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ON CONFORMALLY FLAT SPACES WITH DEFINITE RICCI CURVATURE II

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1. Introduction. There is a formal similarity between the theory of hypersurfaces and conformally flat d-dimensional spaces of constant scalar curvature provided $d \geqq 3$. For, then, the symmetric linear transformation field Q defined by the Ricci tensor satisfies the "Codazzi equation"

$$(\nabla_X Q)Y = (\nabla_Y Q)X .$$

This observation together with the technique and results in [2] and [3] yields the following statement.

THEOREM. *Let M be a compact conformally flat manifold with definite Ricci curvature. If the scalar curvature r is constant and* $\operatorname{tr} Q^2 \leqq r^2/d-1$, $d \geqq 3$, *then M is a space of constant curvature.*

The corresponding result for hypersurfaces is due to M. Okumura [3].

COROLLARY. *A 3-dimensional compact conformally flat manifold of constant scalar curvature whose sectional curvatures are either all negative or all positive is a space of constant curvature.*

Note that, in general $\operatorname{tr} Q^2 \geqq r^2/d$ with equality, if and only if, M is an Einstein space.

Examples of compact negatively curved space forms are given in the paper by A. Borel [1].

2. Definitions and formulas. Let (M, g) be a Riemannian manifold with metric tensor g. The curvature transformation $R(X, Y)$, $X, Y \in M_m$— the tangent space at $m \in M$, and g are related by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] ,$$

where ∇_X is the operation of covariant differentiation with respect to X defined in terms of the Levi-Civita connection. In terms of a basis $X_1, \cdots, X_d$ of M_m we set

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$$R_{ijkh}=g(R(X_i, X_j)X_k, X_h)\,,$$
$$R_{ij}=\mathrm{tr}\,(X_k \rightarrow R(X_i, X_k)X_j)\,,$$
$$t_{i_1\cdots i_p}=t(X_{i_1}, \cdots, X_{i_p})\,,$$
$$\nabla_i t_{i_1\cdots i_p}=(\nabla_{X_i} t)(X_{i_1}, \cdots, X_{i_p})\,.$$

We denote the scalar curvature by r, that is $r=\mathrm{tr}\,Q$, where $Q=(R^i{}_j)$, $R^i{}_j=g^{ik}R_{jk}$. The manifold (M, g) is *conformally flat* if g is conformally related to a locally flat metric. The *Weyl conformal curvature tensor* defined by

$$C^i{}_{jkh}=R^i{}_{jkh}-\frac{1}{d-2}(R_{jk}\delta^i{}_h-R_{jh}\delta^i{}_k+g_{jk}R^i{}_h-g_{jh}R^i{}_k)+\frac{r}{(d-1)(d-2)}(g_{jk}\delta^i{}_h-g_{jh}\delta^i{}_k) \tag{2.1}$$

consequently vanishes, so if (M, g) is conformally flat

$$R^i{}_{jkh}=\frac{1}{d-2}(R_{jk}\delta^i{}_h-R_{jh}\delta^i{}_k+g_{jk}R^i{}_h-g_{jh}R^i{}_k)-\frac{r}{(d-1)(d-2)}(g_{jk}\delta^i{}_h-g_{jh}\delta^i{}_k)\,. \tag{2.2}$$

From (2.1) and the second Bianchi identity

$$\nabla_i C^i{}_{jkh}=(d-3)C_{jkh}\,, \tag{2.3}$$

where

$$C_{jkh}=\frac{1}{d-2}(\nabla_h R_{jk}-\nabla_k R_{jh})-\frac{1}{2(d-1)(d-2)}(g_{jk}\nabla_h r-g_{jh}\nabla_k r)\,. \tag{2.4}$$

For $d=3$ it can be shown that if (M, g) is conformally flat, then $C_{ijk}=0$.

3. The Laplacian of the square length of the Ricci tensor. The following formula may be found in [2]:

$$\frac{1}{2}\Delta\,\mathrm{tr}\,Q^2=g^{ab}\nabla_a R^{ij}\nabla_b R_{ij}+R^{ij}g^{ab}\nabla_a(\nabla_b R_{ij}-\nabla_i R_{bj})+\frac{1}{2}R^{ij}\nabla_i\nabla_j r+K\,, \tag{3.1}$$

where $\mathrm{tr}\,Q^2=R^{ij}R_{ij}$ and

$$K=R^{ik}(R^j{}_i R_{jk}+R^{hj}R_{ijhk})\,. \tag{3.2}$$

If r=const., the third term on the *r. h. s.* of (3.1) vanishes. If, furthermore, M is conformally flat and $d\geqq 3$, then from (2.3) and (2.4), the second term on the *r. h. s.* of (3.1) also vanishes. Substituting (2.2) into the *r. h. s.* of (3.2), we obtain

$$\frac{1}{2}\Delta \operatorname{tr} Q^2 = K + g(\nabla Q, \nabla Q)\,,$$

where

(3.3) $$(d-1)(d-2)K = d(d-1)\operatorname{tr} Q^3 - r(2d-1)\operatorname{tr} Q^2 + r^3\,.$$

4. Proof of Theorem. Put

$$S = Q - \frac{r}{d} I\,,$$

where I is the identity. Since $\operatorname{tr} S^2 \geqq 0$,

$$\operatorname{tr} Q^2 \geqq \frac{r^2}{d}\,,$$

equality holding if and only if M is an Einstein space. Since the scalar curvature is constant, the Laplacian Δf^2 of the function $f^2 = \operatorname{tr} S^2$, $f \geqq 0$, satisfies

$$\Delta f^2 = \Delta \operatorname{tr} Q^2\,,$$

so that

(4.1) $$\frac{1}{2}\Delta f^2 = K + g(\nabla Q, \nabla Q)\,.$$

From the definition of S, we get

(4.2) $$\operatorname{tr} S = 0\,,$$

(4.3) $$\operatorname{tr} Q^2 = \operatorname{tr} S^2 + \frac{r^2}{d}\,,$$

(4.4) $$\operatorname{tr} Q^3 = \operatorname{tr} S^3 + \frac{3r}{d}\operatorname{tr} S^2 + \frac{r^3}{d^2}\,.$$

Substituting (4.3) and (4.4) in (3.3), we obtain

(4.5) $$(d-1)(d-2)K = d(d-1)\left(\operatorname{tr} S^3 + \frac{3r}{d} f^2 + \frac{r^3}{d^2}\right) - r(2d-1)\left(f^2 + \frac{r^2}{d}\right) + r^3\,.$$

LEMMA. *Let a_i, $i=1, \cdots, d$ be real numbers such that*

$$\sum_{i=1}^{d} a_i = 0\,, \qquad \sum_{i=1}^{d} a_i^2 = k^2\,, \qquad k = \text{const.} \geqq 0\,.$$

Then,

$$-\frac{d-2}{\sqrt{d(d-1)}}\,k^3 \leqq \sum_{i=1}^{d} a_i^3 \leqq \frac{d-2}{\sqrt{d(d-1)}}\,k^3\,.$$

Applying the lemma to the eigenvalues of S, (4.5) yields the following inequality

$$(d-1)K \geqq f^2(r - \sqrt{d(d-1)}\,f)\,.$$

Thus, since $f \leqq r/\sqrt{d(d-1)}$, $\Delta f^2 \geqq 0$, from which since M is compact, $f^2=$const., so $\operatorname{tr} Q^2=$const. It follows from (3.1) that $\nabla Q=0$. Theorem 1 of [2] then gives the desired result.

In case the sectional curvatures are all positive the corollary is due to M. Tani [4].

The condition that the Ricci tensor is definite is essential. For, if $M=M_1 \times N$ where M_1 has constant curvature and N is 1-dimensional, then M is conformally flat, r is constant, and $\operatorname{tr} Q^2=r^2/d-1$.

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