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THE HAUSDORFF DIMENSION OF THE SET OF DISSIPATIVE POINTS FOR A CANTOR-LIKE MODEL SET FOR SINGLY CUSPED PARABOLIC DYNAMICS

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Abstract

In this paper we introduce and study a certain intricate Cantor-like set $\mathscr{C}$ contained in unit interval. Our main result is to show that the set $\mathscr{C}$ itself, as well as the set of dissipative points within $\mathscr{C}$, both have Hausdorff dimension equal to 1. The proof uses the transience of a certain non-symmetric Cauchy-type random walk.

1. Introduction

In this paper we estimate the Hausdorff dimension of the set $\mathscr{C}_\infty$ of dissipative points within a certain Cantor-like subset $\mathscr{C}$ of the unit interval $[0,1) \subset \mathbf{R}$. Our motivation for considering the sets $\mathscr{C}$ and $\mathscr{C}_\infty$ stems from the investigations in [16] of the geometry of limit sets of Kleinian groups with singly cusped parabolic dynamics. However, for the purposes of this paper this link can safely be considered to be irrelevant. Nevertheless, to give at least a taste of this link we have included a brief summary of it in an appendix at the end of this paper. Let us remark that intuition coming from Kleinian groups has historically played a very important role in the development of Real and Complex Dynamics, and this paper can be seen as adding to this tradition.

Let us begin with by giving the slightly intricate, but nevertheless down-to-earth fractal geometric construction of the sets $\mathscr{C}$ and $\mathscr{C}_\infty$. For this we have to define certain families of fundamental intervals by induction as follows. We start with the unit interval $[0,1)$, and then partition the left half of $[0,1)$ into the infinitely many intervals

$$I_1 := \left[0, \frac{3}{\pi^2}\right), \quad \text{and} \quad I_{k+1} := \left[\frac{3}{\pi^2}\sum_{l=1}^{k} l^{-2}, \frac{3}{\pi^2}\sum_{l=1}^{k+1} l^{-2}\right), \quad \text{for } k \in \mathbf{N}.$$

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The family of these first-level intervals will be denoted by C_1. Note that the right half $\left[\frac{1}{2}, 1\right)$ of the unit interval, which is clearly not captured by C_1, should be interpreted as 'the hole at the first level'. The second step is to partition each element $I_{k_1} \in C_1$ as follows. By starting from the left endpoint of I_{k_1}, we partition the left half of I_{k_1} into infinitely many mutually adjacent intervals

$$I_{k_1 k_1+1}, \ldots, I_{k_1 k_1+l}, \ldots,$$

where the diameters of these intervals are given by

$$|I_{k_1 k_1+l}| = \frac{3}{\pi^2} \frac{|I_{k_1}|}{l^2}, \quad \text{for } l \in \mathbf{N}.$$

Similarly, by starting from the right endpoint of I_{k_1} we insert into the right half of I_{k_1} the (k_1+1) mutually adjacent intervals

$$I_{k_1 k_1}, I_{k_1 k_1-1}, \ldots, I_{k_1 0},$$

with diameters given by

$$|I_{k_1 k_1-l}| = \begin{cases} \dfrac{3}{\pi^2} \dfrac{|I_{k_1}|}{(2k_1)^2} & \text{for } l = 0 \\ \dfrac{3}{\pi^2} \dfrac{|I_{k_1}|}{l^2} & \text{for } l \in \{1, \ldots, k_1\}. \end{cases}$$

The family of these second-level intervals will be denoted by C_2. Note that in this way we have perforated each $I_{k_1} \in C_1$ such that there is a 'hole' in I_{k_1} with diameter of order $|I_{k_1}|/k_1$.

We then proceed by induction as follows. Suppose that for $n \geq 2$ the n-th level interval $I_{k_1 \cdots k_n}$ has been constructed. The $(n+1)$-th level intervals arising from $I_{k_1 \cdots k_n}$ are then obtained as follows. There are two cases to consider. The first case is that $k_n = 0$, and here the partition only continues in the left half of $I_{k_1 \cdots k_{n-1} 0}$. More precisely, in this case we start from the left endpoint of $I_{k_1 \cdots k_{n-1} 0}$ and partition the left half of $I_{k_1 \cdots k_{n-1} 0}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \cdots k_{n-1} 0 1}, \ldots, I_{k_1 \cdots k_{n-1} 0 l}, \ldots,$$

with diameters given by

$$|I_{k_1 \cdots k_{n-1} 0 l}| = \frac{3}{\pi^2} \frac{|I_{k_1 \cdots k_n}|}{l^2}, \quad \text{for } l \in \mathbf{N}.$$

In the second case we have that $k_n \in \mathbf{N}$, and here we start from the left endpoint of $I_{k_1 \cdots k_n}$ and partition the left half of $I_{k_1 \cdots k_n}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \cdots k_n k_n+1}, \ldots, I_{k_1 \cdots k_n k_n+l}, \ldots.$$

The diameters of these intervals are

$$(1) \qquad |I_{k_1\cdots k_n k_n+l}| = \frac{3}{\pi^2}\frac{|I_{k_1\cdots k_n}|}{l^2}, \quad \text{for } l \in \mathbf{N}.$$

Similarly, by starting from the right endpoint of $I_{k_1\cdots k_n}$ we insert into the right half of $I_{k_1\cdots k_n}$ the $(k_n + 1)$ mutually adjacent intervals

$$I_{k_1\cdots k_n k_n}, I_{k_1\cdots k_n k_n-1}, \ldots, I_{k_1\cdots k_n 0},$$

with diameters given by

$$(2) \qquad |I_{k_1\cdots k_n k_n-l}| = \begin{cases} \dfrac{3}{\pi^2}\dfrac{|I_{k_1\cdots k_n}|}{(2k_n)^2} & \text{for } l = 0 \\ \dfrac{3}{\pi^2}\dfrac{|I_{k_1\cdots k_n}|}{l^2} & \text{for } l \in \{1, \ldots, k_n\}. \end{cases}$$

The so obtained set of intervals of the $(n+1)$-th level will be denoted by C_{n+1}. That is,

$$C_n = \{I_{k_1\cdots k_n} : k_1 \in \mathbf{N}, k_{i+1} \in \mathbf{N}_0 \text{ for } i \in \mathbf{N}, \text{ and if } k_i = 0 \text{ then } k_{i+1} \neq 0\}.$$

Again, note that by this we have perforated $I_{k_1\cdots k_n}$ such that in the first case 'the hole' is precisely the right half of $I_{k_1\cdots k_n}$, whereas in the second case the diameter of the hole is of order $|I_{k_1\cdots k_n}|/k_n$. Also, let us emphasize that by construction, the state 0 necessarily has to renew itself. That is, the generation following the interval $I_{k_1\cdots k_{n-1}0}$ is given by $\{I_{k_1\cdots k_{n-1}0k_{n+1}} : k_{n+1} \in \mathbf{N}\}$. Moreover, note that the system can only be stationary at states $k_n \in \mathbf{N}$, which means that if $I_{k_1\cdots k_n}$ is a given interval of some level n then $I_{k_1\cdots k_n k_n}$ exists if and only if $k_n \neq 0$. Finally, note that we always assume that the intervals $I_{k_1\ldots k_n}$are half open, namely closed to the left and open to the right.

With this inductive construction of the generating intervals $I_{k_1\cdots k_n}$ at hand, the Cantor-like set $\mathscr{C}$ is defined by

$$\mathscr{C} := \bigcap_{n\in\mathbf{N}} \bigcup_{I\in C_n} I.$$

Next, we define the set of dissipative points in $\mathscr{C}$. For this we require the following canonical coding of the elements in $\mathscr{C}$. A finite or infinite sequence $(k_1, k_2, \ldots)$ is called admissable if $I_{k_1\cdots k_n} \in C_n$, for all $n \in \mathbf{N}$. Clearly, the diameter of $I_{k_1\cdots k_n}$ tends to zero as n tends to ∞, for every fixed infinite admissable sequence $(k_n)_{n\in\mathbf{N}}$, and therefore,

$$\bigcap_{n=1}^{\infty} I_{k_1\cdots k_n} \text{ is a singleton.}$$

In particular, each $x \in \mathscr{C}$ is coded uniquely by an infinite admissable sequence, and this gives rise to the bijection

$$\rho : \Sigma \to \mathscr{C}, \quad (k_1, k_2, \ldots) \mapsto \bigcap_{n=1}^{\infty} I_{k_1\cdots k_n},$$

where Σ refers to the set of all admissable sequences. Using this coding, the set $\mathscr{C}_\infty \subset \mathscr{C}$ of dissipative points is then given by

$$\mathscr{C}_\infty := \left\{ x \in \mathscr{C} : x = \rho(k_1, k_2, \ldots) \text{ and } \lim_{n \to \infty} k_n = \infty \right\}.$$

The following theorem gives the main result of this paper. Here, $\dim_H$ refers to the Hausdorff dimension.

Main Theorem. *For $\mathscr{C}$ and $\mathscr{C}_\infty$ as defined above, we have*

$$\dim_H(\mathscr{C}_\infty) = \dim_H(\mathscr{C}) = 1.$$

Remark 1.1. Note that our analysis does not allow to draw any conclusion for the Lebesgue measure of $\mathscr{C}$ and/or $\mathscr{C}_\infty$. However, recent studies in the theory of Kleinian groups (cf. [1] [6], and also [7]) have confirmed the Ahlfors Conjecture, and applying this loosely to our situation here, this strongly suggests that $\mathscr{C}$ and $\mathscr{C}_\infty$ are both of 1-dimensional Lebesgue measure equal to 0. However, here we can only conjecture that the Lebesgue measure of $\mathscr{C}$ and $\mathscr{C}_\infty$ is equal to zero, and it would be desirable to have an elementary proof of this conjecture.

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2. Proof of the Main Theorem

Since $\mathscr{C}_\infty \subset \mathscr{C} \subset [0, 1)$, we have

$$\dim_H(\mathscr{C}_\infty) \leq \dim_H(\mathscr{C}) \leq 1.$$

Therefore, our strategy will be to construct a family of probability measures μ_α on $\mathscr{C}_\infty$, for $1/2 < \alpha < 1$, such that the Hausdorff dimension $\dim_H(\mu_\alpha)$ of the measure μ_α tends to 1 for α tending to 1. Clearly, this will then be sufficient for the proof of the Main Theorem.

2.1. The family of measures μ_α

Let $1/2 < \alpha \leq 1$ be fixed. We then define a set function μ_α on the intervals $I_{k_1 \cdots k_n}$ by induction in the following way. Define $I_0 := [0, 1)$ and set $I_{0k} := I_k$ for all $k \in \mathbf{N}$. Then let

$$\mu_\alpha(I_0) := 1,$$

and define $\mu_\alpha(I_{k_1\cdots k_n k_{n+1}})$ for each finite admissable sequence $(k_1, \ldots, k_{n+1})$ as follows. With $\zeta(s) := \sum_{m=1}^{\infty} m^{-s}$ referring to the Riemann zeta function, we define for $k_n \neq k_{n+1}$,

$$\mu_\alpha(I_{k_1\cdots k_n k_{n+1}}) := \begin{cases} \dfrac{\mu_\alpha(I_{k_1\cdots k_n})}{2\zeta(2\alpha)} \left(\dfrac{1}{|k_{n+1} - k_n|^{2\alpha}} + \dfrac{1}{(k_{n+1} + k_n)^{2\alpha}} \right) & \text{for } k_{n+1} \neq 0 \\ \dfrac{\mu_\alpha(I_{k_1\cdots k_n})}{2\zeta(2\alpha)} \dfrac{1}{k_n^{2\alpha}} & \text{for } k_{n+1} = 0. \end{cases}$$

Also, for $k_n = k_{n+1}$ let

$$\mu_\alpha(I_{k_1\cdots k_n k_n}) := \frac{\mu_\alpha(I_{k_1\cdots k_n})}{2\zeta(2\alpha)} \frac{1}{(2k_n)^{2\alpha}}.$$

On the first sight, this definition of the set function μ_α might appear to be slightly artificial. However, in the next section we will see that this definition reflects the transition probabilities of a certain (transient) random walk on $\mathbf{N}_0$, and therefore is rather canonical. Before we come to this, let us first state the following consistency property for μ_α. This property can also be deduced using the random walk of Section 2.2. Nevertheless, the following gives an elementary proof of this consistency property.

LEMMA 2.1. *For each finite admissable sequence* $(k_1, \ldots, k_n)$, *we have*

$$\mu_\alpha(I_{k_1\cdots k_n}) = \sum_{k_{n+1} \geq 0} \mu_\alpha(I_{k_1\cdots k_n k_{n+1}}).$$

Proof. For $k_n = 0$, we have

$$\sum_{\substack{k_{n+1} \geq 0 \\ k_{n+1} \neq 0}} \mu_\alpha(I_{k_1\cdots k_{n-1} 0 k_{n+1}}) = \frac{\mu_\alpha(I_{k_1\cdots k_{n-1} 0})}{2\zeta(2\alpha)} \sum_{l=1}^{\infty} \frac{2}{l^{2\alpha}} = \mu_\alpha(I_{k_1\cdots k_{n-1} 0}).$$

If $k_n \neq 0$, then we compute

$$\begin{aligned} &\sum_{k_{n+1} \geq 0} \mu_\alpha(I_{k_1\cdots k_n k_{n+1}}) \\ &\quad = \sum_{\substack{k_{n+1} > 0 \\ k_{n+1} \neq k_n}} \frac{1}{2\zeta(2\alpha)} \left(\frac{1}{|k_{n+1} - k_n|^{2\alpha}} + \frac{1}{(k_{n+1} + k_n)^{2\alpha}} \right) \mu_\alpha(I_{k_1\cdots k_n}) \\ &\qquad + \frac{1}{2\zeta(2\alpha) k_n^{2\alpha}} \mu_\alpha(I_{k_1\cdots k_n}) + \frac{1}{2\zeta(2\alpha)(2k_n)^{2\alpha}} \mu_\alpha(I_{k_1\cdots k_n}) \end{aligned}$$

$$= \frac{\mu_\alpha(I_{k_1\cdots k_n})}{2\zeta(2\alpha)} \left(\sum_{\substack{k_{n+1}>0 \\ k_{n+1}\neq k_n}} \left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}} + \frac{1}{(k_{n+1}+k_n)^{2\alpha}} \right) + \frac{1}{k_n^{2\alpha}} + \frac{1}{(2k_n)^{2\alpha}} \right)$$

$$= \frac{\mu_\alpha(I_k 1\cdots k_n)}{2\zeta(2\alpha)} \left(\sum_{k_{n+1}>k_n} \frac{1}{(k_{n+1}-k_n)^{2\alpha}} + \sum_{0<k_{n+1}<k_n} \frac{1}{(k_n-k_{n+1})^{2\alpha}} \right.$$

$$\left. + \sum_{0<k_{n+1}<k_n} \frac{1}{(k_{n+1}+k_n)^{2\alpha}} + \sum_{k_{n+1}>k_n} \frac{1}{(k_{n+1}+k_n)^{2\alpha}} + \frac{1}{k_n^{2\alpha}} + \frac{1}{(2k_n)^{2\alpha}} \right)$$

$$= \frac{\mu_\alpha(I_{k_1\cdots k_n})}{2\zeta(2\alpha)} \cdot 2 \sum_{l=1}^{\infty} \frac{1}{l^{2\alpha}} = \mu_\alpha(I_{k_1\cdots k_n}). \qquad \square$$

The following is an immediate consequence of the previous lemma.

Corollary 2.2. *The measure μ_α is a probability measure on $\mathscr{C}$.*

2.2. The associated random walk

In this section we show that the measure μ_α can be interpreted in terms of a certain random walk. In particular, this will give that μ_α has the Markov property. For this, let the random variables $\tilde{X}_n^\alpha$ be defined by the probability (with respect to μ_α) being in the interval $I_{k_1\cdots k_n k_{n+1}}$ given that in the previous step the process has been in the interval $I_{k_1\cdots k_n}$. That is, the random variables $\tilde{X}_n^\alpha$ is given as follows.

- For $k_{n+1} > 0$ such that $k_{n+1} \neq k_n$, let

$$\mathbf{P}(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n k_{n+1}} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}) := \frac{\mu_\alpha(I_{k_1\cdots k_n k_{n+1}})}{\mu_\alpha(I_{k_1\cdots k_n})}$$
$$= \frac{1}{2\zeta(2\alpha)} \left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}} + \frac{1}{(k_{n+1}+k_n)^{2\alpha}} \right).$$

- For $k_{n+1} > 0$ such that $k_{n+1} = k_n$, let

$$\mathbf{P}(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n k_n} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}) := \frac{\mu_\alpha(I_{k_1\cdots k_n k_n})}{\mu_\alpha(I_{k_1\cdots k_n})} = \frac{1}{2\zeta(2\alpha)} \frac{1}{(2k_n)^{2\alpha}}.$$

- If $k_{n+1} = 0$, then

$$\mathbf{P}(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n 0} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}) := \frac{\mu_\alpha(I_{k_1\cdots k_n 0})}{\mu_\alpha(I_{k_1\cdots k_n})} = \frac{1}{2\zeta(2\alpha)} \frac{1}{k_n^{2\alpha}}.$$

Clearly, these conditional probabilities do not depend on $k_1, \ldots, k_{n-1}$. Hence, we can define an associated random walk X_n^α on $\mathbf{N}_0$ by the following transition probabilities.

- For $l, m \in \mathbf{N}_0$, let

$$\mathbf{P}(X_{n+1}^\alpha = l \mid X_n^\alpha = m) := \begin{cases} \frac{1}{2\zeta(2\alpha)}\left(\frac{1}{|m-l|^{2\alpha}} + \frac{1}{(m+l)^{2\alpha}}\right) & \text{for } l \neq 0,\ l \neq m \\ \frac{1}{2\zeta(2\alpha)}\frac{1}{(2m)^{2\alpha}} & \text{for } l \neq 0,\ l = m \\ \frac{1}{2\zeta(2\alpha)}\frac{1}{m^{2\alpha}} & \text{for } l = 0,\ m \neq 0 \\ 0 & \text{for } l = m = 0. \end{cases}$$

The random walk X_n^α is very closely connected to our original geometric setting, since it allows to recover the measure μ_α as follows.

$$\begin{aligned} &\mathbf{P}(X_1^\alpha = k_1, \ldots, X_n^\alpha = k_n) \\ &\quad = \mathbf{P}(X_n^\alpha = k_n \mid X_{n-1}^\alpha = k_{n-1}) \cdot \mathbf{P}(X_{n-1}^\alpha = k_{n-1} \mid X_{n-2}^\alpha = k_{n-2}) \\ &\qquad \cdot \ldots \cdot \mathbf{P}(X_1^\alpha = k_1 \mid X_0^\alpha = 0) \\ &\quad = \frac{\mu_\alpha(I_{k_1\cdots k_n})}{\mu_\alpha(I_{k_1\cdots k_{n-1}})}\frac{\mu_\alpha(I_{k_1\cdots k_{n-1}})}{\mu_\alpha(I_{k_1\cdots k_{n-2}})}\cdots\frac{\mu_\alpha(I_{k_1})}{\mu_\alpha([0,1))} = \mu_\alpha(I_{k_1\cdots k_n}). \end{aligned}$$

The aim now is to show that the random walk X_n^α is transient. This will then allow us to deduce that μ_α is non-trivial on $\mathscr{C}_\infty$.

THEOREM 2.3. *For each* $1/2 < \alpha < 1$, *the random walk* X_n^α *on* $\mathbf{N}_0$ *is transient. That is, we have* $\mathbf{P}$*-almost surely*,

$$\lim_{n\to\infty} X_n^\alpha = \infty.$$

Proof. Let $1 < \beta < 2$ be fixed, and consider the Cauchy-type random walk Y_n^β on $\mathbf{Z}$, given by the transition probabilities

$$\mathbf{P}(Y_{n+1}^\beta = m + l \mid Y_n^\beta = m) := \frac{1}{2\zeta(\beta)}\frac{1}{|l|^\beta} \quad \text{for } n \in \mathbf{N},\ m \in \mathbf{Z} \text{ and } l \in \mathbf{Z}\backslash\{0\}.$$

It is well known that Y_n^β is symmetric, and that Y_n^β is transient if and only if $\beta < 2$. Let $\tilde{Y}_n^\beta$ denote the non-negative random walk $|Y_n^\beta|$ which arises from Y_n^β. Clearly, $\tilde{Y}_n^\beta$ can be thought of as being a mirror image of Y_n^β, in the sense that the non-positive part of Y_n^β gets reflected at the origin, and hence becomes non-negative. Moreover, note that since Y_n^β is a symmetric random walk, this modification of Y_n^β to $\tilde{Y}_n^\beta$ does not effect the probabilities of any of the sample paths. Therefore, it immediately follows that the transience of Y_n^β implies that $\tilde{Y}_n^\beta$ is transient. Using the fact that by symmetry of Y_n^β we have $\mathbf{P}(Y_n^\beta = m) = \mathbf{P}(Y_n^\beta = -m)$, we now compute the transition probabilities of the random walk $\tilde{Y}_n^\beta$ on $\mathbf{N}_0$ as follows. For $l, m \in \mathbf{N}_0$ such that $l \neq 0$, and $m \neq l$, we have

$$\begin{aligned}
\mathbf{P}(\tilde{Y}^{\beta}_{n+1} = l \mid \tilde{Y}^{\beta}_{n} = m) &= \mathbf{P}(|Y^{\beta}_{n+1}| = l \mid |Y^{\beta}_{n}| = m) \\
&= \frac{\mathbf{P}(|Y^{\beta}_{n+1}| = l,\, Y^{\beta}_{n} = m \text{ or } Y^{\beta}_{n} = -m)}{\mathbf{P}(Y^{\beta}_{n} = m \text{ or } Y^{\beta}_{n} = -m)} \\
&= \frac{\mathbf{P}(|Y^{\beta}_{n+1}| = l,\, Y^{\beta}_{n} = m)}{2\mathbf{P}(Y^{\beta}_{n} = m)} + \frac{\mathbf{P}(|Y^{\beta}_{n+1}| = l,\, Y^{\beta}_{n} = -m)}{2\mathbf{P}(Y^{\beta}_{n} = -m)} \\
&= \frac{\mathbf{P}(|Y^{\beta}_{n+1}| = l,\, Y^{\beta}_{n} = m)}{\mathbf{P}(Y^{\beta}_{n} = m)} = \frac{1}{2\zeta(\beta)}\left(\frac{1}{|m-l|^{\beta}} + \frac{1}{(m+l)^{\beta}}\right).
\end{aligned}$$

Similarly, we obtain for $l = m \neq 0$,

$$\mathbf{P}(\tilde{Y}^{\beta}_{n+1} = m \mid \tilde{Y}^{\beta}_{n} = m) = \frac{1}{2\zeta(\beta)} \frac{1}{(2m)^{\beta}},$$

and for $l = 0$ and $m \neq l$,

$$\mathbf{P}(\tilde{Y}^{\beta}_{n+1} = 0 \mid \tilde{Y}^{\beta}_{n} = m) = \frac{1}{2\zeta(\beta)} \frac{1}{m^{\beta}}.$$

Finally, note that we immediately have

$$\mathbf{P}(\tilde{Y}^{\beta}_{n+1} = 0 \mid \tilde{Y}^{\beta}_{n} = 0) = 0.$$

This shows that the transition probabilities of $\tilde{Y}^{\beta}_{n}$ coincide with the ones of $X^{\beta/2}_{n}$. Therefore, since $\tilde{Y}^{\beta}_{n}$ is transient, it follows that $X^{\beta/2}_{n}$ is transient. This finishes the proof of the theorem. □

As already announced before, Theorem 2.3 has the following important implication.

COROLLARY 2.4. *For every* $1/2 < \alpha < 1$, *we have*

$$\mu_{\alpha}(\mathscr{C}_{\infty}) = 1.$$

Remark 2.5. Note that the proof of Theorem 2.3 relies heavily on the fact that $1/2 < \alpha < 1$. Namely, for instance for $\alpha = 1$ the associated random walk is recurrent, and consequently the measure μ_{α} vanishes on $\mathscr{C}_{\infty}$.

2.3. Approximating the essential support of μ_{α}

In order to prepare our estimate of the lower pointwise dimension of μ_{α}, we need a further approximation of the essential support of this measure. We will see that μ_{α}-almost surely the diameters of the coding intervals of an element of $\mathscr{C}_{\infty}$ do not shrink too fast. For this we define, for $\gamma \in \mathbf{R}$,

$$\mathscr{C}^{\gamma}_{\infty} := \left\{ x \in \mathscr{C}_{\infty} : x = \rho(k_1, k_2, \ldots) \text{ such that } \limsup_{n\to\infty} \frac{|k_{n+1} - k_n|}{n^{\gamma}} \leq 1 \right\}.$$

LEMMA 2.6. *For each* $1/2 < \alpha < 1$ *and* $\gamma > 1/(2\alpha - 1)$, *we have*

$$\mu_\alpha(\mathscr{C}^\gamma_\infty) = 1.$$

Proof. The proof is an easy consequence of the Borel-Cantelli lemma. Indeed, first note that for $\beta = 2\alpha$ and $k \in \mathbf{N}$ we have

$$\mathbf{P}(|Y^\beta_n - Y^\beta_{n-1}| \geq k) \geq \mathbf{P}(|\tilde{Y}^\beta_n - \tilde{Y}^\beta_{n-1}| \geq k).$$

The latter is an immediate consequence of the fact that the random walk Y^β_n has the same distribution as $\tilde{Y}^\beta_n$, but without reflections at 0. Hence, it suffices to prove the lemma for the symmetric random walk Y^β_n. For this we define

$$p^\gamma_n := \mathbf{P}(|Y^\beta_n - Y^\beta_{n-1}| \geq n^\gamma).$$

We then have, for each $n \in \mathbf{N}$ and with $c(\beta) > 0$ referring to some universal constant,

$$p^\gamma_n = \frac{1}{\zeta(\beta)} \sum_{k=n^\gamma}^{\infty} \frac{1}{k^\beta} \leq c(\beta) \frac{1}{n^{\gamma(\beta-1)}}.$$

Since the series $\sum_{n=1}^\infty p^\gamma_n$ converges for $\gamma > 1/(\beta - 1)$, the Borel-Cantelli Lemma implies that $\mathbf{P}$-almost surely there are at most finitely many n which satisfy the inequality

$$|Y^\beta_n - Y^\beta_{n-1}| \geq n^\gamma.$$

This shows that for μ_α-almost every $x = \rho(k_1, k_2, \ldots) \in \mathscr{C}$ we have

$$\limsup_{n\to\infty} \frac{|k_{n+1} - k_n|}{n^\gamma} \leq 1. \qquad \square$$

2.4. The lower pointwise dimension on fundamental intervals

The main result of this section will be the following estimate for the lower pointwise dimension of the measure μ_α restricted to the fundamental intervals $I_{k_1\cdots k_n}$.

PROPOSITION 2.7. *For each* $\varepsilon > 0$ *there exists* $1/2 < \alpha < 1$ *and* $\gamma > 1/(2\alpha - 1)$ *such that for every* $x = \rho(k_1, k_2, \ldots) \in \mathscr{C}^\gamma_\infty$,

$$\liminf_{n\to\infty} \frac{\log \mu_\alpha(I_{k_1\cdots k_n})}{\log|I_{k_1\cdots k_n}|} \geq \alpha - \varepsilon.$$

Furthermore, in here we have that α *tends to* 1 *for* ε *tending to* 0.

Proof. Since we are interested in the asymptotic behaviour of $I_{k_1\cdots k_n}$ for points $x = \rho(k_1, k_2, \ldots) \in \mathscr{C}^\gamma_\infty$, Corollary 2.4 and Lemma 2.6 imply that we can assume without loss of generality that $k_n > 0$ and $|k_{n+1} - k_n| \leq n^\gamma$, for all $n \in \mathbf{N}$. Furthermore, for ease of exposition we only consider sequences which do not contain repetitions. That is, we assume that $k_n \neq k_{n+1}$, for all $n \in \mathbf{N}$. The case with repetitions can be dealt with in similar way and is left to the reader. Using the definition of μ_α, we then have

$$\begin{aligned}
\frac{\log \mu_\alpha(I_{k_1\cdots k_n k_{n+1}})}{\log|I_{k_1\cdots k_n k_{n+1}}|} &= \frac{-\log(2\zeta(2\alpha))}{\log|I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\left(\dfrac{1}{|k_{n+1}-k_n|^{2\alpha}} + \dfrac{1}{(k_n+k_{n+1})^{2\alpha}}\right)\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log|I_{k_1\cdots k_n k_{n+1}}|} \\
&\geq \frac{-\log(2\zeta(2\alpha))}{\log|I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\dfrac{2}{|k_{n+1}-k_n|^{2\alpha}}\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log|I_{k_1\cdots k_n k_{n+1}}|} \\
&= \frac{-\log(2\zeta(2\alpha))}{\log|I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\dfrac{2\cdot 2^\alpha\zeta(2)^\alpha|I_{k_1\cdots k_{n+1}}|^\alpha}{|I_{k_1\cdots k_n}|^\alpha}\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log|I_{k_1\cdots k_n k_{n+1}}|} \\
&= \frac{\log\dfrac{2^\alpha\zeta(2)^\alpha}{\zeta(2\alpha)}}{\log|I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log(|I_{k_1\cdots k_{n+1}}|^\alpha)}{\log|I_{k_1\cdots k_{n+1}}|} + \frac{\log\dfrac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|} \\
&= \alpha + \frac{\log\dfrac{2^\alpha\zeta(2)^\alpha}{\zeta(2\alpha)}}{\log|I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\dfrac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|}.
\end{aligned}$$

Since

$$|I_{k_1\cdots k_{n+1}}| < \left(\frac{1}{2\zeta(2)}\right)^n, \tag{3}$$

it follows for each $\kappa > 0$ and for all n sufficiently large,

$$\frac{\log\dfrac{2^\alpha\zeta(2)^\alpha}{\zeta(2\alpha)}}{\log|I_{k_1\cdots k_n k_{n+1}}|} > -\kappa. \tag{4}$$

Clearly, we even have that the limit of the latter expression is equal to 0. This settles the second term in the final line in the above calculation. The third term is more subtle, and for this we proceed as follows. Using (1) and (2), we derive with the convention $k_0 \equiv 0$,

$$|I_{k_1\cdots k_n}|^\alpha = \prod_{i=0}^{n-1}\left(\frac{1}{2^\alpha\zeta(2)^\alpha}\,\frac{1}{|k_{i+1}-k_i|^{2\alpha}}\right).$$

Similarly, using the recursive definition of μ_α, we obtain

$$\mu_\alpha(I_{k_1\cdots k_n}) = \prod_{i=0}^{n-1}\left(\frac{1}{2\zeta(2\alpha)}\left[\frac{1}{|k_{i+1}-k_i|^{2\alpha}} + \frac{1}{(k_{i+1}+k_i)^{2\alpha}}\right]\right).$$

Hence,

$$
\frac{\log \dfrac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|} = \frac{\log \dfrac{\prod_{i=0}^{n-1}\left(\dfrac{1}{2\zeta(2\alpha)}\left[\dfrac{1}{|k_{i+1}-k_i|^{2\alpha}}+\dfrac{1}{(k_{i+1}+k_i)^{2\alpha}}\right]\right)}{\prod_{i=0}^{n-1}\left(\dfrac{1}{2^\alpha\zeta(2)^\alpha}\,\dfrac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)}}{\log\left(\prod_{i=0}^{n}\left(\dfrac{1}{2^\alpha\zeta(2)^\alpha}\,\dfrac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)\right)}
$$

$$
= \frac{\log\left(\left[\dfrac{2^\alpha\zeta(2)^\alpha}{2\zeta(2\alpha)}\right]^n \prod_{i=0}^{n-1}\left[1+\left(\dfrac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right)^{2\alpha}\right]\right)}{\log\left(\left[\dfrac{1}{2^\alpha\zeta(2)^\alpha}\right]^{n+1}\prod_{i=0}^{n}\dfrac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)}
$$

$$
= \frac{n\alpha\log(2\zeta(2)) - n\log(2\zeta(2\alpha)) + \sum_{i=0}^{n-1}\log\left(1+\left[\dfrac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{-(n+1)\alpha\log(2\zeta(2)) + \sum_{i=0}^{n}\log\dfrac{1}{|k_{i+1}-k_i|^{2\alpha}}}
$$

$$
= \frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \dfrac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\dfrac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\dfrac{n+1}{n}\alpha\log(2\zeta(2)) + \dfrac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}.
$$

Let $\kappa > 0$ be fixed. We then distinguish the following two cases.

First, if for some $n \in \mathbf{N}$ we have

$$
\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right) < \kappa,
$$

then we obtain for α sufficiently close to 1,

$$
\begin{aligned}
&\frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \dfrac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\dfrac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\dfrac{n+1}{n}\alpha\log(2\zeta(2)) + \dfrac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}} \\
&\quad \geq -(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))) - \frac{\kappa}{\alpha\log(2\zeta(2)) + \dfrac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}} \\
&\quad \geq -(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))) - \kappa \geq -2\kappa.
\end{aligned}
$$

Here we made use of the fact that

$$\lim_{\alpha\to 1}(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))) = 0,$$

which implies $\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) < \kappa$, for all α sufficiently close to 1. Also, note that in here the lower bound on α depends only on κ and not on n. In particular, we also have that α tends to 1 as κ tends to 0.

Before we start with the discussion of the second case, first note that since $0 < \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \le 1$, we clearly always have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right) \le \log 2. \tag{5}$$

Furthermore, since k_n tends to infinity, there exists $j(\kappa) \in \mathbf{N}$ such that

$$\log\frac{\kappa^2 k_i^2}{4} > \frac{2\log 2}{\kappa^2}, \quad \text{for all } i \ge j(\kappa). \tag{6}$$

Let us now come to the second case. That is, we now assume that for some $n \in \mathbf{N}$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right) \ge \kappa.$$

Since $x > \log(1+x)$ for all $x > 0$, we then have

$$\frac{1}{n}\sum_{i=0}^{n-1}\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \ge \kappa.$$

Let us make the following two observations. Firstly, using the fact that $0 < \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \le 1$, we can apply Chebyshev's Inequality, which gives that for n suffciently large,

$$\operatorname{card}\mathscr{I}_n > \kappa n,$$

where

$$\mathscr{I}_n := \left\{ i \in [j(\kappa), n] : \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \ge \frac{\kappa}{2}\right\}.$$

Secondly, note that for $1/2 < \alpha < 1$ the following implication holds.

$$\text{If}\quad \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \ge \frac{\kappa}{2}, \quad \text{then} \quad |k_{i+1}-k_i| > \frac{\kappa}{2}k_i.$$

Combining these two observations with (6), we then compute

$$\frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha} \geq \frac{\alpha}{n}\sum_{i\in\mathscr{I}_n}\log|k_{i+1}-k_i|^2 \geq \frac{\alpha}{n}\sum_{i\in\mathscr{I}_n}\log\frac{\kappa^2 k_i^2}{4}$$
$$\geq \frac{\alpha}{n}\sum_{i\in\mathscr{I}_n}\frac{2\log 2}{\kappa^2} > \frac{\log 2}{\kappa^2 n}\operatorname{card}\mathscr{I}_n > \frac{\log 2}{\kappa}.$$

Inserting this into our estimate above and using (5), it follows

$$\frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\frac{n+1}{n}\log(2\zeta(2)) + \frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}$$
$$\geq \frac{-\log 2}{\log(2\zeta(2))+\frac{\log 2}{\kappa}} = -\kappa\frac{\log 2}{\kappa\log(2\zeta(2))+\log 2}$$
$$\geq -\kappa.$$

This finishes the second case.

Combining the latter results with (4), and putting $\varepsilon := 3\kappa$, we have now shown that

$$\liminf_{n\to\infty}\frac{\log\frac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|} \geq -\varepsilon,$$

and hence,

$$\liminf_{n\to\infty}\frac{\log\mu_\alpha(I_{k_1\cdots k_n})}{\log|I_{k_1\cdots k_n}|} \geq \alpha-\varepsilon.$$

Since α tends to 1 for ε tending to 0, the proof is complete. □

2.5. The proof of the Main Theorem

Recall that the lower pointwise dimension $\underline{d}_\nu(x)$ of a Borel measure ν on $\mathbf{R}$ at a point $x\in\mathbf{R}$ is given by

$$\underline{d}_\nu(x) := \liminf_{r\to 0}\frac{\log\nu(B(x,r))}{\log r},$$

where $B(x,r)$ refers to the interval centred at x with diameter equal to $2r$. The idea is to apply the well-known Mass Distribution Principle of Frostman [9] and Billingsley [4] (see also e.g. [8]).

In order to be able to apply the Mass Distribution Principle, we still require the following straight forward generalization of Furstenberg's Lemma [10].

LEMMA 2.8. *Let ν be a Borel measure on* $\mathbf{R}$, *and let* (r_n) *be a sequence of positive numbers for which* $\lim_{n\to\infty} r_n = 0$ *and* $\lim_{n\to\infty}(\log r_{n+1}/\log r_n) = 1$. *We then have for every* $x \in \mathbf{R}$,

$$\underline{d}_\nu(x) = \liminf_{n\to\infty} \frac{\log \nu(B(x, r_n))}{\log r_n}.$$

Proof. For $r > 0$ we define $n = n(r) := \max\{k \in \mathbf{N} : r_k \geq r\}$. The assertion of the lemma is then an immediate consequence of the following simple calculation.

$$\frac{\log \nu(B(x,r))}{\log r} \geq \frac{\log \nu(B(x,r_n))}{\log r_{n+1}} = \frac{\log r_n}{\log r_{n+1}} \frac{\log \nu(B(x,r_n))}{\log r_n}. \qquad \square$$

Proof of the Main Theorem. Let $\varepsilon > 0$ be given, and then fix $1/2 < \alpha < 1$ and $\gamma > 1/(2\alpha - 1)$ as in Proposition 2.7. By Lemma 2.6 we have that in order to find a lower bound for $\dim_H(\mu_\alpha)$ it is sufficient to give an estimate for $\underline{d}_{\mu_\alpha}(x)$ from below, for each $x = \rho(k_1, k_2, \ldots) \in \mathscr{C}_\infty^\gamma$. For this note that Proposition 2.7 implies

$$\liminf_{n\to\infty} \frac{\log \mu_\alpha(I_{k_1\cdots k_n})}{\log|I_{k_1\cdots k_n}|} \geq \alpha - \varepsilon.$$

In order to deduce the desired lower bound for $\underline{d}_{\mu_\alpha}(x)$, we then use (3) and the definition of $\mathscr{C}_\infty^\gamma$, which gives for $r_n := |I_{k_1\cdots k_n}|$,

$$\lim_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} = 1 + \lim_{n\to\infty} \frac{\log \frac{|I_{k_1\cdots k_n k_{n+1}}|}{|I_{k_1\cdots k_n}|}}{\log|I_{k_1\cdots k_n}|}$$
$$\leq 1 + \lim_{n\to\infty} \frac{\gamma \log n}{n \log(2\zeta(2))} = 1.$$

Therefore, Lemma 2.8 implies that for each $x \in \mathscr{C}_\infty^\gamma$,

$$\underline{d}_{\mu_\alpha}(x) \geq \alpha - \varepsilon.$$

Combining this with Corollary 2.2, Corollary 2.4 and Lemma 2.6, we have by the Mass Distribution Principle,

$$\dim_H(\mathscr{C}) \geq \dim_H(\mathscr{C}_\infty) \geq \dim_H(\mathscr{C}_\infty^\gamma) \geq \dim_H(\mu_\alpha) \geq \alpha - \varepsilon.$$

Finally, note that by Proposition 2.7 we have that α tends to 1 for ε tending to 0. This completes the proof of the theorem. $\square$

3. Appendix

The geometry of horosperically tame Kleinian groups, revisited

In this appendix we give a brief discussion of the concepts 'singly cusped' and 'horospherical tameness'. The aim is to give a motivation for why the set $\mathscr{C}$

can be considered to be a 1-dimensional model for the geodesic dynamic in the Nielsen region of a 3-manifold with a simply degenerated ending arising from a singly cusped parabolic fixed point. For further details we refer to [13] [14] [16] [17].

Let G be a finitely generated Kleinian group and $M = \mathbf{H}^3/G$ the associated (oriented) hyperbolic 3-manifold. Recall that $L(G)$, the *limit set* of G, is the derived set of some arbitrary point in hyperbolic 3-space $\mathbf{H}^3$, that is the set of accumulation points of the G-orbit of that point. As usual, let $\Omega(G) = \hat{\mathbf{C}} - L(G)$ denote the set of ordinary points in the boundary at infinity $\hat{\mathbf{C}}$ of hyperbolic space. An element of $L(G)$ is called *radial limit point* if it admits a conical approach by orbit points from inside hyperbolic space [2] [15]. In order to introduce the concept 'singly cusped parabolic fixed points', we recall that a parabolic fixed point p of G has rank 1 or rank 2 depending on the type of its stabiliser G_p in G. Namely, p has *rank* 1 if G_p is isomorphic to a finite extension of $\mathbf{Z}$ (and so is necessarily cyclic or infinite dihedral), whereas p has *rank* 2 if G_p is isomorphic to a finite extension of $\mathbf{Z}^2$. Moreover, a rank 1 parabolic fixed point p is called *doubly cusped* if and only if there exists a pair of disjoint open discs in $\Omega(G)$ tangent at p, and p is called *bounded* if and only if it is either of rank 2 or else is doubly cusped. Then, by a well-known result of Beardon and Maskit [3] we have that a finitely generated Kleinian group G is geometrically finite if and only if every point of $L(G)$ is either a radial limit point or a bounded parabolic fixed point. In contrast to this, we now give the definition of a singly cusped rank 1 parabolic fixed points. By the result of Beardon and Maskit it will be clear that groups with such points are necessarily geometrically infinite, that is they are not geometrically finite. In particular, the class of singly cusped parabolic points provides interesting examples of geometrically infinite ends for hyperbolic 3-manifolds.

- A parabolic fixed point p of a Kleinian group G is called *singly cusped* if there exists an open disc $h_p \subset \Omega(G)$ with p on its boundary such that $L(G)$ intersects every open disc in $\hat{\mathbf{C}}$ which has p on its boundary and which is disjoint from h_p.

Let $N(G)$ refer to the Nielsen region of G, that is the convex hull in $\mathbf{H}^3$ of the limit set $L(G)$. The quotient $C(G) = N(G)/G$ is called the *convex core* of $M = \mathbf{H}^3/G$. It is standard to divide the Nielsen region and the convex core into two parts, called the thick part and the thin part. Given a positive number ε, the *ε-thick part* of $N(G)$ consists of all those points $x \in N(G)$ for which the hyperbolic ball of radius 2ε centred at x does not contain $g(x)$, for all $g \in G\backslash\{id.\}$. Likewise, the *ε-thick part* of $C(G)$ consists of those points $x \in C(G)$ whose ε-neighbourhood in $C(G)$ is an embedded ball. In both cases, the *ε-thin part* is defined to be the complement of the ε-thick part. By a well known result of Margulis [12], generalising a theorem of Leutbecher [11], there exists a universal constant ε_0, called the *Margulis constant*, such that the ε_0-thin part of $N(G)$ is contained in a disjoint union of tubes around the axes of loxodromic elements with translation length at most $2\varepsilon_0$ and a set of pairwise disjoint horoballs each tangent to $\hat{\mathbf{C}}$ at some point of the orbit $G(p)$. Note that the set $\{H_{g(p)} : g \in G/G_p\}$ of these so-called *Leutbecher horoballs* is necessarily precisely invariant under the action of G. Before discussing these horoballs further, let us now first introduce the class of horospherically tame Kleinian groups.

- A group G is said to be *horospherically tame* if there exists $\tau > 0$ such that $N(G)$ is contained in the union over all $g \in G$ of the hyperbolic τ-neighbourhoods $H^{\tau}_{g(p)}$ of the horoballs $H_{g(p)}$. In other words, the whole of $C(G)$ is contained in the hyperbolic τ-neighbourhood of the ε-thin part of $C(G)$.

For ease of exposition for the rest of this appendix we will assume that G is a horospherically tame group with a singly cusped rank 1 parabolic fixed point at $\{\infty\}$, and that $M = \mathbf{H}^3/G$ has no geometrically infinite ends other than the one arising from $\{\infty\}$ (for the existence of such groups see [16], Section 3). We can assume without loss of generality that our setting is normalised such that G_∞ is generated by the parabolic transformation $z \mapsto z+1$ and the Leutbecher horoball H_∞ is at height 1, that is $H_\infty = \{(z,t) \in \mathbf{H}^3 : t > 1\}$ (cf. [11]). By definition of singly cusped, there exists a horodisc h_∞ in $\Omega(G)$ with $\{\infty\}$ on its boundary. We take this to be the largest precisely invariant horodisc at $\{\infty\}$ in $\Omega(G)$, and we assume without loss of generality that this is normalised such that $h_\infty = \{z \in \mathbf{C} : \Im(z) > 0\}$. Then let h^*_∞ be the hyperbolic half space with ideal boundary h_∞. The pair (h_∞, H_∞) will be called the *horopair* at $\{\infty\}$. We then consider the image of the above configuration under the coset gG_∞. More precisely, for an element $g : z \mapsto (az+b)/(cz+d)$ (with $a,b,c,d \in \mathbf{C}$ and $ad - bc = 1$) of G not in the stabiliser of ∞ (that is $c \neq 0$), we clearly have that the point $g(\infty) = a/c$ is a singly cusped rank 1 parabolic fixed point. The horoball $H_g := g(H_\infty)$ is a Euclidean ball in $\mathbf{H}^3$ with

$$\text{radius } R_g := \frac{1}{2|c|^2} \quad \text{and centre} \quad \left(\frac{a}{c}, \frac{1}{2|c|^2}\right).$$

The horodisc $h_g := g(h_\infty)$ is a Euclidean disc in the Riemann sphere with

$$\text{radius } r_g := \frac{1}{|c\bar{d} - d\bar{c}|} = \frac{1}{2|c|^2 \Im(d/c)} \quad \text{and centre} \quad \frac{a\bar{d} - b\bar{c}}{c\bar{d} - d\bar{c}}.$$

Also, the half space $h^*_g := g(h^*_\infty)$ is the Euclidean hemisphere in $\mathbf{H}^3$ with the same centre and radius as h_g. The pair (h_g, H_g) will be referred to as the *horopair* at $g(\infty)$. We then define the drift of such a horopair (h_g, H_g) and the altitude of H_g as follows.

- The *altitude* α_g of the horoball H_g is defined to be the Euclidean distance of $g(\infty)$ from the boundary of h_∞. (Clearly, $\alpha_g := -\Im(a/c)$ measures how far H_g is along the cusp at ∞.)
- The *drift* δ_g of the horopair (h_g, H_g) measures the relative sizes of H_g and h_g, that is $\delta_g := R_g/r_g = \Im(d/c)$.

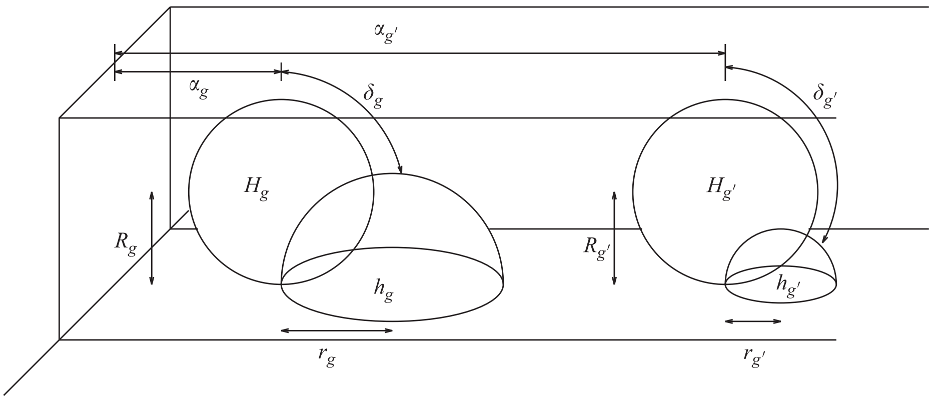

FIGURE 1. Horopairs for a map g with small drift and altitude, and a map g' with large drift and altitude.

Let us now consider in $\mathbf{H}^3 \cup \hat{\mathbf{C}}$ the fundamental domain for G_∞ which contains $0 \in \hat{\mathbf{C}}$, and in here in particular the part B which is in the complement of $H_\infty \cup h_\infty^*$, which is a 'semi-infinite box' (see Figure 1). Since G is horospherically tame, B contains a chain $\{H_{g_{k_1}} : k_1 \in \mathbf{N}\}$ of pairwise disjoint Leutbecher horoballs such that $H_\infty^\tau \cap H_{g_{k_1}}^\tau \neq \emptyset$ and $H_{g_{k_1}}^\tau \cap H_{g_{k_1+1}}^\tau \neq \emptyset$, for all $k_1 \in \mathbf{N}$. Clearly, the radii $R_{g_{k_1}}$ are all of comparable size roughly equal to $1/2$. An elementary calculation involving hyperbolic geometry (see [16], Lemma 2.2) then gives that if the altitude $\alpha_{g_{k_1}}$ of $H_{g_{k_1}}$ is of order say n then the drift $\delta_{g_{k_1}}$ must be of order $1/n$, and hence also $r_{g_{k_1}}$ has to be of order $1/n$. Now, the shade of each of these horoballs $H_{g_{k_1}}$ (when viewed from $\{\infty\}$) again contains a chain of pairwise disjoint horoballs $\{H_{g_{k_1 k_2}} : k_2 \in \mathbf{N}\}$ such that $H_{g_{k_1}}^\tau \cap H_{g_{k_1 k_2}}^\tau \neq \emptyset$ and $H_{g_{k_1 k_2}}^\tau \cap H_{g_{k_1 k_2+1}}^\tau \neq \emptyset$, for all $k_2 \in \mathbf{N}$. Clearly, these are ordered along the 'big horoball' $H_{g_{k_1}}$ following an obvious parabolic hierarchy, and the shadow in $\mathbf{C}$ (when viewed from $\{\infty\}$) of the union of their hyperbolic τ-neighbourhoods covers the shadow of $H_{g_{k_1}} \backslash h_{g_{k_1}}^*$. Obviously, this procedure can be continued indefinitely. In particular, it should be clear from this description why we refer to the Cantor-like set $\mathscr{C}$ which we considered in the previous sections as to the 1-dimensional model for the geodesic dynamics within the convex hull of the limit of a horospherically tame Kleinian group.

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