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KODAI MATH. J.
30 (2007), 140–146

ON A RESULT OF UEDA CONCERNING UNICITY OF MEROMORPHIC FUNCTIONS

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Abstract

This paper studies the problem of the uniqueness of meromorphic functions sharing three values. The results in this paper improve some theorems given by H. Ueda, H. X. Yi and other authors.

1. Introduction and the main results

Let f and g be two nonconstant meromorphic functions on the open complex plane $\mathbf{C}$, and let a be a finite value in the complex plane. We say that f and g share the value a CM (IM) provided that $f - a$ and $g - a$ have the same zeros counting multiplicities (ignoring multiplicities), and f, g share ∞ CM (IM) provided that $1/f$, $1/g$ share 0 CM (IM). We also denote by $N_{p)}(r, f)$ (or $\overline{N}_{p)}(r, f)$) the counting function of the poles of f with multiplicities less than or equal to p (ignoring multiplicities), and $N_{(p}(r, f)$ (or $\overline{N}_{(p}(r, f)$) the counting function of the poles of f with multiplicities greater than or equal to p (ignoring multiplicities). The symbol $S(r, f)$ is quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set E of finite Lebesgue measure. It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found in [3 or 7].

Throughout in this paper we denote by f, g two nonconstant meromorphic functions defined on the open complex plane.

In 1983, H. Ueda [6] proved the following two theorems:

Theorem A. *Let f and g be two distinct nonconstant meromorphic functions sharing 0, 1, ∞ CM. If a $(\neq 0, 1)$ is a finite complex number, then*

$$N_{(3}\left(r, \frac{1}{f - a}\right) = S(r, f) \quad \textit{and} \quad N_{(3}\left(r, \frac{1}{g - a}\right) = S(r, g).$$

2000 *Mathematics Subject Classification*: 30D35, 30D30.

Key words and phrases: Meromorphic functions; Small functions; Uniqueness.

Project supported by Harbin Engineering University Foundational Research Foundation (HEUFT05088).

Received June 23, 2006; revised November 16, 2006.

THEOREM B. *If, in addition to the assumptions of Theorem A*, $\overline{E}_{k)}(a,f) = \overline{E}_{k)}(a,g)$, *where* $k \geq 2$ *is a positive integer or* ∞, *then* f *and* g *share* a *CM.*

In 1989, Brosch [2] showed that the conclusion of Theorem B is still valid if the condition "$\overline{E}_{k)}(a,f) = \overline{E}_{k)}(a,g)$" in Theorem B is replaced by "$\overline{E}_{2)}(a,f) \subseteq \overline{E}(a,g)$". Li and Yi [5] proved the following:

THEOREM C (see [5, Theorem 1.1]). *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If there exists a finite complex number* a $(\neq 0,1)$ *such that* a *is not a Picard value of* f *and* $N_{1)}(r, 1/(f-a)) \neq T(r,f) + S(r,f)$, *then* $N_{1)}(r, 1/(f-a)) = ((k-2)/k)T(r,f) + S(r,f)$ *and one of the following cases will hold*:

(i) $f = \dfrac{e^{(k+1)\gamma} - 1}{e^{s\gamma} - 1}$, $g = \dfrac{e^{-(k+1)\gamma} - 1}{e^{-s\gamma} - 1}$, *with* $\dfrac{(a-1)^{k+1-s}}{a^{k+1}} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq \dfrac{k+1}{s}$;

(ii) $f = \dfrac{e^{s\gamma} - 1}{e^{(k+1)\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{e^{-(k+1)\gamma} - 1}$, *with* $a^s(a-1)^{k+1-s} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq \dfrac{k+1}{s}$;

(iii) $f = \dfrac{e^{s\gamma} - 1}{e^{-(k+1-s)\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{e^{(k+1-s)\gamma} - 1}$, *with* $\dfrac{(-a)^s}{(1-a)^{k+1}} = \dfrac{s^s(k+1-s)^{k+1-s}}{(k+1)^{k+1}}$ *and* $a \neq -\dfrac{s}{k+1-s}$;

(iv) $f = \dfrac{e^{k\gamma} - 1}{\lambda e^{s\gamma} - 1}$, $g = \dfrac{e^{-k\gamma} - 1}{(1/\lambda)e^{-s\gamma} - 1}$, *with* $\lambda^k \neq 0,1$ *and* $\dfrac{(a-1)^{k-s}}{\lambda^k a^k} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

(v) $f = \dfrac{e^{s\gamma} - 1}{\lambda e^{k\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{(1/\lambda)e^{-k\gamma} - 1}$, *with* $\lambda^s \neq 0,1$ *and* $\lambda^s a^s (1-a)^{k-s} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

(vi) $f = \dfrac{e^{s\gamma} - 1}{\lambda e^{-(k-s)\gamma} - 1}$, $g = \dfrac{e^{-s\gamma} - 1}{(1/\lambda)e^{(k-s)\gamma} - 1}$, *with* $\lambda^s \neq 0,1$ *and* $\dfrac{(-\lambda a)^s}{(1-a)^k} = \dfrac{s^s(k-s)^{k-s}}{k^k}$;

where γ *is a nonconstant entire function*, s *and* k (≥ 2) *are positive integers such that* s *and* $k+1$ *are mutually prime and* $1 \leq s \leq k$ *in* (i), (ii) *and* (iii), s *and* k *are mutually prime and* $1 \leq s \leq k-1$ *in* (iv), (v) *and* (vi).

Recently the author has proved the following result which is an improvement of some theorems given by S. Ye [8], H. Yi [9] and other authors.

THEOREM D (see [1, Theorems 2 and 3]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM, and let a $(\neq 0, 1)$ be a small meromorphic function of f and g. Then*

$$N_{(3}\left(r, \frac{1}{f-a}\right) = S(r, f) \quad \textit{and} \quad N_{(3}\left(r, \frac{1}{g-a}\right) = S(r, g).$$

Furthermore, if $N\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f)$, *then* $N\left(r, \frac{1}{f-a}\right) = S(r, f)$, *and f and g satisfy one of the following three relations*:

(i) $(f-a)(g+a-1) \equiv a(1-a)$;

(ii) $f + (a-1)g \equiv a$;

(iii) $f \equiv ag$.

Can one replace a in Theorems A–C by a small function of f and g?

In this paper, we give a positive answer for this question by the following two theorems:

THEOREM 1. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If $a(\not\equiv \infty)$ is a nonconstant small meromorphic function of f and g, then $N_{(2}(r, 1/(f-a)) = S(r, f)$ and $N_{(2}(r, 1/(g-a)) = S(r, g)$.*

THEOREM 2. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If $a(\not\equiv \infty)$ is a nonconstant small meromorphic function of f and g such that $\overline{E}_{1)}(a, f) \subseteq \overline{E}(a, g)$, then f and g satisfy one of the relations* (i)–(iii) *in Theorem D.*

We indicate in here that Theorem 2 is not valid when a is a constant. For example, $f(z) = e^{2z} + e^{z} + 1$, $f(z) = e^{-2z} + e^{-z} + 1$ and $a = 3/4$. It is easy to verify that f and g share 0, 1, ∞ CM and $\overline{E}_{1)}(a, f) = \overline{E}_{1)}(a, g)$.

2. Some lemmas

In this section we present some lemmas which will be needed in the sequel.

LEMMA 1 (see [7, Theorem 5.1]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. Then*

$$f \equiv \frac{e^{\alpha}-1}{e^{\beta}-1}, \quad g \equiv \frac{e^{-\alpha}-1}{e^{-\beta}-1}, \tag{2.1}$$

where $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ *and* $e^{\alpha-\beta} \not\equiv 1$, *and*

$$T(r, g) + T(r, e^{\alpha}) + T(r, e^{\beta}) = O(T(r, f)) \quad (r \notin E).$$

This shows that $S(r,f) = S(r,g)$ and we denote them by $S(r)$, unless otherwise stated.

Lemma 2 (see [4, Lemma 7]). *Let f_1 and f_2 be distinct nonconstant meromorphic functions satisfying* $\overline{N}(r,f_i) + \overline{N}\left(r,\dfrac{1}{f_i}\right) = S(r,f_1,f_2)$, $i = 1,2$. *If* $f_1^s f_2^t - 1$ *is not identically zero for all integers* s, t $(|s|+|t|>0)$, *then for any positive number* ε, *we have*

$$N_0(r,1,f_1,f_2) \le \varepsilon T(r) + S(r,f_1,f_2),$$

where $N_0(r,1,f_1,f_2)$ *denotes the reduced counting function of* f_1 *and* f_2 *related to the common* 1*-points,* $T(r) = T(r,f_1) + T(r,f_2)$ *and* $S(r,f_1,f_2) = \max\{S(r,f_1), S(r,f_2)\}$.

Lemma 3 (see [10, Theorem 1]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\varlimsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} > 1/2,$$

then f is a fractional linear transformation of g. Here, $N_0(r)$ denotes the counting function of the zeros of $f-g$ that are not zeros of $f(f-1)$, $1/f$.

3. Proofs of Theorems 1 and 2

1. Proof of Theorem 1. If f is a fractional linear transformation of g, then f has two distinct Picard values, say a_1, a_2. On the other hand, since a is a not constant then $a \not\equiv a_1, a_2$; we can apply the second fundamental theorem to get

$$T(r,f) = \overline{N}_{1)}\left(r,\frac{1}{f-a}\right) + S(r,f),$$

which gives $N_{(2}\left(r,\dfrac{1}{f-a}\right) = S(r,f)$.

Let us now consider that f is not a fractional linear transformation of g. From (2.1), we have

$$f - a = \frac{e^\alpha - ae^\beta + (a-1)}{e^\beta - 1}. \tag{3.1}$$

Assume that $T(r,e^\alpha) = S(r)$, where $S(r)$ is defined as in Lemma 1. If $e^\alpha + a - 1 \equiv 0$, then the conclusion of Theorem 1 follows from (3.1); if $e^\alpha + a - 1 \not\equiv 0$, by the second fundamental theorem we see that

$$T(r,f) = T(r,e^{\beta}) + S(r) = \overline{N}\left(r, \frac{1}{e^{\alpha} + a - 1 - ae^{\beta}}\right) + S(r)$$
$$= \overline{N}\left(r, \frac{1}{f-a}\right) + S(r),$$

which implies $N_{(2}\left(r, \dfrac{1}{f-a}\right) = S(r)$, and the conclusion of Theorem 1 is clear in this case. Similarly, we can prove that the theorem 1 is true when $T(r,e^{\beta}) = S(r)$ or $T(r,e^{\alpha-\beta}) = S(r)$. Consequently, we may suppose that $T(r,e^{\alpha})$, $T(r,e^{\beta})$ and $T(r,e^{\alpha-\beta})$ are not equal to $S(r)$. Set

$$\alpha_1 = \beta' + \frac{a'}{1-a} + \frac{a'}{a}, \quad \alpha_2 = \alpha' + \frac{a'}{1-a}. \tag{3.2}$$

Since a is a small of f and so g, then we have $T(r,\alpha_i) = S(r)$, $i = 1,2$.

Let z_0 be a multiple zero of $f - a$, which is neither any zero of α', β', $\alpha' - \beta'$, a, $a-1$, α_1, α_2, $\alpha_1 - \alpha_2$, nor the pole of a. From (3.1) we obtain

$$e^{\alpha(z_0)} - a(z_0)e^{\beta(z_0)} + a(z_0) - 1 = 0,$$
$$\alpha'(z_0)e^{\alpha(z_0)} - e^{\beta(z_0)}(a'(z_0) + a(z_0)\beta') + a'(z_0) = 0.$$

These two equations give us

$$e^{\alpha} = (a-1)\frac{\beta' - \dfrac{a'}{a(a-1)}}{\alpha' - \dfrac{a'}{a} - \beta'}, \quad e^{\beta} = \left(\frac{a-1}{a}\right)\frac{\alpha' - \dfrac{a'}{a-1}}{\alpha' - \dfrac{a'}{a} - \beta'}. \tag{3.3}$$

Let us now define the following two functions:

$$f_1 = \frac{1}{1-a}\frac{\alpha_1 - \alpha_2}{\alpha_1}e^{\alpha}, \quad f_2 = \frac{a}{1-a}\frac{\alpha_1 - \alpha_2}{\alpha_2}e^{\beta}, \tag{3.4}$$

and consider

$$T_0(r) = T(r,f_1) + T(r,f_2), \quad S_0(r) = o(T_0(r)) \quad (r \to \infty, r \notin E),$$

where E is a set of r of finite linear measure. From this and (3,4), we get $S_0(r) = S(r)$ and

$$\overline{N}(r,f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r) \quad (j = 1,2). \tag{3.5}$$

In view of (3.3) and (3,4), it is obvious that z_0 is a zero of $f_1 - 1$ and $f_2 - 1$. By this and Theorem D, one deduces that

$$\overline{N}_{(2}\left(r, \frac{1}{f-a}\right) \le N_0(r,1,f_1,f_2) + S(r), \tag{3.6}$$

where $N_0(r,1,f_1,f_2)$ is defined as in Lemma 3.

Suppose that $N_0(r,1,f_1,f_2) \neq S(r)$. Then from (3.5), (3.6) and by using Lemma 2, there exist two integers p and q ($|p|+|q|>0$) such that

$$f_1^p f_2^q \equiv 1. \tag{3.7}$$

From (3.4) and (3.7) we have $pq \neq 0$ and

$$\left(\frac{1}{1-a}\right)^p \left(\frac{a}{1-a}\right)^q e^{p\alpha+q\beta} = \left(\frac{\alpha_1}{\alpha_1-\alpha_2}\right)^p \left(\frac{\alpha_2}{\alpha_1-\alpha_2}\right)^q, \tag{3.8}$$

by logarithmic differentiation and by using (3.2), we can easily obtain that

$$p\alpha_2 + q\alpha_1 = \frac{q\alpha_1 + p\alpha_2}{\alpha_1 - \alpha_2} \frac{\left(\dfrac{\alpha_2}{\alpha_1}\right)'}{\dfrac{\alpha_2}{\alpha_1}}.$$

If $q\alpha_1 + p\alpha_2 \not\equiv 0$ then from the last equation and (3.2), we get $\dfrac{a'}{a} + \beta' - \alpha' = \dfrac{\left(\dfrac{\alpha_2}{\alpha_1}\right)'}{\dfrac{\alpha_2}{\alpha_1}}$, which implies that $e^{\beta-\alpha} = c(1/a)(\alpha_2/\alpha_1)$, where c is a constant. This equation gives us $T(r, e^{\beta-\alpha}) = S(r)$, which is impossible. Thus $q\alpha_1 + p\alpha_2 \equiv 0$; this relation and (3.2) yield

$$p\left\{\alpha' + \frac{a'}{1-a}\right\} + q\left\{\beta' + \frac{a'}{1-a} + \frac{a'}{a}\right\} \equiv 0.$$

Taking integration on the above equation to conclude

$$\left(\frac{1}{1-a}\right)^p \left(\frac{a}{1-a}\right)^q e^{q\beta} = Ae^{-p\alpha}, \tag{3.9}$$

where $A \neq 0$ is a constant.

If $p+q=0$ then from (3.4) and (3.7), we get $T(r, e^{\alpha-\beta}) = S(r)$, which is a contradiction. Therefore, $p+q \neq 0$; from this, (3.9) and because a is not a constant, it is obvious that 0, 1 and ∞ are Picard values of a, which is impossible. This contradiction comes from the assumption $N_0(r,1,f_1,f_2) \neq S(r)$. Therefore, the conclusion of Theorem 1 follows from (3.6) and Theorem D. The proof of Theorem 1 is completed.

2. Proof of Theorem 2. Assume that f is a linear transformation of g. Then f and g have two distinct Picard values, and hence

$$T(r,f) = N(r, 1/(f-a)) + S(r),$$

because a is not a constant. If z_0 is a simple zero of $f-a$, then z_0 is a zero of $g-a$, which means that z_0 is zero of $a-A$, where A is a constant. Hence,

$$T(r,f) \le N(r,1/(a-A)) + S(r),$$

which is impossible. This shows that f is not any linear transformation of g.

Suppose that f and g do not satisfy one of the forms (i)–(iii) in Theorem D. Consequently, from Theorem D and Theorem 1, we get

$$T(r,f) = \overline{N}_{1)}\left(r, \frac{1}{f-a}\right) + S(r). \tag{3.10}$$

If z_0 is a simple zero of $f-a$, then z_0 is a zero of $g-a$, and hence, z_0 is a zero of $f-g$. Therefor, it follows from (3.10) that

$$T(r,f) = \overline{N}_{1)}\left(r, \frac{1}{f-a}\right) + S(r) \le N_0(r) + S(r),$$

which is a contradiction with Lemma 3. This proves Theorem 2.

Acknowledgements. The author thanks a lot for the useful suggestion of referees.

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