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ON MEROMORPHIC FUNCTIONS SHARING THREE VALUES AND ONE SET

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Abstract

We give relations of two meromorphic functions sharing 0, 1, ∞ and a set CM.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a discrete set S in $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. In particular if S is a one point set $\{a\}$, then we say also that f and g share a CM.

In [N], R. Nevalinna showed

Theorem A1. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and $a_1, \dots, a_4$ four distinct points in $\hat{\boldsymbol{C}}$.If f and g share $a_1, \dots, a_4$ CM, then f is a Möbius transformation of g and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}$, $a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

If $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, then the fourth point a_4 such that the cross ratio is -1 in some order is one of -1, 2 and $\frac{1}{2}$. Then a part of Theorem A1 can be denoted as following:

Theorem A2. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing 0, 1, ∞ and a CM, where $a \neq 0, 1, \infty$. Then f and g have one of the following relations:*

$$f = g, \quad f = \frac{1}{g}, \quad f = \frac{g}{g-1} \quad \textit{and} \quad f = -g + 1.$$

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Also, in [T] Tohge considered two meromorphic functions sharing 1, -1, ∞ and a two-point set containing none of them.

Theorem B1. *Let f and g be two nonconstant meromorphic functions on $\mathbf{C}$ sharing* 1, -1 *and* ∞ *CM. Let $S = \{a, b\}$, where $a, b \neq 1, -1, \infty$. If f and g share S CM, then they have one the following relations*:

$$f = \pm g, \quad fg = 1, \quad f + g = \pm 2, \quad (f \pm 1)(g \pm 1) = 4$$
$$f \pm 1 = \omega(g \pm 1) \quad \text{and} \quad \left(f + \frac{1+\omega}{1-\omega}\right)\left(g - \frac{1+\omega}{1-\omega}\right) = \frac{4}{3},$$

where $\omega^3 = 1$, $\omega \neq 1$ and double signs in same order respectively.

If we replace the first three points by 0, 1 and ∞, the result is changed as follows:

Theorem B2. *Let f and g be two nonconstant meromorphic functions on $\mathbf{C}$ sharing* 0, 1 *and* ∞ *CM. Let $S = \{a, b\}$ where $a, b \neq 0, 1, \infty$. If f and g share S CM, then they have one the following relations*:

$$f = g, \quad f = -g + 1, \quad f = \frac{g}{2g - 1}, \quad f = -g + 2, \quad f = -g, \quad fg = 1,$$

$$(f - 1)(g - 1) = 1, \quad f = \omega g, \quad f - 1 = \omega(g - 1) \quad \text{and} \quad f = \frac{g}{\left(1 - \frac{1}{\omega}\right)g + \frac{1}{\omega}},$$

where $\omega^3 = 1$ and $\omega \neq 1$.

There are many results on two meromorphic functions sharing three values CM with additional conditions of defects or counting functions for another value, for example [Li] and [LY]. In this paper we consider two nonconstant meromorphic functions sharing three values 0, 1, ∞ and a finite set containing none of them.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions on $\mathbf{C}$ sharing* 0, 1 *and* ∞ *CM. Let S be a finite set in $\mathbf{C}$ defined by the zeros of a monic polynomial $P(z)$ without multiple zeros such that $P(0) \neq 0$, $P(1) \neq 0$. If f and g share S CM, then they have one of the following relations*:

(i) $f = cg$;
(ii) $f - 1 = c(g - 1)$ *i.e.*, $f = cg - c + 1$;
(iii) $\dfrac{f-1}{f} = c\dfrac{g-1}{g}$ *i.e.*, $f = \dfrac{-g}{(c-1)g - c}$;
(iv) $fg = 1$;
(v) $(f - 1)(g - 1) = 1$ *i.e.*, $f = \dfrac{g}{g-1}$;

(vi) $\dfrac{f-1}{f}=\dfrac{g-1}{g}$ *i.e.*, $f=-g+1$;

(vii) *there exist monic polynomials* $\Phi(X)\in \boldsymbol{C}[X]$ *and* $\varphi(z)=z^p(z-1)^q$ *with* $p,q>0$ *and* $(p,q)=1$ *such that*

$$\varphi(f)=\omega\varphi(g),\quad P_0(z)=\Phi(\varphi(z)),$$

where $\omega^t=1$ *for* t *such that the coefficient of* X^t *of* $\Phi(X)$ *is not zero and* $P_0(z)$ *is one of* $P(z), Q(z):=\dfrac{1}{P(0)}z^nP\left(\dfrac{1}{z}\right)$ *and* $R(z):=\dfrac{1}{P(1)}z^nP\left(\dfrac{z-1}{z}\right)$.

Here, c *is a non-zero constant in* (i), (ii) *and* (iii).

2. Representations of rank N and Borel's lemma

In this section we introduce the definition of representations of rank N which is a generalization of representations in [F, §2]. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A=(a_1,a_2,\dots,a_q)$ of elements a_i in G. For a subgroup $\tilde{A}$ of G generated by $a_1,a_2,\dots,a_q$, we can take a basis $\{b_1,\dots,b_t\}$ of $\tilde{A}$. Then each a_i can be uniquely represented as

$$a_j={b_1}^{\mu_{j1}}{b_2}^{\mu_{j2}}\cdots{b_t}^{\mu_{jt}} \tag{2.1}$$

with suitable integers $\mu_{j\tau}$. Let $p_1,\dots,p_t$ be integers and put $\mu_j:=\mu_{j1}p_1+\cdots+\mu_{jt}p_t$. If

$$\prod_{j=1}^q a_j^{\varepsilon_j}=\prod_{j=1}^q a_j^{\varepsilon_j'} \tag{2.2}$$

for integers ε_j and ε_j', then

$$\sum_{j=1}^q \varepsilon_j\mu_j=\sum_{j=1}^q \varepsilon_j'\mu_j. \tag{2.3}$$

For we have, by substiting (2.1) into (2.2),

$$\prod_{k=1}^t b_k^{\sum_{j=1}^q \varepsilon_j\mu_{jk}}=\prod_{k=1}^t b_j^{\sum_{j=1}^q \varepsilon_j'\mu_{jk}}.$$

Since $b_1,\dots,b_t$ are linearly independent over $\boldsymbol{Z}$, we get

$$\sum_{j=1}^q \varepsilon_j\mu_{jk}=\sum_{j=1}^q \varepsilon_j'\mu_{jk}\quad (k=1,\dots,t), \tag{2.4}$$

and hence

$$\sum_{j=1}^{q}\varepsilon_j\mu_j=\sum_{j=1}^{q}\varepsilon_j\sum_{k=1}^{t}p_k\mu_{jk}=\sum_{k=1}^{t}p_k\sum_{j=1}^{q}\varepsilon_j\mu_{jk}$$
$$=\sum_{k=1}^{t}p_k\sum_{j=1}^{q}\varepsilon_j'\mu_{jk}=\sum_{j=1}^{q}\varepsilon_j'\sum_{k=1}^{t}p_k\mu_{jk}=\sum_{j=1}^{q}\varepsilon_j'\mu_j.$$

Let N be a positive integer. We call integers μ_j *representations of rank* N of a_j if (2.3) implies (2.4) for any integers ε_j, ε_j' with $\sum_{j=1}^{q}|\varepsilon_j|\le N$ and $\sum_{j=1}^{q}|\varepsilon_j'|\le N$. In particular we call representations of rank 1, simply, representations.

For the existence of representations of rank N, it is enough to take $p_\tau=p^{\tau-1}$ $(1\le\tau\le t)$ for an integer $p>2N\cdot\max\{|\mu_{jk}|;1\le j\le q,1\le k\le t\}$.

We introduce the following Borel's Lemma, whose proof can be found, for example, on p. 186 of [La].

LEMMA 2.1. *If entire functions $\alpha_0,\alpha_1,\dots,\alpha_n$ without zeros satisfy*
$$\alpha_0+\alpha_1+\cdots+\alpha_n=0,$$
then for each $j=0,1,\dots,n$ there exists some $k\ne j$ such that α_j/α_k is constant.

We use the following Lemma in the proof of Theorem 1.1 which is an application of Lemma 2.1.

LEMMA 2.2. *Let f and g be two nonconstant meromorphic functions sharing* 0, 1 *and* ∞ *CM. If two of* 0, 1 *and* ∞ *are the common Picard exceptional values of f and g, then f and g have one of the following relations*:
$$f=g,\quad fg=1,\quad (f-1)(g-1)=1\quad\text{and}\quad \frac{f-1}{f}\cdot\frac{g-1}{g}=1.$$

Proof. There exist entire functions α_0, α_1 without zeros such that
$$f=\alpha_0 g,\quad f-1=\alpha_1(g-1). \tag{2.10}$$

Fisrt assume that 0 and ∞ are the common Picard exceptional vaules of f and g. Then f and g are entire functions without zeros. We apply Lemma 2.1 to the second equation of (2.10). Since f and g are not constant, we have either $f=\alpha_1 g$, $1=\alpha_1$ or $f=-\alpha_1$, $-1=\alpha_1 g$. The formar implies $f=g$ and the latter $fg=1$.

Next assume that 1 and ∞ are the common Picard exceptional vaules of f and g. Then $f-1$ and $g-1$ are entire functions without zeros. We apply Lemma 2.1 to
$$(f-1)=\alpha_0(g-1)+\alpha_0-1$$
which is induced from the former of (2.10). Since f and g are not constant, we have either $f-1=\alpha_0(g-1)$, $\alpha_0=1$ or $f-1=\alpha_0$, $\alpha_0(g-1)=1$. The formar implies $f=g$ and the latter $(f-1)(g-1)=1$.

Finally assume that 0 and 1 are the common Picard exceptional vaules of f and g. Then $\dfrac{f-1}{f}$ and $\dfrac{g-1}{g}$ are entire functions without zeros. We apply Lemma 2.1 to

$$\frac{f-1}{f} = \frac{1}{\alpha_0} \cdot \frac{g-1}{g} + 1 - \alpha_0$$

which is induced from the former of (2.10). Since f and g are not constant, we have either $\dfrac{f-1}{f} = \dfrac{1}{\alpha_0} \cdot \dfrac{g-1}{g}$, $\dfrac{1}{\alpha_0} = 1$ or $\dfrac{f-1}{f} = -\dfrac{1}{\alpha_0}$, $\dfrac{1}{\alpha_0} \cdot \dfrac{g-1}{g} = -1$. The formar implies $f = g$ and the latter $\dfrac{f-1}{f} \cdot \dfrac{g-1}{g} = 1$. □

Now we investigate the torsion-free abelian multipicative group $G = \mathscr{E}/\mathscr{C}$, where $\mathscr{E}$ is the abelian group of entire functions without zeros and $\mathscr{C}$ is the subgroup of all non-zero constant functions.

Let $\alpha_1, \dots, \alpha_q$ be elements in $\mathscr{E}$. We represent by $[\alpha_j]$ the element of $\mathscr{E}/\mathscr{C}$ with the representative α_j. Take representations μ_j of rank N of $[\alpha_j]$. For $\prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ we define its index by $\sum_{j=1}^{q} \varepsilon_j \mu_j$. The indices depend only on $[\prod_{j=1}^{q} \alpha_j^{\varepsilon_j}]$.

Lemma 2.3. *Assume that there is a relation* $\Psi(\alpha_1, \dots, \alpha_q) \equiv 0$ *where* $\Psi(X_1, \dots, X_q) \in \boldsymbol{C}[X_1, \dots, X_q]$ *is a nonconstant polynomial of degree at most* N *of* $X_1, \dots, X_q$. *Then each term* $aX_1^{\varepsilon_1} \cdots X_q^{\varepsilon_q}$ *of* $\Psi(X_1, \dots, X_q)$ *has another term* $bX_1^{\varepsilon_1'} \cdots X_q^{\varepsilon_q'}$ *such that* $\alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}$ *and* $\alpha_1^{\varepsilon_1'} \cdots \alpha_q^{\varepsilon_q'}$ *have the same indices, where* a *and* b *are non-zero constants.*

Proof. By using Lemma 2.1 each term $aX_1^{\varepsilon_1} \cdots X_q^{\varepsilon_q}$ has another term $bX_1^{\varepsilon_1'} \cdots X_q^{\varepsilon_q'}$ such that $(\alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q})/(\alpha_1^{\varepsilon_1'} \cdots \alpha_q^{\varepsilon_q'})$ is constant. This implies the conclusion of Lemma. □

3. Proof of Theorem 1.1

Proposition 3.1. *Let* f *and* g *be two nonconstant meromorphic functions and* $P(z)$ *a monic polynomial of degree* $n(\geq 1)$ *such that* $P(0), P(1) \neq 0$. *Assume that there exist entire functions without zeros* α_0, α_1, α_2 *such that*

$$f = \alpha_0 g, \quad f - 1 = \alpha_1(g-1) \tag{3.1}$$

and

$$P(f) = \alpha_2 P(g). \tag{3.2}$$

and assume that f *is not a Möbius transformation of* g *of type* (i)∼(vi) *in Theorem* 1.1. *Then one of* α_2/α_0^n, α_2/α_1^n *and* α_2 *is identically equal to* 1.

Proof. Delete f and g from the relations (3.1) and (3.2). Then we have

$$\alpha_2\{P(0)\alpha_0^n + (-1)^n P(1)\alpha_1^n + (*) + 1\} - \{(-1)^n\alpha_0^n\alpha_1^n + (**) + P(1)\alpha_0^n + P(0)(-1)^n\alpha_1^n(*)\} = 0. \tag{3.3}$$

Here the degree of each term of $(*)$ is not greater than n about α_0 and α_1 and not greater than $n-1$ about each α_j $(j = 0, 1)$, and the degree of each term of $(**)$ is not greater than $2n-1$ and not smaller than n about α_0 and α_1 and not greater than n about each α_j $(j = 0, 1)$.

Let μ_0, μ_1, μ_2 be representations of α_0, α_1, α_2 with rank $2n$. By assumption $\mu_j \neq 0$ $(j = 0, 1)$ and $\mu_0 \neq \mu_1$, and we may assume $\mu_0 < \mu_1$. In addition we assume $\mu_2 \neq 0$.

(I) The case of $\mu_0 < \mu_1 \leq \mu_2$. If $0 < \mu_0 < \mu_1 \leq \mu_2$, the minimal indices of each terms in (3.3) may be only $n\mu_0$ and μ_2. Hence we have $n\mu_0 = \mu_2$ by Lemma 2.3. If $\mu_0 < 0 < \mu_1 \leq \mu_2$, the minimal index of each terms in (3.3) is only $n\mu_0$, which contradicts to Lemma 2.3. If $\mu_0 < \mu_1 < 0 < \mu_2$, the minimal index of each terms in (3.3) is only $n(\mu_0 + \mu_1)$, which contradicts to Lemma 2.3. If $\mu_0 < \mu_1 \leq \mu_2 < 0$, the minimal indices of each terms in (3.3) is only $n(\mu_0 + \mu_1)$, which contradicts to Lemma 2.3. So we have $n\mu_0 = \mu_2$ in this case.

(I$'$) The case of $\mu_2 \leq \mu_0 < \mu_1$. We have also $n\mu_0 = \mu_1$ as in the case (I).

(II) The case of $\mu_0 < \mu_2 < \mu_1$. If $0 < \mu_0 < \mu_2 < \mu_1$, the minimal indices of each terms in (3.3) may be only $n\mu_0$ and μ_2. Hence we have $n\mu_0 = \mu_2$ by Lemma 2.3. If $\mu_0 < 0 < \mu_2 < \mu_1$, the minimal index of each terms in (3.3) is only $n\mu_0$, which contradicts to Lemma 2.3. If $\mu_0 < \mu_2 < 0 < \mu_1$, the minimal index of each terms in (3.3) is only $n\mu_1$, which contradicts to Lemma 2.3. If $\mu_0 < \mu_2 < \mu_1 < 0$, the minimal indices of each terms in (3.3) may be only $n(\mu_0 + \mu_1)$ and $n\mu_0 + \mu_2$. Hence we have $n\mu_1 = \mu_2$ by Lemma 2.3. So we have $n\mu_0 = \mu_2$ in this case.

(III) The case of $\mu_2 = \mu_0 < \mu_1$. If $0 < \mu_2 = \mu_0 < \mu_1$, the minimal index of each terms in (3.3) is only μ_2, which contradicts to Lemma 2.3. If $\mu_2 = \mu_0 < 0 < \mu_1$, the minimal index of each terms in (3.3) is only $n\mu_0 + \mu_2$, which contradicts to Lemma 2.3. If $\mu_2 = \mu_0 < \mu_1 < 0$, the minimal indices of each terms in (3.3) may be only $n(\mu_0 + \mu_1)$ and $n\mu_0 + \mu_2$. Hence we have $n\mu_1 = \mu_2$ by Lemma 2.3. So we have $n\mu_1 = \mu_2$ in the case (III).

(III$'$) The case of $\mu_0 < \mu_1 = \mu_2$. We have $n\mu_0 = \mu_1$ as in this case.

Therefore since μ_j's have rank $2n$, one of α_2/α_0^n, α_2/α_1^n and α_2 is constant. Write it by C, then one of

$$P(f) = CP(g), \quad \frac{P(f)}{f^n} = C\frac{P(g)}{g^n}, \quad \frac{P(f)}{(f-1)^n} = C\frac{P(g)}{(g-1)^n}$$

holds. Since f and g, by assumption and Lemma 2.2, take simultaneously each of 0, 1 and ∞ except at most one, we can conclude $C = 1$. □

Remark. Note that we get formalizations

$$P(f) = P(g), \quad f = \alpha_0 g, \quad f - 1 = \alpha_1(g-1)$$

in the case of $\alpha_2 \equiv 1$,

$$Q\left(\frac{1}{f}\right) = Q\left(\frac{1}{g}\right), \quad \frac{1}{f} = \frac{1}{\alpha_0}\frac{1}{g}, \quad \frac{1}{f} - 1 = \frac{\alpha_1}{\alpha_0}\left(\frac{1}{g} - 1\right)$$

in the case $\alpha_2/\alpha_0^n \equiv 1$ and

$$R\left(\frac{1}{1-f}\right) = R\left(\frac{1}{1-g}\right), \quad \frac{1}{1-f} = \frac{1}{\alpha_1}\frac{1}{1-g}, \quad \frac{1}{1-f} - 1 = \frac{\alpha_0}{\alpha_1}\left(\frac{1}{1-g} - 1\right)$$

in the case of $\alpha_2/\alpha_1^n \equiv 1$. Here monic polynomials $Q(z)$ and $R(z)$ of degree n are defined by

$$Q(z) = \frac{1}{P(0)} z^n P(1/z) \quad \text{and} \quad R(z) = \frac{1}{P(1)} z^n P\left(\frac{z-1}{z}\right)$$

which satisfy $Q(0) \neq 0$, $Q(1) \neq 0$, $R(0) \neq 0$, $R(1) \neq 1$.

Proposition 3.2. *Let f and g be two nonconstant meromorphic functions and $P(z)$ a monic polynomial of degree $n(\geq 1)$ such that $P(0) \neq 0$, $P(1) \neq 0$. Assume that there exist entire functions without zeros α_0, α_1 satisfying* (3.1) *and assume that f is not a Möbius transformation of g of type* (i)~(vi) *in Theorem* 1.1. *If in addition*

$$P(f) = P(g) \tag{3.4}$$

holds, then there exist polynomials $\Phi(X) \in \boldsymbol{C}[X]$ and $\varphi(z) = z^p(z-1)^q$ with $p, q > 0$ and $(p, q) = 1$ such that

$$\varphi(f) = \omega\varphi(g) \quad P_0(z) = \Phi(\varphi(z)),$$

where $\omega^t = 1$ for t such that the coefficient of X^t of $\Phi(X)$ is not zero.

Remark. If $n = 1$, (3.4) implies $f = g$. If $n = 2$ and $f \neq g$, (3.4) implies $f + g + a = 0$ for some constant a. However, $a = 0$ or $a = -2$ by Lemma 2.2, which are (i) and (ii), respectively.

Proof. We proceed the proof by induction on n.

Assume that the result holds for polynomials of degree not greater than $n - 1$.

Let $c(\neq 0)$ the constant term of $P(z)$. There exit integers $m_0 \geq 1$ and $k_0 \geq 0$ and a monic polynomial $P_1(z)$ such that

$$P(z) = z^{m_0}(z-1)^{k_0} P_1(z) + c \quad \text{and} \quad P_1(0) \neq 0, P_1(1) \neq 0. \tag{3.5}$$

Then we have

$$P_1(f) = \left(\frac{g}{f}\right)^{m_0}\left(\frac{g-1}{f-1}\right)^{k_0} P_1(g) = \frac{1}{\alpha_0^{m_0}\alpha_1^{k_0}} P_1(g). \tag{3.6}$$

Put $n_1 := \deg P_1(z) = n - m_0 - k_0$. If $n_1 = 0$, then $\alpha_0^{m_0}\alpha_1^{k_0} = 1$, and hence $k_0 \geq 1$ and there is nothing to prove. So we consider the case of $n_1 > 0$. Apply Proposition 3.1 to (3.6) in place of (3.2), then one of the followings holds:

$$\alpha_0^{m_0}\alpha_1^{k_0} = 1; \tag{3.7}$$

$$\frac{1}{\alpha_0^{m_0}\alpha_1^{k_0}} \Big/ \alpha_0^{n_1} = 1 \quad i.e., \quad \alpha_0^{n-k_0}\alpha_1^{k_0} = 1; \tag{3.8}$$

$$\frac{1}{\alpha_0^{m_0}\alpha_1^{k_0}} \Big/ \alpha_1^{n_1} = 1 \quad i.e., \quad \alpha_0^{m_0}\alpha_1^{n-m_0} = 1. \tag{3.9}$$

Since α_0 and α_1 are nonconstant by assumption, all above exponents of α_0 and α_1 are positive. Now assume that (3.8) holds. Then we have

$$Q_1\left(\frac{1}{f}\right) = Q_1\left(\frac{1}{g}\right), \quad \frac{1}{g} = \frac{1}{\alpha_0}\frac{1}{g}, \quad \frac{1}{f} - 1 = \frac{\alpha_1}{\alpha_0}\left(\frac{1}{g} - 1\right)$$

where $Q_1(z) := \dfrac{1}{P_1(0)} z^{n_1} P_1\left(\dfrac{1}{z}\right)$. By applying the same process above to $\dfrac{1}{f}, \dfrac{1}{g}$ and $Q_1(z)$ there exist non-negative integers v_0, v_1 such that $v_0 + v_1 > 0$ and that

$$\left(\frac{1}{\alpha_0}\right)^{v_0}\left(\frac{\alpha_1}{\alpha_0}\right)^{v_1} = 1 \quad i.e., \quad \alpha_1^{v_1} = \alpha_0^{v_0+v_1},$$

which induce with (3.8) that one of α_0 and α_1 is constant. This is a contradiction to the assumption. Hence (3.8) is excluded and so is (III) by the same manner. We have now (3.9) and then $P_1(f) = P(g)$ holds from (3.5). Note that $k_0 > 0$. By the assumption of induction there exists monic plynomials $\Phi_1(X) \in \boldsymbol{C}[X]$ and $\varphi_1(z) = z^{p_1}(z-1)^{q_1}$ with $p_1, q_1 > 0$ and $(p_1, q_1) = 1$ satisfying

$$P_1(z) = \Phi_1(\varphi_1(z)) \quad \text{and} \quad \varphi_1(f) = \omega_1\varphi_1(g)$$

where ω_1 is a radical root of unity such that $\omega_1^t = 1$ if the coefficients of X^t are not zero. Since some power of $\omega_1 = \alpha_0^{p_1}\alpha_1^{q_1}$ is $1 = \alpha_0^{m_0}\alpha_1^{k_0}$, we have $p_1 : q_1 = m_0 : k_0$. For, otherwise, α_0 and α_1 are constant. So there exist an integer N such that

$$m_0 = Np_1, \quad k_0 = Np_1,$$

and then $\omega := \alpha_0^{p_1}\alpha_1^{q_1}$ is a constant such that $\omega^N = 1$. We obtain $p_0 = p_1$, $q_0 = q_1$ and complete the proof by taking $\Phi(X) = X^N\Phi_1(X) + c$ and $\varphi(z) = \varphi_1(z)$ from (3.5) and $P(z) = \{z^{p_1}(z-1)^{q_1}\}^N\Phi_1(\varphi_1(z)) + c$. □

We have proved in place of Theorem 1.1

THEOREM 3.3. *Let f and g be two nonconstant meromorphic functions and $P(z)$ a monic polynomial of degree n such that $P(0), P(1) \neq 0$. Assume that there exist entire functions without zeros α_0, α_1, α_2 satisfying* (3.1) *and* (3.2). *Then f and g have one of the relations* (i)~(vii) *in Theorem* 1.1.

4. The case of cubic polynomials

In this section we consider cubic polynomials $P(z)$. Let $P(z) = z^3 + az^2 + bz + c$ a cubic polynomial without multiple zeros where a, b, c are constants with $c \neq 0$, $a + b + c \neq -1$. Let S be the zero points of $P(z)$.

Assume that two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share 0, 1, ∞ and the set S CM. Let α_0 and α_1 be the entire fuctions without zeros satisfying (3.1).

Further we assume that none of (i)$\sim$(vi) holds. Then by Theorem 1.1 there exists a monic polynomial $\varphi(z) = z^p(z-1)^q$ with relatively prime positive integers p and q such that

$$P_0(z) = \varphi(z) + c \quad \text{and} \quad \varphi(f) = \varphi(g).$$

Note that $\Phi(X) = X + c$ in this case.

If $P_0(z) = P(z)$, then $P(z) = z^2(z-1) + c$ or $P(z) = z(z-1)^2 + c$. In the former

$$f = \frac{\alpha_0(\alpha_0 + 1)}{\alpha_0^2 + \alpha_0 + 1}, \quad g = \frac{\alpha_0 + 1}{\alpha_0^2 + \alpha_0 + 1}$$

with $\alpha_0^2\alpha_1 = 1$. In the latter

$$f = \frac{1}{\alpha_1^2 + \alpha_1 + 1}, \quad g = \frac{\alpha_1^2}{\alpha_1^2 + \alpha_1 + 1}$$

with $\alpha_0\alpha_1^2 = 1$.

If $P_0(z) = Q(z)$, then $P(z) = z^3 - cz + c$ or $P(z) = z^3 + cz^2 - 2cz + c$. In the former

$$f = \frac{\alpha_0^2 + \alpha_0 + 1}{\alpha_0 + 1}, \quad g = \frac{\alpha_0^2 + \alpha_0 + 1}{\alpha_0(\alpha_0 + 1)}$$

with $\alpha_0^2 = \alpha_1$. In the latter

$$f = \frac{\alpha_1^2 + \alpha_1\alpha_0 + \alpha_0^2}{\alpha_0^2}, \quad g = \frac{\alpha_1^2 + \alpha_1\alpha_0 + \alpha_0^2}{\alpha_1^2}$$

with $\alpha_1^2 = \alpha_0^3$.

If $P_0(z) = R(z)$, then $P(z) = z^3 - 3z^2 + bz - 1$ or $P(z) = z^3 + az^2 + 3z - 1$. In the former

$$f = \frac{\alpha_1^2}{\alpha_1 + 1}, \quad g = \frac{1}{\alpha_1(\alpha_1 + 1)}$$

with $\alpha_0 = \alpha_1^3$. In the latter

$$f = -\frac{\alpha_0(\alpha_0 + \alpha_1)}{\alpha_1^2}, \quad g = -\frac{\alpha_1(\alpha_0 + \alpha_1)}{\alpha_0^2}$$

with $\alpha_0^2 = \alpha_1^3$.

Now we identify the space of all monic polynomials of degree n with $\boldsymbol{C}^n$ by the corresondence $z^n + \sum_{j=1}^{n} a_j z^{n-j}$ with $(a_1, \dots, a_n)$. Let X_n be the subspace of all monic polynomials $P(z)$ of degree n such that there exist two distinct nonconstant meromorphic functions f and g on $\boldsymbol{C}$ satisfying (3.1) and (3.2) for some entire functions α_j $(j = 0, 1, 2)$ and not any of (i)~(vi) in Theorem 1.1. Then X_3 has dimension one under the above identification. As well if n is an odd prime number, so does X_n.

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