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KODAI MATH. J.
30 (2007), 394–408

ANALYTIC COMPUTATION OF SOME AUTOMORPHISM GROUPS OF RIEMANN SURFACES

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Abstract

Equations for the locus of Riemann Surfaces of genus three with a nonabelian automorphism group generated by involutions are determined from vanishings of Riemann's theta function.

Torelli's Theorem implies that all of the properties of a non-hyperelliptic compact Riemann Surface (complex algebraic curve) X are determined by its period matrix Ω. This paper shows how to compute the group $\operatorname{Aut} X$ of conformal automorphisms of a surface X of genus three using Ω, in the case when the group is nonabelian and generated by its involutions.

The connection between Ω and X is Riemann's theta function $\theta(z, \Omega)$. Accola ([1], [2], [3]), building on classical results about hyperelliptic surfaces, found relationships between the *theta divisor* $\Theta = \{z \in \operatorname{Jac}(X) : \theta(z, \Omega) = 0\}$ and $\operatorname{Aut} X$. In the case of genus three, certain vanishings of θ at quarter-periods of $\operatorname{Jac}(X)$ imply that X has an automorphism σ of degree two (or *involution*) such that $X/\langle\sigma\rangle$ has genus one (making σ an *elliptic-hyperelliptic involution*).

This work derives equations in the moduli space of surfaces of genus three for many of the loci consisting of surfaces with a given automorphism group. It is a two-step process. First, topological arguments determine the order of the dihedral group generated by two non-commuting involutions. Then, combinatorial arguments about larger groups generated by involutions determine the theta vanishings corresponding to each.

Much of the work here is based on the author's 1981 PhD dissertation [7] at Brown University. It appears now because of renewed interest in these questions, some of which is inspired by questions in coding theory: See [3], [5]. The research was directed by R. D. M. Accola, and Joe Harris was also a valuable resource. The author extends his (belated) thanks to them.

1. Preliminaries and notation

In all that follows, X is a compact Riemann Surface (or complex algebraic curve) of genus three with automorphism group $\operatorname{Aut} X$, period matrix Ω, jacobian

2000 *Mathematics Subject Classification*. 14H37 (Primary), 14H42 (Secondary).
Received December 7, 2006.

$\mathrm{Jac}(X)$, and theta-divisor Θ. Use $H_1(X,R)$ to denote singular homology with coefficient ring R.

Choice of a symplectic basis $\langle A_1,\dots,A_g,B_1,\dots,B_g\rangle$ for $H_1(X,\mathbf{R})$ and a normalized basis $\langle \omega_1,\dots,\omega_g\rangle$ for $H^{1,0}(X)$ leads to a period matrix of the form $[I|\Omega]$ and an *Abel–Jacobi map* $\mu : X \to \mathrm{Jac}(X)$. Every point of $\mathrm{Jac}(X)$ is a translate of the μ-image of a divisor class on X. This approach, among others, defines a map $\tilde{\pi} : \mathrm{Jac}(Y) \to \mathrm{Jac}(X)$, by lifting divisors using the holomorphic map $\pi : X \to Y$.

The point $\Omega a + b$ in $\mathrm{Jac}(X)$, where a and b are row vectors in $(\mathbf{R}/\mathbf{Z})^g$, is written $\begin{pmatrix} a \\ b \end{pmatrix}$. A point $\begin{pmatrix} a \\ b \end{pmatrix}$ of order n is called it a $\frac{1}{n}$*-period*, so $a,b \in \left(\frac{1}{n}\mathbf{Z}/\mathbf{Z}\right)^g$. Such a point corresponds to a cycle in $H_1(X,\mathbf{Z}/n\mathbf{Z})$, using the chosen basis. For half-periods, this identification makes the intersection product $*\cdot*$ on $H_1(x,\mathbf{Z}/2\mathbf{Z})$ into a sign-valued symmetric bilinear pairing, called the *Weil pairing*. Let ε and η be half-periods, and use the same symbols for their images in $H_1(X,\mathbf{Z}/2\mathbf{Z})$; then

$$|\varepsilon,\eta| = (-1)^{\varepsilon\cdot\eta}.$$

2. Theta vanishings and involutions

Let $\pi : X \to X/\langle\sigma\rangle$ denote the quotient map.

Lemma 1 [1]. *Let X be of genus three, let Θ be the theta divisor on* $\mathrm{Jac}(X)$, *and let $\sigma \in \mathrm{Aut}\,X$ be an elliptic-hyperelliptic involution. Then there is a half-period $e_1 \in \mathrm{Jac}(X)$ such that for any $\zeta \in \mathrm{Jac}(X/\langle\sigma\rangle)$, $\tilde{\pi}(\zeta) + e_1 \in \Theta$.*

Theorem 1. *Suppose that there are two quarter-periods $f_1, f_2 \in \Theta$ such that*:
(i) $f_1 \neq \pm f_2$;
(ii) $2f_1 = 2f_2 \neq 0$;
(iii) $|2f_1, f_1 + f_2| = 1$; *and*
(iv) $\theta(f_1,\Omega) = 0 = \theta(f_2,\Omega)$.
Then Aut X *contains an elliptic-hyperelliptic involution, and the theta-vanishings f_1 and f_2 arise as in Lemma* 1.

Definition 1. The conditions (i)–(iii) are the *admissibility conditions.*

Definition 2. The half-periods $2f_1$, $f_1 + f_2$, and $f_1 - f_2$ are the *derived half-periods* due to the involution.

3. Topology of surfaces with dihedral groups of automorphisms

When X has two non-commuting involutions, Aut X contains a dihedral group. This group acts on the homology of X in the usual way. Each involution's action has an invariant subspace, and this section shows how the relative position of these subspaces with respect to the Weil pairing depends on whether the dihedral group has a non-trivial center. In the case of genus three, the only dihedral groups that appear on non-hyperelliptic surfaces are D_3 and D_4, where only the latter has a nontrivial center, so the dihedral group generated by two non-commuting involutions is determined.

The first step is to determine the homomorphism on homology induced by an involution.

Definition 3. A branched cover $\pi : X_1 \to X_0$ of Riemann surfaces of respective genera g_1 and g_0 is *completely ramified* if the maximal unbranched cover $X' \to X_0$ through which π factors as $X_1 \to X' \to X_0$ is $X' \cong X_0$.

The key to understanding the topological effects of automorphisms is:

Lemma 2 (Accola). *Let $\pi : X_1 \to X_0$ be a completely ramified abelian cover of degree n. There there is a disk $\Delta_0 \subset X_0$ such that*

(a) *all of the ramification of π occurs over Δ_0; and*

(b) *$X_1 - \pi^{-1}(\Delta_0)$ has n connected components homeomorphic to $X_0 - \Delta_0$.*

If σ is an involution on X, then $\pi : X \to X/\langle\sigma\rangle$ is completely ramified. Figure One illustrates the situation when σ is an elliptic-hyperelliptic involution. The figure shows a symplectic homology basis for X that was chosen as

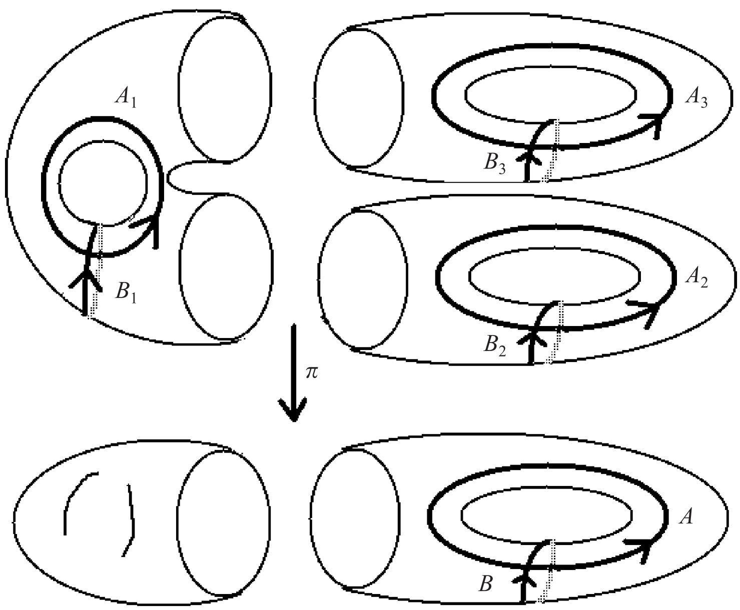

Figure 1. σ-basis

follows: since π branches at four points, the covering $\pi^{-1}(\Delta_0) \to \Delta_0$ corresponds to the finite part of the classical situation of an elliptic curve with a two-to-one mapping to $\mathbf{P}^1$, and the cycles A_1 and B_1 were picked in the classical manner (see, *e.g.*, [4, Chapter 10]). The homeomorphisms between $X/\langle\sigma\rangle - \Delta_0$ and each component of $X - \pi^{-1}(\Delta_0)$ are then used to lift a basis for $H_1(X/\langle\sigma\rangle - \Delta_0, \mathbf{Z}/2\mathbf{Z})$ to X.

Definition 4. A *σ-basis* for X is a homology basis picked as above.

4. Dihedral groups determined by theta vanishings

The two main theorems are below.

Theorem A. *Let $\{f_1, f_2\}$ and $\{f_3, f_4\}$ be two admissible sets of quarter-period vanishings with $2f_1 = 2f_2 \neq 2f_3 = 2f_4$. Then if $\langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4\rangle$ has type $(0, 2)$ the two corresponding involutions generate D_3. Conversely, if two involutions generate a D_3, the corresponding quarter-period vanishings generate a group of type $(0, 2)$.*

Theorem B. *Let $\{f_1, f_2\}$ and $\{f_3, f_4\}$ be two admissible sets of quarter-period vanishings with $2f_1 = 2f_2 \neq 2f_3 = 2f_4$. Then if $\langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4\rangle$ has type $(2, 1)$ the two corresponding involutions generate D_4. Conversely, if two involutions generate a D_4, the corresponding quarter-period vanishings generate a group of type $(2, 1)$.*

The *type* of a subgroup is defined by the following purely algebraic lemma, taken from [4], p. 294.

Lemma 3. *Let G be the subgroup of* $\mathrm{Jac}(X)$ *generated by the distinct half-periods $\varepsilon_1, \ldots, \varepsilon_r$. Then G has a basis $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{2n}\}$ with $m + 2n = r$, $m + n \leq g$, and, for all i and j, $|\alpha_i, \alpha_j| = 1 = |\alpha_i, \beta_j|$ and $|\beta_i, \beta_j| = -1$.*

Such a subgroup has *rank* r and *type* (m, n).

Lemma 4. *Let $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ be a σ-basis for $H_1(X, \mathbf{Z})$. Then the action of σ on $H_1(X, \mathbf{Z})$ is given by the matrix*

$$\sigma_* = \begin{pmatrix} -1 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & 1 & 0 & & & \\ & & & -1 & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{pmatrix}.$$

Proof. This is a routine computation.

Notice that the action of σ_* on $H_1(X, \mathbf{Z})$ has two isotypic components, one corresponding to $+1$ and the other corresponding to -1. By abuse of language, call the former "the" invariant subspace.

LEMMA 5. *The invariant subspace of $H_1(X, \mathbf{Q})$ has integral generators A, B such that $A \cdot B$ is even.*

Proof. The subspace generated by $A_2 + A_3$ and $B_2 + B_3$ is one such, and it is trivial to check that any integral change of basis preserves the parity of the intersection product.

LEMMA 6. *Let $\pi : X \to X/\langle\sigma\rangle$ be the quotient map. Let A, B be a symplectic basis for $H_1(X/\langle\sigma\rangle, \mathbf{Z})$ and let $\{A_1, \dots, B_3\}$ be the corresponding σ-basis for X. Then $\hat{\pi}(A) = A_2 + A_3$ and $\hat{\pi}(B) = B_2 + B_3$.*

Proof. This is clear from the picture, or see [1, p. 44].

LEMMA 7. *The derived half-periods due to σ correspond to the image in $H_1(X, \mathbf{Z}/2\mathbf{Z})$ of the invariant subspace of $H_1(X, \mathbf{Z})$.*

Proof. Choose a σ-basis for X. By Lemma 5, $\tilde{\pi} : \mathrm{Jac}((X/\langle\sigma\rangle)) \to \mathrm{Jac}(X)$ is given by

$$\tilde{\pi}(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} 0 & a & a \\ 0 & b & b \end{pmatrix}.$$

By the remark following Definition 2, the derived half-periods are $\{\tilde{\pi}(e) : 2e = 0\}$, *i.e.*, they are

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

These correspond to the invariant cycles $B_2 + B_3$, $A_2 + A_3$, and $B_2 + B_3 + A_2 + A_3$, respectively.

For the rest of this section, suppose X is compact and non-hyperelliptic of genus three, and that the common hypotheses of Theorems A and B are true; in other words, let $\{f_1, f_2\}$ and $\{f_3, f_4\}$ be two admissible sets of quarter-period vanishings with $2f_1 = 2f_2 \neq 2f_3 = 2f_4$. Theorem 1 implies that $\mathrm{Aut}\, X$ contains two distinct elliptic-hyperelliptic involutions σ and τ. Suppose that $\langle\sigma, \tau\rangle$ is a dihedral group of order 6 or 8 (the other possibilities are order 4 and 12; the former is discussed in a later section, and the latter only occurs on hyperelliptic surfaces).

Let $\langle V, R : V^2 = R^n = 1, VRV = R^{-1} \rangle$ be a presentation of the group D_n generated by σ and τ. Each involution is elliptic-hyperelliptic [2]. The strategy is to explicitly compute the V- and VR-invariant subspaces of $H_1(X, \mathbf{Z})$ and use the results from section two to distinguish the two cases.

4 (a). Case D_3

Suppose that the dihedral group is D_3. Then $E = X/\langle R \rangle$ has genus one, while $X/D_3 \cong \mathbf{P}^1$.

The branched covering $X \to E$ is completely ramified with two branch points p and q. Following Lemma 2, let $\tilde{p} = \pi(p)$ and let $\tilde{q} = \pi(q)$. Take three copies of E on which an oriented path $\tilde{\tau}$ has been constructed from $\tilde{p}$ to $\tilde{q}$, and join the three copies of E along $\tilde{\tau}$ by observing the convention that the lift to X of any oriented arc on E meeting $\tilde{\tau}$ with positive orientation will jump from sheet i to sheet $i+1$ (modulo 3) above the intersection. This is illustrated in Figure Two, in which v is a closed loop. The automorphism R permutes the three sheets cyclically.

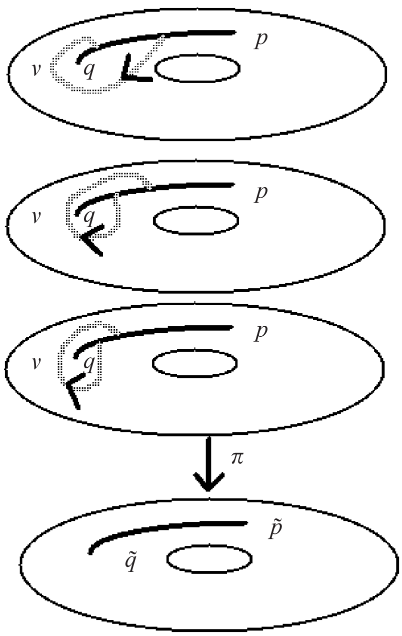

FIGURE 2. Three-sheeted cover

Pick an R-basis for $H_1(X, \mathbf{Z})$ by letting A and B be generators for $H_1(E, \mathbf{Z})$ and letting A_i and B_i denote the copy of A and B on sheet i. With respect to this basis R has the matrix

$$\begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 1 & 0 & 0 & & & \\ & & & 0 & 1 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \end{pmatrix}.$$

The complete linear system $g_2^1 = |\tilde{p} + \tilde{q}|$ defines a map $E \to \mathbf{P}^1$ of degree 2, and thus an involution $\tilde{V} : E \to E$ that can be lifted to X. Notice that if V is an involution in the D_3 on X, then $V(p)$ is also a branch point of π, because $RV(p) = VR^2(p) = V(p)$; $R^2(p) = p$ because p is a branch point. Thus $V(p) = q$. Therefore, $\tilde{V}$ lifts to an involution in D_3 which is denoted by V.

Now, compute the action of V on homology by lifting $\tilde{V}$. Pick a basepoint p_0 for the fundamental group of $E' = E - \{\tilde{p}, \tilde{q}\}$, and label the points $\{p_1, p_2, p_3\}$ in $\pi^{-1}(p_0)$ so that $R(p_1) = p_2$, $R(p_2) = p_3$, and $R(p_3) = p_1$; use p_1 for the basepoint of the fundamental group of $X' = X - \{p, q\}$. Notice that $\pi : X' \to E'$ is a covering.

Choose an arc σ on E' joining p_0 and $\tilde{V}(p_0)$; different choices of σ will lift $\tilde{V}$ to different involutions in D_3. Let $x \in X'$; to compute $V(x)$, draw an arc v on X from p_1 to x, and let $\tilde{v}$ be the pointwise image of v under π. Then $V(x)$ is the endpoint of the lift of the arc on E' obtained by tracing σ and then $\tilde{V}(\tilde{v})$.

When x is on the same π-sheet as p_1, the number of times that $\tilde{v}$ crosses $\tilde{\tau}$ is a multiple of 3, and $\tilde{V}(\tilde{v})$ crosses $\tilde{\tau} = \tilde{V}(\tilde{v})$ the same number of times. Thus, $\tilde{V}$ lifts directly to this sheet.

However, if x is on the same sheet as p_2, the number of times that v crosses τ is congruent to 1 (mod 3), and $\tilde{V}(\tilde{v})$ crosses $\tilde{\tau}$ with the opposite orientation. This is because $\tilde{V}$ can be written $\tilde{V}(z) = -z + b$ in the group law of E, reversing the sense of $\tilde{v}$ when $\tilde{V}$ is applied. From this it follows that $V(x)$ is on the same sheet as p_3, that is, V first acts like $\tilde{V}$ on sheet 2, then exchanges sheets 2 and 3.

The action $V_* : H_1(X, \mathbf{Z}) \to H_1(X, \mathbf{Z})$ is

$$\begin{aligned} A_1 &\mapsto -A_1 & B_1 &\mapsto -B_1 \\ A_2 &\mapsto -A_3 & B_1 &\mapsto -B_3 \\ A_3 &\mapsto -A_2 & B_1 &\mapsto -B_2 \end{aligned}$$

and the invariant subspace is spanned by $A_2 - A_3$ and $B_2 - B_3$.

One computes $(VR)_*$ and $(VR^2)_*$ by functoriality, and finds that the $(VR)_*$ invariant subspace is $\langle A_1 - A_3, B_1 - B_3 \rangle$, and that the $(VR^2)_*$-invariant subspace is $\langle A_1 - A_2, B_1 - B_2 \rangle$. Compute the images of these subspaces in $H_1(X, \mathbf{Z}/2\mathbf{Z})$ to find the following table of derived half-periods:

V	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
VR	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
VR^2	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

Now, pick any two involutive generators, *e.g.* V and VR, and look at the group generated by their derived half-periods, which has rank four. In each case, this group has type $(0,2)$. For example, the subgroup $\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$ has the basis

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\beta_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \beta_4 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Thus, condition (iv) of Theorem A is satisfied whenever $D_3 \subset \operatorname{Aut} X$.

4 (b). Case D_4

Suppose that $D_4 = \langle V, R : V^2 = R^4 = 1, VRV = R^3 \rangle \subset \operatorname{Aut} X$. It follows from [2 (7)] that every involution in this group has quotient genus 1, and that R is fixed-point free, since $X/\langle R \rangle$ has genus 1. Since R^2 is in the center of D_4, all other elements descend to automorphisms of $E = X/\langle R^2 \rangle$; let $\tilde{R}$, $\tilde{V}$, and $\widetilde{VR}$ be the involutions in $\operatorname{Aut} E$ that correspond respectively to R, V, and VR. The strategy is to examine their behavior on E and then lift them back to X to compute their action on homology.

It is clear that $\tilde{V}\tilde{R} = \widetilde{VR}$, and that R is fixed-point free. Thus,

$$\tilde{R}(z) = z + b_R,$$

where the addition is in E and $2b_R = 0$. The other involutions have the form

$$\tilde{V}(z) = -z + b'_V$$

$$\widetilde{VR}(z) = -z + b'_{VR};$$

one has $b'_V + b_R = b'_{VR}$.

Choose the origin of E so that $2b'_V = 0 = 2b_V$. Let b_V be any half-period on E, and pick $\underline{O} \in E$ such that $b'_V = 2\underline{O} + b_V$; then $\tilde{V}(z) = z + b'_V = (2\underline{O} - z) + b'_V$. By making $\underline{O}$ the origin, this becomes

$$\tilde{V}(z) = -z + b_V$$

with respect to the new group law; similarly,

$$\widetilde{VR}(z) = -z + b_V + b_R,$$

and $2b_{VR} = 2(b_V + b_R) = 0$.

We may further assume, by proper choice of the generators for the period lattice for E, that b_V, b_R, and b_{VR} are as shown in figure three:

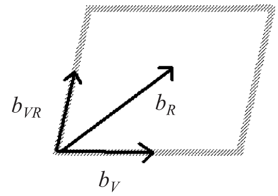

FIGURE 3. Half-periods on E

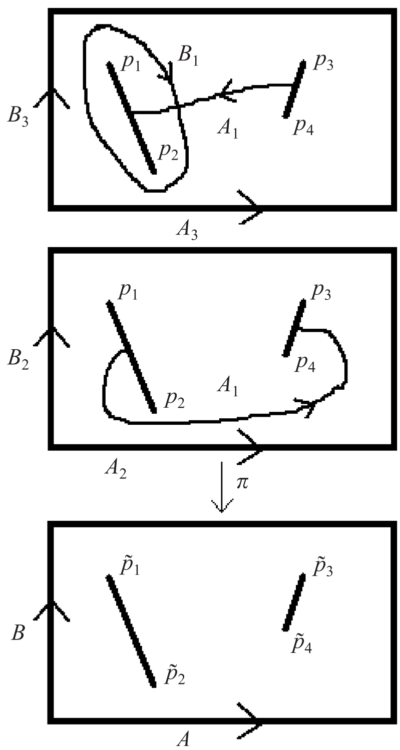

FIGURE 4. Constructing X from E

The map $\pi : X \to E$ branches at four points $p_1, \ldots, p_4$; let $\tilde{p}_i = \pi(p_i)$. One can check (renumbering if necessary) that $p_2 = V(p_1)$, $p_3 = VR(p_1)$, and $p_4 = R^{-1}(p_1)$, so $\tilde{p}_2 = \tilde{V}(\tilde{p}_1)$, $\tilde{p}_3 = \widetilde{VR}(\tilde{p}_1)$, and $\tilde{p}_4 = \tilde{R}(\tilde{p}_1)$. To reconstruct X from E, take two copies of E on which linear cuts have been made joining $\tilde{p}_1$ to $\tilde{p}_2$ and $\tilde{p}_3$ to $\tilde{p}_4$, and join the sheets along the corresponding edges. (Any other choice of cuts would do, due to the rigid configuration of $\tilde{p}_1, \ldots, \tilde{p}_4$.) See Figure Four, which is isomorphic with Figure One.

Let $\underline{O}$ be the basepoint for the fundamental group of $E' = E - \{\tilde{p}_i\}$, and choose a point p_0 over $\underline{O}$ to be the basepoint on $X' = X - \{p_i\}$. To lift $\tilde{V}$, draw an arc σ on E' from $\underline{O}$ to $\tilde{V}(\underline{O}) = b_V$; σ can be taken to be the base of the rectangle representing E in figure four. For $x \in X'$, draw an arc v on X' from p_0 to x, and let $\tilde{v}$ be the pointwise image of v under π. Then $V(x)$ is the endpoint of the lift of σ followed by $\tilde{V}(\tilde{v})$.

The $\tilde{V}$-image of a cut is another cut, so $\tilde{V}(\tilde{v})$ crosses the same number of cuts as $\tilde{v}$, and $V(x)$ is on the same π-sheet as x, *i.e.*, $\tilde{V}$ lifts directly to each sheet. The computation of $V_* : H_1(X, \mathbf{Z}) \to H_1(X, \mathbf{Z})$ is done using diagrams. Figure 5 shows that $V(A_1)$ is homologous to $A_1 - A_2 + A_3$:

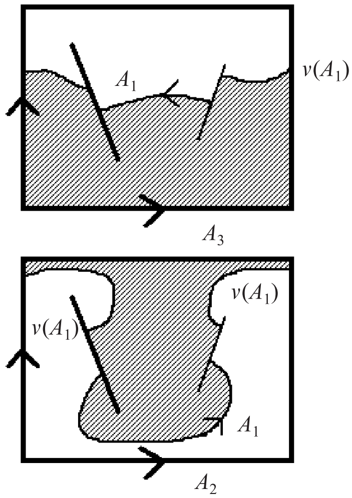

FIGURE 5. V-image of A_1

Similar figures show:

$$V_*(A_2) = -A_2$$
$$V_*(A_3) = -A_3$$
$$V_*(B_1) = B_1$$
$$V_*(B_2) = -B_1 - B_2$$
$$V_*(B_3) = B_1 - B_3.$$

The action of $\widetilde{VR}$ is very different, because the image of a cut is no longer a cut. The unions of the cuts and their $\widetilde{VR}$-images partitions E into two regions. Points in the region containing the origin of the group law of E stay in that region, so arcs in that region lift directly to an arc on one sheet or the other of X. Arcs from the origin into the other region lift into arcs that jump sheets. Thus we find that

$$\begin{aligned} VR_*(A_1) &= A_1 \\ VR_*(A_2) &= A_1 - A_2 \\ VR_*(A_3) &= -A_1 - A_3 \\ VR_*(B_1) &= B_1 + B_2 - B_3 \\ VR_*(B_2) &= -B_2 \\ VR_*(B_3) &= -B_3. \end{aligned}$$

Functoriality shows that

$$\begin{aligned} R_*(A_1) &= -A_1 + A_2 - A_3 \\ R_*(A_2) &= -A_1 + A_2 \\ R_*(A_3) &= A_1 + A_3 \\ R_*(B_1) &= B_1 - B_2 + B_3 \\ R_*(B_2) &= B_1 + B_3 \\ R_*(B_3) &= -B_1 + B_2. \end{aligned}$$

Lemma 6 leads to the following table of derived half-periods for the involutions:

V	$\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}$
VR	$\begin{pmatrix}0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&0&0\end{pmatrix}$
VR^2	$\begin{pmatrix}1&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&1&1\end{pmatrix}$
R^2	$\begin{pmatrix}0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&1&1\end{pmatrix}$
VR^3	$\begin{pmatrix}0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\1&1&1\end{pmatrix}$

A routine calculation shows that the derived half-periods due to any two generators of D_4 generate a group of rank 4 and type $(2,1)$. For example, in the line for R^2, let $\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, and let $\alpha_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

It is important to notice that commuting involutions always share a derived half-period. To see this, let $\mathscr{M}_2 = \{X : \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \subset \operatorname{Aut} X\}$. This follows from the results of Wiman [7], who derived normal forms for the canonical curves of surfaces of genus three with involutions. The normal form for a surface in $\mathscr{M}_2$ is $X^4 + Y^4 - Z^4 + aX^2Y^2 + bY^2Z^2 + cZ^2X^2 = 0$, with the involutions given by sign change of one homogeneous coordinate. The normal form for curves in $\mathscr{M}_4$ is the same, with the condition that $a = c$, so these involutions become commuting involutions on the surface on $\mathscr{M}_4$. Since the derived half-periods vary continuously with X, the commuting involutions in D_4 share a half-period.

Thus, is a surface X has quarter-period vanishings satisfying the admissibility conditions (i), (ii), and (iii), there are two non-commuting involutions in Aut X; if the group of derived half-periods has type $(2,1)$, then the involutions generate a D_4; if not, they generate a D_3.

This concludes the proof of Theorems A and B.

Also note the following:

Corollary 1. *Each locus $\mathscr{M}_3$ and $\mathscr{M}_4$ is the image in $\mathscr{M}$ of a subvariety of $\mathscr{A}_3$ which is a set-theoretic complete intersection.*

The following will be useful in the next section:

Corollary 2. *Two elliptic-hyperelliptic involutions commute if and only if they have a common derived half-period.*

5. Larger automorphism groups

Now consider a non-hyperelliptic Riemann surface of genus 3 with a non-abelian automorphism group generated by involutions. Following [6], there are four such groups in addition to the dihedral groups from the previous section. They are:

$$\mathscr{T} = \langle S, T, U : S^2 = T^2 = U^2 = (STU)^4 = 1, STU = TUS = UST \rangle;$$

$$S_4 = \langle S, T : S^2 = T^4 = (ST)^3 = 1 \rangle,$$

the symmetric group on 4 letters;

$$\mathscr{F} = \langle S, T : S^2 = T^3 = (ST)^8 = (ST^{-1}ST)^3 = 1 \rangle,$$

which has order 96; and

$$\mathscr{K} \cong \operatorname{Aut}(\mathbf{Z}/2\mathbf{Z}^3) = PSL(2, \mathbf{Z}/2\mathbf{Z}) = \langle S, T : S^2 = T^3 = (ST)^7 = (ST^{-1}ST)^4 = 1 \rangle,$$

the famous simple group of order 168.

A fifth non-abelian group of order 48 appears on a non-hyperelliptic surface, but it is not generated by its involutions. It contains $\mathscr{T}$ as a subgroup.

All of these groups contain subgroups isomorphic to D_4, and the basic strategy to understand the locus of surfaces with one of these automorphism groups is to analyze how involutions not in D_4 interact with those in D_4.

Begin with the group $\mathscr{T}$. Its 7 involutions are S, T, U, USU, STS, TUT, and $(SU)^2$, and its center is generated by STU. There are three subgroups isomorphic to D_4: $\langle S, T\rangle$, $\langle T, U\rangle$, and $\langle S, U\rangle$. If H is one of these subgroups and ι is an involution not in H, one easily checks that ι commutes with the central involution of H. In contrast, S_4 also has 3 subgroups isomorphic to D_4, given in cycle notation as $\langle (13), (12)(34)\rangle$, $\langle (12), (14)(23)\rangle$, and $\langle (14), (13)(24)\rangle$. Each of these contains the abelian subgroup of order four generated by (12)(34) and (13)(24), which is a normal subgroup of S_4; and all three are conjugate 2-Sylow subgroups. This, if H is one of these dihedral subgroups and σ is an involution not in H, σ does not commute with the central involution in H.

These differences enable one to distinguish the two groups from the derived half-periods. Recall that each elliptic-hyperelliptic involution causes θ to vanish at 12 quarter-periods $\{\pm f_1, \ldots, \pm f_6\}$; for each $D_4 \subset \operatorname{Aut} X$, then there is a *paradigm* of vanishings:

V	$f_1,\ f_2$		
VR^2	$f_5,\ f_2$		
R^2	$f_5,\ f_1$	$f_6,\ f_3$	$g_1,\ g_2$
VR		$f_4,\ f_6$	
VR^3	$h_1,\ h_2$	$f_3,\ f_4$	

The table is derived using Corollary 2: since, *e.g.*, V commutes with VR^2, they share a quarter-period vanishing. Each entry in the paradigm contains a pair x, y of quarter-periods that satisfy $2x = 2y$ and $|2x, x+y| = 1$. The entries $f_1, \ldots, f_4$ satisfy the hypotheses of Theorem B, so $D_4 \subset \operatorname{Aut} X$ when $\theta(f_i) = 0$, $i = 1, \ldots, 4$.

If $\mathscr{T} \subseteq \operatorname{Aut} X$, then $D_4 \subset \operatorname{Aut} X$, so θ vanishes at all entries in the paradigm. Each involution of $\mathscr{T}$ not in this D_4 commutes with R^2, so it shares a derived half-period with R^2; thus, θ vanishes at a quarter-period g_3 with $2g_1 = 2g_2 = 2g_3$. On the other hand, if θ vanishes at f_1, f_2, f_3, f_4, and g_3, then, by Theorem B, $\operatorname{Aut} X$ contains a D_4, and there is an involution $\sigma \in \operatorname{Aut} X$ that shares a derived half-period with R^2, and thus commutes with R^2. By Lemma 8, the group $\langle D_4, \sigma\rangle$ cannot be S_4. This proves

THEOREM 2. *The group $\mathscr{T} \subset \operatorname{Aut} X$ if an d only if there is a set of five quarter-periods f_1, f_2, f_3, f_4, and g_3, as indicated in the paradigm, where θ vanishes.*

Now, suppose that $S_4 \subset \mathrm{Aut}\, X$; then $\mathrm{Aut}\, X$ contains a D_4 and some involution σ not in the D_4. One may assume that $D_4 = \langle (13), (12)(34) \rangle$, that $\sigma = (12)$, and that $V = (13, VR) = (12)(34)$. Each entry in the paradigm is a θ-vanishing, so *a fortiori*, θ vanishes at h_1 and h_2. Furthermore, because of σ, there is a θ-vanishing at some h_3 not in the paradigm, and $2h_1 = 2h_2 = 2h_3$, since σ commutes with VR.

Notice that h_1, h_2, f_1, and f_2 satisfy the hypotheses of Theorem B, so the vanishing of θ at these four quarter-periods implies the existence of an involution not in D_4 that commutes with VR; by Lemma 7, then, $\mathscr{T} \not\subseteq \langle D_4, \sigma \rangle$, so $\langle D_4, \sigma \rangle \cong S_4 \subseteq \mathrm{Aut}\, X$.

It is also possible to derive information about the type of the subgroups generated by the half-periods, using the dihedral subgroups of order 6 in S_4. Either $\{h_1, h_3\}$ or $\{h_2, h_3\}$ is a vanishing set due to (12); renumber (if necessary) so that $\{h_1, h_3\}$ is this set. Then since $\langle (12), (13) \rangle = \langle \sigma, V \rangle \cong D_3$, $\langle 2h_1, h_1 + h_3, 2f_1, f_1 + f_2 \rangle$ has type $(0, 2)$. Thus, if $S_4 \subseteq \mathrm{Aut}\, X$, then θ vanishes at five quarter-periods f_1, f_2, h_1, h_2, and h_3 that satisfy:

(i) $2f_1 = 2f_2$, $2h_1 = 2h_2$;
(ii) $|2f_1, f_1 + f_2| = 1 = |2h_1, h_1 + h_2|$;
(iii) $\langle 2f_1, f_1 + f + 2, 2h_1, h_1 + h_2 \rangle$ has type $(2, 1)$;
(iv) $2h_1 = 2h_3$;
(v) $|2h_1, h_1 + h_3| = 1$; and
(vi) $\langle 2f_1, f_1 + f_2, 2h_1, h_1 + h_3 \rangle$ has type $(0, 2)$.

Theorem 3. *The symmetric group* $S_4 \subseteq \mathrm{Aut}\, X$ *if and only if* θ *vanishes at quarter-periods* f_1, f_2, h_1, h_2, *and* h_3 *satisfying* (i)–(vi) *above.*

Remark. In fact, it is possible to construct a presentation of S_4 using the information in (i)–(vi) and Theorems A and B.

Let $\mathscr{M}_{\mathscr{T}}$ be the locus of surfaces X with $\mathscr{T} \subseteq \mathrm{Aut}\, X$, and let $\mathscr{M}_{24}$ denote the locus of surfaces X with $S_4 \subseteq \mathrm{Aut}\, X$. Then an argument analagous to that of Corollary 1 gives:

Corollary 3. $\mathscr{M}_{24}$ *is the image in* $\mathscr{M}$ *of a subvariety of* $\mathscr{A}_3$ *that is a* (*set-theoretic*) *complete intersection.*

This argument requires some modification in the case of $\mathscr{M}_{\mathscr{T}}$: moving through $\mathscr{M}_{24} \subset \mathscr{M}_4$ can change the central involution in the D_4, and since the choice of the fifth vanishing g_3 depends on the central involution, the vanishing of θ at $f_1, \ldots, f_4$, g_3 does not guarantee that $\mathscr{T} \subseteq \mathrm{Aut}\, X$ near $\mathscr{M}_{24}$. Thus, the equations $\theta(f_i, \Omega) = 0 = \theta(g_3, \Omega)$ only define the inverse image of $\mathscr{M}_{\mathscr{T}}$ locally.

Corollary 4. *The locus* $\mathscr{M}_{\mathscr{T}}$ *is the image in* $\mathscr{M}$ *of a subvariety of* $\mathscr{A}_3$ *that is a* (*set-theoretic*) *locally complete intersection.*

Similar though tedious analysis of the groups $\mathscr{F}$ and $\mathscr{K}$ yields

THEOREM 4. *The group* $\mathscr{F} \subseteq \operatorname{Aut} X$ *if and only if* θ *vanishes at quarter periods* $f_1, \dots, f_6$ *such that*:

(i) $2f_1 = 2f_2$, $2f_3 = 2f_4$;
(ii) $|2f_1, f_1 + f_2| = 1 = |2f_3, f_3 + f_4|$;
(iii) $\langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4 \rangle$ *has type* $(2, 1)$;
(iv) $2f_5 = 2f_6$;
(v) $|2f_5, f_5 + f_6| = 1$;
(vi) $\langle 2f, f_1 + f_2, 2f_3, f_3 + f_4, 2f_5, f_5 + f_6 \rangle$ *has rank* 6;
(vii) *if* $a, d \in \langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4 \rangle$ *satisfy* $|a, x| = 1 = |d, x|$ *for all* $x \in \langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4 \rangle$, *then* $\langle a, d, 2f_5, f_5 + f_6 \rangle$ *has type* $(2, 1)$,

Note. a and d in condition (vii) are the derived half-periods due to $f_1, \dots, f_4$.

THEOREM 5. *Consider the conditions* (i)–(vi) *above and*
(vii′) *if* $a, d \in \langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4 \rangle$ *satisfy* $|a, x| = 1 = |d, x|$ *for all* $x \in \langle 2f_1, f_1 + f_2, 2f_3, f_3 + f_4 \rangle$, *then* $\langle a, d, 2f_5, f_5 + f_6 \rangle$ *has type* $(0, 2)$.

Then $\mathscr{K} \subseteq \operatorname{Aut} X$ if and only if θ vanishes at quarter periods $f_1, \dots, f_6$ such that satisfy (i) through (vi) and (vii′).

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