq p\sum_{\alpha}(N(\Phi^{\alpha}))^2 \leq p\sum_{\alpha,\beta}(\operatorname{tr}\Phi^{\alpha}\Phi^{\beta})^2. \tag{2.43}$$

By putting (2.42) and (2.43) into (2.41), we arrive to (2.34). □

The following proposition appeared in [6] and [7], for $p = 1$ and $c > 0$.

Proposition 2.2. *Let M^n be a complete spacelike submanifold in $\mathbf{Q}^{n+p}_p(c)$ with constant normalized scalar curvature R, $R \leq c$. If the mean curvature H of M^n is bounded, then there is a sequence of points $\{p_k\} \in M^n$ such that* $\lim_{k\to\infty} nH(p_k) = n\sup H$, $\lim_{k\to\infty}|\nabla nH(p_k)| = 0$ *and* $\limsup_{k\to\infty}(\square(nH)(p_k)) \leq 0$.

Proof. Choose a local orthonormal frame field $e_1, \dots, e_n$ at $p \in M^n$ such that $h_{ij}^{n+1} = \lambda_i^{n+1} \delta_{ij}$. Thus, by (2.32), $\square(nH) = \sum_i (nH - \lambda_i^{n+1})(nH)_{ii}$, $\forall i$.

As $R \le c$, (2.10) implies that $S - n^2H^2 = n(n-1)(R-c) \le 0$ and $\sum_{\alpha, i, j} (h_{ij}^\alpha)^2 = S \le n^2H^2$. Hence $(\lambda_i^{n+1})^2 \le S \le n^2H^2$, which shows that

$$0 \le nH - \left|\lambda_i^{n+1}\right|, \quad \forall i. \tag{2.44}$$

From (2.8) and (2.44), we have

$$R_{ijij} = c - \sum_\alpha (h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2) \ge c - pn^2H^2. \tag{2.45}$$

Since H is bounded, it follows from (2.45) that the sectional curvatures are bounded from below. Therefore, we may apply Lemma 2.3 to nH, obtaining a sequence of points $\{p_k\} \in M^n$ such that

$$\lim_{k\to\infty} nH(p_k) = n \sup H; \quad \lim_{k\to\infty} |\nabla nH(p_k)| = 0 \\ \text{and} \quad \limsup_{k\to\infty} (nH_{ii}(p_k)) \le 0. \tag{2.46}$$

By evaluating (2.44) at points p_k of the sequence above, we get

$$\begin{aligned} 0 &\le nH(p_k) - |\lambda_i^{n+1}(p_k)| \le nH(p_k) - \lambda_i^{n+1}(p_k) \\ &\le nH(p_k) + |\lambda_i^{n+1}(p_k)| \le 2nH(p_k). \end{aligned} \tag{2.47}$$

Using once more that H is bounded, from (2.47) we infer that $\{nH(p_k) - \lambda_i^{n+1}(p_k)\}$ is non-negative and bounded.

By applying $\square(nH)$ at p_k, taking the limit and using (2.46) and (2.47), we get

$$\limsup_{k\to\infty} (\square(nH)(p_k)) \le \sum_i \limsup_{k\to\infty} (nH - \lambda_i^{n+1})(p_k) \limsup_{k\to\infty} (nH_{ii}(p_k)) \le 0. \quad \square$$

Set h^ξ the second fundamental form with respect to the normal direction ξ. Given an isometric immersion $\psi : M_s^n \to \mathbf{Q}_t^{n+p}(c)$, the *first normal space* of ψ at $p \in M_s^n$, $N_1(p)$, is defined to be the orthogonal complement of the set $\{\xi \in T_pM^\perp; h^\xi = 0\}$.

We recall the following indefinite version of a theorem due to Erbacher (see [15] and [16]).

Theorem 2.1. *Let $\psi : M_s^n \to \mathbf{Q}_t^{n+p}(c)$ be an isometric immersion of a connected indefinite Riemannian manifold into a space form. If there exists a k-dimensional parallel normal subbundle $L(p)$ which contains the first normal space $N_1(p)$ for all $p \in M_s^n$, then there exists a $(n+p-k)$-dimensional totally geodesic submanifold Q^{n+p-k} of $\mathbf{Q}_t^{n+p}(c)$ such that $\psi(M_s^n) \subset Q^{n+p-k}$, i.e., ψ admits a reduction of codimension to k.*

3. Proofs of the results

Proof of Theorem 1.1. The following relations may be readily deduced from the *Gauss equation* (2.10) and the formula (2.30):

$$H^2 = \frac{S - n(n-1)(R-c)}{n^2}, \tag{3.1}$$

$$|\Phi|^2 = \frac{(n-1)S + n(n-1)(R-c)}{n}. \tag{3.2}$$

$$|\Phi|^2 = n(n-1)(R - c + H^2). \tag{3.3}$$

Set $P(H, |\Phi|) = \frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\Phi| + n(c - H^2)$ and $Q(H, R) = \frac{n(n-1-p)}{p} H^2 + \frac{n(n-1)}{p}(R-c) + nc - n(n-2)H\sqrt{R - c + H^2}$. If $P_R(x)$ given by (1.4), by virtue of (3.1), (3.2) and (3.3), it is straightforward to verify that

$$P_R(S) = P(H, |\Phi|) = Q(H, R). \tag{3.4}$$

Taking into account our assumption, we will show that if $p \geq 2$, then H is bounded. From (3.1), (3.4) and that R is constant, we get $P_R(\sup S) = Q(\sup H, R)$ and, therefore,

$$\begin{aligned} 0 \leq Q(\sup H, R) = \frac{n(n-1-p)}{p} \sup H^2 + \frac{n(n-1)}{p}(R-c) \\ + nc - n(n-2) \sup H \sqrt{R - c + \sup H^2}. \end{aligned} \tag{3.5}$$

Thus,

$$\begin{aligned} (n-2) \sup H \sqrt{R - c + \sup H^2} \leq \frac{(n-1-p)}{p} \sup H^2 \\ + \frac{(n-1)}{p}(R-c) + c. \end{aligned} \tag{3.6}$$

Squaring the last inequality, we obtain

$$\begin{aligned} &(\sup H^2)^2 \left[(n-2)^2 - \left(\frac{n-1-p}{p} \right)^2 \right] \\ &+ \sup H^2 \left[(n-2)^2(R-c) - 2\frac{(n-1-p)}{p} R - 2\left(\frac{n-1-p}{p} \right)^2 (R-c) \right] \\ &- \left[\left(\frac{n-1-p}{p} \right)(R-c) + R \right]^2 \leq 0. \end{aligned} \tag{3.7}$$

Solving inequation (3.7), we arrive to

$$\sup H^2 \le A^{-1}\left[\left(-(n-2)^2 + 2\left(\frac{n-1-p}{p}\right)^2\right)(R-c) + 2\left(\frac{n-1-p}{p}\right)R + \sqrt{\triangle}\right], \tag{3.8}$$

where

$$\triangle = (n-2)^2\left[(R-c)^2(n-2)^2 + 4\left(\frac{n-1-p}{p}\right)(R-c)R + 4R^2\right]$$

and

$$A = 2\left[(n-2)^2 - \left(\frac{n-1-p}{p}\right)^2\right].$$

Hence, H is bounded.

By our assumptions, Proposition 2.1 and equality (3.4), we may write

$$\square(nH) \ge |\Phi|^2 P_R(S). \tag{3.9}$$

Moreover, as H is bounded, we may apply Proposition 2.2 to obtain a sequence of points p_k in M^n such that

$$\lim_{k\to\infty}(nH(p_k)) = n \sup H \quad \text{and} \tag{3.10}$$

$$\limsup_{k\to\infty}(\square(nH)(p_k)) \le 0. \tag{3.11}$$

As R is constant, it is clear from (3.1) and (3.3) that $\lim_{k\to\infty} S(p_k) = \sup S$ and $\lim_{k\to\infty}|\Phi|(p_k) = \sup|\Phi|$.

By evaluating inequality (3.9) at the points p_k of the sequence obtained by Proposition 2.2 and taking $\limsup_{k\to\infty}$, it gives

$$0 \ge \limsup_{k\to\infty}(\square(nH)(p_k)) \ge \sup|\Phi|^2 P_R(\sup S). \tag{3.12}$$

As we are assuming $P_R(\sup S) \ge 0$, we infer from (3.12)

$$\sup|\Phi|^2 P_R(\sup S) = \limsup_{k\to\infty}(\square(nH)(p_k)) = 0. \tag{3.13}$$

Then $\sup|\Phi|^2 = 0$ or $P_R(\sup S) = 0$. If $p \ge 2$, we shall prove that $\sup|\Phi| = 0$.

If the equality holds in (3.12), all the estimates employed to derive this inequality are, actually, equalities. In this way, the inequalities used to prove Proposition 2.1 become equalities. In particular, from (2.42) and (2.43), we deduce that

$$\limsup_{k\to\infty}(N(\Phi^{n+1}(p_k))) = \limsup_{k\to\infty}(|\Phi|^2(p_k)) = \sup|\Phi|^2. \tag{3.14}$$

$$\sup|\Phi|^4 = p\sum_\alpha \limsup_{k\to\infty}(N(\Phi^\alpha))^2(p_k) = p\sum_\alpha\left(\limsup_{k\to\infty} N(\Phi^\alpha)(p_k)\right)^2. \tag{3.15}$$

Let $C^\alpha = \limsup_{k\to\infty}(N(\Phi^\alpha)(p_k))$, $\alpha \geq n+1$. Notice that $\sum_\alpha C^\alpha = \sup|\Phi|^2$. It follows from (3.14) that $C^\alpha = 0$, for all $\alpha \geq n+2$, which together with (3.15) yield $\sup|\Phi|^4 = p\sum_\alpha (C^\alpha)^2 = p(C^{n+1})^2 = p\sup|\Phi|^4$. Since $p \geq 2$, we conclude that $\sup|\Phi| = 0$.

Keeping in mind Theorem 2.1, it is easy to see that the codimension of the immersion can be reduced which contradicts our initial hypothesis. It shows that the substantial codimension of the immersion x is one.

If $p = 1$ and $R < \left(\frac{n-2}{n}\right)c$, when $c \geq 0$, from (3.7) we obtain

$$\sup H^2 \leq \frac{[(n-2)(R-c)+R]^2}{(n-2)(-nR+(n-2)c)}. \tag{3.16}$$

Therefore, H is bounded and we can follow the pattern of the preceeding proof to conclude that $\sup|\Phi| = 0$ or $P_R(\sup S) = 0$.

It follows from (3.2) that $S \leq n(c-R)$ and the equality holds if, and only if, M^n is totally umbilical. Moreover, $P_R(\sup S) = 0$ if, and only if,

$$\sup S = \frac{n[(-nR+(n-2)c)(n-2)(n-1)(R-c)+n((n-1)(R-c)+c)^2]}{(-nR+(n-2)c)(n-2)}.$$

This completes our proof. □

Remark 3.1. As already pointed out in the Introduction, in [7] we proved that a complete spacelike hypersurface in $\mathbf{S}_1^{n+1}(1)$, $n \geq 3$, with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leq R \leq 1$ and with bounded mean curvature is totally umbilical.

Remark 3.2. When $R = c = 0$ and $p = 1$, $P_R(S)$ is the zero polynomial and cylinders over plane curves are non totally umbilical examples of hypersurfaces in $\mathbf{R}_1^{n+1}$ with vanishing scalar curvature. However, if we assume in addition that the mean curvature H is constant, we claim that M^n is either totally umbilical or isometric to a cylinder $\mathbf{R}^{n-k} \times \mathbf{S}^k$, $1 \leq k \leq n-1$.

Indeed, by Proposition 2.1 we may write

$$\begin{aligned} 0 = \square(nH) &\geq \sum_{i,j,k} h_{ijk}^2 + |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(c-H^2)\right) \\ &= \sum_{i,j,k} h_{ijk}^2 + |\Phi|^2 P_R(S) = \sum_{i,j,k} h_{ijk}^2. \end{aligned}$$

It yields $h_{ijk} \equiv 0$, $\forall i,j,k$. Consequently, M^n is a hypersurface of $\mathbf{R}_1^{n+1}$ with constant principal curvatures and, according to the congruence theorem of Abe-Koike-Yamaguchi [1], M^n is either totally umbilical or isometric to a cylinder $\mathbf{R}^{n-k} \times \mathbf{S}^k$, $1 \leq k \leq n-1$.

Proof of Corollary 1.1. As R is constant and $0 < R < \left(\frac{n-2}{n}\right)c$, by Lemma 2.2 it means that $\square$ is positive definite and so $\square$ is a second order elliptic operator. By virtue of our assumptions and Proposition 2.1, we get $\square(S) \geq 2nH\square(nH) \geq 0$. Since $\sup S$ is attained on M^n, by applying the *Maximum Principle* to elliptic equations (see [17]), we obtain that S is constant on M^n and it is clear from (3.1) that H is also constant.

As H is constant and $P_R(S) \geq 0$ by assumption, from (3.9) we have $0 = \square(nH) \geq |\Phi|^2 P_R(S) \geq 0$. Hence the equality in Proposition 2.1 holds and all the inequalities used to prove this proposition become equalities. It turns out that

$$\sum_{i,j,k} h_{ijk}^2 = 0 \quad \text{and} \quad |\mathrm{tr}(\Phi^{n+1})^3| = \frac{(n-2)}{\sqrt{n(n-1)}}|\Phi|^3. \tag{3.17}$$

Lemma 2.1 and that equality show that M^n is a spacelike hypersurface of $\mathbf{S}_1^{n+1}$ with at most two constant principal curvatures everywhere. Hence, it follows from the congruence theorem of Abe-Koike-Yamaguchi [1] that M^n is either totally umbilical or isometric to the hyperbolic cylinder $\mathbf{H}^1(\sinh r) \times \mathbf{S}^{n-1}(\cosh r)$, which finishes our proof. $\square$

Remark 3.3. We give now a brief exposition of a method developed hy Hu, Scherfner and Zhai [20] of constructing hypersurfaces with constant scalar curvature in $\mathbf{S}_p^{n+1}$ with two principal curvatures and constant scalar curvature R satisfying $0 < R < \left(\frac{n-2}{n}\right)c$. It will follow from Example 1 that the assumptions in Corollary 1.1 on the $\sup S$ can not be dropped.

Example 1. Let us consider $\mathbf{S}_1^{n+1}(1)$ as $\mathbf{S}_1^{n+1}(1) \subset \mathbf{R}_1^{n+2} = \mathbf{R}_1^n \times \mathbf{R}^2$, and denote the standard immersion by $x : \mathbf{H}^{n-1}(-1) \hookrightarrow \mathbf{R}^n$, with $\{\bar{e}_1, \dots, \bar{e}_n\}$ being a local orthonormal frame field in $\mathbf{R}^n$ such that $\{\bar{e}_1, \dots, \bar{e}_{n-1}\}$ is tangent to $\mathbf{H}^{n-1}(-1)$ and $x = \bar{e}_n$ is the timelike normal vector field.

Let us take a plane curve ξ in $\mathbf{R}^2 = \mathbf{C}$ with a given supporting function $h(\theta) \geq 0$.

The generic point $q(\theta)$ of ξ is expressed as

$$q(\theta) = e^{i(\theta - \pi/2)}(h(\theta) + ih'(\theta)). \tag{4.3}$$

The Frenet frame of ξ is given by

$$\bar{e}_{n+1} = e^{i\theta}, \quad \bar{e}_{n+2} = e^{i(\theta+\pi/2)} \tag{4.4}$$

and the arc length u of ξ is given by

$$du = \{h(\theta) + h''(\theta)\}\, d\theta. \tag{4.5}$$

Using $\bar{e}_{n+1}$ and $\bar{e}_{n+2}$, we have

$$q = h'\bar{e}_{n+1} - h\bar{e}_{n+2}, \quad dq = \bar{e}_{n+1}\, du. \tag{4.6}$$

Supposing ξ is in the outside of the unit circle, we define a function $\rho(\theta) > 0$ by

$$\rho^2 = \|q\|^2 - 1 = h^2 + (h')^2 - 1. \tag{4.7}$$

From now on, we assume $h + h'' > 0$. Define a spacelike hypersurface M^n by

$$\varphi : \mathbf{H}^{n-1}(-1) \times \mathbf{R} \to \mathbf{S}_1^{n+1}(1) \subset \mathbf{R}^{n+2},$$

with

$$\varphi = \rho\bar{e}_n + q = \rho\bar{e}_n + h'\bar{e}_{n+1} - h\bar{e}_{n+2}. \tag{4.8}$$

From a standard computation, we obtain the following second order differential equation

$$\begin{aligned} n(h^2-1)[R(h^2-1)+1]\frac{d^2h}{d\theta^2} - 2h\left(\frac{dh}{d\theta}\right)^2 \\ + h(h^2-1)[nR(h^2-1)+n-2] = 0. \end{aligned} \tag{3.18}$$

Therefore, by assuming R constant, we can solve (3.18) and determine h.

Proof of Corollary 1.2. Since $(\sup S)^2 = \sup S^2$, $(\sup H)^2 = \sup H^2$ and $(\sup|\Phi|)^2 = \sup|\Phi|^2$ from (3.4), we get

$$P_R(\sup S) = P(\sup H, \sup|\Phi|). \tag{3.19}$$

Consider the quadratic polynomial $P(x) = \dfrac{x^2}{p} - \dfrac{n(n-2)}{\sqrt{n(n-1)}} \sup Hx + n(c - \sup H^2)$ and denote by Δ its discriminant. It is easily seen that the assumption $\sup H^2 < \dfrac{4(n-1)c}{(n-2)^2 p + 4(n-1)}$ yields $\Delta < 0$. Then (3.19) shows that $P_R(\sup S) = P(\sup H, \sup|\Phi|) > 0$. By applying Theorem 1.1, we conclude that M^n is totally umbilical. □

Proof of Theorem 1.2. By using (2.13), it is easily checked that $\sum_{\alpha,\beta,i,j,k} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha)$, thus (2.18) implies

$$\begin{aligned} \frac{1}{2}\Delta S &= \frac{1}{2}\sum_{\alpha,i,j} \Delta(h_{ij}^\alpha)^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 + n\sum_{\alpha,i,j} h_{ij}^\alpha H_{ij}^\alpha + \frac{1}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha) \\ &\quad + \sum_{\alpha,i,j,k,m} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{\alpha,i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}. \end{aligned} \tag{3.20}$$

As R is constant, by (2.10), we have $\triangle S = \triangle(nH)^2$. From (2.37), we get also

$$\square(nH) = \frac{1}{2}\triangle S - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1} H_{ij}. \tag{3.21}$$

Note that the normalized mean curvature vector h is parallel, from (2.26) we have $\sum_{\alpha,i,j} h_{ij}^{\alpha} H_{ij}^{\alpha} = \sum_{i,j} h_{ij}^{n+1} H_{ij}$, which together with (3.20) and (3.21) imply

$$\square(nH) = \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}. \tag{3.22}$$

Next, we will obtain a pointwise estimate for the last two terms. For each fixed α, let λ_i^{α} be an eigenvalue of h^{α}, i.e. $h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij}$, and denotes by $\inf K$ the infimum of the sectional curvatures at a point p of M^n. Then

$$\begin{aligned} 2\left(\sum_{i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{i,j,k,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}\right) &= \sum_{i,k}(-2\lambda_i^{\alpha}\lambda_k^{\alpha})R_{ikik} + \sum_{i,k}((\lambda_i^{\alpha})^2 + (\lambda_k^{\alpha})^2)R_{ikik} \\ &= \sum_{i,k}(\lambda_i^{\alpha} - \lambda_k^{\alpha})^2 R_{ikik} \geq (\inf K)\sum_{i,k}(\lambda_i^{\alpha} - \lambda_k^{\alpha})^2 \\ &= (\inf K)(2nN(h^{\alpha}) - 2n^2(H^{\alpha})^2) = 2n(\inf K)N(\Phi^{\alpha}). \end{aligned} \tag{3.23}$$

Therefore

$$\sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + \sum_{\alpha,i,j,k,m} h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk} \geq n(\inf K)\sum_{\alpha} N(\Phi^{\alpha}) = n(\inf K)|\Phi|^2. \tag{3.24}$$

In view of $R_{ijij} \geq 0$, from (3.22), (3.23) and Lemma (2.2), we get

$$\square(nH) \geq \sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + n(\inf K)|\Phi|^2 + \frac{1}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha}) \geq 0. \tag{3.25}$$

As M^n is compact and $\square$ is self-adjoint, from (3.25), we deduce that

$$0 \geq \int_M \left(\sum_{\alpha,i,j,k} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{\alpha,\beta} N(h^{\alpha}h^{\beta} - h^{\beta}h^{\alpha})\right) dM.$$

It turns out that $h^{\alpha}h^{\beta} = h^{\beta}h^{\alpha}$ $\forall \alpha,\beta$ and so the normal bundle of M^n is flat. Furthermore, we have the equality $\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 = n^2|\nabla H|^2$, hence, from Lemma 2.2 we obtain that either H is constant or M^n lies in a totally geodesic subspace $\mathbf{S}_1^{n+1}(c)$ of $\mathbf{S}_p^{n+p}(c)$ and, in this case, the matrix h^{n+1} has rank 1.

If H is constant, then M^n has mean parallel vector and flat normal bundle thus, according to Theorem 1 in [3], we conclude that M^n is totally umbilical.

Otherwise, h^{n+1} has rank 1 and, from (3.23), we may write

$$2\left(\sum_{i,j,k,m} h_{ij}^{n+1}h_{km}^{n+1}R_{mijk} + \sum_{i,j,k,m} h_{ij}^{n+1}h_{mi}^{n+1}R_{mkjk}\right) = \sum_{i,k}(\lambda_i^{n+1} - \lambda_k^{n+1})^2 R_{ikik} = 2n^2(n-1)H^2. \tag{3.26}$$

Inserting (3.26) into (3.22) and taking into account that $\sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 = n^2|\nabla H|^2$, $N(h^\alpha h^\beta - h^\beta h^\alpha) = 0$ and the self-adjointness of $\Box$, we obtain

$$0 = \int_M \Box(nH)\,dM = n^2(n-1)\int_M H^2\,dM. \tag{3.27}$$

It shows that $H \equiv 0$, which leads to a contradiction. Consequently, M^n is totally umbilical. Since the sphere $\mathbf{S}^n(c_1)$ is the only compact totally umbilical spacelike submanifold of $\mathbf{S}_p^{n+p}(c)$, our proof is finished. $\Box$

Proof of Corollary 1.3. Corollary 1.3 follows immediately from *Myers' Theorem* and Theorem (1.2).

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