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ON THE ORDER OF A ZERO OF THE THETA FUNCTION

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1. Introduction. In this paper we shall examine the bounds of the order of a zero of the theta function attached to a compact Riemann surface with a non-trivial conformal automorphism.

At first, we shall give an estimate at the vector of Riemann constants, whose base point is a fixed point of an automorphism. Recently, Farkas [2] has given an estimate from below and some equality conditions when the genus is congruent to one modulo the order of the automorphism. In this paper we shall consider both lower and upper bounds.

Secondly, we shall give an estimate at half periods. Accola [1] has examined them when the surface has an automorphism of order 2 and Farkas [2] has also given some estimates. We shall examine them when an automorphism satisfies a certain condition on its fixed points. Our estimates contain Accola's one and our proof is a modified one of Farkas'. Furthermore, if the genus of the orbit surface of the cyclic group generated by an automorphism is one, we shall give another estimate.

Thirdly, we shall give examples of Riemann surfaces which attain the bounds of the estimates at the vector of Riemann constants.

Lastly, we shall state closing remarks.

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2. Preliminary. In this section we state notations and known results. Let S be a compact Riemann surface of genus g ($\geqq 2$) with a canonical homology basis $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$. Let $\phi_1, \cdots, \phi_g$ be the basis for the space of abelian differentials of the first kind on S, which are normalized so that $\int_{\alpha_i}\phi_j=\delta_{ij}\pi i$ where δ_{ij} is the Kronecker δ. Let $\Omega=(\omega_{ij})$ denote the matrix where $\omega_{ij}=\int_{\beta_i}\phi_j$. It is known that Ω is symmetric with negative definite real part. Then the first order theta function is defined by

$$\theta(z;\ \Omega)=\sum_{m}\exp(2{}^t mz+{}^t m\Omega m)$$

where the variable z is $g\times 1$ vector and m runs over all $g\times 1$ vectors with

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integer entries. Let ϕ be the map of S into $J(S)$, the Jacobian variety of S, defined by $\phi(P)=\int_{P_0}^{P}\phi$ for some fixed point P_0 in S where $\phi={}^t(\phi_1, \cdots, \phi_g)$. The map ϕ can be extended to divisors on S so that $\phi(D)=n_1\phi(P_1)+\cdots+n_k\phi(P_k)$ where $D=P_1^{n_1}\cdots P_k^{n_k}$ is a divisor on S. It is known [4] that the zeros of $\theta(z;\Omega)$ are well defined on $J(S)$ and the set of the zeros of $\theta(z;\Omega)$ is exactly $W^{g-1}+K(P_0)$. Here W^{g-1} denotes the image of all positive divisors of degree $g-1$ on S under ϕ, and $K(P_0)$ is a vector in $J(S)$ which is called the vector of Riemann constants. It is also known as Riemann's vanishing theorem that $\phi(D)+K(P_0)$ is a zero of order $i(D)$. Here D is a positive divisor of degree $g-1$ and $i(D)$ denotes the dimension of the space of abelian differentials of the first kind on S whose divisors are multiples of D.

If $D=P_0^{g-1}$, then $\phi(D)=0$. Hence $K(P_0)$ is always a zero of $\theta(z;\Omega)$ whose order is $i(P_0^{g-1})$. Let $l(D)$ denote the dimension of the space of meromorphic functions on S whose divisors are multiples of D^{-1}. If D is of degree $g-1$, then the Riemann-Roch theorem implies that $l(D)=i(D)$.

Let T be a conformal automorphism of S with $t(>0)$ fixed points. Let $\langle T\rangle$ denote the cyclic group generated by T, and let N be its order. Let $S/\langle T\rangle$ be the surface obtained by identifying the equivalent points on S under the element of $\langle T\rangle$ and π the natural projection of S into $S/\langle T\rangle$.

In order to estimate the order of the zero of $\theta(z;\Omega)$ at $K(P)$, where P is a fixed point of T, we have only to make a study of $l(P^{g-1})$. By virtue of this reason we shall state our theorems associated with $K(P)$ in terms of $l(P^{g-1})$. But those theorems can be restated at once in terms of the order of the zero of $\theta(z;\Omega)$.

3. The lower bounds. Throughout the following theorems, suppose that S is a compact Riemann surface of genus $g(\geqq 2)$, T is a conformal automorphism of S with fixed points, t denotes the number of those fixed points, P denotes any one of those fixed points, N is the order of $\langle T\rangle$, $\tilde{g}$ is the genus of $S/\langle T\rangle$ and that π denotes the projection of S into $S/\langle T\rangle$.

THEOREM 1. *Let N be prime. If $g\equiv k \pmod N$, $(0<k<N)$, then*

$$l(P^{g-1})\geqq \frac{g-k}{N}+1-\tilde{g}. \tag{1}$$

If $g\equiv 0 \pmod N$, then

$$l(P^{g-1})\geqq \frac{g}{N}-\tilde{g}. \tag{2}$$

(Farkas proved for $k=1$.)

Proof. Let $\tilde{P}$ be $\pi(P)$, the projection of P. Let $\tilde{f}$ be a meromorphic function on $S/\langle T\rangle$ with pole only at $\tilde{P}$. There is a meromorphic function f on S such that $f=\tilde{f}\circ\pi$. Then the order of pole of f at P is N times of that of $\tilde{f}$

at $\tilde{P}$. It is easy to see that there are at least $[(g-1)/N]-\tilde{g}+1$ linearly independent meromorphic functions on $S/\langle T\rangle$ with pole only at $\tilde{P}$ whose order $\leqq(g-1)/N$. Here $[s]$ denotes the integer part of s. If $g\equiv k \pmod N$, then $[(g-1)/N]=(g-k)/N$. If $g\equiv 0 \pmod N$, then $[(g-1)/N]=g/N-1$. This completes the proof.

4. The case N=2. In this section we consider the upper bounds of $l(P^{g-1})$ for the case $N=2$, and give some equality statements. Farkas [2] already showed that if $N=2$, $k=1$, $g\geqq 2\tilde{g}+1$ and $\tilde{P}$ is not a Weierstrass point of $S/\langle T\rangle$, then $l(P^{g-1})=(g-1)/2+1-\tilde{g}$. But he made a computational mistake, so this is not true. For example, let S be the surface defined by $y^3=x^2(x^2-1)(x^4+1)^2$ and let T be the automorphism defined by $T(x,y)=(-x,y)$. We choose as P the point corresponding to $(x,y)=(0,0)$. Then we have $l(P^4)=2$, while $g=5$ and $\tilde{g}=2$. It is easily seen if we choose a basis of differentials on S so that dx/y, $x(x^4+1)dx/y^2$, xdx/y, $x^2(x^4+1)dx/y^2$ and x^2dx/y. It is also obvious that $\tilde{P}$ is not a Weierstrass point.

It is known that every zero of Riemann's theta function is of order less than or equal to $(g+1)/2$ [4]. Hence, if $\tilde{g}=0$ and $N=2$, then equality holds in Theorem 1. Therefore, we shall only consider the case $\tilde{g}>0$.

THEOREM 2. *Suppose that* $N=2$, $g\equiv 1 \pmod 2$ *and that* $\tilde{g}>0$. *Then*

$$l(P^{g-1})=\frac{g+1}{2}-\tilde{g}, \quad \text{if } g\geqq 4\tilde{g}-1, \tag{3}$$

$$l(P^{g-1})\leqq \tilde{g}+1, \quad \text{if } g=4\tilde{g}-3, \tag{4}$$

$$l(P^{g-1})\leqq 2\tilde{g}-\frac{g+5}{4}, \quad \text{if } g\leqq 4\tilde{g}-5,\ g\equiv 3 \pmod 4 \tag{5}$$

and

$$l(P^{g-1})\leqq 2\tilde{g}-\frac{g+3}{4}, \quad \text{if } g\leqq 4\tilde{g}-5,\ g\equiv 1 \pmod 4. \tag{6}$$

Furthermore, if $\tilde{P}=\pi(P)$ *is not a Weierstrass point of* $S/\langle T\rangle$, *then*

$$l(P^{g-1})\leqq \tilde{g}, \quad \text{if } g=4\tilde{g}-3 \tag{7}$$

and

$$l(P^{g-1})\leqq \tilde{g}-1, \quad \text{if } g\leqq 4\tilde{g}-5. \tag{8}$$

Proof. In this and the sequential proofs, we consider meromorphic functions on S (or $S/\langle T\rangle$) with no pole but P (or $\tilde{P}$). For brevity, we call the order of the pole of such a meromorphic function at P (or $\tilde{P}$) "the order of the function".

For any positive integer n, there is a function of order n on $S/\langle T\rangle$ if and only if there is a function of order $2n$ on S. Indeed, let f be a function of order $2n$ on S and let z be a local parameter at P such that $T(z)=-z$ and $f(z)=z^{-2n}+\cdots$. Put $F=f+f\circ T$. Then $F(z)=2z^{-2n}+\cdots$ at P and $F=F\circ T$. Hence, there is a function of order n on $S/\langle T\rangle$. Only if part is trivial.

By Weierstrass' gap theorem there are exactly $n-\tilde{g}$ linearly independent functions on $S/\langle T\rangle$ of order less than n for every $n\geqq 2\tilde{g}$. If $g\geqq 4\tilde{g}-1$, then $g-1\geqq 2(2\tilde{g}-1)$. Therefore, there are exactly $(g+1)/2-\tilde{g}$ linearly independent functions of even order which are not greater than $g-1$.

Suppose there is a function f of odd order k such that $k\leqq g-2$. We can choose a local parameter z at P such that $T(z)=-z$ and $f(z)=z^{-k}+\cdots$. Put $F=f-f\circ T$. Then $F(z)=2z^{-k}+\cdots$ at P and $F\circ T=-F$. Therefore, $F=0$ for every fixed point of T but P. Hence $k\geqq t-1$. Since $t=2g-4\tilde{g}+2$, we have $g-2\geqq k\geqq 2g-4\tilde{g}+1$. Therefore, $4\tilde{g}-3\geqq g$. Hence,

$$l(P^{g-1})=\frac{g+1}{2}-\tilde{g}, \qquad \textit{if} \quad g\geqq 4\tilde{g}-1\,.$$

Before proving the rest of this theorem, we have the following observation:

"*If there is a function f_1 of order $k_1=t-1=2g-4\tilde{g}+1$, then there is not a function of order k_1+2*"

Suppose there is a function f_2 of order k_1+2. Since $F_1=f_1-f_1\circ T$ has exactly k_1 simple zeros at all the fixed points of T but P and these fixed points are also zeros of $F_2=f_2-f_2\circ T$, $F=F_2/F_1$ is a function of order 2. Hence, there is a function of order one on $S/\langle T\rangle$. Since $\tilde{g}>0$, this is absurd.

We proceed the rest of the proof. Let $g\leqq 4\tilde{g}-3$. By Clifford's theorem [7] there are at most $[(g+3)/4]$ linearly independent functions of order less than or equal to $(g-1)/2$ on $S/\langle T\rangle$. Hence, there are at most $[(g+3)/4]$ linearly independent even order functions of order not greater than $g-1$ on S. By the same reason as the case of $g\geqq 4\tilde{g}-1$, the possible order of an odd order function is not less than $2g-4\tilde{g}+1$. Let m be the number of linearly independent odd order functions of order less than $g-1$. By the observation, we have

$$m\geqq\frac{g-2-(2g-4\tilde{g}+1)}{2}=2\tilde{g}-\frac{g+3}{2}, \qquad \text{if} \quad g\leqq 4\tilde{g}-5$$

and

$$m\leqq 1, \qquad \text{if} \quad g=4\tilde{g}-3\,.$$

Hence, we have (4), (5) and (6).

If $\pi(P)$ is not a Weierstrass point of $S/\langle T\rangle$ and if $g\leqq 4\tilde{g}-3$, then there are at most $(g+1)/2-\tilde{g}$ linearly independent functions of order not greater than $(g-1)/2$ on $S/\langle T\rangle$. The bounds of the number of linearly independent odd order function is the same as above. Thus we have (7) and (8).

THEOREM 3. *Suppose that $N=2$, $g\equiv 0 \pmod 2$ and that $\tilde{g}>0$. Then*

$$l(P^{g-1})=\frac{g}{2}-\tilde{g}, \qquad \textit{if} \quad g\geqq 4\tilde{g}, \tag{9}$$

(10) $$l(P^{g-1}) \leqq \tilde{g}+1, \quad \textit{if} \quad g=4\tilde{g}-2,$$

(11) $$l(P^{g-1}) \leqq 2\tilde{g}-1-\frac{g}{4}, \quad \textit{if} \quad g \leqq 4\tilde{g}-4, \quad g \equiv 0 \pmod 4$$

and

(12) $$l(P^{g-1}) \leqq 2\tilde{g}-\frac{1}{2}-\frac{g}{4}, \quad \textit{if} \quad g \leqq 4\tilde{g}-4, \quad g \equiv 2 \pmod 4.$$

If $\tilde{P}$ is not a Weierstrass point of $S/\langle T\rangle$, then

(13) $$l(P^{g-1}) \leqq \tilde{g}, \quad \textit{if} \quad g=4\tilde{g}-2 \quad \textit{or if} \quad g=2\tilde{g}$$

and

(14) $$l(P^{g-1}) \leqq \tilde{g}-1, \quad \textit{if} \quad 2\tilde{g}+2 \leqq g \leqq 4\tilde{g}-4.$$

Proof. The tool of the proof is similar to that of the preceding theorem. Since $g-1$ is odd, if $g \geqq 4\tilde{g}$ then there are exactly $g/2-\tilde{g}$ linearly independent functions of even order, which is not greater than $g-1$. Since $g-1 \geqq k \geqq 2g-4\tilde{g}+1$, we have (9). If $g \leqq 4\tilde{g}-2$, then there are at most $[(g+2)/4]$ linearly independent functions of order less than or equal to $(g-1)/2$ on $S/\langle T\rangle$. By the observation in the preceding proof, we have

$$m \leqq \frac{g-1-(2g-4\tilde{g}+1)}{2} = 2\tilde{g}-\frac{g}{2}-1, \quad \text{if} \quad g \leqq 4\tilde{g}-4$$

and

$$m \leqq 1, \quad \text{if} \quad g=4\tilde{g}-2.$$

Thus we have (10), (11) and (12).

If $\pi(P)$ is not a Weierstrass point of $S/\langle T\rangle$ and if $2\tilde{g}+2 \leqq g \leqq 4\tilde{g}-2$, then there are at most $g/2-\tilde{g}$ linearly independent functions of order not greater than $(g-1)/2$ on $S/\langle T\rangle$. In the case $g=2\tilde{g}$, while $g/2-\tilde{g}=0$ there is always a non-zero constant function. Thus we have (13) and (14).

5. The case $N \geqq 3$. In this section we shall consider the upper bounds of $l(P^{g-1})$ for the case $N \geqq 3$. Farkas [2] showed that if $N=3$, $g \equiv 1 \pmod 3$ and $\tilde{g}=0$ then equality occurs in Theorem 1. We shall prove it without the hypothesis $g \equiv 1$.

THEOREM 4. *Let N be a prime number, $N \geqq 3$ and let $\tilde{g}=0$. If $g \equiv k \pmod N$, $(0<k<N)$, then*

(15) $$l(P^{g-1}) \leqq \frac{N+1}{4}\left(\frac{g-k}{N}+1\right)+\min\left\{\frac{(k+1)N-3k+1}{4N}, \ \frac{(k-1)(N+1)}{4N}\right\}.$$

If $g \equiv 0 \pmod N$, then

(16) $$l(P^{g-1}) \leqq \frac{(N+1)g+(N-1)^2}{4N}.$$

Proof. Assume $0<k<N$. Since $\tilde{g}=0$, there are $(g-k)/N$ linearly independent functions on S of order $N, 2N, \cdots, g-k$.

Suppose there is a function f on S of order j such that $j\not\equiv 0 \pmod N$, $j\leqq g-1$. Choose a local parameter z at P such that $T(z)=\varepsilon z$ and $f(z)=z^{-j}+\cdots$ where $\varepsilon^N=1$ and $\varepsilon\neq 1$. Put

$$F=f+\varepsilon^j f\circ T+\varepsilon^{2j}f\circ T^2+\cdots+\varepsilon^{(N-1)j}f\circ T^{N-1}.$$

Since $f\circ T(z)=\varepsilon^{-j}z^{-j}+\cdots$, $F(z)=Nz^{-j}+\cdots$ at P. Since $F=\varepsilon^j F\circ T$, every fixed point of T but P is a zero of F. Let F_1 and F_2 be functions of order k_1 and k_2, respectively, such that $F_1=\varepsilon^{k_1}F_1\circ T$ and $F_2=\varepsilon^{k_2}F_2\circ T$. If $k_1\not\equiv k_2 \pmod N$, then for each fixed point of T but P the order of the zero of F_1 at the point is different from that of F_2. If there is a function of order j, then functions of order $j+N$, $j+2N, \cdots$ also exist.

Let $k_1<k_2<\cdots<k_{N-1}$ be integers such that $k_i\not\equiv k_j \pmod N$ for $i\neq j$ and that $k_i\not\equiv 0 \pmod N$ for every i and that there exist a function of order k_i but of order k_i-N for each $i=1,\cdots,N-1$. Then we have

$$k_1+\cdots+k_m\geqq\frac{m(m+1)}{2}(t-1) \tag{17}$$

$$=\frac{m(m+1)}{2}\left(\frac{2g}{N-1}+1\right),$$

for each $m=1,\cdots,N-1$. Let M be the number of linearly independent functions of order less than g but of order $N, 2N,\cdots, g-k$. Then we have

$$NM\leqq\max_m (g-1+N)+\cdots+(g-m+N)-(k_1+\cdots+k_m) \tag{18}$$

where m runs from 1 to $N-1$. Combining with (17) and (18) we have

$$NM\leqq\frac{N-3}{4}(g-k+N)+\frac{(k+1)N-3k+1}{4}. \tag{19}$$

If $(N-1)/2\geqq k$, then we have

$$NM\leqq\max_m \{(g-1+N)+\cdots+(g-m+N)-(m-k+1)-(k_1+\cdots+k_m)\} \tag{20}$$

$$\leqq\frac{N-3}{4}(g+N-k)+\frac{(k-1)(N+1)}{4}.$$

Since $(k+1)N-3k+1\leqq(k-1)(N+1)$ for $k\geqq(N+1)/2$, we have

$$M\leqq\frac{N-3}{4}\left(\frac{g-k}{N}+1\right)+\min\left\{\frac{(k+1)N-3k+1}{4N},\ \frac{(k-1)(N+1)}{4N}\right\}. \tag{21}$$

Since $l(P^{g-1})=(g-k)/N+1+M$, we have (15).

If $g\equiv 0 \pmod N$, then we can use the preceding discussion for $k=N$. Thus we have (16).

COROLLARY 1. *Let N be a prime number, $N\geqq 3$ and let $\tilde{g}=0$. If $g\equiv 1$ or $g\equiv 2 \pmod N$, then*

$$l(P^{g-1})\leqq \frac{N+1}{4}\left(\frac{g-k}{N}+1\right), \quad (k=1,\ 2). \tag{22}$$

COROLLARY 2. *Let N be a prime number so that $N+1$ is a multiple of 4. Let $\tilde{g}=0$. If $g=1, 2, 3$ or $4 \pmod N$, then*

$$l(P^{g-1})\leqq \frac{N+1}{4}\left(\frac{g-k}{N}+1\right), \quad (k=1, \cdots, 4). \tag{23}$$

COROLLARY 3. *Let $N=3$ and $\tilde{g}=0$. If $g\equiv 1$ or $2 \pmod 3$, then*

$$l(P^{g-1})=\frac{g-k}{3}+1, \quad (k=1,\ 2). \tag{24}$$

If $g\equiv 0 \pmod 3$, then

$$l(P^{g-1})=\frac{g}{3}. \tag{25}$$

(For the case $k=1$ was already obtained by Farkas).

Next, we shall consider the case $\tilde{g}>0$. Suppose $g\equiv k \pmod N$, $(0<k<N)$. If $g\leqq 2N\tilde{g}-2N+k$, then by Clifford's theorem there are at most $[(g-k)/2N]+1$ linearly independent functions of order less than or equal to $g-1$, so that each one of whose order is divisible by N. If $\pi(P)$ is not a Weierstrass point, then the number of such functions is $(g-k)/N+1-\tilde{g}$. If $g>2N\tilde{g}-2N+k$, then the number of such functions is always $(g-k)/N+1-\tilde{g}$.

Let $k_1<k_2<\cdots<k_{N-1}$ be integers such that $k_i\not\equiv k_j \pmod N$ for $i\neq j$ and that $k_i\not\equiv 0 \pmod N$ for each i and that there exists a function of order k_i but of order k_i-sN for each $i=1, \cdots, N-1$ and each $s=1, 2, \cdots$. Then the same reason as in the proof of Theorem 4 implies

$$k_1+\cdots+k_m\geqq \frac{m(m+1)}{2}(t-1).$$

Put

$$A=(g+N-1)+\cdots+(g+N-m)-(k_1+\cdots+k_m)-(m-k+1), \quad \text{if} \quad m\geqq k \tag{26}$$

and

$$A=(g+N-1)+\cdots+(g+N-m)-(k_1+\cdots+k_m), \quad \text{if} \quad m<k. \tag{27}$$

Put $A_0=\max_m A$, where m runs from 1 to $N-1$.

If $k\leqq n=[(g+N)/t]\leqq N-2$, then

$$A_0\leqq n(g+N)-\frac{n(n+1)}{2}t-(n+1)+k. \tag{28}$$

If $n<k$, then

$$A_0 \leqq n(g+N)-\frac{n(n+1)}{2}t. \tag{29}$$

If $n \geqq N-1$, then

$$\begin{aligned} A_0 &\leqq (N-1)(g+N)-\frac{(N-1)N}{2}t-N+k \\ &= N^2\tilde{g}-N-g+k. \end{aligned} \tag{30}$$

By a similar way of the observation in the proof of Theorem 2, we can observe that if $g-1-k_i \geqq N$ then there are at most $[(g-1-k_i)/N]$ linearly independent functions of order less than or equal to $g-1$, so that each one of whose order is congruent to k_i.

Put

$$B=n(g+N)-\frac{n(n+1)}{2}t+\min\{0,\ k-n-1\}, \quad \text{if} \quad n \leqq N-2 \tag{31}$$

and

$$B=N^2\tilde{g}-N-g+k, \quad \text{if} \quad n \geqq N-1. \tag{32}$$

Let B_0 be the number of linearly independent functions of order less than g, each one of which is not a multiple of N. Put $n'=\min\{n, N-1\}$. If $B-n'N \geqq N$, then $B_0 \leqq [B/N]-1$. If $B-(n'-1)N-(g+N-1-k_1) \geqq N$, then $B_0 \leqq [B/N]-2$. If $B-(n'-2)N-(g+N-1+g+N-2-k_1-k_2) \leqq N$, then $B_0 \leqq [B/N]-3$ and so on. Since

$$\begin{aligned} &(g+N-1)+(g+N-2)+\cdots+(g+N-s)-(k_1+k_2+\cdots+k_s) \\ &\leqq s(g+N)-\frac{s(s+1)}{2}t, \end{aligned}$$

we shall consider the equation

$$B-s(g+N)+\frac{s(s+1)}{2}t-(n'+1-s)N=0. \tag{33}$$

When (33) has a real solution, let s_1 denote the least one. When (33) has no real solution, let $s_1=N-1$. Put

$$s_0=\min\{N-1,\ \max\{0,\ [s_1]\}\}. \tag{34}$$

Then we have

$$B_0 \leqq [B/N]-s_0.$$

Put $k=N$. Then the preceding discussion can be applied when $g \equiv 0 \pmod{N}$. But if $t=2$, then it is necessary a trivial modification in the first part of this discussion.

Summing up, we have

THEOREM 5. *Let N be a prime number, $N\geqq 3$ and let $\tilde{g}>0$. If $g\equiv k(\bmod N)$, $(0<k<N)$, then*

$$l(P^{g-1})\leqq C+[B/N]-s_0. \tag{35}$$

Here B is defined by (31) *and* (32) *and*

$$C=\left[\frac{g-k}{2N}\right]+1, \tag{36}$$

if $g\leqq 2Ng-2N+k$ and $\pi(P)$ is a Weierstrass point, and

$$C=\frac{g-k}{N}+1-\tilde{g}, \quad \textit{otherwise}. \tag{37}$$

If $g\equiv 0$ $(\bmod N)$ and $t>2$, then (35) *holds if B and C are defined by $k=N$. If $g\equiv 0$ $(\bmod N)$ and $t=2$, then*

$$l(P^{g-1})\leqq 1+\left[\frac{B}{N}\right]-s_0, \tag{38}$$

where B is defined by $k=N$.

6. Estimates at half periods. In this section we shall give an estimate at a half period of $J(S)$.

Let $P_1, \cdots, P_t$ be the fixed points of T. Assume that N, the order of T, is prime. Let z_i be an arbitrary local parameter at P_i $(i=1, \cdots, t)$. There is an ε_i such that $T(z_i)=\varepsilon_i z_i+\cdots$ at P_i where $\varepsilon_i{}^N=1$, $\varepsilon_i\neq 1$. Such an ε_i is independent of the choice of z_i. Consider the following condition:

$$\varepsilon=\varepsilon_1=\varepsilon_2=\cdots=\varepsilon_{t-1}. \tag{A}$$

Then we have

THEOREM 6. *Let N be prime. Assume that T satisfies the condition* (A). *If $g\equiv k$ $(\bmod N)$, $(0<k<N)$, then $\theta(z;\Omega)$ vanishes to order at least*

$$\frac{g-k}{N}+1-\tilde{g} \tag{39}$$

at the $4^{\tilde{g}}$ half periods of $J(S)$.

If $g\equiv 0$ $(\bmod N)$, then $\theta(z;\Omega)$ vanishes to order at least

$$\frac{g}{N}-\tilde{g} \tag{40}$$

at the $4^{\tilde{g}}$ half periods of $J(S)$.

To prove this theorem we need two lemmas.

LEMMA 1 (Lewittes [4], Rauch and Farkas [6]). *Let Δ be a positive divisor* *of degree* $2g-2$. *Then* Δ *is the divisor of a differential if and only if* $\phi(\Delta)+2K(P_0)\equiv 0$ *on* $J(S)$. *Therefore, if* D *is a positive divisor of degree* $g-1$ *such that* D^2 *is the divisor of a differential, then* $\phi(D)+K(P_0)$ *is a half period of* $J(S)$.

LEMMA 2. *If* T *satisfies the condition* (A), *then there is a differential* ω *on* S *such that* $\omega\circ T=\varepsilon\omega$.

Proof. Let H_j be the space of abelian differentials of the first kind such that $\theta\circ T=\varepsilon^j\theta$ for each θ in H_j $(j=0, 1, \cdots, N-1)$. Let n_j be the dimension of H_j. By Lewittes [3] $n_0=\tilde{g}$, the genus of $S/\langle T\rangle$ and $\sum_{j=0}^{N-1} n_j=g$. Assuming the condition (A) and applying Lewittes' method, we can see that if $n_j\neq 0$, then

$$n_j=\left[\frac{2g-2-(j-1)(t-1)}{N}\right]+1-\tilde{g}.$$

Since $2g-2=N(2\tilde{g}-2)+(N-1)t$ and $t>0$, we have $2g>N\tilde{g}$. Therefore, we have

$$\sum_{j=2}^{N-1} n_j \leqq \frac{(N-2)(2g-2)-\sum_{j=2}^{N-1}(j-1)t+N-2}{N}+(N-2)(1-\tilde{g})$$

$$=\frac{N-2}{N}g<g-\tilde{g}.$$

Hence, $n_1\neq 0$. Thus $H_1\neq\phi$.

Proof of Theorem 6. Suppose that $g\equiv k$, $(0<k<N)$ and that $\tilde{g}\geqq 1$. Let ω be a differential on S such that $\omega\circ T=\varepsilon\omega$. The existence of such an ω is guaranteed by Lemma 2. Then the divisor of ω is of the form

$$P_1^{r_1N}P_2^{r_2N}\cdots P_{t-1}^{r_{t-1}N}P_t^{r_tN+2k-2}\Delta\Delta_1\cdots\Delta_{(N-1)}. \tag{41}$$

Here $r_1, r_2, \cdots, r_{t-1}$ are non-negative integers, r_t is an integer $\geqq -1$, Δ is a divisor of degree $2(g-k)/N-\sum_{i=1}^{t} r_i$ and $\Delta_i=T^i(\Delta)$. Put $\tilde{P}_i=\pi(P_i)$, $i=1, \cdots, t$. Let $\tilde{\phi}$ be the canonical map of $S/\langle T\rangle$ into $J(S/\langle T\rangle)$. We may choose $\tilde{P}_t$ as the base point of $\tilde{\phi}$. Consider the $4^{\tilde{g}}$ equations

$$\tilde{\phi}(X)\equiv\frac{1}{2}\tilde{\phi}(\tilde{P}_1^{r_1}\tilde{P}_2^{r_2}\cdots\tilde{P}_{t-1}^{r_{t-1}}\tilde{\Delta}). \tag{42}$$

Here X is a positive divisor of degree $(g-k)/N$, $\tilde{\Delta}=\pi(\Delta)$ and $(1/2)\tilde{\phi}(\cdot)$ denotes a point α of $J(S/\langle T\rangle)$ such that $\tilde{\phi}(\cdot)\equiv 2\alpha$. The number of such α's is $4^{\tilde{g}}$. By the Jacobi inversion problem, each equation is solved with $(g-k)/N-\tilde{g}$ free points. Let $\tilde{D}$ be the solution of one of (42).

By Abel's theorem,

$$\tilde{P}_1^{r_1}\tilde{P}_2^{r_2}\cdots\tilde{P}_{t-1}^{r_{t-1}}\tilde{P}_t^{r_t}\tilde{\Delta}\tilde{D}^{-2}$$

is the divisor of a function $\tilde{f}$ on $S/\langle T\rangle$. There is a function f on S such that $f=\tilde{f}\circ\pi$. The divisor of f is

$$P_1^{r_1N}P_2^{r_2N}\cdots P_{t-1}^{r_{t-1}N}P_t^{r_tN}\Delta\Delta_1\cdots\Delta_{(N-1)}D^{-2}D_1^{-2}\cdots D_{(N-1)}^{-2}$$

where D is a divisor such that $\pi(D)=\tilde{D}$. Since (41) is the divisor of a differential, $D^2D_1^2\cdots D^2_{(N-1)}P_t^{2k-2}$ is also the divisor of a differential. Therefore, by Lemma 1 $\phi(DD_1\cdots D_{(N-1)}P_t^{k-1})+K(P_t)$ is a half period of $J(S)$. Since $\tilde{D}$ is solved with $(g-k)/N-\tilde{g}$ free point,

$$i(DD_1\cdots D_{(N-1)}P_t^{k-1})\geqq(g-k)/N+1-\tilde{g}.$$

Hence by Riemann's vanishing theorem and Lemma 1 we have (39). Putting $k=N$ we have (40) at once.

Suppose $\tilde{g}=0$. Let ω be a differential on S such that $\omega\circ T=\varepsilon\omega$. Such an ω is of the form (41). Since $\tilde{g}=0$, there is a differential ω' such that the divisor of ω' is P_t^{2g-2}. Therefore, $K(P_t)$ is a half period of $J(S)$. Hence, by Theorem 1 we have (39) and (40).

COROLLARY. (Accola [1]). *Let* $N=2$.

If g *is odd, then* $\theta(z;\,\Omega)$ *vanishes to order at least*

$$\frac{g+1}{2}-\tilde{g} \tag{43}$$

at the $4^{\tilde{g}}$ *half periods of* $J(S)$.

If g *is even, then* $\theta(z;\,\Omega)$ *vanishes to order at least*

$$\frac{g}{2}-\tilde{g} \tag{44}$$

at the 4^{g} *half periods of* $J(S)$.

Proof. Since $N=2$, the condition (A) is always satisfied by $\varepsilon=-1$.

For $\tilde{g}=1$, we have another estimate which is better than Theorem 6 when t is large enough.

THEOREM 7. *Assume that* $\tilde{g}=1$, $N\geqq 3$ *and that* T *satisfies the condition* (A). *Then* $\theta(z;\,\Omega)$ *vanishes at a half period of* $J(S)$ *to order at least*

$$1+\frac{N+1}{4N}(g-1), \qquad \textit{if} \quad g\equiv 1 \pmod N \tag{45}$$

and

$$1+\sum_{j=1}^{(N-1)/2}\left[\frac{(2g-N-1)j/(N-1)-1}{N}\right] \tag{46}$$

$$=1+\sum_{j=1}^{(N-1)/2}\left[\frac{j(t-1)-1}{N}\right], \qquad \textit{if} \quad g\not\equiv 1 \pmod N.$$

To prove this theorem we need a lemma which is a counterpart of Lemma 2. Let H_j be the same as in the proof of Lemma 2. Let n_j be the dimension of H_j. Then we have

LEMMA 3. *Assume* $\tilde{g}=1$. *If* T *satisfies the condition* (A), *then* $n_j\neq 0$ *for all* $j=0, 1, \cdots, N-1$.

Proof. Put

$$m_j=\left[\frac{2g-2-(j-1)(t-1)}{N}\right] \quad \text{for} \quad j=1, \cdots, N-1 .$$

If $n_j\neq 0$, then $n_j=m_j$ by Lewittes' method [3]. By the condition (A), $t\not\equiv 1 \pmod N$. Hence,

$$\begin{aligned}\sum_{j=1}^{N-1} m_j &= \frac{(2g-2)(N-1)-\sum_{j=1}^{N-1}(j-1)t}{N} \\ &= g-1 .\end{aligned}$$

Since $t\geqq 2$ and $2g-2=(N-1)t$, we have $m_j\neq 0$ for all $j=1, \cdots, N-1$. It is obvious that $n_0=1$ and $\sum_{j=0}^{N-1} n_j=g$. Hence, we obtain that $n_j\neq 0$ for all $j=0, 1, \cdots, N-1$.

Proof of Theorem 7. Assume $g\equiv 1 \pmod N$. By the Riemann-Hurwitz relation we have $t\equiv 0 \pmod N$. Let ω be a differential on S such that $\omega\circ T=\omega$. Since $\tilde{g}=1$, the divisor of ω is of the form $P_1^{N-1}P_2^{N-1}\cdots P_t^{N-1}$. Since N is a prime number ($\geqq 3$), $N-1$ is even. Put

$$D=P_1^{\frac{N-1}{2}}P_2^{\frac{N-1}{2}}\cdots P_t^{\frac{N-1}{2}} .$$

If ψ is a differential in H_j, then the divisor of ψ is of the form $P_1^{j-1}P_2^{j-1}\cdots P_t^{j-1}\Delta\Delta_1\cdots\Delta_{(N-1)}$, where Δ is a divisor of degree $2(g-1)(N-j)/N(N-1)$ and $\Delta_i=T^i(\Delta)$. In this case $n_j=2(g-1)(N-j)/N(N-1)$. For every ψ in $H_j (j=(N+1)/2, \cdots, N-1)$, the divisor of ψ is a multiple of D. Hence, we have

$$i(D)\geqq 1+\sum_{j=(N+1)/2}^{N-1} n_j=1+\frac{N+1}{4N}(g-1) .$$

Therefore, we have (45).

Assume $g\not\equiv 1 \pmod N$. By the same reasoning there is a differential ω on S whose divisor is of the form $P_1^{N-1}P_2^{N-1}\cdots P_t^{N-1}$. Put again

$$D=P_1^{\frac{N-1}{2}}P_2^{\frac{N-1}{2}}\cdots P_t^{\frac{N-1}{2}} .$$

If ψ is a differential in H_j, then the divisor of ψ is of the form $P_1^{j-1}P_2^{j-1}\cdots P_{t-1}^{j-1}P_t^{\alpha_j}\Delta\Delta_1\cdots\Delta_{(N-1)}$ where $0\leqq\alpha_j\leqq N-2$ and Δ is a divisor of degree

$$n_j=[((N-j)t+j-1)/N] .$$

Let H_j' be the subspace of H_j such that every divisor of a differential in H_j' is a multiple of D. Since $\alpha_j \geqq 0$, the dimension of H_j' is at least n_j-1 if $j=(N+1)/2, \cdots, N-1$. Hence, we have

$$i(D) \geqq 1+\sum_{j=(N+1)/2}^{N-1}(n_j-1)=1+\sum_{j=1}^{(N-1)/2}\left[\frac{j(t-1)-1}{N}\right].$$

This completes the proof of Theorem 7.

7. Examples. In this section we show several examples in order to see that some of the estimates obtained cannot be improved. As is shown by Farkas, in certain cases equality holds in Theorem 1. In fact, the cases $N=2$, $g \geqq 4\tilde{g}$ and $N=3$, $\tilde{g}=0$ are such cases. There are consequences of Theorem 2, Theorem 3 and Corollary 3 to Theorem 4.

While equality does not always hold in other cases, we shall show an example which attains the lower bound.

EXAMPLE 1. *Let S be defined by*

$$y^N=(x-a_0)(x-a_{N-1})^{N-2}\prod_{j=1}^{N-2}(x-a_j)^j, \tag{47}$$

where a_j, $j=0, \cdots, N-1$ are complex numbers which are different from each other. Let T be an automorphism of S defined by $T(x, y)=(x, e^{2\pi i/N}y)$. If P is the point corresponding to $(x, y)=(a_0, 0)$, then

$$l(P^{g-1})=\frac{g-1}{N}+1. \tag{48}$$

This relates to (1) for $k=1$, $\tilde{g}=0$.

Indeed, we can choose as a basis for the space of abelian differentials on S,

$$\frac{(x-a_0)^i(x-a_{N-1})^{N-3-\left[\frac{m(N-2)-1}{N}\right]}\prod_{j=2}^{N-2}(x-a_j)^{j-1-\left[\frac{mj-1}{N}\right]}dx}{y^{N-m}} \tag{49}$$

where $m=1, \cdots, N-1$ if $i=0, \cdots, (N-5)/2$ and $m=1, \cdots, (N-1)/2$ if $i=(N-3)/2$. Hence, we have $i(P^{N(N-3)/2})=(N-1)/2$. Since $g=(N-1)(N-2)/2$, we have (48).

The next two examples have respect to Theorem 4.

EXAMPLE 2. *Let S be defined by*

$$y^N=\prod_{j=0}^{N-1}(x-a_j) \tag{50}$$

where a_j, $j=0, \cdots, N-1$ are complex numbers which are different from each other. Suppose $T(x, y)=(x, e^{2\pi i/N}y)$. If P is the point corresponding to $(x, y)=(a_0, 0)$, then

$$l(P^{g-1})=\frac{N+1}{4}\left(\frac{g-1}{N}+1\right). \tag{51}$$

This shows that Theorem 4 gives the sharp bound for $k=1$.

Indeed, as a basis for differentials we can choose

$$\frac{(x-a_0)^n dx}{y^{N-m}}, \tag{52}$$

where $n=0, \cdots, N-3$ and $m=1, \cdots, N-2-n$. Therefore, we have $i(P^{N(N-3)/2})=(N^2-1)/8$. Since $g=(N-1)(N-2)/2$, we have (51).

For $k \neq 1$, the author fails to give an example which attains the upper bound in (15) or (16). It is, however, shown that the estimate is not improved in the following sense.

"For each k, $(N+1)/4$, the factor of the first term of (15), cannot be improved".

This is shown in the following example. It seems to the author that this example gives the least upper bounds for that estimate.

EXAMPLE 3. *Let S be defined by*

$$y^N=(x-a_0)^{2k-1} \prod_{j=1}^{mN-2k+1} (x-a_j) \tag{53}$$

where $m(\geqq 2)$ is an integer, $k>0$ and $a_j (j=0, \cdots, mN-2k+1)$ are complex numbers which are different form each other. Let T be an automorphism of S such that $T(x, y)=(x, e^{2\pi i/N}y)$. If P is a point which corresponds to $(x, y)=(a_0, 0)$ then

$$l(P^{g-1})=\frac{N^2-1}{8}m+K(N, k), \tag{54}$$

where $K(N, k)$ is a constant which depends only on N and k. Since

$$g=(m(N-1)/2-k)N+k,$$

it is easily seen that $(N+1)/4$ cannot be improved if we take an m large enough.

Indeed, as a basis for differentials we choose

$$\frac{(x-a_0)^n dx}{y^s}, \tag{55}$$

where n and s are integers such that

$$ms-n-2 \geqq 0, \tag{56}$$

$$N-1+nN-s(2k-1) \geqq 0 \tag{57}$$

and

$$1 \leqq s \leqq N-1. \tag{58}$$

Considering an inequality

$$N-1+nN-s(2k-1)\geqq g-1=\frac{(mN-2k)(N-1)}{2}-1, \tag{59}$$

we can see that the number of pairs of n and s which satisfy (56), (58) and (59) is equal to $l(P^{g-1})$. Thus we have (54).

The remaining examples have respect to Theorem 2 and Theorem 3.

EXAMPLE 4. *Let S be defined by*

$$y^7=x^4(x^2-1)^3(x^2+1)^2. \tag{60}$$

Suppose $T(x,y)=(-x,y)$. If P is $(x,y)=(0,0)$, then $\pi(P)$ is not a Weierstrass point of $S/\langle T\rangle$ and $l(P^{g-1})=3$. Since $g=9$ and $\tilde{g}=3$, this example relates to (7).

Indeed, choose as a basis for differentials

$$\begin{gathered}\frac{x^2(x^2-1)^2(x^2+1)dx}{y^5},\ \frac{x(x^2-1)dx}{y^3},\ \frac{dx}{y},\\ \frac{x^3(x^2-1)(x^2+1)dx}{y^6},\ \frac{x^2(x^2-1)(x^2+1)dx}{y^4},\ \frac{xdx}{y^2},\\ \frac{x^2(x^2-1)dx}{y^3},\ \frac{x^4(x^2-1)^2(x^2+1)dx}{y^6},\ \frac{x^2dx}{y^2}.\end{gathered} \tag{61}$$

Then we find that the gap sequence at P is $\{1,2,3,4,5,6,9,11,13\}$. Thus we have $l(P^{g-1})=3$. That $\pi(P)$ is not a Weierstrass point is seen by the facts that $S/\langle T\rangle$ is defined by $Y^7=X^2(X-1)^3(X+1)^2$ and that $T_1(X,Y)=((X+1)/(3X-1), 2Y/(1-3X))$ is the hyperelliptic involution of $S/\langle T\rangle$.

On the other hand if P is a point which corresponds to one of $(x,y)=(\infty,\infty)$ then the gap sequence at P is $\{1,2,3,4,5,6,7,\text{odd},\text{odd}\}$. Therefore, we have $l(P^{g-1})=2$.

EXAMPLE 5. *Let S be defined by*

$$y^7=x^3(x^4-1). \tag{62}$$

Suppose $T(x,y)=(-x,-y)$. If P is $(x,y)=(0,0)$ or one of $(x,y)=(\infty,\infty)$, then $\pi(P)$ is a Weierstrass point on $S/\langle T\rangle$ and $l(P^{g-1})=4$. This relates to (4).

Indeed, choose as a basis for differentials

$$\frac{dx}{y^2},\ \frac{xdx}{y^4},\ \frac{x^2dx}{y^6},\ \frac{xdx}{y^3},\ \frac{x^2dx}{y^5},\ \frac{x^2dx}{y^4},\ \frac{x^3dx}{y^6},\ \frac{x^3dx}{y^5},\ \frac{x^4dx}{y^6}. \tag{63}$$

Then we find that the gap sequence at P is $\{1,2,3,5,6,9,10,13,17\}$. Thus we have $l(P^{g-1})=4$. That $\pi(P)$ is a Weierstrass point can be seen as the preceding example.

EXAMPLE 6. *Let S be defined by*

(64) $$y^6=x^4-4x^{10}+3x^{12}.$$

Suppose $T(x, y)=(-x, y)$. *If* P *is one of the points corresponding to* $(x, y)=(0, 0)$, *then* $\pi(P)$ *is a Weierstrass point of* $S/\langle T\rangle$ *and* $l(P^{g-1})=4$. *Since* $g=11$ *and* $\tilde{g}=4$, *this relates to* (5).

Indeed, choose as a basis for differentials

(65) $$\frac{dx}{y}, \frac{xdx}{y^2}, \frac{(x^2+x^4)dx}{y^3}, \frac{(x^3+x^5)dx}{y^4}, \frac{x^2dx}{y^2}, \frac{(x^4+x^6)dx}{y^4},$$
$$\frac{(x^5+x^7)dx}{y^5}, \frac{(x^6+x^8)dx}{y^5}, \frac{(x^6-x^2+y^3)dx}{y^4}, \frac{(x^7-x^3+xy^3)dx}{y^5},$$
$$\frac{(x^8-x^4+x^2y^3)dx}{y^5}.$$

Then we find that the gap sequence at P is $\{1, 2, 3, 4, 5, 7, 8, 11, 13, 14, 17\}$. Thus we have $l(P^{g-1})=4$. Since the gap sequence at $\pi(P)$ is $\{1, 2, 4, 7\}$, $\pi(P)$ is a Weierstrass point.

EXAMPLE 7. *Let* S *be defined by*

(66) $$y^5=x(x^4-1).$$

Suppose $T(x, y)=(-x, -y)$. *Then* $S/\langle T\rangle$ *is of genus* 2. *If* P *is* $(x, y)=(0, 0)$, *then* $\pi(P)$ *is a Weierstrass point of* $S/\langle T\rangle$ *and* $l(P^{g-1})=3$. *This relates to* (10).

Indeed, $S/\langle T\rangle$ is defined by $Y^5=X^3(X^2-1)$ and $l(P^{g-1})$ is obtained by (51).

EXAMPLE 8. *Let* S *be defined by*

(67) $$y^5=x^4(x^2-1)(x^2+1)^2.$$

Suppose $T(x, y)=(-x, y)$. *If* P *is* $(x, y)=(0, 0)$, *then* $\pi(P)$ *is not a Weierstrass point of* $S/\langle T\rangle$ *and* $l(P^{g-1})=2$. *This relates to* (13).

Indeed, choose as a basis for differentials

(68) $$\frac{dx}{y}, \frac{xdx}{y^2}, \frac{x^2dx}{y^3}, \frac{x^3dx}{y^4}, \frac{x^2dx}{y^2}, \frac{x^4dx}{y^4}.$$

Then we find that the gap sequence at P is $\{1, 2, 3, 4, 7, 9\}$. That $\pi(P)$ is not a Weierstrass point is induced by that $S/\langle T\rangle$ is defined by $Y^5=X^2(X-1)(X+1)^2$.

8. Remarks.

1) If S admits an automorphism T of prime order such that $S/\langle T\rangle$ is of genus zero, then S can be defined by an equation of the form

(69) $$y^N=\prod_{i=1}^{t}(x-a_i)^{\alpha_i},$$

where N is the order of T, $\alpha_i>0$ $(i=1,\cdots,t)$, $\sum_{i=1}^{t}\alpha_i\equiv 0 \pmod N$. In (69), T is represented such as $T(x, y)=(x, \varepsilon y)$, where ε is a primitive N-th root of the unity. For each i, let ζ_i be a local parameter at $(a_i, 0)$ and let β_i be a solution of $\alpha_i\beta_i\equiv 1 \pmod N$. Then we have

(70) $$T(\zeta_i)=\varepsilon^{\beta_i}\zeta_i+\cdots$$

Suppose further that T satisfies the condition (A) in Section 6. Then the above argument tells us that S is represented as

(71) $$y^N=(x-a_t)^{2k-1}\prod_{i=1}^{t-1}(x-a_i)$$

for a suitable integer k.

Thus, by Example 2 and Example 3, we obtain better estimates than (39) and (40) for $\tilde{g}=0$.

2) When does an N-valued function on $S/\langle T\rangle$ lift to a single valued function on S? In general, it seems a difficult question. Unfortunately, the author cannot prove Theorem 6 without the condition (A). If $\tilde{g}=0$, Theorem 6 is not true if the condition (A) fails. $y^3=(x^3-1)(x^3+1)^2$ is a counterexample, which is shown by Farkas [2]. For $\tilde{g}>0$, the author does not know whether Theorem 6 is true or not if the condition (A) fails. The author has no idea to resolve some ambiguity in lifting of an N-valued function on $S/\langle T\rangle$, if the condition (A) is not assumed.

3) In Theorem 7, if $t\geqq N$, it is clear that the estimates (45) and (46) are better than (39) and (40) in Theorem 6. If $N=3$ and $k=0$, then (45) is $(g-1)/3+1$ which is better than (39). If $N=3$ and $k=0$, then (46) is $g/3$ which is better than (40).

4) For $\tilde{g}=1$, $N=2$ and $g=5$ or $g\geqq 7$, Theorem 6 (or Corollary) is best possible. Indeed, Martens [5] showed that if $g=5$ or $g\geqq 7$ and if there is a zero of $\theta(z;\Omega)$ of order greater than $(g-1)/2$, then S is hyperelliptic. But, if $g>3$, a surface cannot be both hyperelliptic and elliptic-hyperelliptic (Farkas [2]).

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