K. KURIBAYASHI AND M. YOKOTANI
KODAI MATH. J.
30 (2007), 19–40

ITERATED CYCLIC HOMOLOGY

Katsuhiko Kuribayashi and Masaaki Yokotani

Abstract

From the viewpoint of rational homotopy theory, we introduce an iterated cyclic homology of connected commutative differential graded algebras over the rational number field, which is regarded as a generalization of the ordinary cyclic homology. Let $\boldsymbol{T}$ be the circle group and $\mathscr{F}(\boldsymbol{T}^l, X)$ denote the function space of continuous maps from the l-dimensional torus $\boldsymbol{T}^l$ to an l-connected space X. It is also shown that the iterated cyclic homology of the differential graded algebra of polynomial forms on X is isomorphic to the rational cohomology algebra of the Borel space $E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X)$, where the $\boldsymbol{T}$-action on $\mathscr{F}(\boldsymbol{T}^l, X)$ is induced by the diagonal action of $\boldsymbol{T}$ on the source space $\boldsymbol{T}^l$.

1. Introduction

Let $\boldsymbol{T}$ be the circle group and $\mathscr{F}(\boldsymbol{T}, X)$ the function space equipped with the $\boldsymbol{T}$-action induced from the multiplication on the source space $\boldsymbol{T}$. The results of Goodwillie [5] and of Burghelea and Fiedorowicz [2] assert that the cyclic homology of the singular chain $C_*(\Omega X; R)$ on the Moore loop space of a path-connected space X can be identified with the homology of the Borel space $E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}, X)$ with coefficients in R, where R denotes a commutative ring with unit. Jones [8] has proved that the (negative) cyclic homology of the singular cochain $C^*(X; R)$ of a simply-connected space X is isomorphic to the cohomology of $E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}, X)$ with coefficients in R as a module over $H^*(B\boldsymbol{T}; R) = R[u]$. In the case where R is the rational number field $\boldsymbol{Q}$, the cyclic homology of $C^*(X; \boldsymbol{Q})$ is isomorphic to that of the differential graded commutative algebra $A_{PL}(X)$ of rational polynomial forms on X. In [11], Vigué-Poirrier and Burghelea have proved that a complex which computes the cyclic homology of $A_{PL}(X)$ is quasi-isomorphic to a Sullivan minimal model for $E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}, X)$ if $\dim \pi_i(X) \otimes \boldsymbol{Q}$ is finite for any i. Thus the model allows us to calculate the cyclic homology of *a space* explicitly [11, Theorem B]. Under such a background, it is natural to generalize the cyclic homology of a differential graded commutative algebra

2000 *Mathematics Subject Classification*: 13D03, 55P62.

Key words and phrases. Borel cohomology, Function spaces, Iterated cyclic homology, Sullivan minimal model.

Received August 11, 2005; revised June 26, 2006.

(DGA) over $\boldsymbol{Q}$ requiring that the homology is isomorphic to the cohomology $H^*(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X); \boldsymbol{Q})$ when $A_{PL}(X)$ for an appropriate l-connected space X is chosen as the input DGA. Here the $\boldsymbol{T}$-action on $\mathscr{F}(\boldsymbol{T}^l, X)$ is defined by $(f \cdot a)(t_1, \dots, t_l) = f(at_1, \dots, at_l)$ for $a \in \boldsymbol{T}$, $f \in \mathscr{F}(\boldsymbol{T}^l, X)$ and $(t_1, \dots, t_l) \in T^l$.

The purpose of this paper is to define such a generalized cyclic homology and to investigate fundamental properties of the homology. We shall refer to the homology as *the iterated cyclic homology* with iteration degree l and denote it by $HC_*^{\{l\}}(A, d)$.

In what follows, we assume that a DGA (A, d) is unital and locally finite in the sense that A^i is of finite dimension for any i. A DGA is said to be l-connected if $A^i = 0$ for $i < 0$, $A^0 = \boldsymbol{Q}$ and $H^i(A, d) = 0$ for any $0 < i \le l$.

Roughly speaking, the construction of the iterated cyclic homology of a DGA is as follows: First, for any l-connected DGA (A, d), the iterated Hochschild homology $HH_*^{\{l\}}(A, d)$ is defined using an appropriate complex $(\wedge W, \delta)$. The complex is indeed a minimal model for the function space $\mathscr{F}(\boldsymbol{T}^l, X)$ if (A, d) is l-connected and X is the spatial realization $|(A, d)|$ of the DGA (A, d).

We next introduce a derivation β on $\wedge W$ with degree -1. By perturbing the differential δ on $\wedge W \otimes \boldsymbol{Q}[u]$ with β, we define the complex $(\wedge W \otimes \boldsymbol{Q}[u], \delta + u\beta)$ which gives the iterated cyclic homology $HC_*^{\{l\}}(A, d)$. It will be readily seen that $HC_*^{\{l\}}(A, d)$ is equipped a $\boldsymbol{Q}[u]$-algebra structure. Moreover we see that if $l = 1$, then $HC_*^{\{l\}}(A, d)$ is the ordinary cyclic homology of (A, d). We wish to emphasize that $HC_*^{\{l\}}(A, d)$ does not necessarily consist of elements with non-negative degree and that $HC_0^{\{1\}}(A, d)$ is not a 1-dimensional vector space in general (see Section 4).

The definition of the derivation β is quite algebraic. However, if the input DGA (A, d) is l-connected, then the derivation β is related to a model for the $\boldsymbol{T}$-action on $\mathscr{F}(\boldsymbol{T}^l, |(A, d)|)$. The relationship, which is described more precisely in Proposition 5.5, is the key to completing the proof of the following theorem.

THEOREM 1.1. *Suppose that X is an l-connected space with* $\dim \pi_k(X) \otimes \boldsymbol{Q} < \infty$ *for any k. Then, as a $\boldsymbol{Q}[u]$-algebra,*

$$HC_*^{\{l\}}(A_{PL}(X)) \cong H^*(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X); \boldsymbol{Q}).$$

Here the $\boldsymbol{Q}[u]$-algebra structure on $H^*(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X); \boldsymbol{Q})$ is induced from the projection p of the Borel fibration $\mathscr{F}(\boldsymbol{T}^l, X) \to E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X) \xrightarrow{p} B\boldsymbol{T}$.

We prove Theorem 1.1 by induction on the iteration degree l modifying the proof of [11, Theorem A] due to Vigué-Poirrier and Burghelea. So a new idea does not appear in our proof. However we dare to repeat the augment of their proof in order to state exactly the key proposition (Proposition 5.5). We also expect that an idea inspired by the repeat may enable us to define more general cyclic homology of a connected DGA (A, d), which is isomorphic to the Borel cohomology of the form $H^*(E_H \times_H \mathscr{F}(G, X); \boldsymbol{Q})$. Here G is a Lie group, H is a subgroup of $G, X = |(A, d)|$ and the action by H on $\mathscr{F}(G, X)$ is induced by the product on G. Regrettably, such consideration is not made in this paper. We also mention that Proposition 5.5 is proved by applying the algebraic model for the evaluation map $m(ev) : \mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T}^l \to X$ considered in [9].

We now direct our attention to an algebraic property of the iterated cyclic homology. The result [11, Corollary 2] asserts that, for any 1-connected DGA (A, d) with $H(A, d) \neq \boldsymbol{Q}$, the sequence consisting of $\dim HC_i^{\{1\}}(A, d)$ for $i \geq 0$ is bounded if and only if the cohomology algebra $H^*(A, d)$ can be generated by a single class. As for the iterated cyclic homology with iteration degree greater than 1, we have the following theorem.

Theorem 1.2. *For any integer $l \geq 2$ and any l-connected DGA (A, d) such that $H^*(A, d) \neq \boldsymbol{Q}$, the sequences consisting of* $\dim HC_i^{\{l\}}(A, d)$ *and of* $\dim HH_i^{\{l\}}(A, d)$ *for $i \geq 0$ are unbounded as $i \to \infty$, respectively.*

This paper is organized as follows. In Section 2, after defining explicitly the iterated cyclic homology, we show that the homology can be regarded as a functor from the category of DGA's to the category of $\boldsymbol{Q}[u]$-algebras. In Section 3, fundamental properties of the iterated cyclic homology are described. We also introduce a natural transformation from $HC_*^{\{l\}}$ to $HC_{*-1}^{\{l+1\}}$. By applying it, we prove Theorem 1.2. Section 4 is devoted to computing the iterated cyclic homology with iteration degree 2 of the polynomial algebra generated by a single element. In Section 5, Theorem 1.1 is proved. We present in the last section the proof of Proposition 5.5.

2. Definition of the iterated cyclic homology

The free algebra generated by a graded vector space V will be denoted by $\wedge V$ or $\boldsymbol{Q}[V]$. Let $(\wedge V, d)$ be a free DGA and (B, d_B) a DGA. Let B_* denote the differential graded coalgebra defined by $B_q = \mathrm{Hom}(B^{-q}, \boldsymbol{Q})$ for $q \leq 0$ together with the coproduct D and the differential d_{B*} which are dual to the multiplication of B and to the differential d_B, respectively. Let I be the ideal of the free algebra $\boldsymbol{Q}[\wedge V \otimes B_*]$ generated by $1 \otimes 1 - 1$ and all elements of the form

$$a_1 a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_i')(a_2 \otimes \beta_i''),$$

where $a_1, a_2 \in \wedge V$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta_i' \otimes \beta_i''$. Observe that $\boldsymbol{Q}[\wedge V \otimes B_*]$ is a DGA with the differential $d := d \otimes 1 \pm 1 \otimes d_{B*}$.

The result [1, Theorems 3.3] asserts that the differential $d \otimes 1 \pm 1 \otimes d_{B*}$ respects the ideal, that is, $(d \otimes 1 \pm 1 \otimes d_{B*})(I) \subset I$. Moreover, the result [1, Theorem 3.5] implies that the composition map

$$\rho_V : \boldsymbol{Q}[V \otimes B_*] \hookrightarrow \boldsymbol{Q}[\wedge V \otimes B_*] \to \boldsymbol{Q}[\wedge V \otimes B_*]/I$$

is an isomorphism of graded algebras. Thus we can define a differential δ on $\boldsymbol{Q}[V \otimes B_*]$ by $\rho_V^{-1} \tilde{d} \rho_V$, where $\tilde{d}$ is the differential on $\boldsymbol{Q}[\wedge V \otimes B_*]/I$ induced by d.

We apply this construction to the case that B is the DGA $\wedge(t_1, \dots, t_l)$ together with the trivial differential, where $|t_i| = 1$ for any i. In what follows, we fix the DGA $(\boldsymbol{Q}[\wedge V \otimes B_*], \delta)$.

Let $D^{(m-1)}$ be the $(m-1)$-fold iterated coproduct on B_* and the dual base of $T_J = t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l}$ $(J = (\varepsilon_1, \dots, \varepsilon_l))$ shall be denoted by $T_{J*} = (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*$. Observe that if $d(v) = v_1 \cdots v_m$, then (2.1):

$$\begin{aligned}
\delta(v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) &= \sum_J (-1)^{\varepsilon(J)} v_1 \cdots v_m \cdot T_{J_1 *} \otimes \cdots \otimes T_{J_m *} \\
&= \sum_J (-1)^{\varepsilon(J) + \varepsilon(v_1, \dots, v_m, T_{J_1 *}, \dots, T_{J_m *})} v_1 \otimes T_{J_1 *} \cdots v_m \otimes T_{J_m *},
\end{aligned}$$

where $(-1)^{\varepsilon(v_1, \dots, v_m, T_{J_1 *}, \dots, T_{J_m *})} v_1 T_{J_1} \cdots v_m T_{J_m} = v_1 \cdots v_m T_{J_1} \cdots T_{J_m}$ in the graded algebra $\wedge V \otimes B$ and $D^{(m-1)}((t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) = \sum_J (-1)^{\varepsilon(J)} T_{J_1 *} \otimes \cdots \otimes T_{J_m *}$. We define a derivation $\beta : \boldsymbol{Q}[V \otimes B_*] \to \boldsymbol{Q}[V \otimes B_*]$ with degree -1 by

$$\beta(v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) = \sum_k (-1)^{|v| + \varepsilon_1 + \cdots + \varepsilon_{k-1}} v \otimes (t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k + 1} \cdots t_l^{\varepsilon_l})_*.$$

Proposition 2.1. *$\beta^2 = 0$ and $\delta\beta + \beta\delta = 0$.*

Proof. By a straightforward computation, we can check that the first equality holds. As for the second one, it suffices to prove that $(\delta\beta + \beta\delta)(v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) = 0$ when $dv = v_1 \cdots v_m$. The formula (2.1) makes it possible to verify the equality. □

The iterated Hochschild homology $HH_*^{\{l\}}(\wedge V, d)$ and the iterated cyclic homology $HC_*^{\{l\}}(\wedge V, d)$ with iteration degree l for a free DGA $(\wedge V, d)$ are defined as the homologies of the complexes

$$(\mathscr{C}_*^{\{l\}}(\wedge V), \delta_l) = (\boldsymbol{Q}[V \otimes B_*], \delta) \quad \text{and} \quad (\mathscr{E}_*^{\{l\}}(\wedge V), \mathscr{D}_l) = (\mathscr{C}_*^{\{l\}}(\wedge V) \otimes \boldsymbol{Q}[u], \delta + u\beta),$$

respectively, where $|u| = 2$. Observe that $(\mathscr{E}_*^{\{l\}}(\wedge V), \mathscr{D}_l)$ has a DGA structure over $\boldsymbol{Q}[u]$, which is induced by the multiplication on $\boldsymbol{Q}[u]$. It is readily seen that the iterated cyclic homology inherits the $\boldsymbol{Q}[u]$-algebra structure. When the iteration degree l is clear from the context, frequently, the suffix is dropped in the notation as $\mathscr{C}_*(\wedge V)$ for $\mathscr{C}_*^{\{l\}}(\wedge V)$.

Remark 2.2. Let $(\wedge V, d)$ be a minimal model for an l-connected space X. We can choose the DGA $B = \wedge(t_1, \dots, t_l)$ with the trivial differential as a model for the l-dimensional torus $\boldsymbol{T}^l$. Applying the construction in [1], we have a minimal model of the form $(\boldsymbol{Q}[V \otimes B_*], \delta)$ for the function space $\mathscr{F}(\boldsymbol{T}^l, X)$. The model is nothing but the complex $(\mathscr{C}_*(\wedge V), \delta_l)$ which gives the iterated Hochschild homology of $(\wedge V, d)$.

We define maps on the iterated Hochschild and cyclic homologies, which are induced from a DGA map between free DGA's. Let $\varphi : (\wedge V, d) \to (\wedge W, d)$ be a DGA map. Then we can define a DGA map

$$\mathscr{C}(\varphi) : (\boldsymbol{Q}[V \otimes B_*], \delta) \to (\boldsymbol{Q}[W \otimes B_*], \delta)$$

by $\mathscr{C}(\varphi) = \rho_W^{-1}(\widetilde{\varphi \otimes 1})\rho_V$ with the isomorphisms $\rho_V : \boldsymbol{Q}[V \otimes B_*] \to \boldsymbol{Q}[\wedge V \otimes B_*]/I$ and $\rho_W : \boldsymbol{Q}[W \otimes B_*] \to \boldsymbol{Q}[\wedge W \otimes B_*]/I$ mentioned above. The same argument as in the proof of Proposition 2.1 works well to show that $\beta(\mathscr{C}(\varphi) \otimes 1) = (\mathscr{C}(\varphi) \otimes 1)\beta$. Thus we have a DGA map

$$\mathscr{E}(\varphi) = \mathscr{C}(\varphi) \otimes 1 : (\boldsymbol{Q}[V \otimes B_*] \otimes \boldsymbol{Q}[u], \delta + u\beta) \to (\boldsymbol{Q}[W \otimes B_*] \otimes \boldsymbol{Q}[u], \delta + u\beta).$$

Accordingly the DGA map φ gives a morphism of graded algebras

$$H(\mathscr{C}(\varphi)) : HH_*^{\{l\}}(\wedge V, d) \to HH_*^{\{l\}}(\wedge W, d)$$

and a morphism of $\boldsymbol{Q}[u]$-algebras

$$H(\mathscr{E}(\varphi)) : HC_*^{\{l\}}(\wedge V, d) \to HC_*^{\{l\}}(\wedge W, d).$$

Let $m_V : (\wedge V, d) \to (A, d)$ and $m_W : (\wedge W, d) \to (A, d)$ be minimal models for a connected DGA (A, d). For any elements $x \in HC_*^{\{l\}}(\wedge V, d)$ and $y \in HC_*^{\{l\}}(\wedge W, d)$, we write $x \sim y$ if $H(\mathscr{E}(\varphi_{VW}))(x) = y$ for some isomorphism $\varphi_{VW} : (\wedge V, d) \to (\wedge W, d)$ such that the diagram

$$\begin{tikzcd} & (A, d) & \\ (\wedge V, d) \arrow[ur, "m_V"] \arrow[rr, "\varphi_{VW}"'] & & (\wedge W, d) \arrow[ul, "m_W"'] \end{tikzcd}$$

is homotopy commutative. Observe that the isomorphism, which makes the triangle homotopy commutative, is determined uniquely up to homotopy. It is readily seen that $\sim$ is an equivalence relation.

We now define the iterated cyclic homology $HC_*^{\{l\}}(A, d)$ of a connected DGA (A, d) with iteration degree l by

$$HC_*^{\{l\}}(A, d) = \coprod_{\mathscr{M}_A \ni m_V : (\wedge V, d) \to (A, d)} HC_*^{\{l\}}(\wedge V, d) \Big/ \sim.$$

Here $\mathscr{M}_A$ denotes the set consisting of all minimal models for (A, d). It follows that, for any element $m_V : (\wedge V, d) \to (A, d)$ in $\mathscr{M}_A$, the inclusion map $HC_*^{\{l\}}(\wedge V, d) \hookrightarrow \coprod_{\mathscr{M}_A \ni m_V} HC_*^{\{l\}}(\wedge V, d)$ induces a bijection

$$\eta_{m_V} : HC_*^{\{l\}}(\wedge V, d) \to HC_*^{\{l\}}(A, d).$$

The $\boldsymbol{Q}[u]$-algebra structure on $HC_*^{\{l\}}(A, d)$ can be defined by forcing the bijection η_{m_V} becomes an isomorphism of $\boldsymbol{Q}[u]$-algebras for the given minimal model $m_V : (\wedge V, d) \to (A, d)$. Take another minimal model $m_W : (\wedge W, d) \to (A, d)$. Then we have a commutative diagram:

$$\begin{tikzcd} & HC_*^{\{l\}}(A,d) & \\ HC_*^{\{l\}}(\wedge V,d) \arrow[ur, "\eta_{m_V}"] \arrow[rr, "H(\mathscr{E}(\varphi_{VW}))"'] & & HC_*^{\{l\}}(\wedge W,d). \arrow[ul, "\eta_{m_W}"'] \end{tikzcd}$$

This implies that the $\boldsymbol{Q}[u]$-algebra structure on $HC_*^{\{l\}}(A,d)$ is uniquely determined without depending on the choice of minimal models.

In order to show that the cyclic homology can be viewed as a functor, we need a lemma.

LEMMA 2.3. *Let $\varphi_0, \varphi_1 : (\wedge V, d) \to (\wedge W, d)$ be DGA maps between free DGA's. If φ_0 is homotopic to φ_1, then $H(\mathscr{C}(\varphi_0)) = H(\mathscr{C}(\varphi_1))$ and $H(\mathscr{E}(\varphi_0)) = H(\mathscr{E}(\varphi_1))$.*

Proof. For $i = 0, 1$, let $\varepsilon_i : \wedge(t, dt) \to \boldsymbol{Q}$ be a DGA map defined by $\varepsilon_i(t) = i$. Let $H : \wedge V \to \wedge W \otimes \wedge(t, dt)$ be a homotopy from φ_0 to φ_1, namely, a DGA map satisfying $(1 \otimes \varepsilon_i)H = \varphi_i$ for $i = 0, 1$. We consider the following diagram in the category of DGA's:

$$\begin{tikzcd} \boldsymbol{Q}[\wedge V \otimes B_*]/I \arrow[r, "\widetilde{H \otimes 1_{B_*}}"] & \boldsymbol{Q}[\wedge W \otimes \wedge(t,dt) \otimes B_*]/I \arrow[d, "1_{\wedge W} \otimes \varepsilon_i \otimes 1_{B*}"] & \boldsymbol{Q}[(W \oplus \boldsymbol{Q}\{t,dt\}) \otimes B_*] \arrow[l, "\rho_{W \oplus \boldsymbol{Q}\{t,dt\}}"', "\cong"] \arrow[d, "\tilde{\varepsilon}_i"] \\ \boldsymbol{Q}[V \otimes B_*] \arrow[u, "\rho_V", "\cong"'] & \boldsymbol{Q}[\wedge W \otimes B_*]/I & \boldsymbol{Q}[W \otimes B_*] \arrow[l, "\rho_W"', "\cong"] \end{tikzcd}$$

in which $\tilde{\varepsilon}_i$ is a DGA map defined by $\rho_W^{-1}(1_{\wedge W} \otimes \varepsilon_i \otimes 1_{B*})\rho_{W \oplus \boldsymbol{Q}\{t,dt\}}$. Put $B_*^+ = B_*/\boldsymbol{Q}$. Let J_1 be the ideal of $\wedge(t \otimes 1)$ generated by the element $(t \otimes 1)(1 - t \otimes 1)$. Since $\wedge(t \otimes 1, dt \otimes 1)$ is decomposed as $\boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \oplus J_1 \oplus dJ_1$, it follows that

$$\begin{aligned} &\boldsymbol{Q}[(W \oplus \boldsymbol{Q}\{t, dt\}) \otimes B_*] \\ &\quad = \boldsymbol{Q}[W \otimes B_*] \otimes \wedge(t \otimes 1, dt \otimes 1) \otimes \boldsymbol{Q}[t \otimes B_*^+ \oplus dt \otimes B_*^+] \\ &\quad = \boldsymbol{Q}[W \otimes B_*] \otimes (\boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \oplus J_1 \oplus dJ_1) \otimes (\boldsymbol{Q} \oplus J_2) \\ &\quad = \boldsymbol{Q}[W \otimes B_*] \otimes \boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \\ &\qquad \oplus (\boldsymbol{Q}[W \otimes B_*] \otimes \boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \otimes J_2) \\ &\qquad \oplus (\boldsymbol{Q}[W \otimes B_*] \otimes (J_1 \oplus dJ_1)) \oplus (\boldsymbol{Q}[W \otimes B_*] \otimes (J_1 \oplus dJ_1) \otimes J_2) \\ &\quad = \boldsymbol{Q}[W \otimes B_*] \otimes \boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \oplus \mathscr{J}. \end{aligned}$$

Here J_2 is the ideal of $\boldsymbol{Q}[t \otimes B_*^+ \oplus dt \otimes B_*^+]$ generated by $t \otimes B_*^+$ and $dt \otimes B_*^+$, and $\mathscr{J}$ denotes the vector space $(\boldsymbol{Q}[W \otimes B_*] \otimes (J_1 \oplus dJ_1)) \oplus (\boldsymbol{Q}[W \otimes B_*] \otimes \wedge(t \otimes 1, dt \otimes 1) \otimes J_2)$. Thus we have

$$\mathscr{E}_*^{\{l\}}(\wedge W) = (\tilde{\varepsilon}_i \otimes 1)(\boldsymbol{Q}[W \otimes B_*] \otimes \boldsymbol{Q}\{1, t \otimes 1, dt \otimes 1\} \otimes \boldsymbol{Q}[u] \oplus \mathscr{J} \otimes \boldsymbol{Q}[u]).$$

Moreover a simple calculation enables us to deduce that $\mathscr{D}_l((J_1 \oplus dJ_1) \otimes \boldsymbol{Q}[u]) \subset \mathscr{J} \otimes \boldsymbol{Q}[u]$ and hence $\mathscr{D}_l(\mathscr{J} \otimes \boldsymbol{Q}[u]) \subset \mathscr{J} \otimes \boldsymbol{Q}[u]$. Since $1 \otimes t_{i*} = 1 \cdot 1 \otimes t_{i*} = (1 \otimes t_{i*})(1 \otimes 1) + (1 \otimes 1)(1 \otimes t_{i*})$ modulo I, it follows that $1 \otimes t_{i*} = 0$ in $\boldsymbol{Q}[\wedge W \otimes B_*]/I$. This fact implies that $1 \otimes b = 0$ in $\boldsymbol{Q}[\wedge W \otimes B_*]/I$ for any $b \in B_*^+$ and hence $(\varepsilon_i \otimes 1_{B_*})(t \otimes b) = 0$ for $b \in B_*^+$ and $i = 0, 1$. Thus we see that the DGA map $\mathscr{E}(H) = (\rho^{-1}_{W \oplus \boldsymbol{Q}\{t, dt\}} \otimes 1_{\boldsymbol{Q}[u]})(H \widetilde{\otimes 1}_{B_*} \otimes 1_{\boldsymbol{Q}[u]})(\rho_V \otimes 1_{\boldsymbol{Q}[u]})$ defines a linear map $h_{HC} : \mathscr{E}_*^{\{l\}}(\wedge V) \to \mathscr{E}_*^{\{l\}}(\wedge W)$ of degree -1 such that, for any $x \in \mathscr{E}_*^{\{l\}}(\wedge V)$,

$$\mathscr{E}(H)(x) = \mathscr{E}(\varphi_0)(x) + (\mathscr{E}(\varphi_1)(x) - \mathscr{E}(\varphi_0)(x))t - (-1)^{\deg x} h_{HC}(x)\, dt + \xi(x),$$

where $\xi(x)$ is an appropriate element of $\mathscr{J} \otimes \boldsymbol{Q}[u]$. Since $\mathscr{D}_l \mathscr{E}(H) = \mathscr{E}(H)\mathscr{D}_l$, we have $\mathscr{D}_l h_{HC} + h_{HC}\mathscr{D}_l = \mathscr{E}(\varphi_1) - \mathscr{E}(\varphi_0)$.

The same argument works well to show that $H(\mathscr{C}(\varphi_0)) = H(\mathscr{C}(\varphi_1))$ if φ_0 is homotopic to φ_1. □

We define a map between the iterated cyclic homologies induced from a DGA map. Let $\varphi : (A, d_A) \to (B, d_B)$ be a DGA map and let $\tilde{\varphi}_i : (\wedge V_i, d_i) \to (\wedge W_i, d_i')$ be models for φ $(i = 1, 2)$. Since there are isomorphisms of DGA's, $\varphi_{V_1 V_2} : (\wedge V_1, d_1) \xrightarrow{\cong} (\wedge V_2, d_2)$ and $\varphi_{W_1 W_2} : (\wedge W_1, d_1') \xrightarrow{\cong} (\wedge W_2, d_2')$ such that $\tilde{\varphi}_2 \varphi_{V_1 V_2} \sim \varphi_{W_1 W_2} \tilde{\varphi}_1$, it follows from Lemma 2.3 that $H(\mathscr{E}(\varphi_{W_1 W_2}))H(\mathscr{E}(\tilde{\varphi}_1)) = H(\mathscr{E}(\tilde{\varphi}_2))H(\mathscr{E}(\varphi_{V_1 V_2}))$. This fact allows us to define a map $HC(\varphi) : HC_*^{\{l\}}(A, d_A) \to HC_*^{\{l\}}(B, d_B)$ by $HC(\varphi)(x) = H(\mathscr{E}(\tilde{\varphi}_1))(x)$ for $x \in HC_*^{\{l\}}(\wedge V_1, d_1)$.

In similar fashion, we can define the iterated Hochschild homology $HH_*^{\{l\}}(A, d)$ of a DGA (A, d) with iteration degree l using minimal models for (A, d). Moreover it follows that a DGA map $\varphi : (A, d_A) \to (B, d_B)$ induces an algebraic map $HH(\varphi) : HH_*^{\{l\}}(A, d_A) \to HH_*^{\{l\}}(B, d_B)$.

Let $\mathscr{DGA}$ and $\mathscr{GA}$ be the categories of connected DGA's and the graded algebras, respectively. Let $\boldsymbol{Q}[u]$-$\mathscr{GA}$ denote the category of graded $\boldsymbol{Q}[u]$-algebras.

Theorem 2.4. *The iterated cyclic homology and the iterated Hochschild homology define functors* $HC_*^{\{l\}} : \mathscr{DGA} \to \boldsymbol{Q}[u]$-$\mathscr{GA}$ *and* $HH_*^{\{l\}} : \mathscr{DGA} \to \mathscr{GA}$, *respectively.*

Proof. The result follows from Lemma 2.3. □

One might regard that Theorem 2.4 follows from Theorem 1.1 by using the realization of a given DGA. This is valid if the DGA is an l-connected. We wish to stress that the 0-connectedness of DGA's is only assumed in Theorem 2.4.

Remark 2.5. As mentioned in Introduction, for a connected DGA (A, d), the iterated cyclic homology $HC_*^{\{l\}}(A, d)$ has an element with negative degree in general. If (A, d) is s-connected and l is a positive integer less than $s + 1$, then $HC_i^{\{l\}}(A, d) = 0$ for any nonzero integer i less than $s + 1 - l$. Moreover we see that $HC_0^{\{l\}}(A,d)$ contains $\boldsymbol{Q}$ as a direct summand. The iterated Hochschild homology enjoys the same property.

3. A natural transformation from $HC_*^{\{l\}}$ to $HC_{*-1}^{\{l+1\}}$

As is seen in the previous section, the iterated Hochschild and cyclic homologies are regarded as functors from the category $\mathscr{DGA}$ to the category $\mathscr{GA}$. By restricting the functors to each dimension, we can get functors $HH_i^{\{l\}}$ and $HC_i^{\{l\}}$ from $\mathscr{DGA}$ to the category of vector spaces over $\boldsymbol{Q}$. In this section, we define a natural transformation $\tau : HC_i^{\{l\}} \to HC_{i-1}^{\{l+1\}}$ and prove Theorem 1.2 by utilizing the natural transformation.

We begin by introducing the Connes exact sequence for the iterated Hochschild and cyclic homologies. Let $(\wedge V, d)$ be a connected free DGA. With the notations in the previous section, we have a short exact sequence:

$$0 \leftarrow \mathscr{C}_*^{\{l\}}(\wedge V) \xleftarrow{\pi} \mathscr{E}_*^{\{l\}}(\wedge V) \xleftarrow{i} \mathscr{E}_{*-2}^{\{l\}}(\wedge V) \leftarrow 0$$

in which i and π are defined by $i(\sum_{i\geq 0} w_i u^i) = \sum_{i\geq 0} w_i u^{i+1}$ and $\pi(\sum_{i\geq 0} w_i u^i) = w_0$, respectively, where $w_i \in \mathscr{C}_*^{\{l\}}(\wedge V)$. The short exact sequence gives rise to a long exact sequence, which is called the Connes exact sequence,

$$\cdots \leftarrow HC_{*-1}^{\{l\}}(\wedge V,d) \xleftarrow{B} HH_*^{\{l\}}(\wedge V,d) \xleftarrow{\tilde{\pi}} HC_*^{\{l\}}(\wedge V,d) \xleftarrow{S} HC_{*-2}^{\{l\}}(\wedge V,d) \leftarrow \cdots.$$

We observe that $B([w]) = [\beta w]$ for any cycle $w \in \mathscr{C}_*^{\{l\}}(\wedge V)$. Since the maps B, $\tilde{\pi}$ and S are natural for DGA maps between free DGA's, it follows that, in the above long exact sequence, $(\wedge V, d)$ can be replaced by any connected DGA (A,d). We also obtain natural transformations $B : HH_*^{\{l\}} \to HC_{*-1}^{\{l\}}$, $\tilde{\pi} : HC_*^{\{l\}} \to HH_*^{\{l\}}$ and $S : HC_{*-2}^{\{l\}} \to HC_*^{\{l\}}$.

Let B_s denote the exterior algebra $\wedge(t_1, \dots, t_s)$. We define a derivation $\tau : \boldsymbol{Q}[\wedge V \otimes B_{l*}] \to \boldsymbol{Q}[\wedge V \otimes B_{l+1*}]$ of degree -1 by

$$\tau(a \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) = (-1)^{|a|+\varepsilon_1+\cdots+\varepsilon_l} a \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l} t_{l+1})_*$$

for $a \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_* \in \wedge V \otimes B_{l*}$.

LEMMA 3.1. (i) $(d \otimes 1) \circ \tau = -\tau \circ (d \otimes 1)$.
(ii) $\tau(I) \subset I$, *where I denotes the ideal defined in Section* 2.

Proof. It is straightforward to check (i) and (ii). □

Lemma 3.1 allows us to obtain a derivation $\mathscr{C}(\tau) = \rho_V^{-1}\tilde{\tau}\rho_V : \mathscr{C}_i^{\{l\}}(\wedge V) \to \mathscr{C}_{i-1}^{\{l+1\}}(\wedge V)$ which satisfies the condition that $\mathscr{C}(\tau)\delta_l = -\delta_{l+1}\mathscr{C}(\tau)$. In particular, we have $\mathscr{C}(\tau)(v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_*) = (-1)^{|v|+\varepsilon_1+\cdots+\varepsilon_l} v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l} t_{l+1})_*$ for $v \otimes (t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l})_* \in V \otimes B_{l*}$. Define a derivation $\mathscr{E}(\tau) : \mathscr{E}_i^{\{l\}}(\wedge V) \to \mathscr{E}_{i-1}^{\{l+1\}}(\wedge V)$ by $\mathscr{E}(\tau) = \mathscr{C}(\tau) \otimes 1$. It is not hard to verify that $\beta\mathscr{C}(\tau) = -\mathscr{C}(\tau)\beta$. Thus we see that $\mathscr{D}_{l+1}\mathscr{E}(\tau) = -\mathscr{E}(\tau)\mathscr{D}_l$. From naturality of the maps $\mathscr{C}(\tau)$ and $\mathscr{E}(\tau)$, it follows that the maps induce natural transformations $\tau_{HH} : HH_*^{\{l\}} \to HH_{*-1}^{\{l+1\}}$ and $\tau_{HC} : HC_*^{\{l\}} \to HC_{*-1}^{\{l+1\}}$. Thus we have the following theorem. The proof is left to the reader.

THEOREM 3.2. *The diagram*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathscr{C}_*^{\{l\}}(\wedge V) & \xleftarrow{\pi} & \mathscr{E}_*^{\{l\}}(\wedge V) & \xleftarrow{i} & \mathscr{E}_{*-2}^{\{l\}}(\wedge V) & \longleftarrow & 0 \\ & & \Big\downarrow {\scriptstyle \mathscr{C}(\tau)} & & \Big\downarrow {\scriptstyle \mathscr{E}(\tau)} & & \Big\downarrow {\scriptstyle \mathscr{E}(\tau)} & & \\ 0 & \longleftarrow & \mathscr{C}_{*-1}^{\{l+1\}}(\wedge V) & \xleftarrow{\pi} & \mathscr{E}_{*-1}^{\{l+1\}}(\wedge V) & \xleftarrow{i} & \mathscr{E}_{*-3}^{\{l+1\}}(\wedge V) & \longleftarrow & 0 \end{array}$$

is commutative and hence, as natural transformations, $\tau_{HC}B = -B\tau_{HH}$, $\tau_{HH}\tilde{\pi} = \tilde{\pi}\tau_{HC}$ *and* $\tau_{HC}S = S\tau_{HC}$.

The rest of this section is devoted to proving Theorem 1.2. We mention here that an idea of the proof of [12, Theorem] due to Vigué-Poirrier and Sullivan underlies our proof of the theorem. We first prove the following proposition, which asserts that Theorem 1.2 holds for a special case.

PROPOSITION 3.3. *Let l be an integer greater than or equal to* 2 *and let* $(\wedge V, d)$ *be a minimal model for an l-connected DGA* (A, d). *Assume that* $(\wedge V, d)$ *has a sub DGA* $(\wedge Z, d)$ *of the form* $(\wedge a, 0)$ *or* $(\wedge(a, q), dq = a^{n+1})$ *and that there is a DGA map* $p : (\wedge V, d) \to (\wedge Z, d)$ *such that* $p\iota = id$, *where* $\iota : (\wedge Z, d) \to (\wedge V, d)$ *denotes the inclusion map. Then the sequence* $\dim HC_i^{\{l\}}(A, d)$ *is unbounded as* $i \to \infty$.

Proof. By virtue of Theorem 3.2, we have a commutative diagram:

$$\begin{array}{ccc} HH_{*+1}^{\{s\}}(\wedge Z, d) & \xrightarrow{\tau_{HH}} & HH_*^{\{s+1\}}(\wedge Z, d) \\ {\scriptstyle HH(\iota)} \Big\downarrow & & \Big\downarrow {\scriptstyle HH(\iota)} \\ HH_{*+1}^{\{s\}}(A, d) & \xrightarrow{\tau_{HH}} & HH_*^{\{s+1\}}(A, d) \\ {\scriptstyle B} \Big\downarrow & & \Big\downarrow {\scriptstyle B} \\ HC_*^{\{s\}}(A, d) & \xrightarrow{\tau_{HC}} & HC_{*-1}^{\{s+1\}}(A, d) \\ {\scriptstyle \tilde{\pi}} \Big\downarrow & & \Big\downarrow {\scriptstyle \tilde{\pi}} \\ HH_*^{\{s\}}(A, d) & \xrightarrow{\tau_{HH}} & HH_{*-1}^{\{s+1\}}(A, d) \\ {\scriptstyle HH(p)} \Big\downarrow & & \Big\downarrow {\scriptstyle HH(p)} \\ HH_*^{\{s\}}(\wedge Z, d) & \xrightarrow{\tau_{HH}} & HH_{*-1}^{\{s+1\}}(\wedge Z, d) \end{array}$$

for any $s < l$. Let μ_s denote the composite map $HH(p)\tilde{\pi}BHH(\imath) : HH_{*+1}^{\{s\}}(\wedge Z, 0) \to HH_*^{\{s\}}(\wedge Z, d)$.

Lemma 3.4. *There exists a graded subspace* $T = \bigoplus_{n \geq 1} T^n$ *of* $HH_{*+1}^{\{2\}}(\wedge Z, d)$ *such that*

$$\limsup_{n \to \infty} \dim \underbrace{\tau_{HH} \cdots \tau_{HH}}_{(s-1)\text{-}times} \mu_2(T^n) = \infty.$$

Thus Proposition 3.3 *follows from Lemma* 3.4. *In fact the sequence*

$$\dim \underbrace{\tau_{HC} \cdots \tau_{HC}}_{(s-1)\text{-}times} BHH(\imath)(T^n)$$

is unbounded. □

Proof of Lemma 3.4. Let $\bar{v}_i$ and $\bar{\bar{v}}$ denote the elements $v \otimes (t_i)_*$ in $\mathscr{C}_*^{\{2\}}(\wedge Z, d)$ for $i = 1, 2$ and the element $v \otimes (t_1 t_2)_*$, respectively. Then it follows that

$$(\mathscr{C}_*^{\{2\}}(\wedge Z, d), \delta) = (\wedge(a, \bar{a}_1, \bar{a}_2, \bar{\bar{a}}), 0)$$

when $d \equiv 0$. If $d \not\equiv 0$, then we see that

$$(\mathscr{C}_*^{\{2\}}(\wedge Z, d), \delta) = (\wedge(a, \bar{a}_1, \bar{a}_2, \bar{\bar{a}}, q, \bar{q}_1, \bar{q}_2, \bar{\bar{q}}), \delta),$$

where $\delta(q) = a^{n+1}$, $\delta(\bar{\bar{q}}) = (n+1)a^n \bar{\bar{a}} + 2\binom{n+1}{2} a^{n-1} \bar{a}_1 \bar{a}_2$, $\delta(\bar{q}_i) = (n+1)a^n \bar{a}_i$ and $\delta(a) = \delta(\bar{a}_i) = \delta(\bar{\bar{a}}) = 0$ for $i = 1, 2$. Now define a subspace T of $HH_*^{\{2\}}(\wedge Z, d)$ as follows:

$$T = \begin{cases} \boldsymbol{Q}\{\bar{a}_1^{k_1} \bar{a}_2^{k_2}\}_{k_1, k_2 \geq 1} & \text{when } |a| \text{ is odd and } d \equiv 0, \\ \boldsymbol{Q}\{a^{k_1} \bar{\bar{a}}^{k_2}\}_{k_1, k_2 \geq 1} & \text{when } |a| \text{ is even and } d \equiv 0, \\ \boldsymbol{Q}\{\bar{a}_1 \bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2}\}_{k_1, k_2 \geq 1} & \text{when } d \not\equiv 0. \end{cases}$$

Since $\mu_2(\bar{a}_1^{k_1} \bar{a}_2^{k_2}) = (k_1 \bar{a}_1^{k_1 - 1} \bar{a}_2^{k_2} - k_2 \bar{a}_1^{k_1} \bar{a}_2^{k_2 - 1})\bar{\bar{a}}$ and $\mu_2(a^{k_1} \bar{\bar{a}}^{k_2}) = k_1 a^{k_1 - 1}(\bar{a}_1 + \bar{a}_2)\bar{\bar{a}}^{k_2}$, it follows that T is the required subspace of $HH_*^{\{2\}}(\wedge Z, 0)$ in the case where $d \equiv 0$.

Suppose that $d \not\equiv 0$. For a positive integer $s < l$, let $\mathscr{I}_s$ be the ideal of $\mathscr{C}_*^{\{s+1\}}(\wedge Z)$ generated by the elements $a \otimes 1_*$ and $a \otimes (t_1^{\varepsilon_1} \cdots t_{s+1}^{\varepsilon_{s+1}})_* \cdot a \otimes (t_1^{\varepsilon_1'} \cdots t_{s+1}^{\varepsilon_{s+1}'})_*$ with $\varepsilon_1 + \cdots + \varepsilon_{s+1} < s+1$ and $\varepsilon_1' + \cdots + \varepsilon_{s+1}' < s+1$. Then it follows that elements $\delta(q \otimes (t_1^{\varepsilon_1} \cdots t_{s+1}^{\varepsilon_{s+1}})_*)$ are in the ideal $\mathscr{I}_s$ (see (2.1)) and hence $\operatorname{Im} \delta$ is a vector subspace of $\mathscr{I}_s$. By definition we see that

$$\mu_2(\bar{a}_1 \bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2}) = (-\bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2} - \bar{a}_1 \bar{q}_1^{k_1} \bar{q}_2^{k_2})\bar{\bar{a}} + (k_1 \bar{a}_1 \bar{q}_1^{k_1 - 1} \bar{a}_2 \bar{q}_2^{k_2} - k_2 \bar{a}_1 \bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2 - 1})\bar{\bar{q}}.$$

Therefore $\underbrace{\tau_{HH} \cdots \tau_{HH}}_{(s-1)\text{-times}} \mu_2(\bar{a}_1 \bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2})$ has a term

$$(-1)^{[s/2]}(-\bar{q}_1^{k_1} \bar{a}_2 \bar{q}_2^{k_2} - \bar{a}_1 \bar{q}_1^{k_1} \bar{q}_2^{k_2})(a \otimes (t_1 t_2 \cdots t_{s+1})_*).$$

We can conclude that elements $\tau_{HH}\cdots\tau_{HH}\mu_2(\bar{a}_1\bar{q}_1^{k_1}\bar{a}_2\bar{q}_2^{k_2})$ $(k_1, k_2 \geq 1)$ are not in the ideal $\mathscr{I}_s$ and are homologically independent. Thus we have Lemma 3.4. $\square$

Before proving Theorem 1.2, we fix terminology. Let $(\wedge W, d)$ be a DGA and Γ a basis of W. Let x be an element of W. We write $dx = \sum_i \lambda_i v_{i_1} v_{i_2} \cdots v_{i_{k(i)}}$, where v_{i_j} is in Γ, $\lambda_i \in \boldsymbol{Q}$ and $v_{i_1} v_{i_2} \cdots v_{i_{k(i)}} \neq v_{j_1} v_{j_2} \cdots v_{j_{k(j)}}$ if $i \neq j$. Then we shall say that an element y of Γ is detected by x with the differential d if y appears in some term of dx as a factor; that is, $y = v_{i_j}$ for some i_j.

Proof of Theorem 1.2. We intend to prove the unboundedness of the sequence of $\dim HC_i^{\{l\}}(A, d)$. The result concerning the iterated Hochschild homology follows from the proof.

Let $(\wedge V, d) \to (A, d)$ be a minimal model for a DGA (A, d). Theorem 1.2 follows from Proposition 3.3 if there exists a non-trivial element y with odd degree in V such that $V^i = 0$ for $i < \deg y$. In fact, the element y is a cycle. Therefore we see that the inclusion $(\wedge y, 0) \to (\wedge V, d)$ has a left inverse.

We choose a basis $S = \{x_1, x_2, \ldots, x_n, y_1, x_{n+1}, \ldots, x_r, y_2, \ldots\}$ of V so that $|x_1| \leq \cdots \leq |x_n| < |y_1| < |x_{n+1}| \leq \cdots$, where $|x_i|$ is even and $|y_j|$ is odd. Observe that $d(x_i) = 0$ for $1 \leq i \leq n$. Suppose that there is no element with odd degree in S or $d(y_1) = 0$. Then the result follows from Proposition 3.3.

In what follows, we put $a_{\{k\}} = a \otimes (t_1 \cdots t_k)_*$ for $a \in S$. We assume that $n \geq 2$. Let us deal with the case where l is even. We choose an element $z = x_{1\{l\}}^{k_1} x_{2\{l\}}^{k_2}$ in $\mathscr{E}_*^{\{l\}}(\wedge V, d)$, where k_i is a positive integer for $i = 1, 2$. It is readily seen that z is a $\mathscr{D}_l$-cycle and hence $\pi(z) = x_{1\{l\}}^{k_1} x_{2\{l\}}^{k_2}$ is a δ_l-cycle. Suppose that there exists an element α in $\mathscr{C}_*^{\{l\}}(\wedge V, d) = (\wedge W_l, \delta_l)$ such that $\delta_l(\alpha) = \pi(z)$. Look at the element $\theta = x_{1\{l\}}$ in $\pi(z)$. Since the element θ can be detected by some factor q of a term of α with δ_l, it follows from the definition of δ_l that the factor q is of the form $v \otimes (t_1 \cdots t_l)_*$ in which $v \in V$ detects x_1 with d. We write $d(v) = x_1(\sum_s u_s) + \gamma$, where u_s are elements in $\wedge V$ and the element γ is not in the ideal of $\wedge V$ generated by x_1. Therefore the element $q = v \otimes (t_1 \cdots t_l)_*$ detects factors of u_s, as well, in the term which contains θ. This is a contradiction because $\pi(z)$ does not contain any elements of the form $b \otimes 1$ with $b \in \wedge V$. By the same argument as above, we see that the element $x_{2\{l\}}$ in $\pi(z)$ is not detected by an element of $\wedge W_l$ with δ_l. In consequence, the elements $\pi(x_{1\{l\}}^{k_1} x_{2\{l\}}^{k_2})$ $(k_1, k_2 \geq 1)$ are homologically independent in $\mathscr{C}_*^{\{l\}}(\wedge W_l, \delta_l)$ and hence so are $x_{1\{l\}}^{k_1} x_{2\{l\}}^{k_2}$ in $\mathscr{E}_*^{\{l\}}(\wedge V, d)$.

We next consider the case where $n \geq 2$ and l is odd. Choose an element $w = x_{1\{l-1\}}^{k_1} x_{2\{l-1\}}^{k_2}$ which is a δ_l-cycle. We show that $\beta(w)$ is not in $\operatorname{Im} \mathscr{D}_l$. To this end, it suffices that

$$\pi\beta(w) = k_1 x_{1\{l-1\}}^{k_1-1} x_{1\{l\}} x_{2\{l-1\}}^{k_2} + k_2 x_{1\{l-1\}}^{k_1} x_{2\{l\}} x_{2\{l-1\}}^{k_2-1}$$

is not in $\operatorname{Im} \delta_l$. Suppose that there exists an element γ in $\mathscr{C}_*^{\{l\}}(\wedge V, d) = (\wedge W_l, \delta_l)$ such that $\delta_l(\gamma) = \pi\beta(w)$. If the element $x_{1\{l-1\}}$ in $\pi\beta(w)$ can be detected by some factor q of a term of γ with δ_l, then, by definition, the factor q has the form $v \otimes (t_1 \cdots t_{l-1})_*$ or $v \otimes (t_1 \cdots t_l)_*$ for some $v \in V$. This fact enables us to deduce that if the element $x_{1\{l-1\}}$ is detected with δ_l, then v' or $v' \otimes (t_l)_*$ is also detected in the same term, as well. Here v' denotes an element of V which appears as a factor of a term dv. This is a contradiction. By the same argument as above, we see that the elements $x_{1\{l\}}$, $x_{2\{l-1\}}$ and $x_{2\{l\}}$ *in* $\pi\beta(z)$ are never detected with δ_l. Hence it follows that the elements $\beta(x_{1\{l-1\}}^{k_1} x_{2\{l-1\}}^{k_2})$ $(k_1 \geq 1, k_2 \geq 1)$ are homologically independent in $\mathscr{E}_*^{\{l\}}(\wedge V, d)$.

We prove Theorem 1.2 in the case where $n = 1$ and there is at least one element with odd degree. In such a case, if $d(y_1) \neq 0$, then the result follows from Proposition 3.3. This completes the proof. □

4. A computational example

In this section, we compute the iterated cyclic homology with iteration degree 2 of the polynomial algebra $\boldsymbol{Q}[x]$ generated by a single element x.

By definition, we see that

$$HC_*^{\{2\}}(\boldsymbol{Q}[x], 0) = H(\boldsymbol{Q}[x, \bar{\bar{x}}] \otimes \wedge(\bar{x}_1, \bar{x}_2) \otimes \boldsymbol{Q}[u], \delta + u\beta),$$

where $x = x \otimes 1$, $\bar{x}_1 = x \otimes t_{1*}$, $\bar{x}_2 = x \otimes t_{2*}$ and $\bar{\bar{x}} = x \otimes (t_1 t_2)_*$. Since $\beta(x) = \bar{x}_1 + \bar{x}_2$, $\beta(\bar{x}_1) = -\bar{\bar{x}}$, $\beta(\bar{x}_2) = \bar{\bar{x}}$ and $\beta(\bar{\bar{x}}) = \beta(u) = 0$, the differential $\mathscr{D} = \delta + u\beta$ is given by

$$\mathscr{D}(x) = (\bar{x}_1 + \bar{x}_2)u, \quad \mathscr{D}(\bar{x}_1) = -\bar{\bar{x}}u, \quad \mathscr{D}(\bar{x}_2) = \bar{\bar{x}}u, \quad \mathscr{D}(\bar{\bar{x}}) = \mathscr{D}(u) = 0.$$

Put $\alpha = \bar{x}_1 + \bar{x}_2$. It is readily seen that the complex which computes $HC_*^{\{2\}}(\boldsymbol{Q}[x], 0)$ is isomorphic to the complex $\wedge U = (\boldsymbol{Q}[x, \bar{\bar{x}}] \otimes \wedge(\alpha, \bar{x}_2) \otimes \boldsymbol{Q}[u], \mathscr{D}')$ as a DGA, where $\mathscr{D}'(x) = \alpha u$, $\mathscr{D}'(\alpha) = 0$, $\mathscr{D}'(\bar{x}_2) = \bar{\bar{x}}u$, $\mathscr{D}'(\bar{\bar{x}}) = 0$, and $\mathscr{D}'(u) = 0$. With the aid of the manner for computation of a cohomology, which is described in [10, Section 7], we can execute the computation.

Define a weight w on the DGA $\wedge U$ by $w(\alpha) = 1$ and $w(x) = w(\bar{x}_2) = w(\bar{\bar{x}}) = w(u) = 0$. Consider a filtration of the DGA defined by $F_i = \{z \in \wedge U \mid w(z) \geq i\}$. Then it follows that the differential $\mathscr{D}'$ and the product respect the filtration. Thus the filtration gives rise to a spectral sequence converging to $HC_*^{\{2\}}(\boldsymbol{Q}[x], 0)$ as an algebra. The E_0-term is given by

$$E_0 = (\boldsymbol{Q}[\bar{\bar{x}}] \otimes \wedge(\bar{x}_2) \otimes \boldsymbol{Q}[u]) \otimes (\boldsymbol{Q}[x] \otimes \wedge(\alpha)),$$

$$d_0(x) = 0, \quad d_0(\alpha) = 0, \quad d_0(\bar{x}_2) = \bar{\bar{x}}u, \quad d_0(\bar{\bar{x}}) = 0, \quad d_0(u) = 0.$$

Therefore we see that

$$E_1 = \frac{\boldsymbol{Q}[\bar{\bar{x}}, u]}{(\bar{\bar{x}}u)} \otimes \boldsymbol{Q}[x] \otimes \wedge(\alpha)$$

as an algebra. It follows from the definition of the filtration that $d_1(x) = \alpha u$ and $d_1(\alpha) = d_1(\bar{\bar{x}}) = d_1(u) = 0$. Thus we can get

$$E_2 = \frac{\wedge(\beta_0, \beta_1, \ldots) \otimes \boldsymbol{Q}[\gamma_0, \gamma_1, \ldots] \otimes \boldsymbol{Q}[u]}{(\beta_k\beta_{k'}, \beta_k\gamma_l - \beta_{k+l}\gamma_0, \beta_k u, \gamma_l\gamma_{l'} - \gamma_{l+l'}\gamma_0, \gamma_l u)},$$

where $\beta_k = \alpha x^k$ and $\gamma_l = x^l\bar{\bar{x}}$ $(k, l = 0, 1, \ldots)$. There is no element whose filtration degree is increased by more than or equal to 2 by the differential. Thus the spectral sequence collapses at the E_2-term and hence $E_2 \cong E_\infty$ as a bigraded algebra. We have to solve the extension problems. A straightforward calculation enables us to conclude that $\mathscr{D}'(\beta_k) = 0$ and $\mathscr{D}'(\gamma_l + lx^{l-1}\alpha\bar{x}_2) = 0$. The latter equality implies that the element $\tilde{\gamma}_l := \gamma_l + lx^{l-1}\alpha\bar{x}_2$ of $\wedge U$ represents γ_l in the E_∞-term. Moreover we see that, in $\wedge U$, $\beta_k\beta_{k'} = 0$, $\beta_k\tilde{\gamma}_l - \beta_{k+l}\tilde{\gamma}_0 = 0$, $\beta_k u = \mathscr{D}'\left(\dfrac{1}{k+1}x^{k+1}\right)$, $\tilde{\gamma}_l\tilde{\gamma}_{l'} - \tilde{\gamma}_{l+l'}\tilde{\gamma}_0 = 0$ and $\tilde{\gamma}_l u = \mathscr{D}'(x^l\bar{x}_2)$. Thus we have

$$HC_*^{\{2\}}(\boldsymbol{Q}[x], 0) = \frac{\wedge(\beta_0, \beta_1, \ldots) \otimes \boldsymbol{Q}[\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots] \otimes \boldsymbol{Q}[u]}{(\beta_k\beta_{k'}, \beta_k\tilde{\gamma}_l - \beta_{k+l}\tilde{\gamma}_0, \beta_k u, \tilde{\gamma}_l\tilde{\gamma}_{l'} - \tilde{\gamma}_{l+l'}\tilde{\gamma}_0, \tilde{\gamma}_l u)},$$

where $\deg \beta_k = (k+1)\deg x - 1$, $\deg \tilde{\gamma}_l = (l+1)\deg x - 2$ and $\deg u = 2$.

We see that $HC_*^{\{2\}}(\boldsymbol{Q}[x], 0)$ consists of elements with non-negative degree. Observe that $HC_0^{\{2\}}(\boldsymbol{Q}[x], 0) \cong \boldsymbol{Q}$ as long as $\deg x > 2$.

Remark 4.1. By virtue of Theorem 1.1, we can obtain the explicit form of the Borel cohomology $H^*(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^2, BSU(2)); \boldsymbol{Q})$. In fact the cohomology is isomorphic to $HC_*^{\{2\}}(\boldsymbol{Q}[x], 0)$ as an $H^*(B\boldsymbol{T}; \boldsymbol{Q}) = \boldsymbol{Q}[u]$-algebra, where $\deg x = 4$.

More computations of the iterated cyclic homologies will be made in [13].

5. A model for $E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X)$

With the same notations as in Section 2, we construct the extension

$$(\boldsymbol{Q}[u], 0) \underset{j}{\rightarrowtail} (\mathscr{C}_*^{\{l\}}(\wedge V) \otimes \boldsymbol{Q}[u], \delta_l + u\beta) = (\mathscr{E}_*^{\{l\}}(\wedge V), \mathscr{D}_l) \underset{\pi}{\twoheadrightarrow} (\mathscr{C}_*^{\{l\}}(\wedge V), \delta_l)$$

in which $j(u) = u$ and $\pi(\sum_{i \geq 0} w_i u^i) = w_0$. We relate the extension with a topological object in terms of rational homotopy theory. More precisely, we shall establish the following theorem.

Theorem 5.1. *Suppose X is an l-connected space with* $\dim \pi_k(X) \otimes \boldsymbol{Q} < \infty$ *for any k. Let $(\wedge V, d) \xrightarrow{\simeq} A_{PL}(X)$ be a minimal model for X and $\mathscr{F}(\boldsymbol{T}^l, X) \xrightarrow{i} E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X) \xrightarrow{p} B\boldsymbol{T}$ the Borel fibration. Then there exists a commutative diagram* (5.1):

$$\begin{tikzcd}
A_{PL}(B\boldsymbol{T}) \arrow[r, "A_{PL}(p)"] & A_{PL}(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X)) \arrow[r, "A_{PL}(i)"] & A_{PL}(\mathscr{F}(\boldsymbol{T}^l, X)) \\
(\boldsymbol{Q}[u], 0) \arrow[u, "n", "\simeq"'] \arrow[r, tail, "j"'] & (\mathscr{E}_*^{\{l\}}(\wedge V), \mathscr{D}_l) \arrow[u, "m_l", "\simeq"'] \arrow[r, two heads, "\pi"'] & (\mathscr{C}_*^{\{l\}}(\wedge V), \delta_l) \arrow[u, "\bar{m}_l", "\simeq"']
\end{tikzcd}$$

such that n, m_l and $\bar{m}_l$ are quasi-isomorphisms.

As an immediate corollary, we have the following result which yields Theorem 1.1.

Corollary 5.2. *Under the same hypotheses as in Theorem* 5.1, *there are an isomorphism of* $\boldsymbol{Q}[u]$*-algebras*

$$HC_*^{\{l\}}(A_{PL}(X)) \cong H^*(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^l, X); \boldsymbol{Q})$$

and an isomorphism of algebras

$$HH_*^{\{l\}}(A_{PL}(X)) \cong H^*(\mathscr{F}(\boldsymbol{T}^l, X); \boldsymbol{Q}).$$

We argue by induction on the integer l in the diagram (5.1). In the case $l = 1$, The result follows from [11, Theorem A].

Assume that Theorem 5.1 holds for the case $l = N - 1$. We put $(\mathscr{C}^{\{N\}}(\wedge V), \delta_N) = (\wedge W_N, \delta_N)$. Let $n : (\boldsymbol{Q}[u], 0) \xrightarrow{\simeq} A_{PL}(B\boldsymbol{T})$ be a minimal model for $B\boldsymbol{T}$ and $q : (\wedge Z \otimes \boldsymbol{Q}[u], d) \xrightarrow{\simeq} A_{PL}(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X))$ be a Sullivan model for the map $A_{PL}(p)n$. Then we have a commutative diagram:

$$\begin{tikzcd}
A_{PL}(B\boldsymbol{T}) \arrow[r, "A_{PL}(p)"] & A_{PL}(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)) \arrow[r, "A_{PL}(i)"] & A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X)) \\
(\boldsymbol{Q}[u], 0) \arrow[u, "n", "\simeq"'] \arrow[r, tail, "j"'] & (\wedge Z \otimes \boldsymbol{Q}[u], d) \arrow[u, "q", "\simeq"'] \arrow[r, two heads] & (\wedge Z, \bar{d}), \arrow[u, "\bar{q}", "\simeq"']
\end{tikzcd}$$

where the map $\bar{q}$ is obtained from q by reducing the elements in the ideal of $\wedge Z \otimes \boldsymbol{Q}[u]$ generated by u. The result [4, Proposition 15.3] enables us to deduce that the map $\bar{q}$ is a minimal model for $\mathscr{F}(\boldsymbol{T}^N, X)$ (see also [7]). Choose the minimal model $c_N : (\wedge W_N, \delta_N) \xrightarrow{\simeq} A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X))$ for $\mathscr{F}(\boldsymbol{T}^N, X)$ due to Brown and Szczarba (see Remark 2.2). Observe that the quasi-isomorphism c_N is obtained by applying the lifting lemma to quasi-isomorphisms between DGA's (see [1], [9, Section 2]). Then there exists an isomorphism $f : (\wedge W_N, \delta_N) \xrightarrow{\cong} (\wedge Z, d)$ such that $\bar{q}f \sim c_N$. Using the isomorphism $f \otimes 1 : \wedge W_N \otimes \boldsymbol{Q}[u] \to \wedge Z \otimes \boldsymbol{Q}[u]$, we define a differential Δ on $\wedge W_N \otimes \boldsymbol{Q}[u]$ by $\Delta = (f \otimes 1)^{-1} d (f \otimes 1)$. Put $m_N = q(f \otimes 1)$ and $\bar{m}_N = \bar{q}f$. We thus have a commutative diagram

$$\begin{tikzcd}
A_{PL}(B\boldsymbol{T}) \arrow[r, "A_{PL}(p)"] & A_{PL}(E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)) \arrow[r, "A_{PL}(i)"] & A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X)) \\
(\boldsymbol{Q}[u], 0) \arrow[u, "n", "\simeq"'] \arrow[r, tail, "j"'] & (\wedge W_N \otimes \boldsymbol{Q}[u], \Delta) \arrow[u, "m_N", "\simeq"'] \arrow[r, two heads] & (\wedge W_N, \delta_N). \arrow[u, "\bar{m}_N", "\simeq"']
\end{tikzcd}$$

Let $c_{N-1} : (\wedge W_{N-1}, \delta_N) \xrightarrow{\simeq} A_{PL}(\mathscr{F}(\boldsymbol{T}^{N-1}, X))$ be the minimal model due to Brown and Szczarba. Let $\{v_j\}$ be a basis of the vector space V. Since W_{N-1} and W_N have bases $\{v_j \otimes (t_1^{\varepsilon_1} \cdots t_{N-1}^{\varepsilon_{N-1}})_*\}$ and $\{v_j \otimes (t_1^{\varepsilon_1} \cdots t_N^{\varepsilon_N})_*\}$, respectively, we can write $W_N = W_{N-1} \oplus (W_{N-1} \otimes t_{N*})$. To simplify, put $W = W_{N-1}$ and $\overline{W} = W_{N-1} \otimes t_{N*}$. For $w \in W$, let $\bar{w}$ denote the element $\beta_N(w) - \beta_{N-1}(w)$.

Since $\overline{w} = (-1)^{|v|+\varepsilon_1+\cdots+\varepsilon_{N-1}} v_j \otimes (t_1^{\varepsilon_1} \cdots t_{N-1}^{\varepsilon_{N-1}} t_N)_*$ if $w = v_j \otimes (t_1^{\varepsilon_1} \cdots t_{N-1}^{\varepsilon_{N-1}})_*$, it follows that the correspondence $w \mapsto \overline{w}$ gives an isomorphism between W and $\overline{W}$.

Let $\eta : \boldsymbol{T}^N \to \boldsymbol{T}^{N-1}$ be the natural projection on the first $N-1$ factors. Observe that η is a $\boldsymbol{T}$-equivariant map. There exists a commutative diagram

$$\begin{tikzcd}
A_{PL}(\boldsymbol{T}^{N-1}) \arrow[r, "A_{PL}(\eta)"] & A_{PL}(\boldsymbol{T}^N) \\
\wedge(t_1, \ldots, t_{N-1}) \arrow[u, "\simeq"'] \arrow[r, "\tilde{\eta}"'] & \wedge(t_1, \ldots, t_N) \arrow[u, "\simeq"]
\end{tikzcd}$$

such that vertical arrows are quasi-isomorphisms and $\tilde{\eta}(t_i) = t_i$ for $i \le N-1$. Moreover quasi-isomorphisms c_l connecting $A_{PL}(\mathscr{F}(\boldsymbol{T}^l, X))$ with $(\wedge W_l, \delta_l)$ mentioned above are natural with respect to $A_{PL}(\eta)$ and $\tilde{\eta}$ for $l = N$ and $N-1$. These facts enable us to obtain a homotopy commutative diagram:

$$\begin{tikzcd}
A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X)) \arrow[r, "A_{PL}(\eta^{\#})"] & A_{PL}(\mathscr{F}(\boldsymbol{T}^{N-1}, X)) \\
(\wedge W \otimes \wedge \overline{W}, \delta_N) \arrow[u, "c_N", "\simeq"'] \arrow[r, "k"'] & (\wedge W, \delta_{N-1}), \arrow[u, "\simeq", "c_{N-1}"']
\end{tikzcd}$$

where $\eta^{\#} : \mathscr{F}(\boldsymbol{T}^{N-1}, X) \to \mathscr{F}(\boldsymbol{T}^N, X)$ is the map induced by η and k is a DGA map defined by $k(w) = w$ for $w \in W$ and $k(\overline{w}) = 0$ for $\overline{w} \in \overline{W}$.

Consider the following commutative diagram:

$$\begin{tikzcd}
(\boldsymbol{Q}[u], 0) \arrow[rr, tail] \arrow[d, "j"'] & & (\wedge W \otimes \boldsymbol{Q}[u], \mathscr{D}_{N-1}) \arrow[d, "\simeq"', "m_{N-1}"] \\
(\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \Delta) \arrow[r, "m_N", "\simeq"'] & A_{PL}(E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)) \arrow[r, "A_{PL}(1 \times \eta^{\#})"] & A_{PL}(E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^{N-1}, X)).
\end{tikzcd}$$

By virtue of [4, Proposition 14.6], we have a lift q of $A_{PL}(1 \times \eta^{\#}) m_N$ and a diagram (5.2):

$$\begin{tikzcd}
A_{PL}(B\boldsymbol{T}) \arrow[rr] \arrow[dr] & & A_{PL}(E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^{N-1}, X)) \arrow[rr] & & A_{PL}(\mathscr{F}(\boldsymbol{T}^{N-1}, X)) \\
& A_{PL}(E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)) \arrow[ur, "A_{PL}(1 \times \eta^{\#})"] \arrow[rr] & & A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X)) \arrow[ur, "A_{PL}(\eta^{\#})"] & \\
(\boldsymbol{Q}[u], 0) \arrow[uu] \arrow[rr, tail, "m_N"] \arrow[dr] & & (\wedge W \otimes \boldsymbol{Q}[u], \mathscr{D}_{N-1}) \arrow[uu, "m_{N-1}"] \arrow[rr, two heads, "\bar{m}_N"] & & (\wedge W, \delta_{N-1}) \arrow[uu, "\bar{m}_{N-1}"] \\
& (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \Delta) \arrow[uu] \arrow[ur, dotted, "q"] \arrow[rr, two heads] & & (\wedge W \otimes \wedge \overline{W}, \delta) \arrow[uu] \arrow[ur, dotted, "\bar{q}"] &
\end{tikzcd}$$

in which $A_{PL}(1 \times \eta^{\#})m_N \sim m_{N-1}q$, $A_{PL}(\eta^{\#})\overline{m}_N \sim \overline{m}_{N-1}\bar{q}$ and other squares are commutative. Observe that a homotopy between $A_{PL}(1 \times \eta^{\#})m_N$ and $m_{N-1}q$ gives rise to that between $A_{PL}(\eta^{\#})\overline{m}_N$ and $\overline{m}_{N-1}\bar{q}$. Since $\overline{m}_{N-1}$ and c_{N-1} are minimal models for $A_{PL}(\mathscr{F}(\boldsymbol{T}^{N-1}, X))$, there is an isomorphisms g from $(\wedge W, \delta_{N-1})$ to itself such that $c_{N-1}g \sim \overline{m}_{N-1}$. It is readily seen that $k \sim g\bar{q}$ because $\overline{m}_N \sim c_N$. Let $\{w_i\}$ be a basis of W and let $\{\overline{w}_i\}$ be the corresponding basis of $\overline{W}$. Since the linear part of k coincides with that of $g\bar{q}$ (see, for example, [4, Proposition 12.8(ii)]), by induction on the degree of $\{w_i\}$ and $\{\overline{w}_i\}$, we see that

$$g\bar{q}(w_i + z_i) = w_i \quad \text{and} \quad g\bar{q}(\overline{w}_i + y_i) = 0$$

for some decomposable elements z_i and y_i of $\wedge W$. We write $g(w_i) = \sum \mu_{ij} w_j$ modulo decomposable elements. Accordingly there exists a decomposable element x_i such that $g\bar{q}(\sum \mu_{ij} w_j + x_i) = g(w_i)$. Define a map φ from $(\wedge W \otimes \wedge \overline{W})$ to itself by $\varphi(w_i) = \sum \mu_{ij} w_j + x_i$ and $\varphi(\overline{w}_i) = \overline{w}_i + y_i$. Then it follows that φ is an isomorphism of algebras satisfying $\bar{q}\varphi(w_i) = w_i$ and $\bar{q}\varphi(\overline{w}_i) = 0$. By using φ, we can construct a commutative diagram in $\mathscr{DGA}$, (5.3):

$$\begin{tikzcd}
(\boldsymbol{Q}[u], 0) \arrow[r, tail] \arrow[d, equal] & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \Delta) \arrow[r, two heads] & (\wedge W \otimes \wedge \overline{W}, \delta_N) \\
(\boldsymbol{Q}[u], 0) \arrow[r, tail] & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\Delta}) \arrow[r, two heads] \arrow[u, "\cong"', "\varphi \otimes 1"] & (\wedge W \otimes \wedge \overline{W}, \tilde{\delta}_N), \arrow[u, "\cong"', "\varphi"]
\end{tikzcd}$$

where $\tilde{\Delta} = (\varphi \otimes 1)^{-1} \Delta (\varphi \otimes 1)$ and $\tilde{\delta} = \varphi^{-1} \delta \varphi$.

The commutativity of the diagram (5.3) and of the bottom of the diagram (5.2) implies that $q(\varphi \otimes 1)(w_i) = w_i$ and $q(\varphi \otimes 1)(\overline{w}_i) = 0$ modulo the ideal (u) of $\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u]$ generated by u. By induction on the degree of elements of $\wedge W \otimes \wedge \overline{W}$, we can define an isomorphism ψ of algebras from $\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u]$ to itself satisfying the condition that the induced map $\bar{\psi} : \wedge W \otimes \wedge \overline{W} \to \wedge W \otimes \wedge \overline{W}$ is the identity map and $\psi(u) = u$, $q(\varphi \otimes 1)\psi(w_i) = w_i$, $q(\varphi \otimes 1)\psi(\overline{w}_i) = 0$. Thus we have a commutative diagram (5.4):

$$\begin{tikzcd}
(\boldsymbol{Q}[u], 0) \arrow[r, tail] \arrow[d, equal] & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\Delta}) \arrow[r, two heads] & (\wedge W \otimes \wedge \overline{W}, \tilde{\delta}_N) \\
(\boldsymbol{Q}[u], 0) \arrow[r, tail] & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\tilde{\Delta}}) \arrow[r, two heads] \arrow[u, "\cong"', "\psi"] & (\wedge W \otimes \wedge \overline{W}, \tilde{\delta}_N), \arrow[u, "="', "\bar{\psi}"]
\end{tikzcd}$$

where $\tilde{\tilde{\Delta}} = (\psi)^{-1} \tilde{\Delta} \psi$. Observe that the lower sequence is also a model for the fibration $\mathscr{F}(\boldsymbol{T}^N, X) \to E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X) \to B\boldsymbol{T}$.

It remains to prove that there exists an isomorphism of DGA's between $(\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \mathscr{D}_N)$ and $(\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\tilde{\Delta}})$ whose restriction on $\boldsymbol{Q}[u]$ is the identity map. We write

$$\tilde{\tilde{\Delta}}(z) = \tilde{\delta}_N(z) + \sum_{i \geq 1} u^i \theta_i(z)$$

for $z \in \wedge W \otimes \wedge \overline{W}$. Put $\tilde{\tilde{p}} = q(\varphi \otimes 1)\psi$. Since $\mathscr{D}_{N-1}\tilde{\tilde{p}} = \tilde{\tilde{p}}\tilde{\tilde{\Delta}}$ and $\tilde{\tilde{p}}$ is the identity map on W, it follows that, for $w \in W$,

$$\begin{aligned}\delta_{N-1}(w) + u\beta_{N-1}(w) = \mathscr{D}_{N-1}(w) = \mathscr{D}_{N-1}\tilde{\tilde{p}}(w) = \tilde{\tilde{p}}\tilde{\tilde{\Delta}}(w)\\ = \tilde{\tilde{p}}\tilde{\delta}_N(w) + \sum_{i \geq 1} u^i \tilde{\tilde{p}}\theta_i(w).\end{aligned}$$

The equality enables us to conclude that $\tilde{\tilde{p}}\tilde{\delta}_N(w) = \delta_{N-1}(w)$ for $w \in W$. By definition of φ, we see that $\varphi(\wedge W) = \wedge W$ and hence $\tilde{\delta}_N = \varphi^{-1}\delta_N\varphi$ is closed in $\wedge W$. This fact implies that $\tilde{\delta}_N(w) = \delta_N(w)$. Observe that $\delta_N = \delta_{N-1}$ on $\wedge W$. Putting $\tilde{\theta}_1(w) = \theta_1(w) - \beta_{N-1}(w)$ and $\tilde{\theta}_i(w) = \theta_i(w)$ for $i \geq 2$, we can write

$$\tilde{\tilde{\Delta}}(w) = \mathscr{D}_{N-1}(w) + \sum_{i \geq 1} u^i \tilde{\theta}_i(w),$$

where $w \in W$.

Define a map F from $(\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u])$ to itself by $F(u) = u$, $F(w) = w$ for $w \in W$ and $F(\overline{w}) = \sum_{i \geq 1} u^{i-1}\tilde{\theta}_i(w)$ for $\overline{w} \in \overline{W}$.

Proposition 5.3. *If F is an isomorphism, then F fits in the following commutative diagram of DGA's*:

$$\begin{tikzcd}
 & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\tilde{\Delta}}) \\
(\boldsymbol{Q}[u], 0) \arrow[ur, tail] \arrow[dr, tail] & \\
 & (\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \mathscr{D}_N). \arrow[uu, "\cong"', "F"]
\end{tikzcd}$$

Proof. Define $D = F^{-1}\tilde{\tilde{\Delta}}F$. Then it suffices to show that $\mathscr{D}_N = D$. Since $\mathscr{D}_{N-1}(w) \in \wedge W$, it follows that

$$\begin{aligned}D(w) &= F^{-1}(\mathscr{D}_{N-1}(w) + uF(\overline{w}))\\ &= \mathscr{D}_{N-1}(w) + u\overline{w}\\ &= \delta_{N-1}(w) + u\beta_{N-1}(w) + u\overline{w}\\ &= \delta_N(w) + u\beta_N(w) = \mathscr{D}_N(w).\end{aligned}$$

The above fact yields that

$$\begin{aligned}0 = D^2 w = D(\delta_N(w) + u\beta_N(w)) &= \mathscr{D}_N\delta_N(w) + uD\beta_N(w)\\ &= \delta_N\delta_N(w) + u\beta_N\delta_N(w) + uD\beta_N(w)\\ &= u\beta_N\delta_N(w) + uD\beta_N(w).\end{aligned}$$

Thus we have $-\beta_N\delta_N(w) = D\beta_N(w) = D(\overline{w} + \beta_{N-1}(w)) = D(\overline{w}) + D\beta_{N-1}(w)$. Since $\beta_{N-1}w \in \wedge W$, it follows that

$$
\begin{aligned}
D(\overline{w}) &= -\beta_N\delta_N(w) - \mathscr{D}_N\beta_{N-1}(w) \\
&= -\beta_N\delta_N(w) - \delta_N\beta_{N-1}(w) - u\beta_N\beta_{N-1}(w) \\
&= -\delta_N\beta_{N-1}(w) - \beta_N\delta_N(w) + u\beta_N(\overline{w} - \beta_N(w)) \\
&= -\delta_N\beta_{N-1}(w) + \delta_N\beta_N(w) + u\beta_N(\overline{w}) - u\beta_N\beta_N(w) \\
&\quad \text{(by using } \beta_N\delta_N + \delta_N\beta_N = 0) \\
&= \delta_N(\beta_N(w) - \beta_{N-1}(w)) + u\beta_N(\overline{w}) \\
&= \delta_N(\overline{w}) + u\beta_N(\overline{w}) = \mathscr{D}_N(\overline{w}).
\end{aligned}
$$

Then we have the result. □

In order to complete the induction, we have to prove the following proposition.

Proposition 5.4. *The map F is an isomorphism of algebras.*

Proof. Since $\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u]$ is locally finite, it suffices to show that F is surjective. Let $\mu : \mathscr{F}(\boldsymbol{T}^N, X) \times \boldsymbol{T} \to \mathscr{F}(\boldsymbol{T}^N, X)$ be the $\boldsymbol{T}$-action induced from the diagonal $\boldsymbol{T}$-action on the source $\boldsymbol{T}^N$ and let $p_1 : \mathscr{F}(\boldsymbol{T}^N, X) \times \boldsymbol{T} \to \mathscr{F}(\boldsymbol{T}^N, X)$ be the projection in the first factor. We see that $i\mu \sim ip_1$. From the diagrams (5.2), (5.3) and (5.4), we can obtain a homotopy commutative diagram:

$$
\begin{tikzcd}
A_{PL}(E\boldsymbol{T} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)) \arrow[rr, "A_{PL}(i)"] & & A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X)) \\
(\wedge W \otimes \wedge \overline{W} \otimes \boldsymbol{Q}[u], \tilde{\tilde{\Delta}}) \arrow[u, "\simeq"] \arrow[r, "\tilde{\tilde{i}}"'] & (\wedge W \otimes \wedge \overline{W}, \tilde{\delta}_N) \arrow[r, "\cong", "\varphi"'] & (\wedge W \otimes \wedge \overline{W}, \delta_N). \arrow[u, "\simeq"', "c_N"]
\end{tikzcd}
$$

The diagram allows us to choose $\varphi\tilde{\tilde{i}}$ as a model for the inclusion $i : \mathscr{F}(\boldsymbol{T}^N, X) \to E_{\boldsymbol{T}} \times_{\boldsymbol{T}} \mathscr{F}(\boldsymbol{T}^N, X)$. Let $\tilde{\mu}$ be a model for μ associated with the models $c_N : (\wedge W \otimes \wedge \overline{W}, \delta_N) \to A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X))$ and $c_N \otimes 1 : (\wedge W \otimes \wedge \overline{W}, \delta_N) \otimes (\wedge t, 0) \to A_{PL}(\mathscr{F}(\boldsymbol{T}^N, X) \times \boldsymbol{T})$. As in the proof of [11, Proposition 2.2], by using the definition of homotopy given in [7, Chapter 5], we can conclude that

$$\tilde{\mu}\varphi\tilde{\tilde{i}}(w) = \varphi(w) + A(w) + (\lambda\varphi\tilde{\theta}_1(w) + w' + B(w)) \otimes t$$

for any element $w \in W$, where $A(w)$ and $B(w)$ are appropriate decomposable elements in $\wedge W \otimes \wedge \overline{W}$, $w' \in \wedge W$ and $\lambda \in \boldsymbol{Q}$.

To finish the proof, we need the following proposition. The proof is deferred to the next section.

Proposition 5.5. *With the above notation, $\tilde{\mu} = Id - \beta_N \otimes t$.*

Thus we have an equality (5.5):

$$\varphi^{-1}\beta_N\varphi(w) = \lambda\tilde{\theta}_1(w) + w'' + B'(w)$$

with $w'' = \varphi^{-1}(w') \in \wedge W$ and $B'(w) = \varphi^{-1}B(w)$. Fix an integer $k > 0$. Consider the basis $\{w_i\}_{1 \le i \le s}$ for W^k mentioned above. Let $\{\overline{w}_i\}$ be the corresponding basis for $\overline{W}^{k-1}$; that is, $\overline{w}_i = \beta_N(w_i) - \beta_{N-1}(w_i)$. By definition, we have

$$\varphi(\overline{w}_i) = \overline{w}_i + y_i \quad \text{and} \quad \varphi(w_i) = \sum(\mu_{ij}w_j + x_i)$$

with some decomposable elements y_i and x_i of $\wedge W$. Thus it follows that

$$\beta_N\varphi(w_i) = \beta_N\left(\sum \mu_{ij}w_j + x_i\right) = \sum(\mu_{ij}(\beta_{N-1}(w_j) + \overline{w}_j) + \beta(x_i)).$$

Therefore, by applying φ^{-1}, we see that

$$\varphi^{-1}\beta\varphi(w_i) = \sum(\mu_{ij}\overline{w}_j + z_i + d_i),$$

where d_i is a decomposable element in $\wedge W \otimes \wedge\overline{W}$ and $z_i \in \wedge W$. Using the regular matrix $A = (\mu_{ij})$, we write

$$\begin{pmatrix} \varphi^{-1}\beta\varphi(w_1) \\ \vdots \\ \varphi^{-1}\beta\varphi(w_s) \end{pmatrix} = A\begin{pmatrix} \overline{w}_1 \\ \vdots \\ \overline{w}_s \end{pmatrix} + \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}.$$

The equality (5.5) yields that

$$\begin{pmatrix} \overline{w}_1 \\ \vdots \\ \overline{w}_s \end{pmatrix} = A^{-1}\begin{pmatrix} \lambda_1\tilde{\theta}_1(w_1) \\ \vdots \\ \lambda_s\tilde{\theta}_1(w_s) \end{pmatrix} + \begin{pmatrix} z'_1 \\ \vdots \\ z'_s \end{pmatrix} + \begin{pmatrix} d'_1 \\ \vdots \\ d'_s \end{pmatrix}$$

with elements $z'_i \in \wedge W$ and decomposable elements $d'_i \in \wedge W \otimes \wedge\overline{W}$. From this fact, it follows by induction on the degree k of the vector space W that every $\overline{w}_i$ is in the image of F. We have Proposition 5.4. □

6. Proof of Proposition 5.5

Let X be an l-connected space with a minimal model $(\boldsymbol{Q}[V], d)$. Let $\mu : \mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T} \to \mathscr{F}(\boldsymbol{T}^l, X)$ be the $\boldsymbol{T}$-action on $\mathscr{F}(\boldsymbol{T}^l, X)$ defined by

$$\mu(f, a)(a_1, \dots, a_l) = f(aa_1, \dots, aa_l)$$

for $f \in \mathscr{F}(\boldsymbol{T}^l, X)$, $a \in \boldsymbol{T}$ and $(a_1, \dots, a_l) \in \boldsymbol{T}^l$. Let B be the minimal model for $\boldsymbol{T}^l$ of the form $(\wedge(t_1, \dots, t_l), 0)$. We denote by ε an l-tuple $(\varepsilon_1, \dots, \varepsilon_l)$, where ε_i is 0 or 1, and write $a_\varepsilon = t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l}$. Choose $\{a_\varepsilon\}$ as a basis of B. Then we have a minimal model for $\mathscr{F}(\boldsymbol{T}^l, X)$ of the form $(\boldsymbol{Q}[V \otimes B_*], \delta)$. Moreover, it follows from [9, Theorem 4.5] that the map $\hat{u}$

$$\boldsymbol{Q}[V] \to \boldsymbol{Q}[V \otimes B_*] \otimes B,$$

which is defined by

$$\hat{u}(x) = \sum_{\varepsilon} (-1)^{\alpha(|a_\varepsilon|)} (x \otimes a_{\varepsilon *}) \otimes a_\varepsilon$$

for $x \in \boldsymbol{Q}[V]$, is a model for the evaluation map $ev : \mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T}^l \to X$, where $\{a_{\varepsilon *}\}$ is the dual base to $\{a_\varepsilon\}$ and $\alpha(n) = \left[\dfrac{(n+1)}{2}\right]$, the greatest integer in $\dfrac{(n+1)}{2}$.

Proof of Proposition 5.5. Let $\varphi : \boldsymbol{T} \times \boldsymbol{T}^l \to \boldsymbol{T}^l$ be the action on $\boldsymbol{T}^l$ defined by $a(a_1, \dots, a_l) = (aa_1, \dots, aa_l)$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T} \times \boldsymbol{T}^l & \xrightarrow{\mu \times 1} & \mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T}^l \\
\Big\downarrow{\scriptstyle 1\times\varphi} & & \Big\downarrow{\scriptstyle ev} \\
\mathscr{F}(\boldsymbol{T}^l, X) \times \boldsymbol{T}^l & \xrightarrow[ev]{} & X.
\end{array}$$

This means that the composition $ev(1 \times \varphi)$ is the adjunction map to μ. Therefore, in order to prove Proposition 5.5, it suffices to show that the following diagram is commutative:

$$\begin{tikzcd}
(\boldsymbol{Q}[V \otimes B_*]) \otimes \wedge t \otimes B & & (\boldsymbol{Q}[V \otimes B_*]) \otimes B \arrow[ll, "g\otimes 1"'] \\
& \boldsymbol{Q}[V] \arrow[ul, "(1\otimes\tilde{\varphi})\hat{u}"] \arrow[ur, "\hat{u}"'] &
\end{tikzcd}$$

for some model $\tilde{\varphi}$ for φ because $\hat{u}$ is a model for the evaluation map ev, where g denotes the DGA map $Id - \beta \otimes t$. Let $\Delta : \boldsymbol{T} \to \boldsymbol{T}^l$ be the diagonal map and $m : \boldsymbol{T}^l \times \boldsymbol{T}^l \to \boldsymbol{T}^l$ the multiplication. The action $\boldsymbol{T} \times \boldsymbol{T}^l \to \boldsymbol{T}^l$ factors through $\Delta \times 1 : \boldsymbol{T} \times \boldsymbol{T}^l \to \boldsymbol{T}^l \times \boldsymbol{T}^l$ and the map m. Therefore it follows that the map φ has a model $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(a_\varepsilon) = 1 \otimes a_\varepsilon + \sum_k (-1)^{\varepsilon_1 + \cdots + \varepsilon_{k-1}} t \otimes t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k - 1} \cdots t_l^{\varepsilon_l}.$$

Hence we have

$$\begin{aligned}
(1 \otimes \tilde{\varphi})\hat{u}(v) = {} & \sum_\delta (-1)^{\alpha(|a_\delta|)} v \otimes a_{\delta *} \otimes 1 \otimes a_\delta \\
& + \sum_\delta (-1)^{\alpha(|a_\delta|)} \sum_k (-1)^{\delta_1 + \cdots + \delta_{k-1}} (v \otimes a_{\delta *}) \otimes t \otimes t_1^{\delta_1} \cdots t_k^{\delta_k - 1} \cdots t_l^{\delta_l}.
\end{aligned}$$

On the other hand, it is readily seen that

$$
\begin{aligned}
(g \otimes 1)\hat{u}(v) &= ((Id - \beta \otimes t) \otimes 1)\hat{u}(v) \\
&= \sum_{\varepsilon} (-1)^{\alpha(|a_\varepsilon|)} v \otimes a_{\varepsilon *} \otimes 1 \otimes a_\varepsilon \\
&\quad - \sum_{\varepsilon} (-1)^{\alpha(|a_\varepsilon|)} \sum_{k} (-1)^{\varepsilon_k + \cdots + \varepsilon_l} v \otimes (t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k + 1} \cdots t_l^{\varepsilon_l})_* \otimes t \otimes a_\varepsilon.
\end{aligned}
$$

We compare the coefficients $(-1)^{\alpha(|a_\delta|)+\delta_1+\cdots+\delta_{k-1}}$ and $(-1)^{\alpha(|a_\varepsilon|)+\varepsilon_k+\cdots+\varepsilon_l+1}$ when $a_\varepsilon = t_1^{\varepsilon_1} \cdots t_k^{\varepsilon_k} \cdots t_l^{\varepsilon_l} = t_1^{\delta_1} \cdots t_k^{\delta_k - 1} \cdots t_l^{\delta_l}$. By applying the formula such that $\alpha(n+m) \equiv \alpha(n) + \alpha(m) + mn$ modulo 2, we see that

$$
\begin{aligned}
&\alpha(|a_\delta|) + \delta_1 + \cdots + \delta_{k-1} \\
&\quad \equiv \alpha(\delta_1 + \cdots + \delta_l) + \varepsilon_1 + \cdots + \varepsilon_{k-1} \\
&\quad \equiv \alpha(\varepsilon_1 + \cdots + \varepsilon_{k-1} + \varepsilon_k + 1 + \varepsilon_{k+1} + \cdots + \varepsilon_l) + \varepsilon_1 + \cdots + \varepsilon_{k-1} \\
&\quad \equiv \alpha(\varepsilon_1 + \cdots + \varepsilon_l) + \alpha(1) + 1 \cdot (\varepsilon_1 + \cdots + \varepsilon_l) + \varepsilon_1 + \cdots + \varepsilon_{k-1} \\
&\quad \equiv \alpha(|a_\varepsilon|) + 1 + \varepsilon_k + \cdots + \varepsilon_l.
\end{aligned}
$$

This finishes the proof. □

Acknowledgements. The authors thank the referee for suggestions to revise a previous version of this paper.

References

[1] E. H. Brown Jr and R. H. Szczarba, Rational homotopy type of function spaces, Trans. Amer. Math. Soc. **349** (1997), 4931–4951.

[2] D. Burghelea and Fiedorowicz, Cyclic homology and the algebraic K-theory of spaces **II**, Topology **25** (1986), 303–317.

[3] D. Burghelea and M. Vigué-Poirrier, Cyclic homology of commutative algebras **I**, Lecture notes in math. **1318**, Springer-Verlag, New York, 1988, 51–72.

[4] Y. Félix, S. Halperin and J.-C. Thomas, Rational homotopy theory, Graduate texts in mathematics **205**, Springer-Verlag, 2001.

[5] T. G. Goodwillie, Cyclic homology, derivations and the free loopspace, Topology **24** (1985), 187–215.

[6] A. Haefliger, Rational homotopy of space of sections of a nilpotent bundle, Trans. Amer. Math. Soc. **207** (1982), 609–620.

[7] S. Halperin, Lectures on minimal models, Mem. Soc. Math. France **9/10** (1983).

[8] J. D. S. Jones, Cyclic homology and equivariant homology, Invent. Math. **87** (1987), 403–423.

[9] K. Kuribayashi, A rational model for the evaluation map, Georgian Mathematical Journal **13** (2006), 127–141.

[10] K. Kuribayashi, M. Mimura and T. Nishimoto, Twisted tensor products related to the cohomology of the classifying spaces of loop groups, Mem. Amer. Math. Soc. **180** no. 849 (2006).

[11] M. Vigué-Poirrier and D. Burghelea, A model for cyclic homology and algebraic K-theory of 1-connected spaces, J. Differential Geometry **22** (1985), 243–253.

[12] M. Vigué-Poirrier and D. Sullivan, The homology theory of the closed geodesic problem, J. Differential Geometry **11** (1976), 633–644.

[13] M. Yokotani, Computational examples of the iterated cyclic homology with low iteration degree, in preparation.

Katsuhiko Kuribayashi
Department of Mathematical Sciences
Faculty of Science, Shinshu University
Matsumoto, Nagano 390-8621
Japan
E-mail: kuri@math.shinshu-u.ac.jp

Masaaki Yokotani
Division of General Education
Tsuyama National College of Technology
Tsuyama, Okayama 708-8509
Japan
E-mail: yokotani@tsuyama-ct.ac.jp