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ON THE ADELE RINGS OF ALGEBRAIC NUMBER FIELDS

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Introduction.

Let Q be the rational number field, $\bar{Q}$ the algebraic closure of Q and k ($k \subset \bar{Q}$) an algebraic number field of finite degree. Let $\zeta_k(s)$ be the Dedekind zeta-function of k, k_A the adele ring of k and G_k the Galois group of $\bar{Q}/k$ with Krull topology. We adopt similar notations for an algebraic number field k' ($k' \subset \bar{Q}$) of finite degree. If the extension k/Q is a finite Galois extension and if $\zeta_k(s) = \zeta_{k'}(s)$, then $k = k'$ (cf. Lemma 2). The Lemma 7 of [3] shows that $k_A \cong k'_A$ implies $\zeta_k(s) = \zeta_{k'}(s)$ (cf. Corollary of Lemma 3). We also proved that $G_k \cong G_{k'}$ implies $\zeta_k(s) = \zeta_{k'}(s)$ (cf. [6] or [4]). From the above results, it is natural and interesting to consider whether, for any algebraic number fields k and k' of finite degree, $k_A \cong k'_A$ implies $k \cong k'$ and whether $G_k \cong G_{k'}$ implies $k \cong k'$. In Theorem 1, we shall show that there exist algebraic number fields k and k' of finite degree satisfying the following conditions:

1) $\zeta_k(s) = \zeta_{k'}(s)$.
2) $k_A \not\cong k'_A$.

Furthermore, in Theorem 2, we shall show that there exist algebraic number fields k and k' of finite degree satisfying the following conditions:

1) $k_A \cong k'_A$.
2) $G_k \not\cong G_{k'}$.

This also shows that there exist algebraic number fields k and k' of finite degree satisfying the following conditions:

1) $k_A \cong k'_A$.
2) $k \not\cong k'$.

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Notation and terminology.

Throughout this paper, Q and Z denote the rational number field and the rational integer ring respectively. An algebraic number field always means an algebraic number field of finite degree, an integer means a rational integer and a prime number means a rational prime number. For an algebraic number field

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k, we denote by O_k the integer ring of k, by k_A the adele ring of k, by $\zeta_k(s)$ the Dedekind zeta-function of k and by $\mathrm{Spec}(O_k)^\times$ the set of non-zero prime ideals of O_k. For any prime ideal $\mathfrak{p}\in\mathrm{Spec}(O_k)^\times$, $k_\mathfrak{p}$ denotes the completion of k by $\mathfrak{p}$-adic valuation. Let P_∞ be the set of infinite places of k and put $V_k=\mathrm{Spec}(O_k)^\times\cup P_\infty$. An element of V_k is called to be a place of k. For $\mathfrak{p}\in P_\infty$, if $\mathfrak{p}$ is a real place of k, then $k_\mathfrak{p}$ denotes the real number field and if $\mathfrak{p}$ is an imaginary place, then $k_\mathfrak{p}$ denotes the complex number field. We write $N_{k/Q}(\ )$ for the norm of an ideal in k. Let F be a field and for a Galois extension L/F, we denote by $\mathrm{Gal}(L/F)$ the Galois group of L/F. We write $[G;H]$ for the index of a subgroup H in a finite group G. For a complex number s, we denote by $\mathrm{Re}(s)$ the real part of s. The word "isomorphism" for topological groups, topological rings and topological fields, means a topological isomorphism.

1. A Dirichlet series is a series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$, where s is a complex number, and the coefficients a_n are complex numbers. The following lemma is an elementary property of a Dirichlet series:

LEMMA 1. *Let c be a real number. If Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ and $\sum_{n=1}^{\infty} b_n n^{-s}$ converge in the common half plane $\mathrm{Re}(s)>c$, and if their sum functions coinside in a non-empty open set contained in that half-plane, then $a_n=b_n$ for all $n\geqq 1$.*

COROLLARY. *Let $a_1, a_2, \cdots, a_r, a'_1, a'_2, \cdots, a'_{r'}$, be positive integers with $a_1<a_2<\cdots<a_r$ and $a'_1<a'_2<\cdots a'_{r'}$. Let $\mu_1, \mu_2, \cdots, \mu_r, \mu'_1, \mu'_2, \cdots, \mu'_{r'}$ be positive integers and let p be a prime number. If*

$$\prod_{i=1}^{r}\Big(\sum_{\nu=0}^{\infty} p^{-a_i\nu s}\Big)^{\mu_i}=\prod_{i=1}^{r'}\Big(\sum_{\nu=0}^{\infty} p^{-a'_i\nu s}\Big)^{\mu'_i} \tag{1}$$

in the half-plane $\mathrm{Re}(s)>1$, then $r=r'$, $a_i=a'_i$ and $\mu_i=\mu'_i$ for $i=1, \cdots, r$.

Proof. A series $\sum_{\nu=0}^{\infty} p^{-f\nu s}$ converges in the half-plane $\mathrm{Re}(s)>0$, where f is a positive integer. Hence the both-hand sides of the equation (1) converge in the half-plane $\mathrm{Re}(s)>0$. For the expansion $\sum_{n=1}^{\infty} c_n n^{-s}$ of the left-hand side of the equation (1), $c_1=1$, $c_n=0$ for $1<n<p^{a_1}$ and $c_{p^{a_1}}=\mu_1$. For the expansion $\sum_{n=1}^{\infty} c'_n n^{-s}$ of the right-hand side of the equation (1), $c'_1=1$, $c'_n=0$ for $1<n<p^{a'_1}$ and $c_{p^{a'_1}}=\mu'_1$. Hence from Lemma 1 follows $c_n=c'_n$ for all $n\geqq 1$, which shows $a_1=a'_1$ and $\mu_1=\mu'_1$. Cancelling the series $(\sum_{\nu=0}^{\infty} p^{-a_1\nu s})^{\mu_1}$ and $(\sum_{\nu=0}^{\infty} p^{-a'_1\nu s})^{\mu'_1}$ from the equation (1), we have

$$\prod_{i=2}^{r}\Big(\sum_{\nu=0}^{\infty} p^{-a_i\nu s}\Big)^{\mu_i}=\prod_{i=2}^{r'}\Big(\sum_{\nu=0}^{\infty} p^{-a'_i\nu s}\Big)^{\mu'_i}.$$

Repeating our argument inductively, we conclude $r=r'$, $\mu_i=\mu'_i$ and $a_i=a'_i$ for $i=1, \cdots, r$.

LEMMA 2. *Let L be a finite Galois extension of Q, let $G=\mathrm{Gal}(L/Q)$, and let k and k' be subfields of L corresponding to subgroups H and H' respectively. For any element σ of G, let $C(\sigma)=\{\tau^{-1}\sigma\tau \mid \tau\in G\}$. Then the following conditions are equivalent:*

(i) *For every element σ of G, the number* $\mathrm{card}(C(\sigma)\cap H)$ *of the elements of $C(\sigma)\cap H$ is equal to the number* $\mathrm{card}(C(\sigma)\cap H')$ *of the elements of $C(\sigma)\cap H'$.*

(ii) *For every prime number p, the collection of degrees of the factors of p in k is identical with the collection of degrees of the factors of p in k'.*

(iii) *The zeta-functions $\zeta_k(s)$ and $\zeta_{k'}(s)$ are the same.*

Proof. From p. 363 of [1], it suffices to show (ii) from (iii) part. For every prime number p, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_{g_p}$ be distinct prime ideals in k such that $p=\mathfrak{p}_1^{e_1}\cdots\mathfrak{p}_{g_p}^{e_{g_p}}$ and that $N_{k/Q}(\mathfrak{p}_i)=p^{f_{i,p}}$ with $f_{1,p}\leqq f_{2,p}\leqq\cdots\leqq f_{g_p,p}$. We adopt similar notations, viz, $\mathfrak{p}_1', \cdots, \mathfrak{p}'_{g'_p}\, f'_{1,p}, \cdots, f'_{g'_p,p}(f'_{1,p}\leqq\cdots\leqq f'_{g_p,p})$, for k'. Then we have

$$\zeta_k(s)=\prod_{\mathfrak{p}\in\mathrm{Spec}(O_k)}(1-N_{k/Q}(\mathfrak{p})^{-s})^{-1}=\prod_p\prod_{i=1}^{g_p}(1-p^{-f_{i,p}s})^{-1}$$

$$=\prod_p\prod_{i=1}^{g_p}\sum_{\nu=0}^{\infty}p^{-f_{i,p}\nu s} \qquad \text{for} \quad \mathrm{Re}(s)>1,$$

where p runs through all the prime numbers. Similarly we have

$$\zeta_{k'}(s)=\prod_p\prod_{i=1}^{g'_p}\sum_{\nu=0}^{\infty}p^{-f'_{i,p}\nu s} \qquad \text{for} \quad \mathrm{Re}(s)>1.$$

Since $\zeta_k(s)=\zeta_{k'}(s)$, it follows from Lemma 1

$$\prod_{i=1}^{g_p}\sum_{\nu=0}^{\infty}p^{-f_{i,p}\nu s}=\prod_{i=1}^{g'_p}\sum_{\nu=0}^{\infty}p^{-f'_{i,p}\nu s}.$$

Hence by Corollary of Lemma 1, we conclude $g_p=g'_p$ and $f_{i,p}=f'_{i,p}$ for $i=1, 2, \cdots, g_p$.

LEMMA 3. *Let k be an algebraic number field, k_A the adele ring of k, V_k the set of places of k,* $\mathrm{Spec}(O_k)^\times$ *the set of non-zero prime ideals of k, r_1 the number of real places of k and r_2 the number of imaginary places of k. We adopt similar notations for an algebraic number field k'. Then the following conditions are equivalent:*

(i) *k_A and k'_A are isomorphic.*

(ii) *There exists a bijection Φ of V_k onto $V_{k'}$ such that $k_{\mathfrak{p}}$ and $k'_{\Phi(\mathfrak{p})}$ are isomorphic for all $\mathfrak{p}\in V_k$.*

(iii) *There exists a bijection Ψ of* $\mathrm{Spec}(O_k)^\times$ *onto* $\mathrm{Spec}(O_{k'})^\times$ *such that $k_{\mathfrak{p}}$ and $k'_{\Psi(\mathfrak{p})}$ are isomorphic for all* $\mathfrak{p}\in\mathrm{Spec}(O_k)^\times$.

Proof. The Lemma 7 of [3] shows that the conditions (i) and (ii) are equivalent. It is obvious to prove (iii) from (ii) part. Hence it suffices to prove (ii) from (iii) part. We have $k_{\mathfrak{p}}\cong k'_{\Psi(\mathfrak{p})}$ for every $\mathfrak{p}\in\mathrm{Spec}(O_k)^\times$, which proves $N_{k/Q}(\mathfrak{p})$

$=N_{k'/Q}(\Psi(\mathfrak{p}))$ for every $\mathfrak{p}\in \mathrm{Spec}\,(O_k)^{\times}$. Hence we conclude $\zeta_k(s)=\zeta_{k'}(s)$. Since $\zeta_k(s)$ has zero point of the order r_2 at $s=-1$ and since $\zeta_k(s)$ has zero point of the order r_1+r_2 at $s=-2$, we have $r_1=r_1'$ and $r_2=r_2'$. Therefore we have the condition (iii).

COROLLARY. *$k_A\cong k'_A$ implies $\zeta_k(s)=\zeta_{k'}(s)$.*

Proof. It is obvious from the proof of Lemma 3.

From p. 138 of [2] we have the following lemma:

LEMMA 4. *Let $k=Q(\sqrt[8]{3^7})$ and $k'=Q(\sqrt[8]{3\times 2^4})$. Then $\zeta_k(s)=\zeta_{k'}(s)$ and $k\not\cong k'$.*

THEOREM 1. *There exist algebraic number fields k and k' such that the zeta-functions $\zeta_k(s)$ and $\zeta_{k'}(s)$ are the same and that the adele rings k_A and k'_A are not isomorphic.*

Proof. Let $k=Q(\sqrt[8]{3^7})$ and $k'=Q(\sqrt[8]{3\times 2^4})$. We note that $k=Q(\sqrt[8]{3})$ and $k'=Q(\sqrt[8]{3}\times\sqrt{2})$. Let η_8 be a primitive 8-th root of 1 and L the minimal Galois extension of Q containing k. Then $L=Q(\sqrt[8]{3},\eta_8)=Q(\sqrt[8]{3},\sqrt{2},\sqrt{-1})$, which proves $L\supset k'$. We put $G=\mathrm{Gal}\,(L/Q)$, $H=\mathrm{Gal}\,(L/k)$, $H'=\mathrm{Gal}\,(L/k')$ and $N=\mathrm{Gal}\,(L/Q(\eta_8))$. Then $H\cong H'\cong Z/2Z\times Z/2Z$, $N\cong Z/8Z$, N is a normal subgroup of G and G is a semi-direct product of H and N. The quadratic number fields which are contained in L are $Q(\sqrt{3})$, $Q(\sqrt{-3})$, $Q(\sqrt{2})$, $Q(\sqrt{-2})$, $Q(\sqrt{6})$, $Q(\sqrt{-6})$ and $Q(\sqrt{-1})$. In none of them, the ideal (2) splits completely. Let $\mathfrak{P}$ be a prime divisor of the ideal (2) in L, D the decomposition group of $\mathfrak{P}$ with respect to L/Q and F the decomposition field of $\mathfrak{P}$ with respect to L/Q. Suppose $G\neq D$. As G is a 2-group, there exists a maximal proper subgroup N_1 of G such that $N_1\supset D$ and that $[G;N_1]=2$. Let k_1 be the subfield of L corresponding to N_1. The ramification index and the degree of the ideal $\mathfrak{P}\cap k_1$ in k_1/Q are equal to 1. Since k_1/Q is a Galois extension, the ideal (2) splits completely in k_1/Q. This is a contradiction. Hence we have $G=D$. Let $L_{\mathfrak{P}}$ be the completion of L by $\mathfrak{P}$-adic valuation. We put $\mathfrak{p}=\mathfrak{P}\cap k$ and $\mathfrak{p}'=\mathfrak{P}\cap k'$. Let K (resp. K') be topological closure of k (resp. k') in $L_{\mathfrak{P}}$. We should notice $K\cong k_{\mathfrak{p}}$ and $K'\cong k'_{\mathfrak{p}'}$. Since $G=D$, there exists a natural isomorphism φ of $\mathrm{Gal}\,(L_{\mathfrak{P}}/Q_2)$ onto G, where Q_2 is the topological closure of Q in $L_{\mathfrak{P}}$. We have $\varphi(\mathrm{Gal}\,(L_{\mathfrak{P}}/K))=H$ and $\varphi(\mathrm{Gal}\,(L_{\mathfrak{P}}/K'))=H'$. Since $k\not\cong k'$ follows from Lemmr 4, H and H' are not conjugate in G, which shows $K\not\cong K'$. Therefore we have $k_{\mathfrak{p}}\not\cong k'_{\mathfrak{p}'}$. Hence $k_A\not\cong k'_A$ follows from Lemma 3. The Lemma 4 also shows $\zeta_k(s)=\zeta_{k'}(s)$. This completes our proof.

2. Let l be an odd prime number, H an elementary abelian l-group of order l^3, i.e. $H\cong(Z/lZ)^3$, and H' a non-commutative group of order l^3 and of exponent l. An existence of H' with the above properties is shown in p. 151 of [9]. We denote by S_{l^3} the permutation group of the set $\{1, 2, \cdots, l^3\}$. Let $H=\{a_1, a_2, \cdots, a_{l^3}\}$. For an element $a\in H$, we define an element $\sigma\in S_{l^3}$ satisfying $a_ia=a_{\sigma(i)}$ for $i=1,\cdots,l^3$: we write this $\varphi(a)$. We call φ the embedding of H into S_{l^3}. We will identify H with its image in S_{l^3} by φ. We adopt similar

notations for H'.

LEMMA 5. *Let S_{l^3}, H and H' be as above and let A_{l^3} be the alternative group of degree l^3 in S_{l^3}. Then* $\mathrm{card}(C(\sigma)\cap H)=\mathrm{card}(C(\sigma)\cap H')$ *for all $\sigma\in S_{l^3}$ and $H\cup H'\subset A_{l^3}$.*

Proof. Let e be the unit element of S_{l^3} and ϕ the empty set. Put $X=\{\rho\in H\cup H' \mid \rho\neq e\}$. From the definition of embedding of H and H' into S_{l^3}, for $\rho\in X$, we have $\rho(i)\neq i$ for $i=1,\cdots,l^3$ and $\rho^l=e$. So a disjoint cycle decomposition of ρ is the product of l^2 disjoint cycles of length l, i.e. $\rho=(i_{11}\cdots i_{1l})\cdots(i_{l^21}\cdots i_{l^2l})$. Hence ρ is an even permutation, $H\cup H'\subset A_{l^3}$, and two elements ρ_1 and ρ_2 of X have the same cycle decompositions. Hence ρ_1 and ρ_2 are conjugate in S_{l^3}. Therefore for any element $\sigma\in S_{l^3}$ with $\sigma\neq e$, the following assertions hold:

(i) $C(\sigma)\cap H=\phi$ if and only if $C(\sigma)\cap H'=\phi$.

(ii) $C(\sigma)\cap H\neq\phi$ if and only if $\mathrm{card}(C(\sigma)\cap H)=\mathrm{card}(C(\sigma)\cap H')=l^3-1$.

(iii) $C(e)\cap H=C(e)\cap H'=\{e\}$.

Hence we conclude $\mathrm{card}(C(\sigma)\cap H)=\mathrm{card}(C(\sigma)\cap H')$ for every element $\sigma\in S_{l^3}$.

Let $\bar{Q}$ be the algebraic closure of Q, let k and k' be algebraic number fields contained in $\bar{Q}$ and let G_k be the absolute Galois group of k, i.e. the Galois group of $\bar{Q}/k$ with Krull topology. We adopt similar notations for k'.

LEMMA 6. *k and k' being as above and let L be a finite Galois extension of Q such that $L\supset k$. If G_k and $G_{k'}$ are isomorphic, then $L\supset k'$ and* $\mathrm{Gal}(L/k)\cong\mathrm{Gal}(L/k')$.

Proof. Let λ be an isomorphism of G_k onto $G_{k'}$. Since $\lambda(G_L)$ is an open subgroup of $G_{k'}$, there exists a finite extension L' of k' such that $\lambda(G_L)=G_{L'}$. From Satz 12 of [6], we have $L=L'$. This shows $L\supset k'$. The isomorphism λ induces an isomorphism of G_k/G_L onto $G_{k'}/G_L$, which proves $\mathrm{Gal}(L/k)\cong\mathrm{Gal}(L/k')$.

Let n ($\geqq 3$) be an integer and S_n the symmetric group of degree n. A finite Galois extension L of Q whose Galois group is isomorphic to S_n is called to be an S_n-extension of Q. Theorem 1 of Part II in [8] shows the following:

LEMMA 7. *Let an integer n and S_n be as above and let A_n be the alternative group of degree n in S_n. Then there exists a quadratic number field F which satisfies the following conditions:*

(i) *There exists an S_n-extension L of Q containing F.*

(ii) $\mathrm{Gal}(L/F)\cong A_n$.

(iii) *All the prime ideals in F are unramified in L/F.*

THEOREM 2. *There exist algebraic number fields k and k' such that k_A and k'_A are isomorphic and that G_k and $G_{k'}$ are not isomorphic, where k_A (resp. k'_A) is the adele ring of k (resp. k') and where G_k (resp. $G_{k'}$) is the absolute Galois group of k (resp. k').*

Proof. Let l be an odd prime number and put $n=l^3$. We take L and F as in Lemma 7. For an isomorphism φ of S_n onto $\mathrm{Gal}(L/Q)$, we notice $\varphi(A_n)=\mathrm{Gal}(L/F)$. Let H and H' be as in Lemma 5 and k (resp. k') the subfield of L corresponding to $\varphi(H)$ (resp. $\varphi(H')$). From Lemma 5 we have $A_n \supset H \cup H'$ and $k \cap k' \supset F$. Lemma 6 and $H \not\cong H$ show $G_k \not\cong G_{k'}$. Let p be a prime number. Suppose p is unramified in F/Q. From Lemma 7, p is unramified in k/Q and in k'/Q. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_g$ be the prime divisors of p in k and put $N_{k/Q}(\mathfrak{P}_i)=p^{f_i}$ for $i=1, \cdots, g$. By Lemma 5 and Lemma 2, there exist distinct prime ideals $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_g$ in k' such that $p=\mathfrak{P}'_1 \cdots \mathfrak{P}'_g$ and that $N_{k'/Q}(\mathfrak{P}'_i)=p^{f_i}$ for $i=1, \cdots, g$. Hence $k_{\mathfrak{P}_i}$ and $k'_{\mathfrak{P}'_i}$ are unramified extensions of Q_p of degree f_i, which shows $k_{\mathfrak{P}_i} \cong k'_{\mathfrak{P}'_i}$. Suppose p is ramified in F/Q. For the prime divisors $\mathfrak{P}_1, \cdots, \mathfrak{P}_g$ of p in k, we have $p=\mathfrak{P}_1^2 \cdots \mathfrak{P}_g^2$ from Lemma 7. Put $N_{k/Q}(\mathfrak{P}_i)=p^{f_i}$. By Lemma 5 and Lemma 2, there exist distinct prime ideals $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_g$ in k' such that $p=\mathfrak{P}'^2_1 \cdots \mathfrak{P}'^2_g$ and $N_{k'/Q}(\mathfrak{P}'_i)=p^{f_i}$ for $i=1, \cdots, g$. We notice $\mathfrak{P}_i \cap F=\mathfrak{P}'_i \cap F$ and we put $\mathfrak{p}=\mathfrak{P}_i \cap F$. $k_{\mathfrak{P}_i}$ and $k'_{\mathfrak{P}'_i}$ are unramified extensions of $F_{\mathfrak{p}}$ of degree f_i. This shows $k_{\mathfrak{P}_i} \cong k'_{\mathfrak{P}'_i}$. Therefore we conclude $k_A \cong k'_A$. This completes our proof.

COROLLARY 1. *There exist algebraic number fields k and k' satisfying the following conditions:*

(i) $k \not\cong k'$.

(ii) $k_A \cong k'_A$.

Proof. It is obvious from that $k \cong k'$ implies $G_k \cong G_{k'}$.

COROLLARY 2. *There exist algebraic number fields k and k' satisfying the following conditions*

(i) $k \not\cong k'$.

(ii) *There exists a bijection Φ of V_k onto $V_{k'}$ such that $k_{\mathfrak{p}} \cong k'_{\Phi(\mathfrak{p})}$ for any place $\mathfrak{p} \in V_k$.*

Proof. It is obvious from Lemma 3.

COROLLARY 3. *There exist algebraic number fields k and k' such that $G_k \not\cong G_{k'}$ and that $\zeta_k(s)=\zeta_{k'}(s)$.*

Proof. It follows from Corollary of Lemma 3.

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